Towards an Optimized Hydrodynamic Theory for Heavy-Ion Collisions

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In collaboration with D. Bazow, G. Denicol, M. Martinez, J. Noronha, M. Strickland

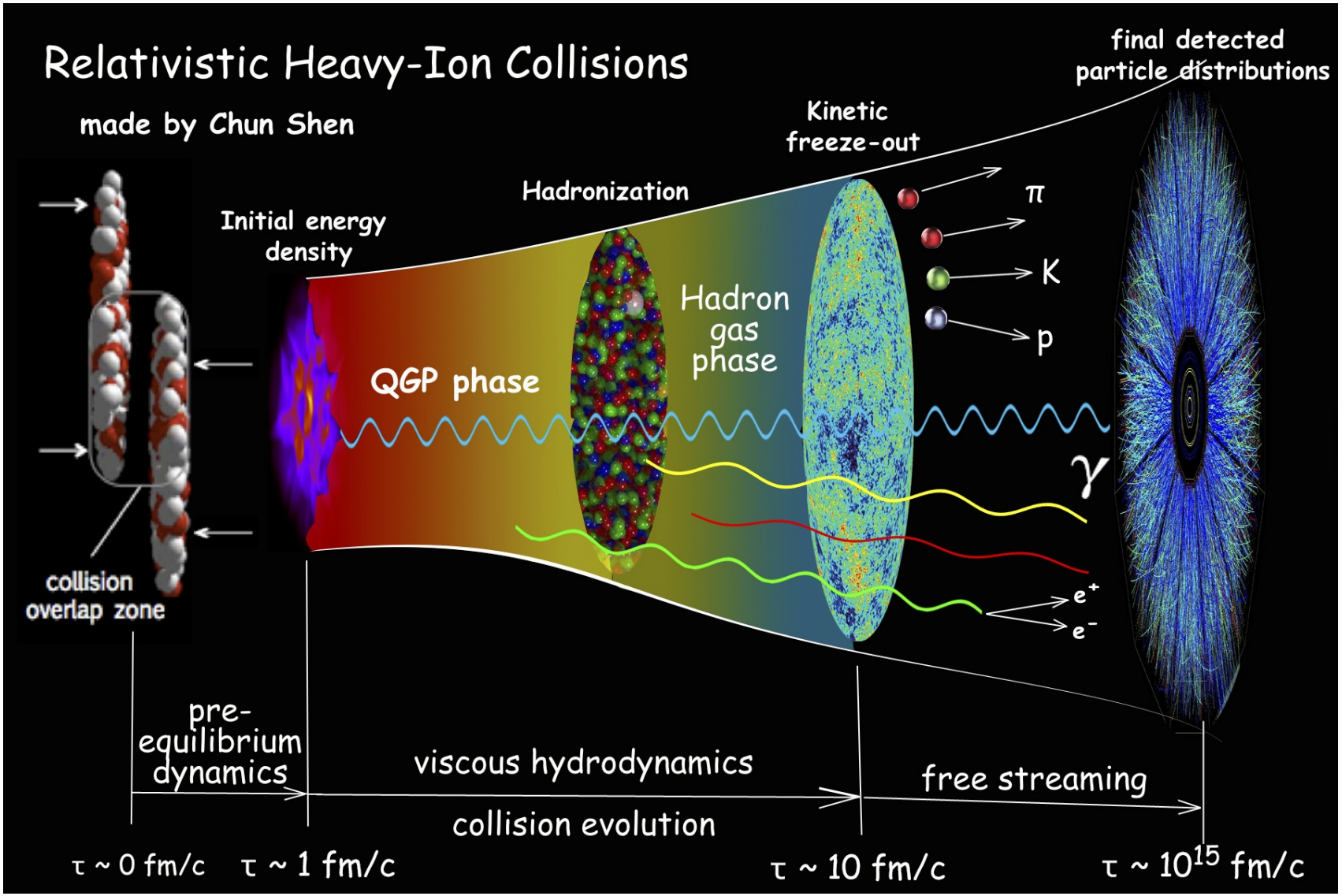
References:

Bazow, UH, Strickland, PRC 90 ('14) 054910;
Denicol, UH, Martinez, Noronha, Strickland, PRL 113 ('14) 202301; PRD 90 ('14) 125026;
Bazow, UH, Martinez, PRC 91 ('15) 064903;
UH, Martinez, NPA 943 ('15) 26;
Nopoush, Strickland, Ryblewski, Bazow, UH, Martinez, PRC 92 ('15) 044912;
Bazow, Denicol, UH, Martinez, Noronha, PRL 116 ('16) 022301; arXiv:1607.05245.

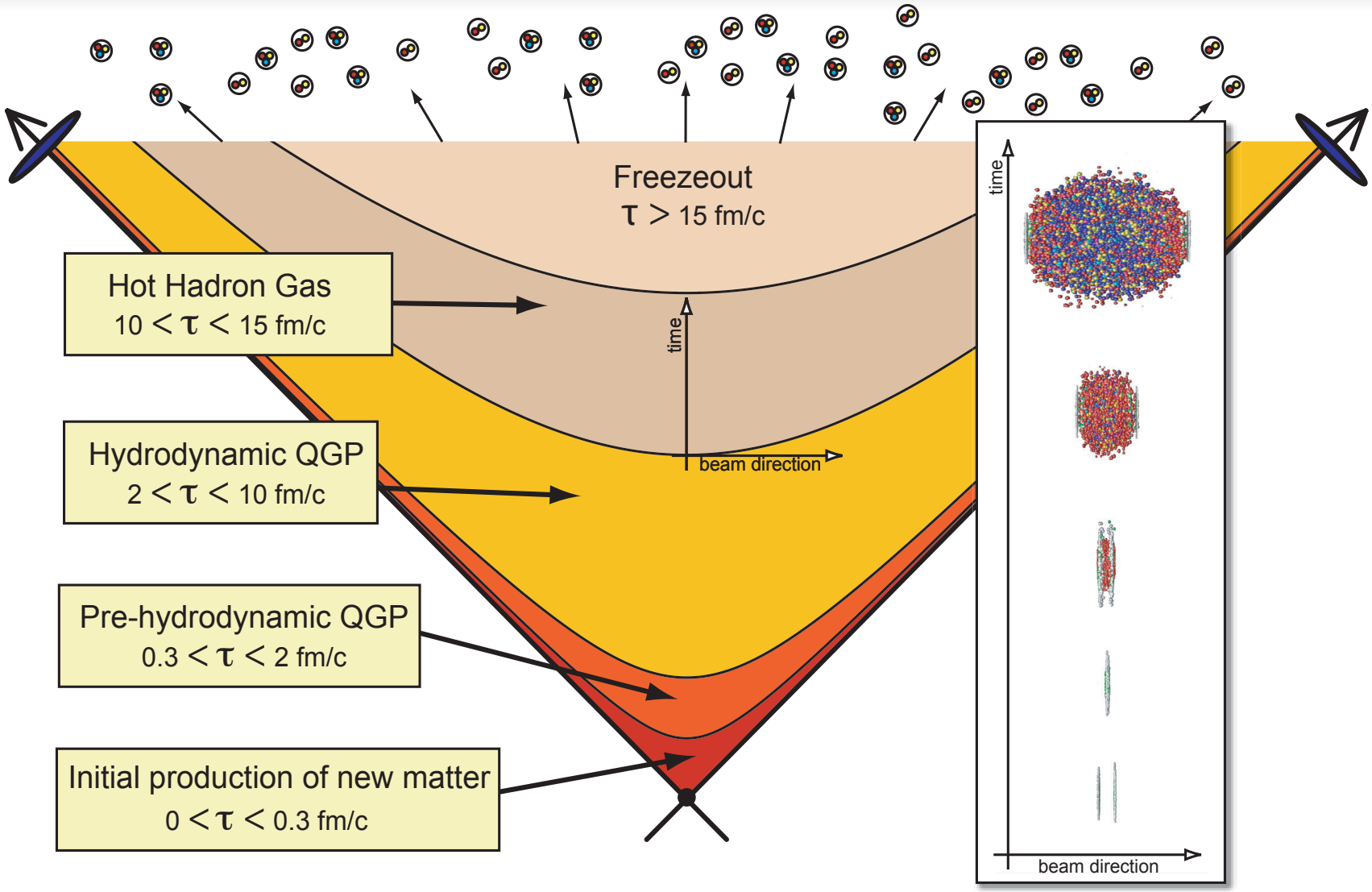
Relativistic Hydrodynamics: Theory and Modern Applications, MITP, Oct. 10-14, 2016

Overview

The Little Bang (credit: Paul Sorensen/Chun Shen)



Prologue



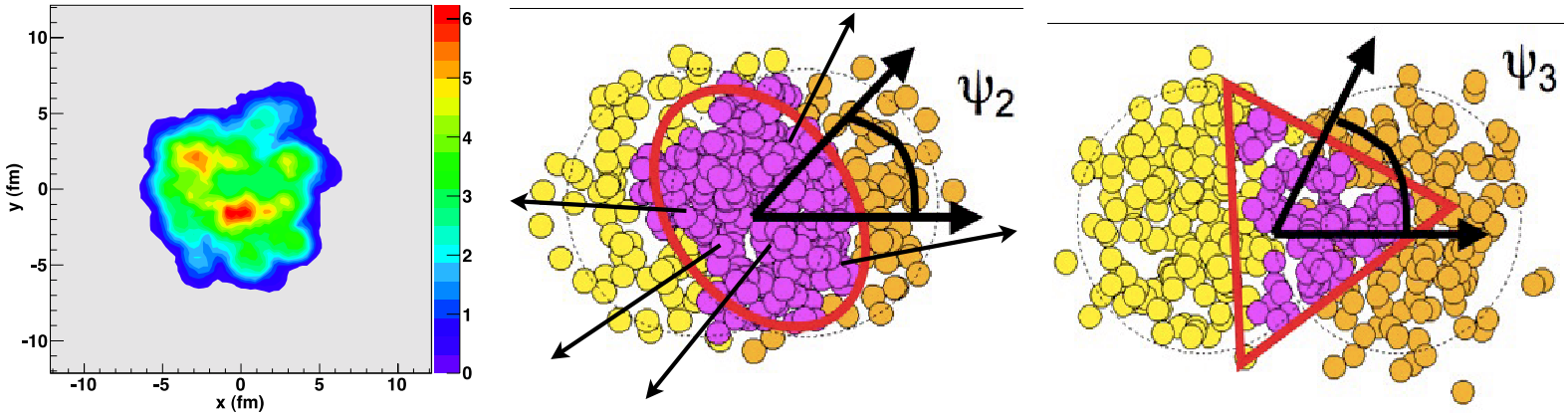
(After M. Strickland, arXiv:1410.5786)

Prologue

- Relativistic viscous hydrodynamics has become the workhorse of dynamical modeling of ultra-relativistic heavy-ion collisions

Event-by-event shape and flow fluctuations rule!

(Alver and Roland, PRC81 (2010) 054905)

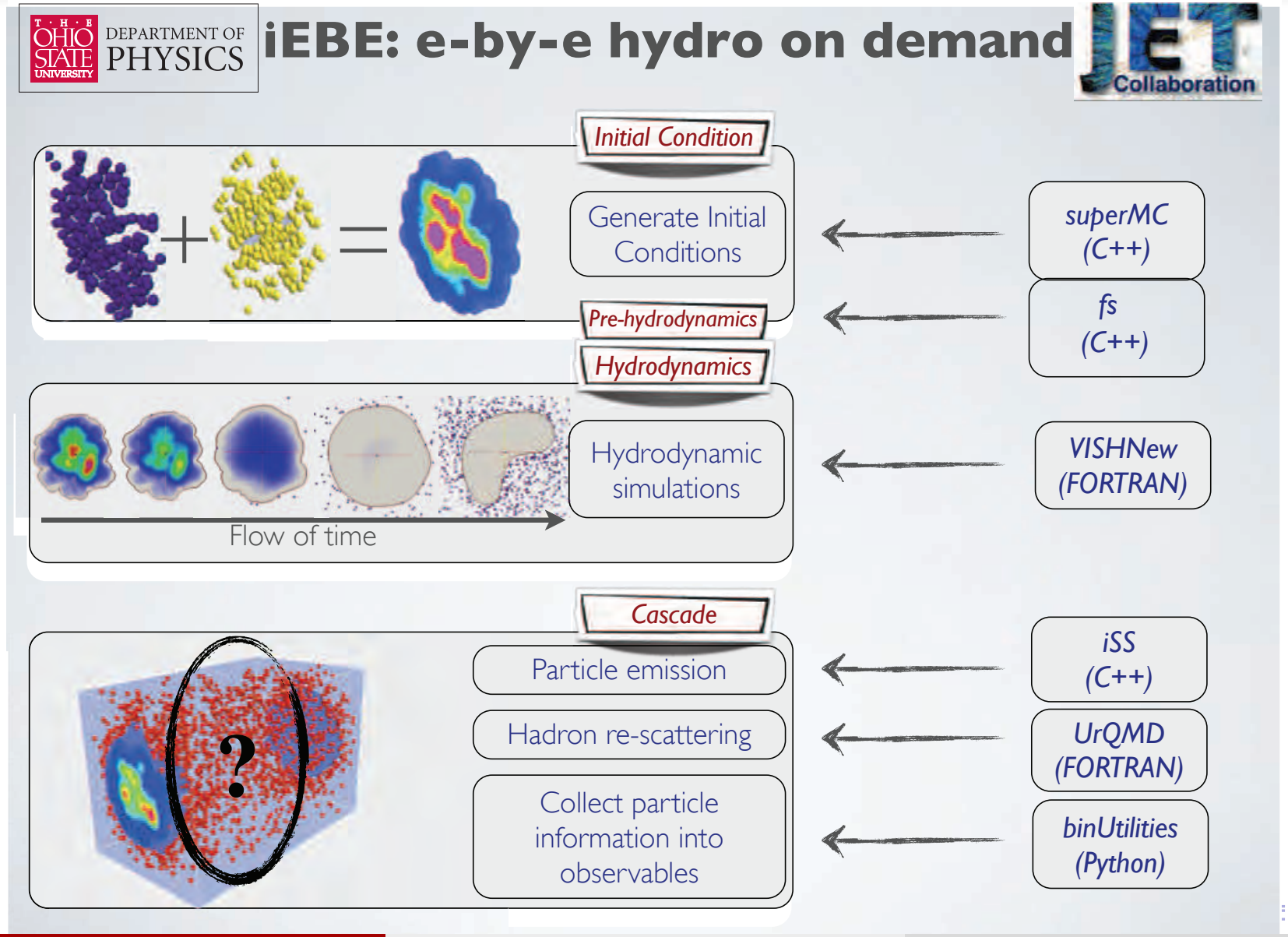


- Each event has a different initial shape and density distribution, characterized by different set of harmonic eccentricity coefficients ε_n
- Each event develops its individual hydrodynamic flow, characterized by a set of harmonic flow coefficients v_n and flow angles ψ_n
- At small impact parameters fluctuations ("hot spots") dominate over geometric overlap effects (Alver & Roland, PRC81 (2010) 054905; Qin, Petersen, Bass, Müller, PRC82 (2010) 064903)

Definition of flow coefficients:

$$\frac{dN^{(i)}}{dy\, p_T dp_T\, d\phi_p}(b) = \frac{dN^{(i)}}{dy\, p_T dp_T}(b) \left(1 + 2\sum_{n=1}^{\infty} \boldsymbol{v_n^{(i)}(y, p_T; b)} \cos\left(n(\phi_p - \Psi_n^{(i)})\right)\right)$$

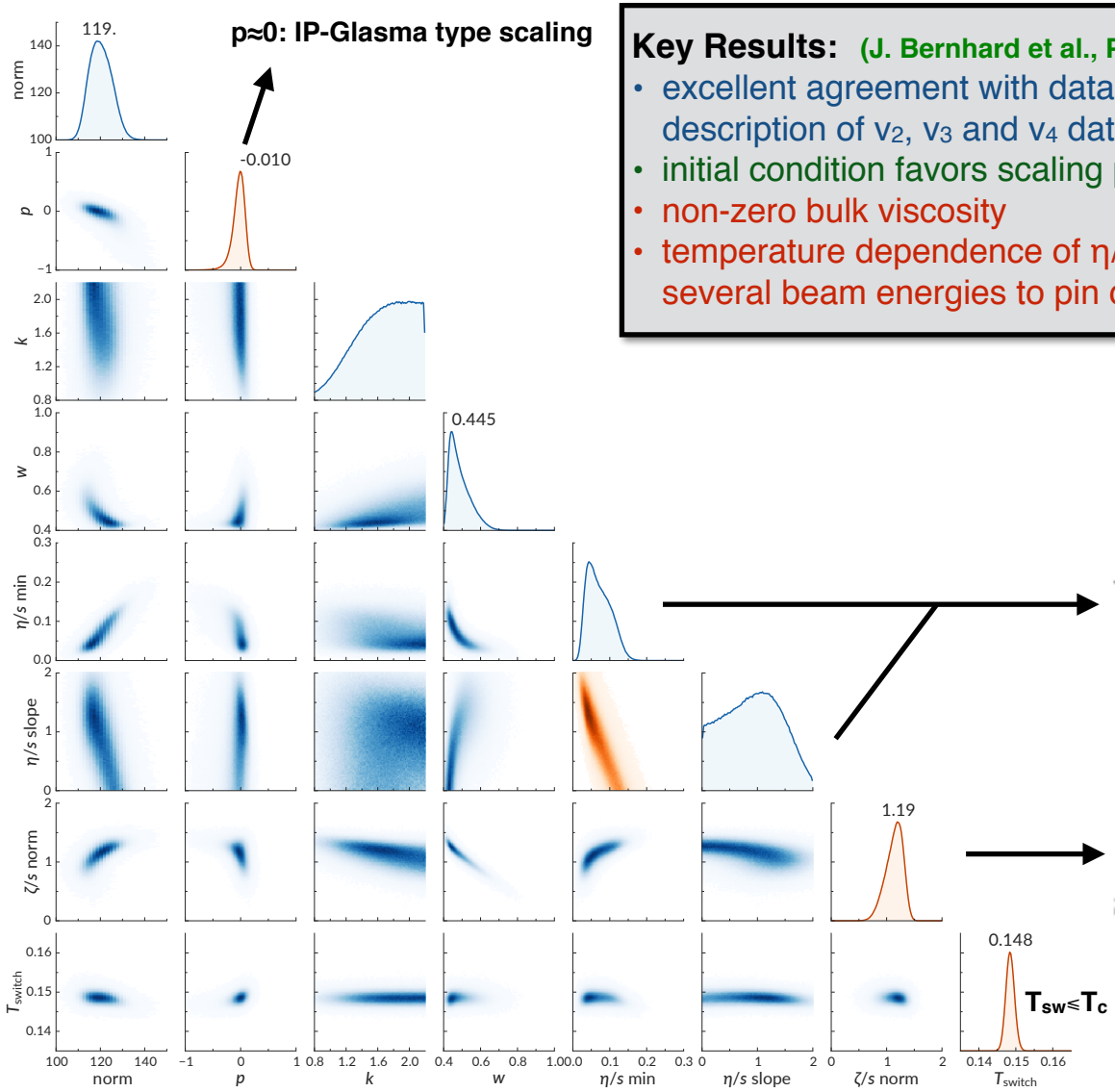
https://u.osu.edu/vishnu: A product of the JET Collaboration



Prologue

- Relativistic viscous hydrodynamics has become the workhorse of dynamical modeling of ultra-relativistic heavy-ion collisions
- It has been successfully used in a Bayesian analysis of LHC Pb+Pb collision data for putting meaningful constraints on the initial conditions and medium properties of QGP created in heavy-ion collisions:

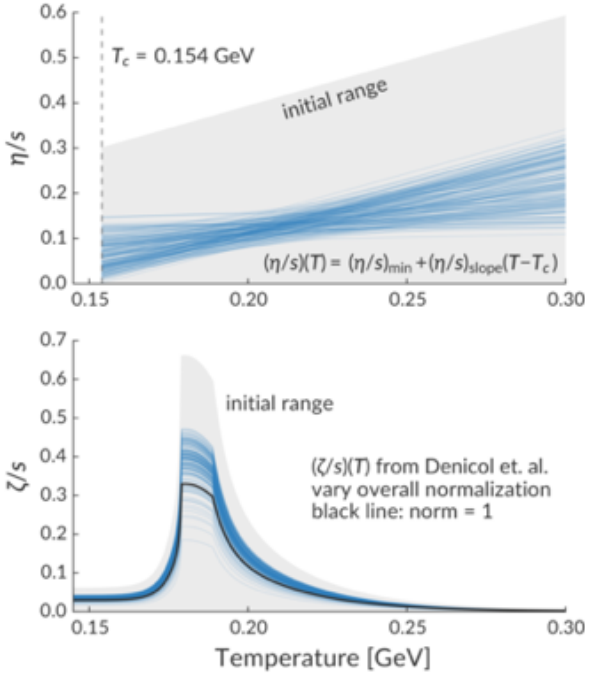
Calibrated Posterior Distribution



Key Results: (J. Bernhard et al., PRC94 (2016) 024907)

- excellent agreement with data, simultaneous description of v_2, v_3 and v_4 data
- initial condition favors scaling properties of IP-Glasma
- non-zero bulk viscosity
- temperature dependence of η/s requires data at several beam energies to pin down

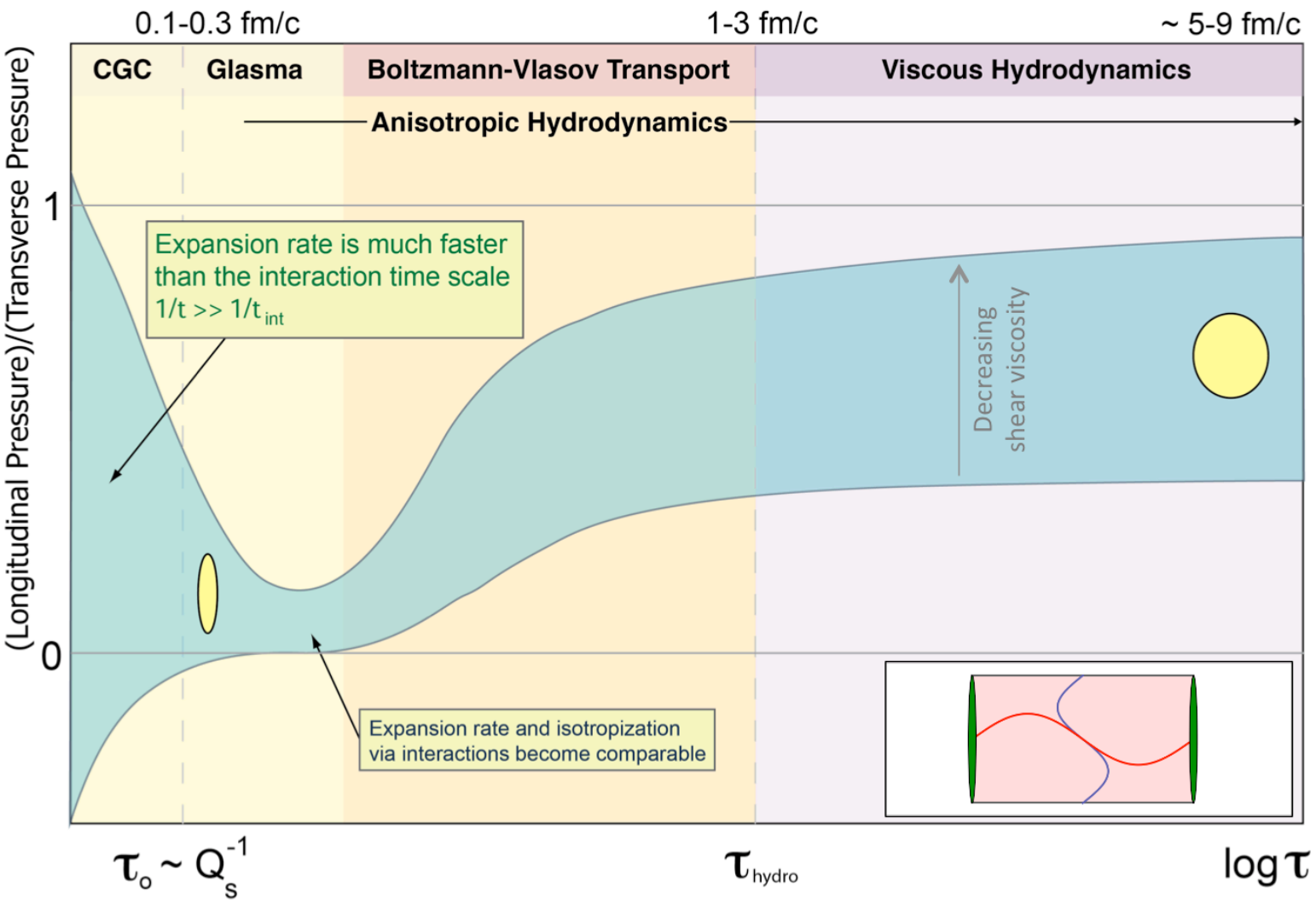
temperature-dependent viscosities from the calibrated posterior:



Prologue

- Relativistic viscous hydrodynamics has become the workhorse of dynamical modeling of ultra-relativistic heavy-ion collisions
- However, the kinematics of ultra-relativistic heavy-ion collisions introduces a complication that severely limits the applicability of standard viscous relativistic fluid dynamics:
  large viscous stresses caused by large initial anisotropies between the longitudinal and transverse expansion rates

Strong initial longitudinal-transverse pressure anisotropies



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- Exact solutions have been found for weakly interacting systems with highly symmetric flow patterns and density distributions:
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  FLRW universe (full Boltzmann collision term)
- Can be used to test different hydrodynamic expansion schemes for the Boltzmann equation

Overview

Kinetic theory vs. hydrodynamics

Both simultaneously valid if weakly coupled and small pressure gradients.

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Boltzmann Equation in Relaxation Time Approximation (RTA):

$$p^\mu \partial_\mu f(x,p) = C(x,p) = \frac{p\cdot u(x)}{\tau_{\rm rel}(x)}\Big(f_{\rm eq}(x,p) - f(x,p)\Big)$$

For conformal systems $\tau_{\rm rel}(x) = c/T(x) = 5\eta/(\mathcal{S}T) \equiv 5\bar{\eta}/T(x)$.

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Macroscopic currents:

$$j^\mu(x) = \int_p p^\mu\, f(x,p) \equiv \langle p^\mu \rangle; \quad T^{\mu\nu}(x) = \int_p p^\mu\, p^\nu\, f(x,p) \equiv \langle p^\mu p^\nu \rangle$$

where $$\int_p \cdots \equiv \frac{g}{(2\pi)^3} \int \frac{d^3p}{E_p} \cdots \equiv \langle \ldots \rangle$$

Hydrodynamics from kinetic theory (I):

Expand the solution $f(x,p)$ of the Boltzmann equation as

$$f(x,p) = f_0(x,p) + \delta f(x,p) \qquad \left(\left|\delta f / f_0\right| \ll 1\right)$$

where f_0 is parametrized through macroscopic observables as

$$f_0(x,p) = f_0\left(\frac{\sqrt{p_\mu \Xi^{\mu\nu}(x) p_\nu} - \tilde{\mu}(x)}{\tilde{T}(x)}\right)$$

where $\quad \Xi^{\mu\nu}(x) = u^\mu(x) u^\nu(x) - \Phi(x)\Delta^{\mu\nu}(x) + \xi^{\mu\nu}(x).$

$u^\mu(x)$ defines the local fluid rest frame (LRF).
$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the spatial projector in the LRF.
$\tilde{T}(x)$, $\tilde{\mu}(x)$ are the effective temperature and chem. potential in the LRF.
$\Phi(x)$ partially accounts for bulk viscous effects in expanding systems.
$\xi^{\mu\nu}(x)$ describes deviations from local momentum isotropy in anisotropically expanding systems due to shear viscosity.

Hydrodynamics from kinetic theory (II):

$u^\mu(x)$, $\tilde{T}(x)$, $\tilde{\mu}(x)$ are fixed by the Landau matching conditions:

$$T^\mu_{\ \nu} u^\nu = \mathcal{E}(\tilde{T}, \tilde{\mu}; \xi, \Phi) u^\mu; \qquad \left\langle u{\cdot}p \right\rangle_{\delta f} = \left\langle (u{\cdot}p)^2 \right\rangle_{\delta f} = 0$$

$\mathcal{E}$ is the LRF energy density. We introduce the true local temperature $T(\tilde{T}, \tilde{\mu}; \xi, \Phi)$ and chemical potential $\mu(\tilde{T}, \tilde{\mu}; \xi, \Phi)$ by demanding $\mathcal{E}(\tilde{T}, \tilde{\mu}; \xi, \Phi) = \mathcal{E}_{\rm eq}(T, \mu)$ and $\mathcal{N}(\tilde{T}, \tilde{\mu}; \xi, \Phi) \equiv \langle u{\cdot}p \rangle_{f_0} = \mathcal{R}_0(\xi, \Phi) \mathcal{N}_{\rm eq}(T, \mu)$ (see cited literature for $\mathcal{R}$ functions).
Writing

$$T^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu} \equiv T_0^{\mu\nu} + \Pi^{\mu\nu}, \qquad j^\mu = j_0^\mu + \delta j^\mu \equiv j_0^\mu + V^\mu,$$

the conservation laws

$$\partial_\mu T^{\mu\nu}(x) = 0, \qquad \partial_\mu j^\mu(x) = \frac{\mathcal{N}(x) - \mathcal{N}_{\rm eq}(x)}{\tau_{\rm rel}(x)}$$

are sufficient to determine $u^\mu(x)$, $T(x)$, $\mu(x)$, but not the dissipative corrections $\xi^{\mu\nu}$, Φ, $\Pi^{\mu\nu}$, and V^μ whose evolution is controlled by microscopic physics.

Hydrodynamics from kinetic theory (III):

Different hydrodynamic approaches can be characterized by the different assumptions they make about the dissipative corrections and/or the different approximations they use to derive their dynamics from the underlying Boltzmann equation:

- **Ideal hydro:** local momentum isotropy ($\xi^{\mu\nu} = 0$), $\Phi = \Pi^{\mu\nu} = V^{\mu} = 0$.

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- **Viscous anisotropic hydrodynamics (vaHydro):** improved **aHydro** that additionally evolves residual dissipative flows $\Pi^{\mu\nu}$, V^μ with **IS** or **DNMR theory**.

Overview

BE for systems with highly symmetric flows: I. Bjorken flow

- Longitudinal boost invariance, transverse homogeneity ("physics on the light cone", no transverse flow) $\Longrightarrow$ $\boldsymbol{u^\mu = (1,0,0,0)}$ in Milne coordinates (τ, r, ϕ, η) where $\tau = (t^2 - z^2)^{1/2}$ and $\eta = \frac{1}{2}\ln[(t-z)/(t+z)]$ $\Longrightarrow$ $\boldsymbol{v_z = z/t}$

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- Metric: $ds^2 = d\tau^2 - dr^2 - r^2 d\phi^2 - \tau^2 d\eta^2$, $\quad g_{\mu\nu} = \mathrm{diag}(1, -1, -r^2, -\tau^2)$

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- Symmetry restricts possible dependence of distribution function $f(x,p)$ (Baym '84, Florkowski et al. '13, '14):

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- RTA BE simplifies to ordinary differential equation

$$\partial_\tau f(\tau; p_\perp, w) = -\frac{f(\tau; p_\perp, w) - f_{\mathrm{eq}}(\tau; p_\perp, w)}{\tau_{\mathrm{rel}}(\tau)}.$$

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- **Solution:**

$$f(\tau; p_\perp, w) = D(\tau, \tau_0) f_0(p_\perp, w) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm rel}(\tau')} D(\tau, \tau') f_{\rm eq}(\tau'; p_\perp, w)$$

where $\quad D(\tau_2, \tau_1) = \exp\left(-\int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_{\rm rel}(\tau'')}\right).$

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  $\Longrightarrow$ $\boldsymbol{u^\mu = (1,0,0,0)}$ in de Sitter coordinates $(\rho, \theta, \phi, \eta)$ where
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- With $T(\tau, r) = \hat{T}(\rho(\tau, r))/\tau$ RTA BE simplifies to the ODE

$$\frac{\partial}{\partial\rho} f(\rho; \hat{p}_\Omega^2, \hat{p}_\varsigma) = -\frac{\hat{T}(\rho)}{c}\left[f\left(\rho; \hat{p}_\Omega^2, \hat{p}_\varsigma\right) - f_{\mathrm{eq}}\left(\hat{p}^\rho/\hat{T}(\rho)\right)\right].$$

BE for systems with highly symmetric flows: II. Gubser flow

- Longitudinal boost invariance, azimuthally symmetric radial dependence ("physics on the light cone" with azimuthally symmetric transverse flow) (Gubser '10, Gubser & Yarom '11)
  $\Longrightarrow$ $\boldsymbol{u^\mu = (1,0,0,0)}$ in de Sitter coordinates (ρ,θ,ϕ,η) where
  $\rho(\tau,r) = -\sinh^{-1}\left(\frac{1-q^2\tau^2+q^2r^2}{2q\tau}\right)$ and $\theta(\tau,r) = \tan^{-1}\left(\frac{2qr}{1+q^2\tau^2-q^2r^2}\right)$.
  $\Longrightarrow$ $\boldsymbol{v_z = z/t}$ and $\boldsymbol{v_r = \frac{2q^2\tau r}{1+q^2\tau^2+q^2r^2}}$ where q is an arbitrary scale parameter.
- Metric: $d\hat{s}^2 = ds^2/\tau^2 = d\rho^2 - \cosh^2\rho\,(d\theta^2 + \sin^2\theta\, d\phi^2) - d\eta^2$,
  $g_{\mu\nu} = \mathrm{diag}(1, -\cosh^2\rho, -\cosh^2\rho\,\sin^2\theta, -1)$
- Symmetry restricts possible dependence of distribution function $f(x,p)$

$$f(x,p) = f(\rho;\hat{p}_\Omega^2,\hat{p}_\eta) \quad \text{where} \quad \hat{p}_\Omega^2 = \hat{p}_\theta^2 + \frac{\hat{p}_\phi^2}{\sin^2\theta} \quad \text{and} \quad \hat{p}_\eta = w.$$

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- **Solution:**
  $f(\rho;\hat{p}_\Omega^2,w) = D(\rho,\rho_0) f_0(\hat{p}_\Omega^2,w) + \frac{1}{c}\int_{\rho_0}^{\rho} d\rho'\,\hat{T}(\rho')\,D(\rho,\rho')\,f_{\mathrm{eq}}(\rho';\hat{p}_\Omega^2,w)$

Hydrodynamic equations for systems with Gubser flow*:

- The exact solution for f can be worked out for any "initial" condition $f_0(\hat{p}_\Omega^2, w) \equiv f(\rho_0; \hat{p}_\Omega^2, w)$. We here use equilibrium initial conditions, $f_0 = f_{\rm eq}$.

*For Bjorken flow, including **(0+1)-d vaHydro**, see UH@QM14

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- By taking hydrodynamic moments, the exact f yields the exact evolution of all components of $T^{\mu\nu}$. Here, $\Pi^{\mu\nu}$ has only one independent component, $\pi^{\eta\eta}$.
- This exact solution of the BE can be compared to solutions of the various hydrodynamic equations in de Sitter coordinates, using identical initial conditions.
  - **Ideal:** $\hat{T}_{\rm ideal}(\rho) = \frac{\hat{T}_0}{\cosh^{2/3}(\rho)}$
  - **NS:** $\frac{1}{\hat{T}}\frac{d\hat{T}}{d\rho} + \frac{2}{3}\tanh\rho = \frac{1}{3}\bar{\pi}^\eta_\eta(\rho)\tanh\rho$ (viscous T-evolution)
    with $\bar{\pi}^\eta_\eta \equiv \hat{\pi}^\eta_\eta/(\hat{T}\hat{\mathcal{S}})$ and $\hat{\pi}^{\eta\eta}_{NS} = \frac{4}{3}\hat{\eta}\tanh\rho$ where $\frac{\hat{\eta}}{\hat{\mathcal{S}}} \equiv \bar{\eta} = \frac{1}{5}\hat{T}\hat{\tau}_{\rm rel}$
  - **IS:** $\frac{d\bar{\pi}^\eta_\eta}{d\rho} + \frac{4}{3}\left(\bar{\pi}^\eta_\eta\right)^2\tanh\rho + \frac{\bar{\pi}^\eta_\eta}{\hat{\tau}_{\rm rel}} = \frac{4}{15}\tanh\rho$
  - **DNMR:** $\frac{d\bar{\pi}^\eta_\eta}{d\rho} + \frac{4}{3}\left(\bar{\pi}^\eta_\eta\right)^2\tanh\rho + \frac{\bar{\pi}^\eta_\eta}{\hat{\tau}_{\rm rel}} = \frac{4}{15}\tanh\rho + \frac{10}{21}\bar{\pi}^\eta_\eta\tanh\rho$
  - **aHydro:** see M. Nopoush et al., PRD 91 (2015) 045007
  - **vaHydro:** in preparation – stay tuned!

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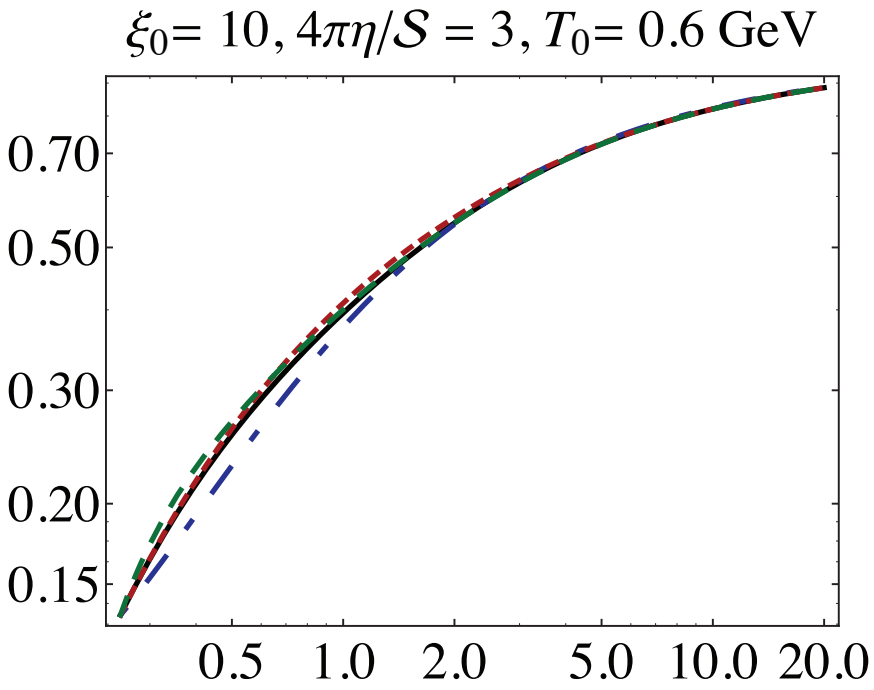
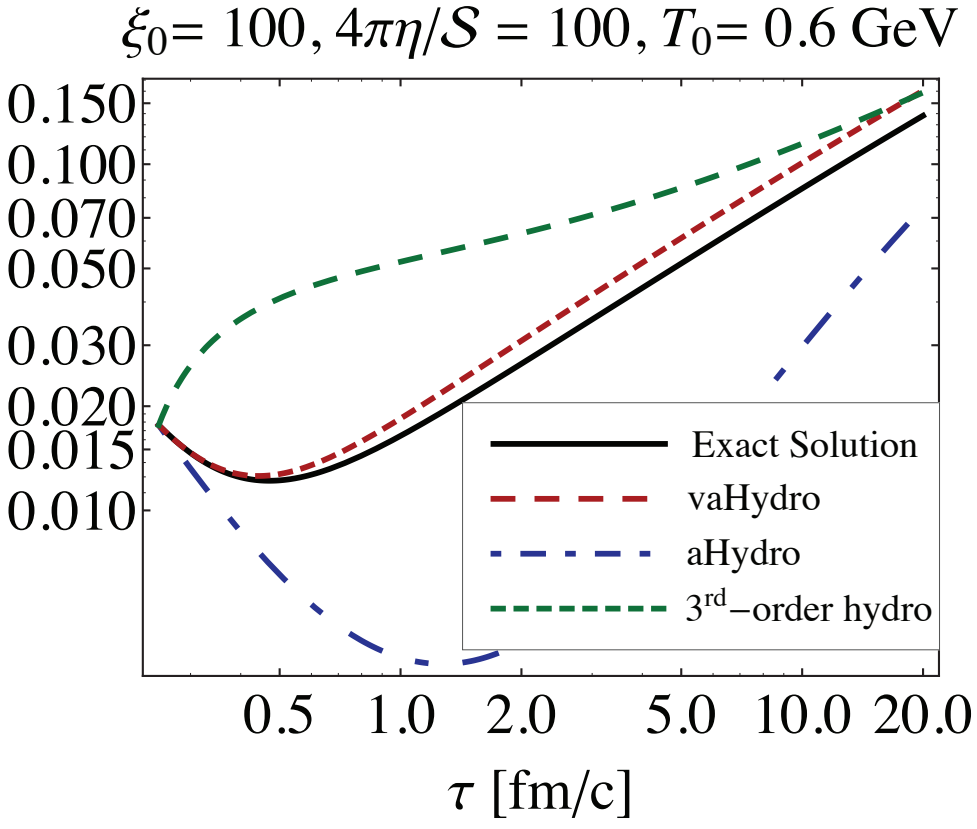
Overview

Bjorken flow (I)

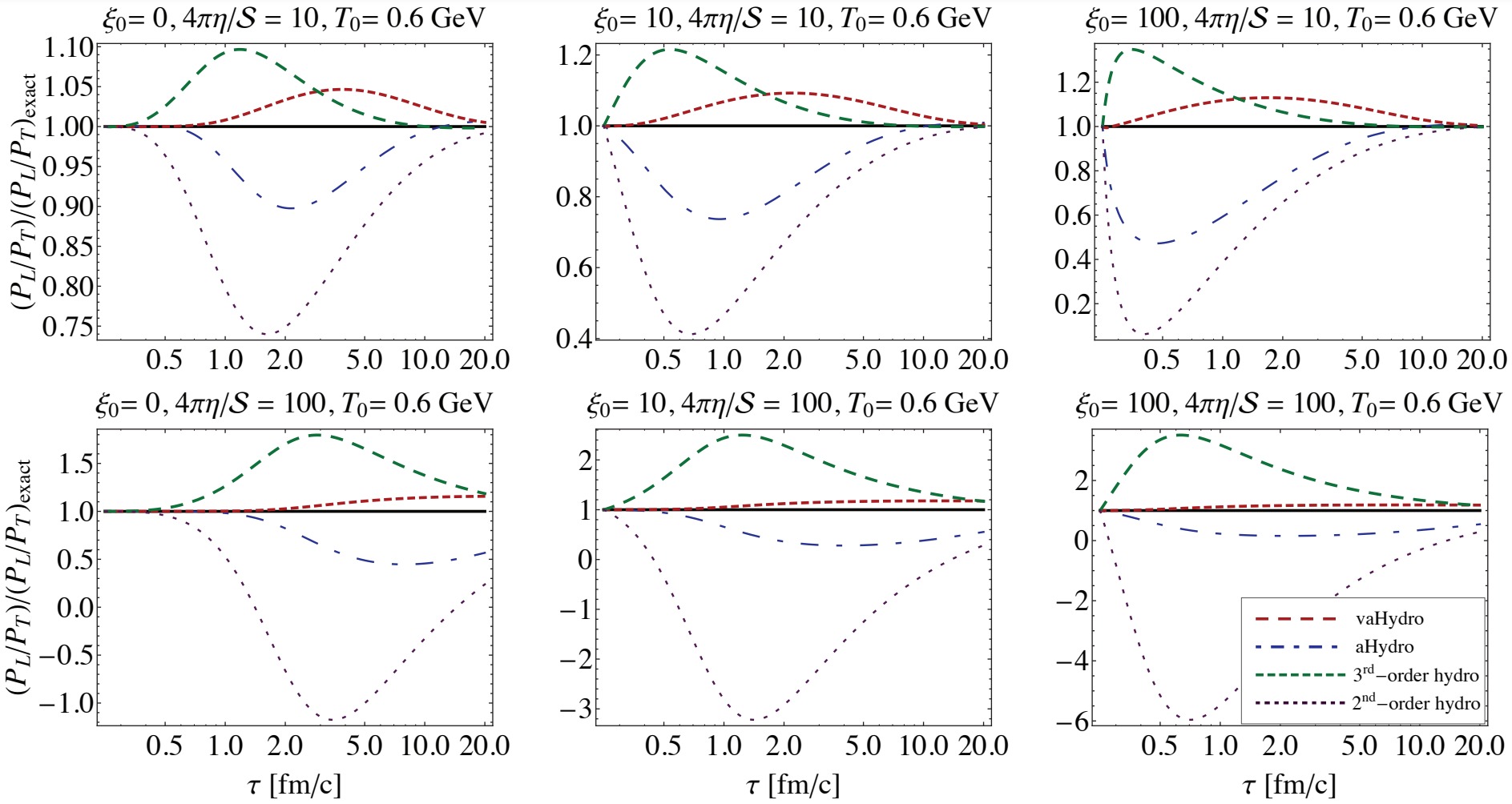
Pressure anisotropy P_L/P_T vs. τ:





In the right plot, IS theory yields negative $P_L/P_T < 0$!

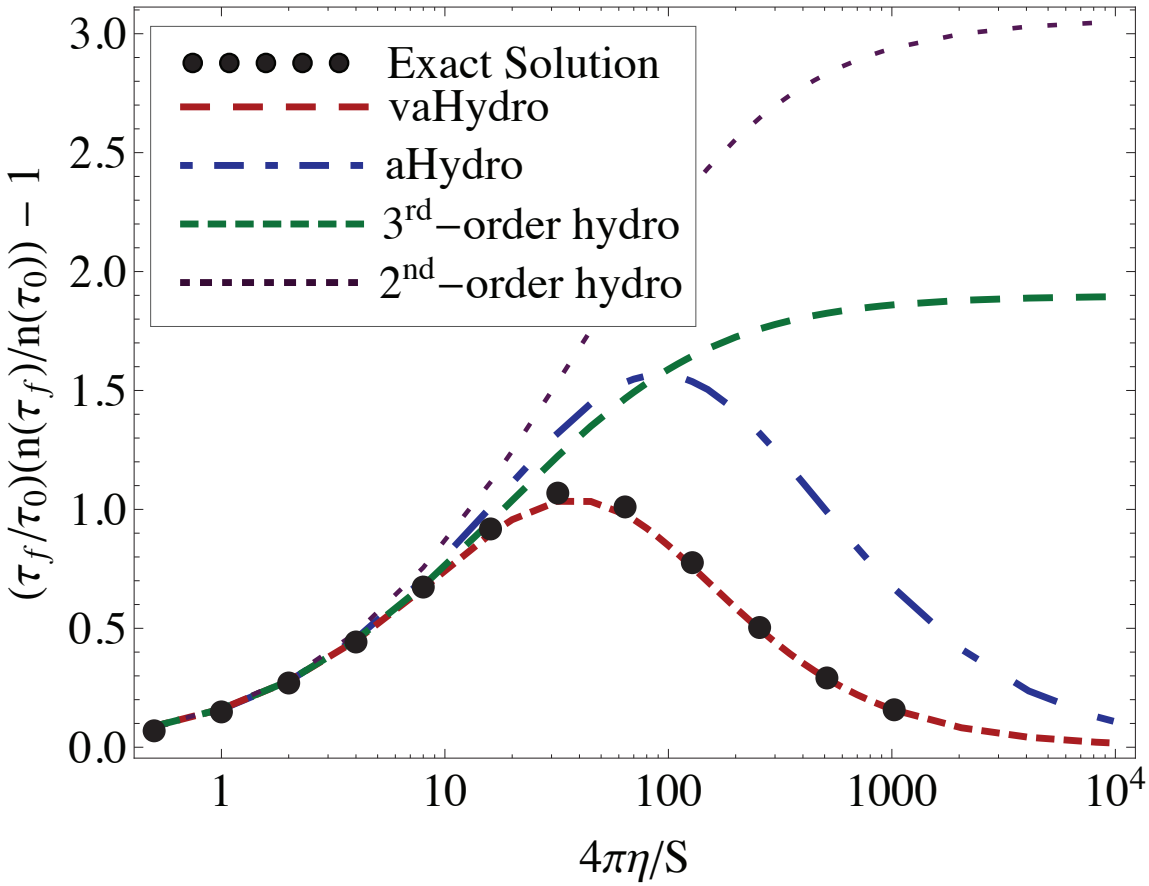
Bjorken flow (II)



vaHydro agrees within a few % with exact result, even for very large $\eta/\mathcal{S}$!

Bjorken flow (III)

Total entropy (particle) production $\frac{n(\tau_f)\cdot\tau_f}{n(\tau_0)\cdot\tau_0} - 1$



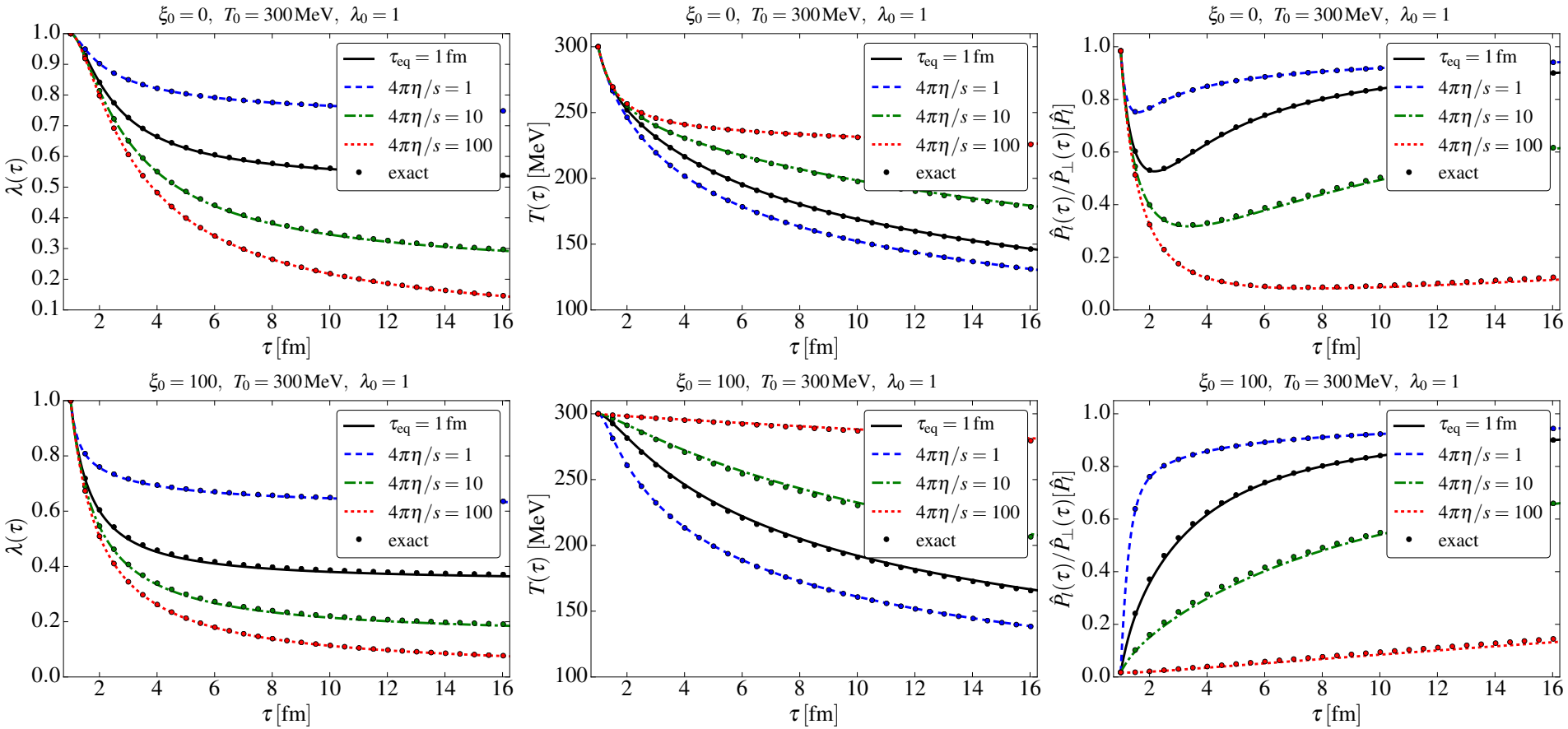
Bjorken flow (IV)

Note: The analysis just presented (from Bazow, UH, Strickland, PRC 90 ('14) 054910) used the zeroth moment of the BE to close the set of equations, and did not enforce the dynamical matching condition for $\mathcal{P}_L - \mathcal{P}_T$ to be fully captured by the dynamical evolution of ξ. As a result of this weakness, a $\delta\tilde{f}$ correction $\tilde{\pi}^{\eta\eta}$ needed to be propagated dynamically to achieve agreement with the exact solution of the BE.

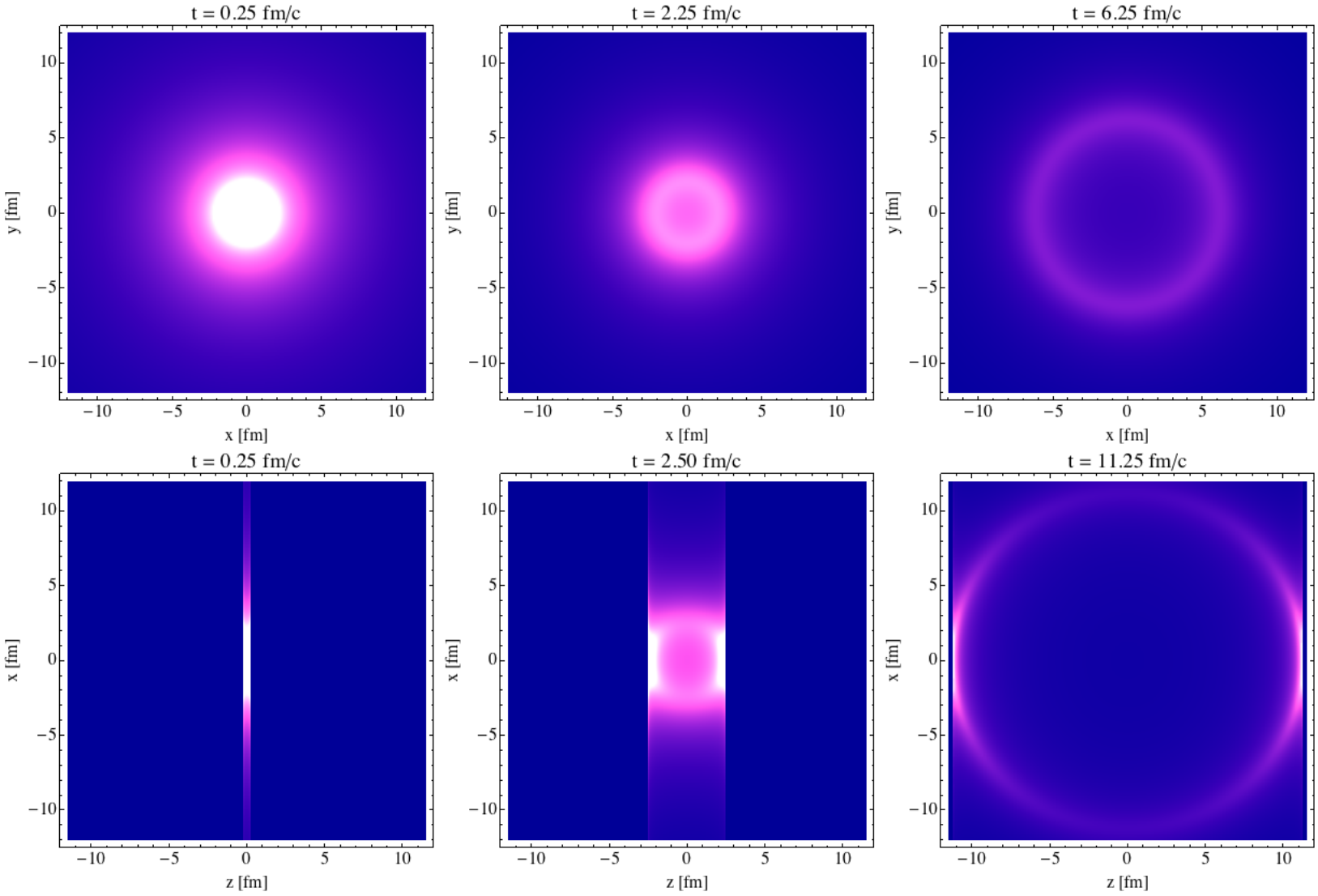
This flaw of Bazow, UH, Strickland, PRC 90 ('14) 054910 was recently fixed in Molnar, Niemi, Rischke, arXiv:1606.09019. If ξ is properly matched to $\mathcal{P}_L - \mathcal{P}_T$, for Bjorken flow **vaHydro exactly reduces to aHydro**, i.e. there are no $\delta\tilde{f}$ correction at all! aHydro then reproduces the hydrodynamic moments of the exact BE solution f.a.p.p. exactly.

Bjorken flow (V)

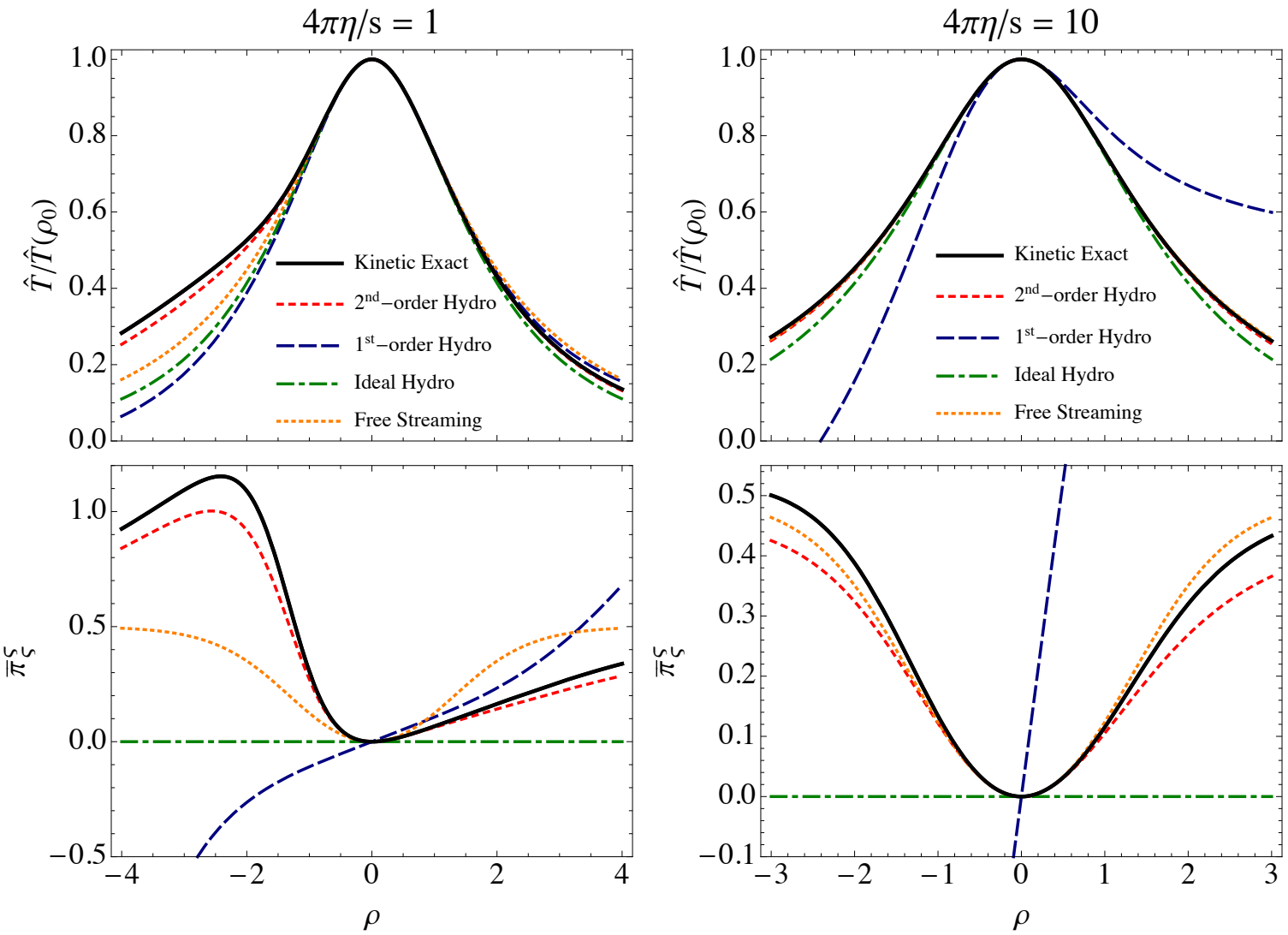
Molnar, Niemi, Rischke, arXiv:1606.09019



Gubser flow I: temperature profile in (x, y) and (x, z)

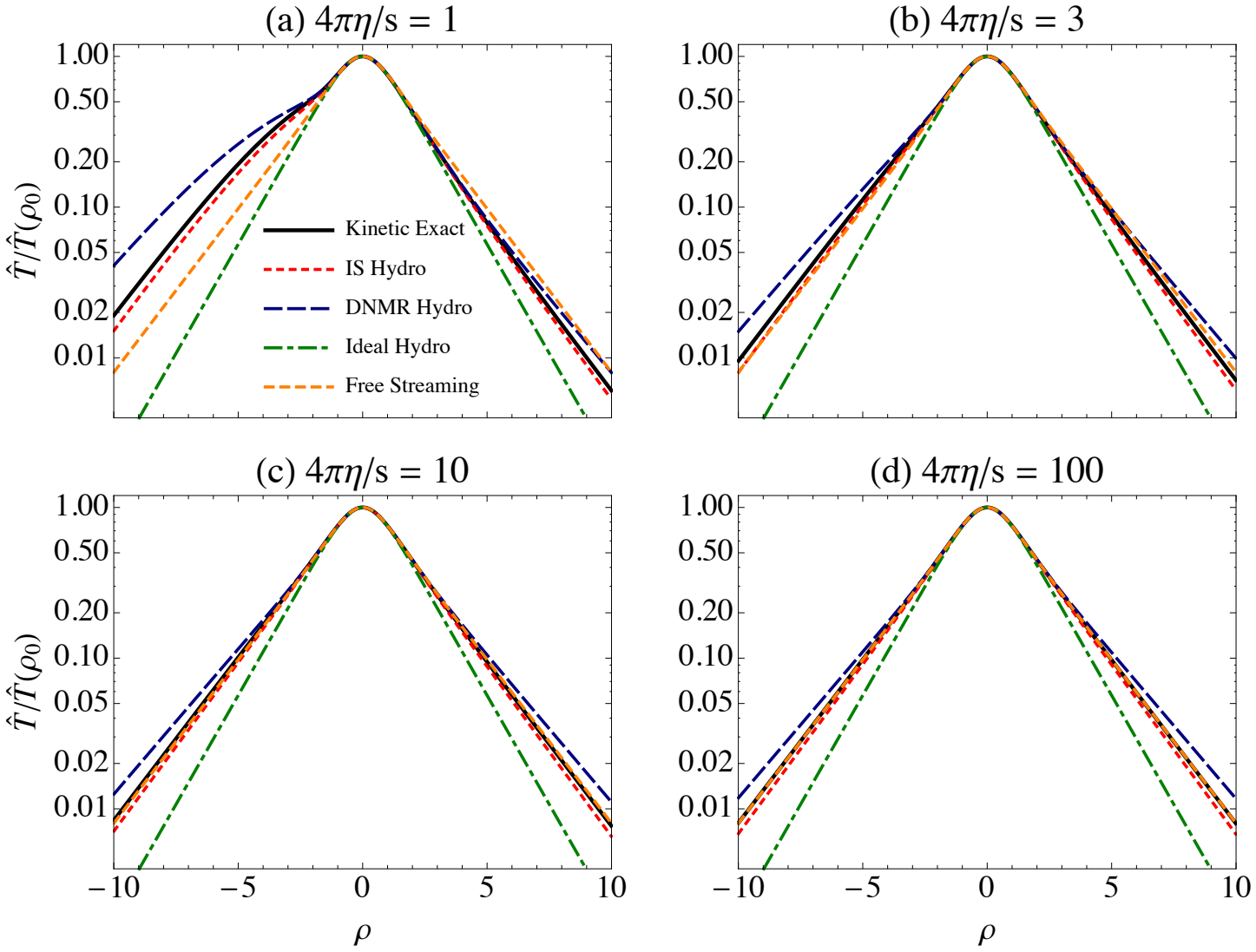


Gubser flow II: ρ-evolution of temperature and shear stress



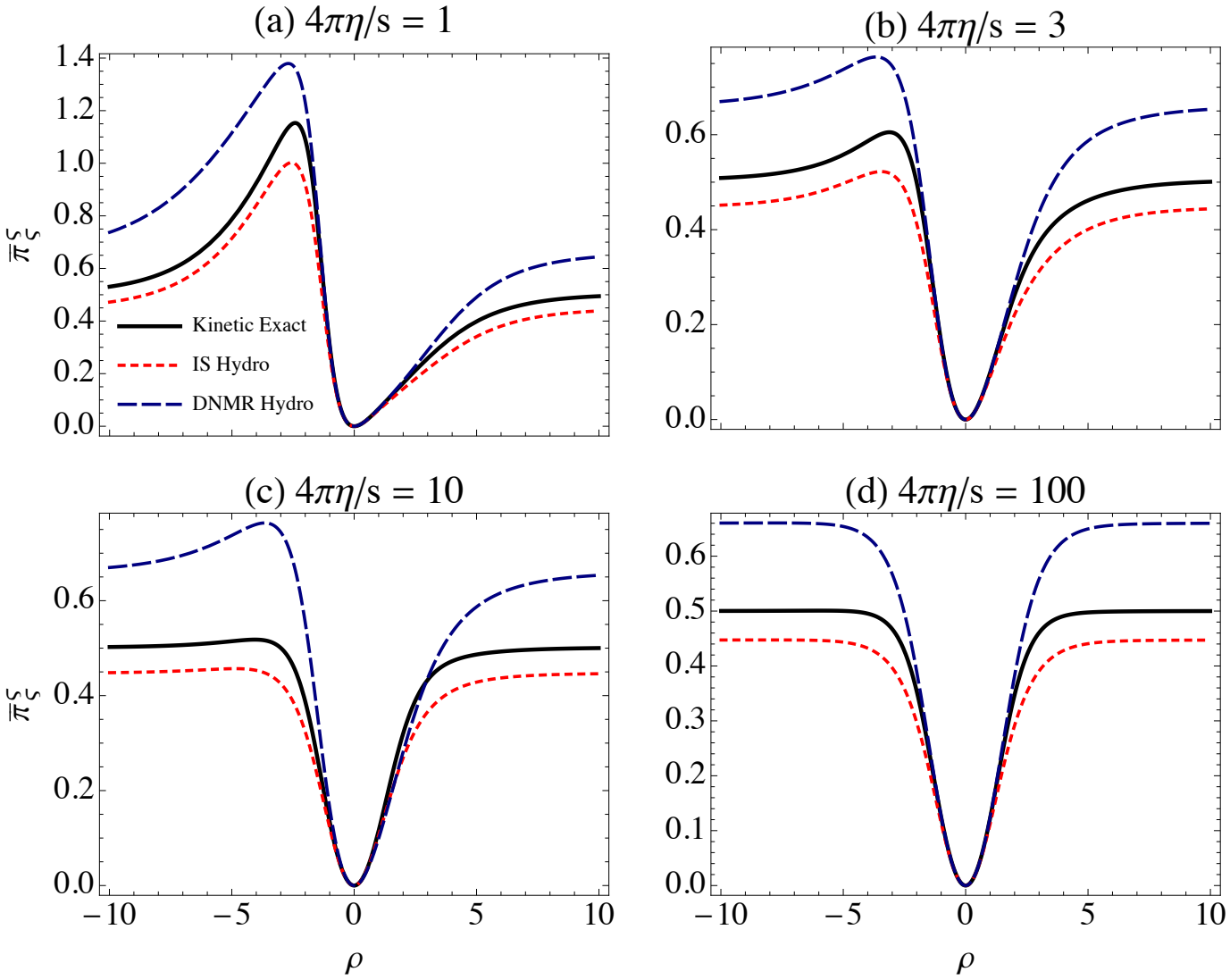
Note: $\bar{\pi}^{\varsigma}_{\varsigma} \equiv \bar{\pi}^{\eta}_{\eta}$! Thermal equil. initial conditions at $\rho_0 = 0$.

Gubser flow III: temperature evolution in de Sitter time ρ



IS seems to work better than DNMR (!?)

Gubser flow IV: shear stress evolution in de Sitter time ρ



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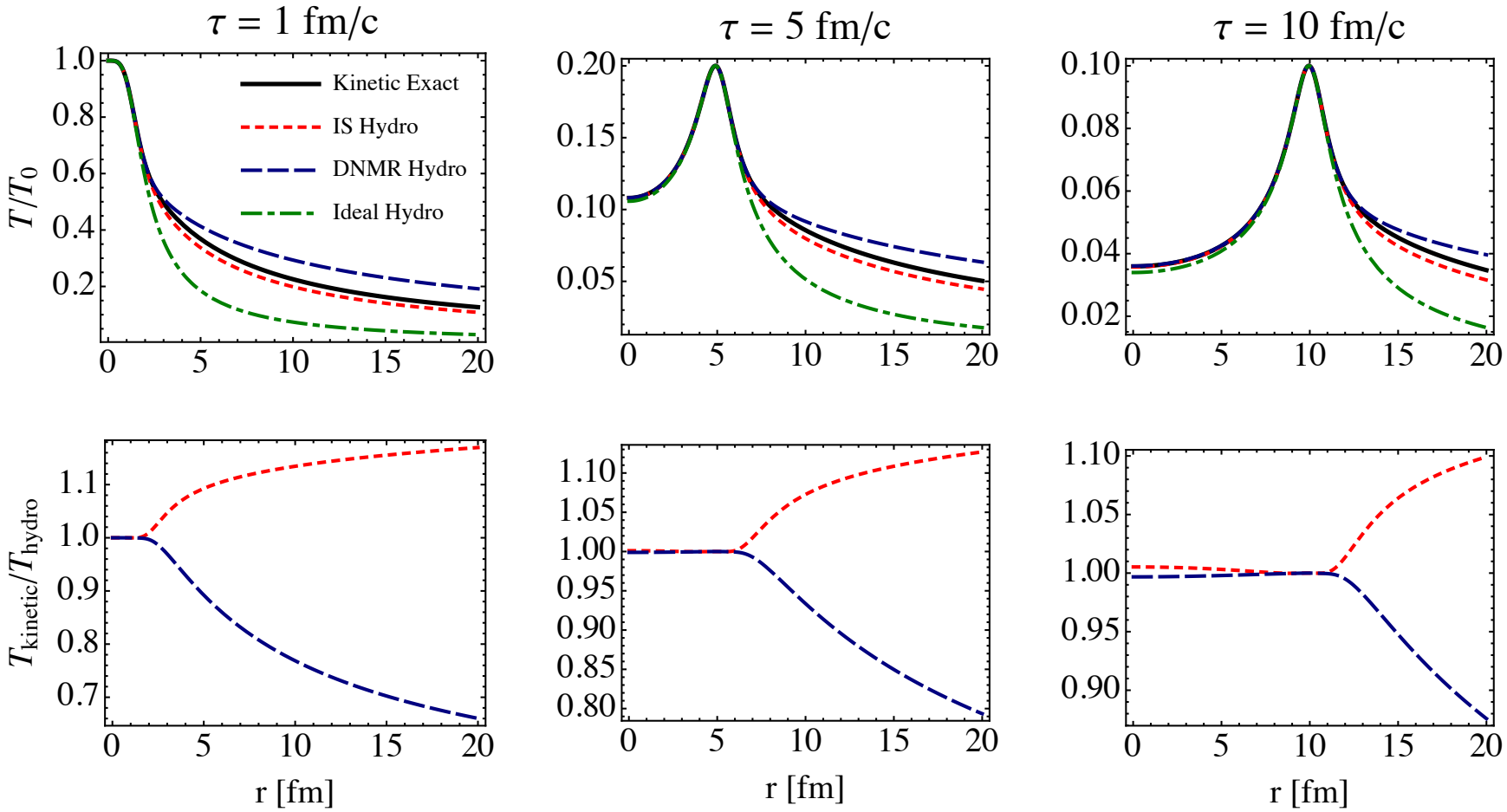
Gubser flow V: temperature evolution in Minkowski space

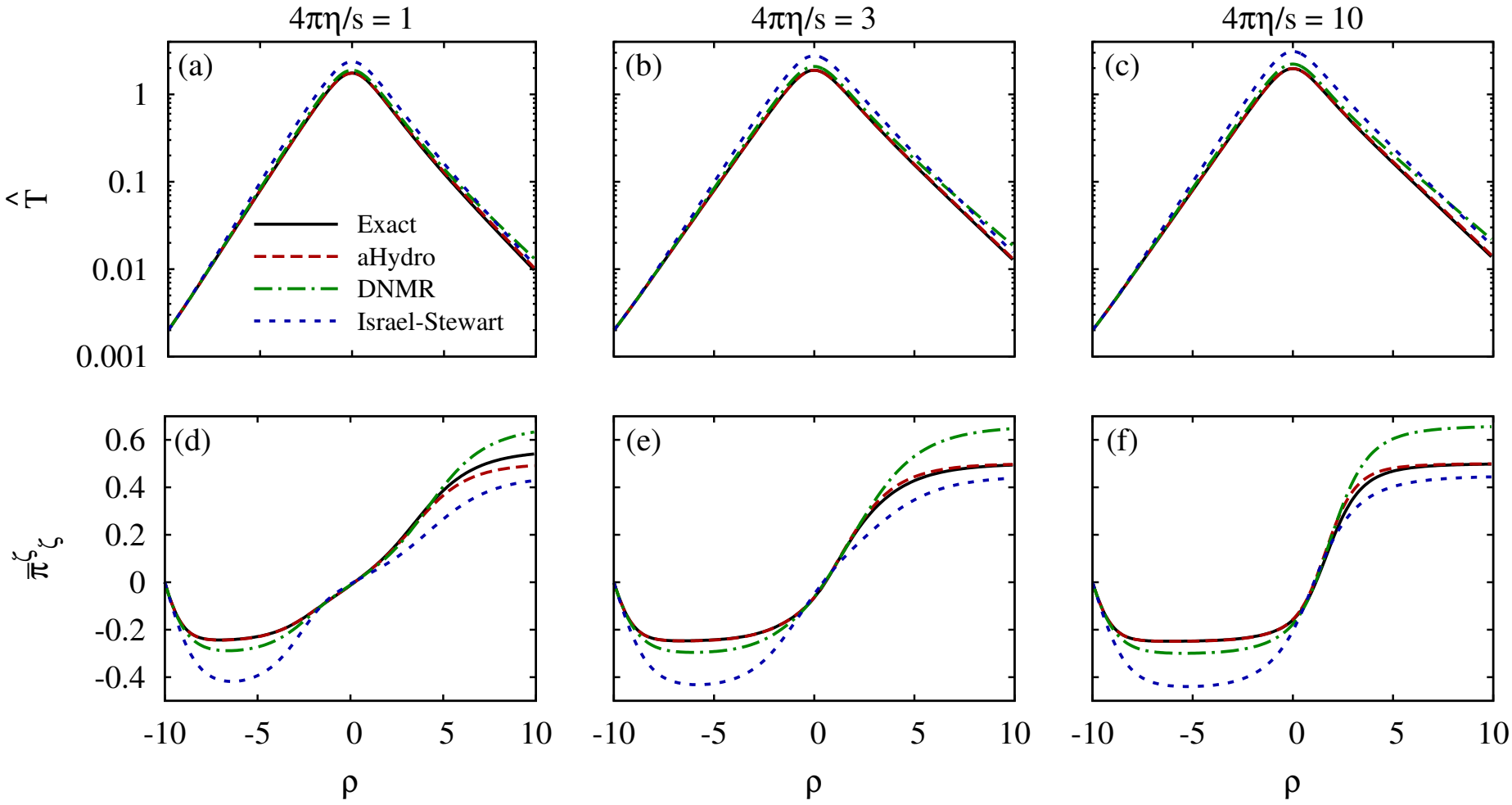


IS seems to work better than DNMR (!?)

Both seem to have problems at large $r \leftrightarrow$ large negative ρ

Gubser flow in aHydro: ρ-evolution of T and shear stress

M. Nopoush, R. Ryblewski, M. Strickland, PRD 91 (2015) 045007



Thermal equil. initial conditions at $\rho_0 \to -\infty$. aHydro works better than IS & DNMR

Overview

Conclusions

- Exact solutions of the Boltzmann equation with a relaxation time collision term for systems undergoing Bjorken or Gubser flow enable **precise tests of hydrodynamic approximation schemes**.
- The new Gubser flow solution of the BE **extends these tests to a situation that resembles heavy-ion collisions** where the created matter undergoes simultaneous longitudinal and transverse expansion.
- When compared with the exact solution, second-order viscous hydrodynamics (IS and DNMR) works better than NS theory, anisotropic hydrodynamics (aHydro) works better than hydrodynamic schemes based on an expansion around local mometum isotropy (IS and DNMR), and viscous anisotropic hydrodynamic (vaHydro) (which treats small viscous corrections as IS or DNMR but resums the largest viscous terms) outperforms aHydro.
  Performance hierarchy: vaHydro $>$ aHydro $>$ DNMR $\sim$ IS $>$ NS $>$ ideal fluid.