Gradient expansion for anisotropic hydrodynamics

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based on WF, R. Ryblewski, M. Spalinski, arXiv:1608.07558

Relativistic hydrodynamics: Theory and modern applications
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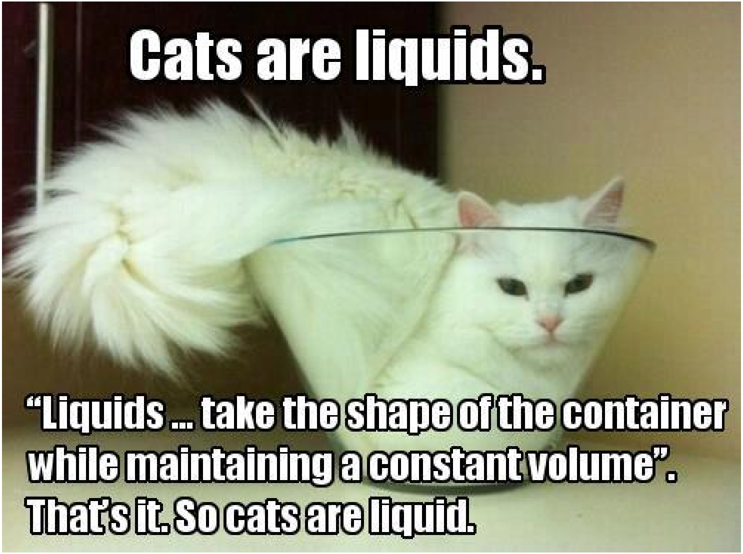
Cats are liquids.
“Liquids ... take the shape of the container
while maintaining a constant volume”.
That's it. So cats are liquid.

Relativistic Navier-Stokes (NS) equations (Eckart, Landau)
— T, u^μ are the only hydrodynamic variables

- energy-momentum conservation

$$\partial_\mu T^{\mu\nu}_{vis} = 0 \qquad T^{\mu\nu}_{vis} = \mathcal{E} u^\mu u^\nu - \Delta^{\mu\nu}(\mathcal{P} + \Pi) + \pi^{\mu\nu}$$

- number of equations: 5 + 6 ($\mathcal{E}, \mathcal{P}, u^\mu(3), \Pi, \pi^{\mu\nu}(5)$)
- number of equations: 4 + 1 (equation of state $\mathcal{E}(\mathcal{P})$)
- 6 extra equations

$$\frac{\Pi}{\tau_\Pi} = -\beta_\Pi \theta, \qquad \theta = \partial_\mu u^\mu \text{ — expansion scalar}$$

$$\frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu}, \qquad \sigma^{\mu\nu} \text{ — shear flow tensor constructed from the velocity field } u^\mu$$

kinetic coefficients: $\tau_\Pi \beta_\Pi = \zeta \rightarrow$ bulk viscosity, $\tau_\pi \beta_\pi = \eta \rightarrow$ shear viscosity

algebraic equations for the shear stress tensor and the bulk pressure, problems with causality

Israel-Stewart (IS) equations — $\Pi, \pi^{\mu\nu}$ promoted to dynamic variables

- energy-momentum conservation

$$\partial_\mu T^{\mu\nu}_{vis} \quad = \quad 0 \qquad T^{\mu\nu}_{vis} = \mathcal{E} u^\mu u^\nu - \Delta^{\mu\nu}(\mathcal{P} + \Pi) + \pi^{\mu\nu}$$

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$$\begin{aligned} \dot{\Pi} + \frac{\Pi}{\tau_\Pi} &= -\beta_\Pi \theta \\ \dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} &= 2\beta_\pi \sigma^{\mu\nu} \end{aligned}$$

$\tau_\Pi \beta_\Pi = \zeta \rightarrow$ bulk viscosity, $\tau_\pi \beta_\pi = \eta \rightarrow$ shear viscosity

non-hydrodynamic modes with the relaxation times τ_Π, τ_π are introduced,
— act as regulators of the theory

Müller-Israel-Stewart (MIS), Muronga

- energy-momentum conservation

$$\partial_\mu T^{\mu\nu}_{vis} \quad = \quad 0 \qquad T^{\mu\nu}_{vis} = \mathcal{E} u^\mu u^\nu - \Delta^{\mu\nu}(\mathcal{P} + \Pi) + \pi^{\mu\nu}$$

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$$\begin{aligned} \dot{\Pi} + \frac{\Pi}{\tau_\Pi} &= -\beta_\Pi \theta - \frac{\zeta T}{2\tau_\Pi} \Pi \, \partial_\lambda \left(\frac{\tau_\Pi}{\zeta T} u^\lambda \right) \\ \dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} &= 2\beta_\pi \sigma^{\mu\nu} - \frac{\eta T}{2\tau_\pi} \pi^{\mu\nu} \, \partial_\lambda \left(\frac{\tau_\pi}{\eta T} u^\lambda \right) \end{aligned}$$

derived from the kinetic theory, (probably) the most popular version used in phenomenological applications

Dissipative hydrodynamics

New approaches add new terms: Baier, Romatschke, Son, Starinets, Stephanov (BRSSS)

- energy-momentum conservation

$$\partial_\mu T^{\mu\nu}_{vis} = 0 \qquad T^{\mu\nu}_{vis} = \mathcal{E}u^\mu u^\nu - \Delta^{\mu\nu}(\mathcal{P} + \Pi) + \pi^{\mu\nu}$$

- number of equations: 5 + 6 ($\mathcal{E}, \mathcal{P}, u^\mu(3), \Pi, \pi^{\mu\nu}(5)$)
- number of equations: 4 + 1 (equations of state $\mathcal{E}(\mathcal{P})$)
- 6 extra equations

$$\begin{aligned} \Pi &= 0 \\ \dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} &= 2\beta_\pi\sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta + \frac{\lambda_1}{\tau_\pi\eta^2}\pi^{\langle\mu}_{\ \lambda}\pi^{\nu\rangle\lambda} \end{aligned}$$

(+ terms including vorticity and curvature)

pure symmetry arguments about the form of $T^{\mu\nu}_{vis}$ due to conformal symmetry and gradient expansion
NS equations used as an additional input to derive the dynamic equations

New approaches add new terms: Denicol, Niemi, Molnar, Rischke (DNMR)
simultaneous expansion in the Knudsen number and inverse Reynolds number
similar results by Jaiswal

- energy-momentum conservation

$$\partial_\mu T^{\mu\nu}_{vis} = 0 \qquad T^{\mu\nu}_{vis} = \mathcal{E} u^\mu u^\nu - \Delta^{\mu\nu}(\mathcal{P} + \Pi) + \pi^{\mu\nu}$$

- number of equations: 5 + 6 ($\mathcal{E}, \mathcal{P}, u^\mu(3), \Pi, \pi^{\mu\nu}(5)$)
- number of equations: 4 + 1 (equations of state $\mathcal{E}(\mathcal{P})$)
- 6 extra equations

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\beta_\Pi \theta - \delta_{\Pi\Pi}\Pi\theta \underline{+\lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} - \delta_{\pi\pi}\pi^{\mu\nu}\theta \underline{-\tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}}$$

here: RTA version of the Boltzmann kinetic equation, neglected vorticity
additional terms (with new kinetic coefficients) appear — shear-bulk coupling

MIS, BRSS, DNMR are examples of **HYDRODYNAMIC EXPANSION** — truncated gradient expansion + less or more heuristic arguments used to construct hydrodynamic equations (hydrodynamic framework) for practical applications

Anisotropic hydrodynamics - reorganized hydrodynamic expansion

standard approach:

$$T^{\mu\nu} = T^{\mu\nu}\left(T, U^\mu, \pi^{\mu\nu}, \Pi\right) = T^{\mu\nu}_{\rm eq}(T, U) + \pi^{\mu\nu} - \Pi\Delta^{\mu\nu} \equiv T^{\mu\nu}_{\rm eq} + \delta T^{\mu\nu}$$

reproduced with the distribution function

$$f(x,p) = f_{\rm eq}(x,p) + \delta f(x,p)$$

anisotropic hydrodynamics approach:

$$T^{\mu\nu} = T^{\mu\nu}\left(T, U^\mu, \xi^{\mu\nu}, \phi, \tilde{\pi}^{\mu\nu}, \tilde{\Pi}\right) = T^{\mu\nu}_{\rm a}(T, U, \xi^{\mu\nu}, \phi) + \tilde{\pi}^{\mu\nu} - \tilde{\Pi}\Delta^{\mu\nu} \equiv T^{\mu\nu}_{\rm a} + \delta\tilde{T}^{\mu\nu}$$

reproduced with the distribution function

$$f(x,p) = f_{\rm a}(x,p) + \delta\tilde{f}(x,p), \quad f_{\rm a} = f_{\rm iso}\left(\frac{\sqrt{p^\mu \Xi_{\mu\nu} p^\nu}}{\lambda}\right)$$

separation into the leading and next-to-leading orders is not unique, depends on $\Xi_{\mu\nu}$

$$f(x,p) = f_{\rm eq}(x,p) + \delta f(x,p) = f_{\rm a}(x,p) + \delta\tilde{f}(x,p) = f_{\rm a1}(x,p) + \delta\tilde{f}_1(x,p) = \ldots$$

Romatschke-Strickland form

appropriate for one-dimensional boost-invariant expansion, accounts for the difference between the longitudinal and transverse pressures

U — flow four-vector, Z — beam four-vector, $U^2 = 1$, $Z^2 = -1$, $U \cdot Z = 0$

$$f_{RS} = \exp\left(-\frac{1}{\lambda}\sqrt{(p\cdot U)^2 + \xi\,(p\cdot Z)^2}\right), \quad U = (t/\tau, 0, 0, z/\tau), \quad Z = (z/\tau, 0, 0, t/\tau)$$

yields the energy-momentum tensor

$$T^{\mu\nu} = (\varepsilon + P_\perp)\, U^\mu U^\nu - P_\perp\, g^{\mu\nu} - (P_\perp - P_\parallel) Z^\mu Z^\nu$$

generalised Romatschke-Strickland form

$$f_{\rm a} = f_{\rm iso}\left(\frac{\sqrt{p^\mu \Xi_{\mu\nu} p^\nu}}{\lambda}\right)$$

1) ANISOTROPIC HYDRODYNAMICS INCLUDES INFINITE NUMBER OF GRADIENTS SIMILAR TO THE EXPONENTIATION METHOD
2) Heinz, viscous anisotropic hydrodynamics, $f = f_{RS} + \delta\tilde{f}(x,p)$ with corrections included

Two expansion methods

Perturbative approach

Bazow, Heinz, Strickland
PRC 90, 044908 (2014)

$$f = f_{RS} + \delta f$$

- the leading order is still described by the Romatschke-Strickland form (accounting for the difference between the longitudinal and transverse pressures)
- advanced methods of traditional viscous hydrodynamics are used to restrict the form of the correction δf and to derive aHydro equations — non-trivial dynamics included in the transverse plane and, more generally, in (3+1)D case

Molnar, Niemi, Rischke
PRD93 (2016) no.11, 114025 and arXiv:1602.00573

Non-perturbative approach

Nopoush, Ryblewski, Strickland,Tinti, WF

$$f = f_{\rm a} + \delta \tilde{f}$$

- all effects due to anisotropy included in the leading order, in the generalised RS form

1. (1+1)D conformal case, two anisotropy parameters
2. (1+1)D non-conformal case, two anisotropy parameters + one bulk parameter
3. full (3+1)D case, five anisotropy parameters + one bulk parameter (shear tensor and bulk pressure)

$$\Xi^{\mu\nu} = u^\mu u^\nu + \xi^{\mu\nu} - \Delta^{\mu\nu}\Phi$$

$$u_\mu \xi^{\mu\nu} = 0 \qquad \xi^\mu_{\ \mu} = 0 \quad (5 \text{ parameters in } \xi^{\mu\nu})$$

Two methods to derive equations of anisotropic hydrodynamics

Moments of the Boltzmann equation

Martinez, Strickland, Ryblewski, WF,...

- appropriate number of the moments is chosen to determine the dynamics of ahydro parameters
- selection based on comparisons with exact solutions of the kinetic equation

Anisotropic matching principle

Tinti, PRC94 (2016) 044902

- the anisotropy parameters should reproduce exactly $T^{\mu\nu}$, generalised Landau matching $T^{\mu\nu} = T_a^{\mu\nu}$
- exact equations for the shear stress tensor and the bulk pressure derived and closed with the assumption $f = f_a$
- see Tinti's talk

Two ahydro equations for on-dimensional Bjorken expansion for the conformal case with $\tau_\pi = c/T$

AH1: L. Tinti, WF, PRC89 (2014) 034907

$$4\,\frac{\mathcal{R}(\xi)}{T}\,\frac{dT}{d\tau} = -\frac{1}{\tau}\left(\mathcal{R}(\xi) + \frac{\mathcal{R}_\parallel(\xi)}{3}\right)$$

$$-\frac{d\xi}{d\tau} + \frac{2(1+\xi)}{\tau} = \frac{\xi\, T\, \mathcal{R}(\xi)^{5/4}}{c}(1+\xi)^{3/2}$$

with Martinez-Strickland functions

$$\mathcal{R}(\xi) = \frac{1}{2}\left(\frac{1}{1+\xi} + \frac{\tanh^{-1}(\sqrt{\xi})}{\sqrt{\xi}}\right), \quad \mathcal{R}_\parallel(\xi) = \frac{3}{\xi}\left(\mathcal{R}(\xi) - \frac{1}{1+\xi}\right)$$

AH2: L. Tinti, PRC94 (2016) 044902

$$4\,\frac{\mathcal{R}(\xi)}{T}\,\frac{dT}{d\tau} = -\frac{1}{\tau}\left(\mathcal{R}(\xi) + \frac{\mathcal{R}_\parallel(\xi)}{3}\right)$$

$$\frac{d\,\Delta P}{d\tau} = -\frac{T\,\Delta P}{c} - \frac{F}{\tau}$$

with

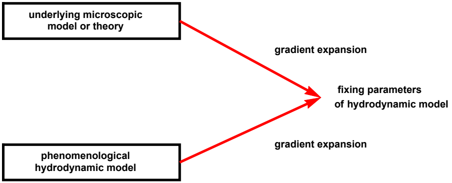
$$\Delta P = -\frac{6k\pi\Lambda^4}{\xi}\left(\frac{\xi+3}{\xi+1} + \frac{(\xi-3)\tan^{-1}(\sqrt{\xi})}{\sqrt{\xi}}\right), \quad F = -2(1+\xi)\frac{\partial\Delta}{\partial\xi}, \quad \Delta = \frac{\Delta P}{P} = \frac{P_\parallel - P_\perp}{P}$$

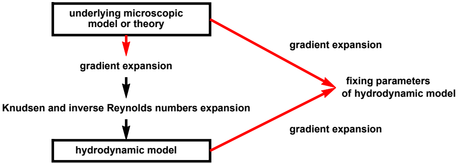
Gradient expansion

M. P. Heller, R. A. Janik, M. Spalinski, P. Witaszczyk
Formal expansion of $T^{\mu\nu}$ in gradients of hydrodynamic variables T and u^μ

$$T^{\mu\nu} \quad = \quad T^{\mu\nu}_{\rm eq} + \text{powers of gradients of } T \text{ and } u^\mu$$

Formal tool to make comparisons between different theories and check their close to equilibrium behaviour, no useful for finding approximate solutions of the theory, unless completed as a transseries





Simple structures for boost-invariant flow with the relaxation time $\tau_\pi = \frac{c}{T}$, for example, T is expanded around the Bjorken flow

$$
\begin{aligned}
T &= T_0 \left(\frac{\tau_0}{\tau}\right)^{1/3} \left(1 + \sum_{n=1}^{\infty} \left(\frac{c}{T_0 \tau_0}\right)^n t_n \left(\frac{\tau_0}{\tau}\right)^{2n/3}\right) \\
\xi(\tau) &= \sum_{n=1}^{\infty} \left(\frac{2c}{\tau_0 T_0}\right)^n \xi_n \left(\frac{\tau_0}{\tau}\right)^{2n/3},
\end{aligned}
$$

Change to $g(w)$

$$
g = \frac{1}{T}\frac{dw}{d\tau}, \qquad w = \tau T, \quad \Delta = \frac{\Delta P}{P} = 3\frac{P_\parallel - P_\perp}{\varepsilon} = 12\left(g - \frac{2}{3}\right)
$$

The gradient expansion for boost-invariant flow takes the form of an expansion

$$
g(w) = \sum_{n=0}^{\infty} g_n w^{-n}, \quad g_0 = \frac{2}{3}
$$

Gradient expansion for viscous hydrodynamics

RTA - gradient expansion for the RTA kinetic-theory model – Heller, Kurkela, Spalinski, arXiv:1609.04803
WF, R. Ryblewski, M. Spalinski, arXiv:1608.07558

n	RTA	BRSSS	DNMR	MIS
0	2/3	2/3	2/3	2/3
1	4/45	4/45	4/45	4/45
2	16/945	16/945	16/945	8/135
3	−208/4725	−1712/99225	-304/33075	112/2025
3	−0.044	−0.017	-0.009	0.055

MIS $$\tau_\Pi \dot{\phi} = \frac{4\eta}{3\tau} - \frac{4\tau_\Pi \phi}{3\tau} - \phi$$ ϕ – shear stress component

BRSS $$\tau_\Pi \dot{\phi} = \frac{4\eta}{3\tau} - \frac{\lambda_1 \phi^2}{2\eta^2} - \frac{4\tau_\Pi \phi}{3\tau} - \phi$$

DNMR $$\tau_\Pi \dot{\phi} = \frac{4\eta}{3\tau} - \frac{38}{21}\frac{\tau_\Pi \phi}{\tau} - \phi$$

RTA with $\tau_\pi = c/T$, the n=1 term controlled by viscosity, $\eta/s = (9/4)g_1$

1) BRSS and DNMR equivalent up to n=2, $\eta/s = c/5$, agrees with the kinetic-theory result
2) BRSS has two free parameters that are fitted to RTA
3) DNMR reproduces RTA, since the kinetic coefficients correspond to the RTA kinetic equation
4) MIS good only for n=1, opposite sign for n=3
5) DNMR and BRSS differ for larger values of n and far away from equilibrium (Ryblewski)
– physics properties should be defined within a given framework

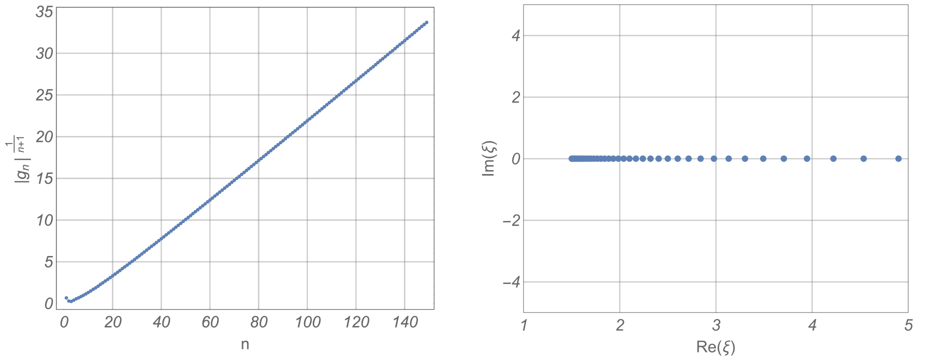
Gradient expansion for anisotropic hydrodynamics

n	RTA	BRSSS	AH I	AH II
0	2/3	2/3	2/3	2/3
1	4/45	4/45	4/45	4/45
2	16/945	16/945	8/945	16/945
3	−208/4725	−1712/99225	−184/4725	−176/6615
3	−0.044	−0.017	−0.039	−0.027

1) AH1, for n=2 too small (by a factor of two) but for n=3 quite close to RTA
2) AH2 reproduces exactly the first three terms of RTA, not too bad for n=3

the series g_n, has vanishing radius of convergence,the Borel transform of g is introduced, analytic continuation using diagonal Padé approximants of order 70 is done

$$g_B(\xi) = \sum_{n=0}^{\infty} \frac{g_n}{n!} \xi^n, \tag{1}$$



the cut along the real axis indicates the presence of a single nonhydrodynamic mode, which is is purely decaying, as in MIS theory (Romatschke, Heller).

SUMMARY

A comparison of the gradient expansions provides an indication of how successful the various hydrodynamic approaches are in reproducing close-to-equilibrium dynamics governed by the underlying microscopic theory

Effective hydrodynamic descriptions tailored to a specific microscopic theory may provide a better picture for a given system than a general framework

The gradient expansions in hydrodynamic theories are divergent, so their usefulness, apart from formal comparisons, lies in the fact that a few leading terms give a reasonable approximation at late times. The theory of asymptotic series provides the concept of optimal truncation, which in the cases considered is of the order of a few terms