How to Implement Noise in Relativistic Hydrodynamics?

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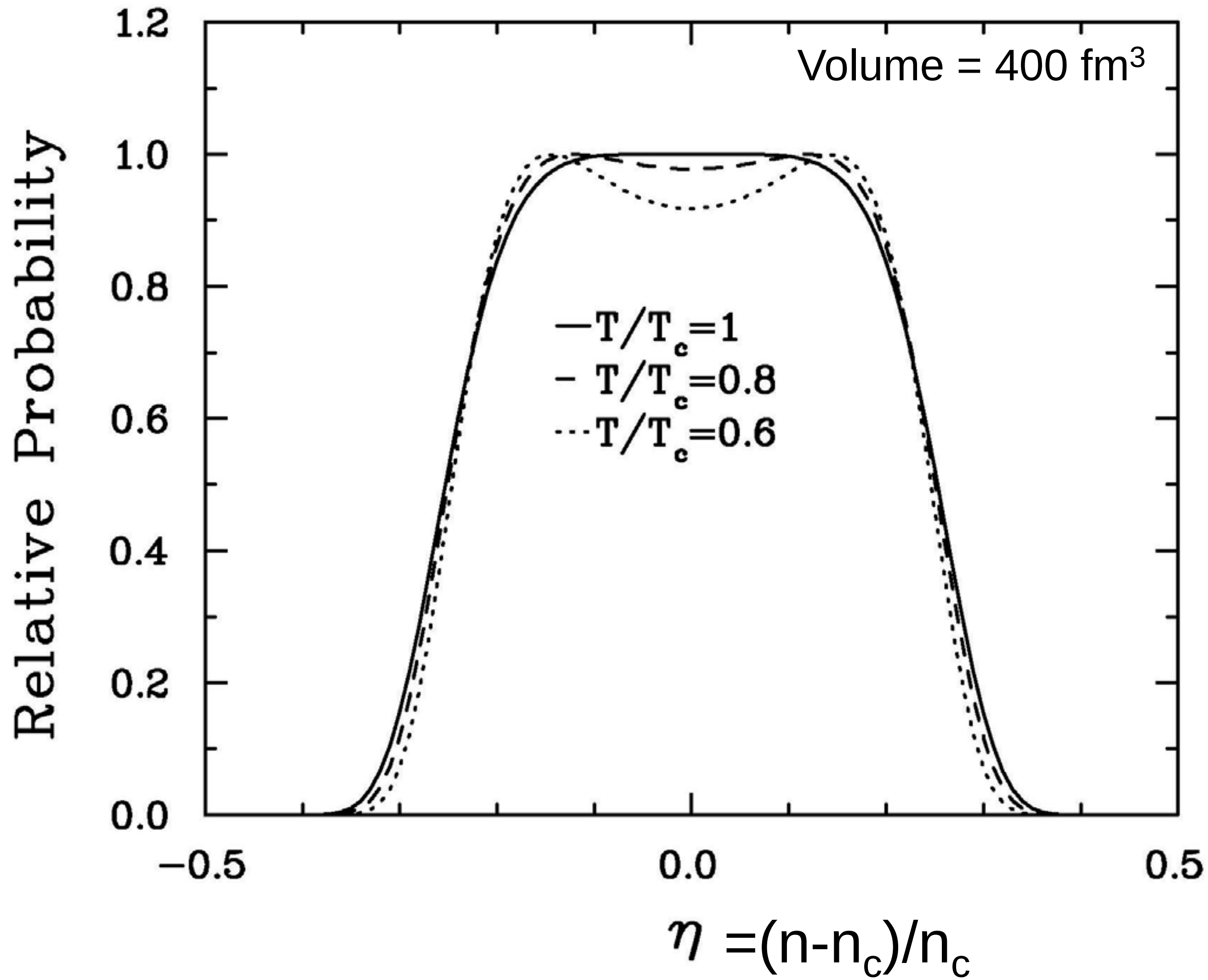
Sources of Fluctuations in High Energy Nuclear Collisions

- Initial state fluctuations
- Hydrodynamic fluctuations due to finite particle number
- Energy and momentum deposition by jets traversing the medium
- Freeze-out fluctuations

Expansion away from equilibrium states using Landau theory



Incorporates correct critical exponents and amplitudes - Kapusta (2010)
Static univerality class: 3D Ising model & liquid-gas transition

But this is for a small system
in contact with a heat and
particle reservoir.

How do you treat fluctuations
in an expanding and cooling
system as in heavy ion collisions?

Kapusta, Muller, Stephanov

Hydrodynamic Fluctuations!

Hydrodynamic fluctuations (noise) have been applied to a wide variety of physical, chemical, and biological systems.

There are fluctuations in high energy heavy ion collisions due to the finite size and finite particle content of the system.

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Dynamics of Liquid Nanojets

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We study the breakup of a liquid jet a few nanometers in diameter, based on a stochastic differential equation derived recently by Moseler and Landman [Science **289**, 1165 (2000)]. In agreement with their simulations, we confirm that noise qualitatively changes the characteristics of breakup, leading to symmetric profiles. Using the path integral description, we find a self-similar profile that describes the most probable breakup mode. As illustrated by a simple physical argument, noise is the driving force behind pinching, speeding up the breakup to make surface tension irrelevant.

PRL 98, 064504 (2007) PHYSICAL REVIEW LETTERS week ending 9 FEBRUARY 2007

Universality Crossover of the Pinch-Off Shape Profiles of Collapsing Liquid Nanobridges in Vacuum and Gaseous Environments

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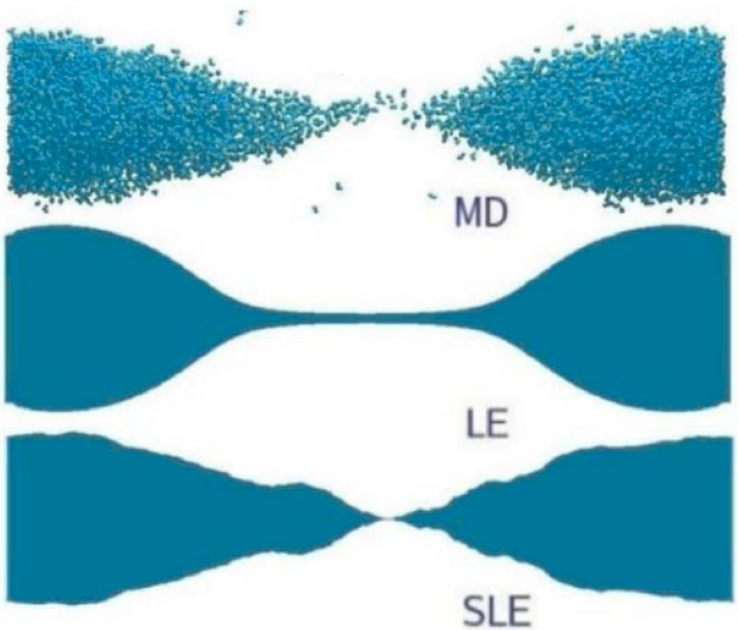
Liquid propane nanobridges were found through molecular dynamics simulations to exhibit in vacuum a symmetric break-up profile shaped as two cones joined in their apexes. With a surrounding gas of sufficiently high pressure, a long-thread profile develops with an asymmetric shape. The emergence of a long-thread profile, discussed previously for macroscopic fluid structures, originates from the curvature-dependent evaporation-condensation processes of the nanobridge in a surrounding gas. A modified stochastic hydrodynamic description captures the crossover between these universal break-up regimes.



Molecular Dynamics

Lubrication Equation

Stochastic Lubrication Equation

Relativistic Dissipative Fluid Dynamics

$$\boxed{\begin{aligned} T^{\mu\nu} &= -Pg^{\mu\nu} + wu^{\mu}u^{\nu} + \Delta T^{\mu\nu} \\ J_B^{\mu} &= n_B u^{\mu} + \Delta J_B^{\mu} \end{aligned}}$$

In the Landau-Lifshitz approach u is the velocity of energy transport.

$$\boxed{\Delta T^{\mu\nu} = \eta\left(\Delta^{\mu}u^{\nu} + \Delta^{\nu}u^{\mu}\right) + \left(\tfrac{2}{3}\eta - \zeta\right)H^{\mu\nu}\partial_{\rho}u^{\rho}}$$

$$H^{\mu\nu} \equiv u^{\mu}u^{\nu} - g^{\mu\nu}, \quad \Delta_{\mu} \equiv \partial_{\mu} - u_{\mu}u^{\beta}\partial_{\beta}, \quad Q_{\alpha} \equiv \partial_{\alpha}T - Tu^{\rho}\partial_{\rho}u_{\alpha}$$

$$\Delta J_B^{\mu} = \chi\left(\frac{n_B T}{w}\right)^2 \Delta^{\mu}\left(\frac{\mu_B}{T}\right), \quad s^{\mu} = su^{\mu} - \frac{\mu_B}{T}\Delta J_B^{\mu}$$

$$\boxed{\partial_{\mu}s^{\mu} = \frac{\eta}{2T}\left(\partial_i u^j + \partial_j u^i - \tfrac{2}{3}\delta^{ij}\partial_k u^k\right)^2 + \frac{\zeta}{T}\left(\partial_k u^k\right)^2 + \frac{\chi}{T^2}\left(\partial_k T + T\dot{u}_k\right)^2}$$

Extend Landau's theory of hydrodynamic fluctuations to the relativistic regime

$$T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + \Delta T^{\mu\nu} + S^{\mu\nu}$$

Stochastic source $$S^{\mu\nu} = S^{\mu\nu}_{\text{vis}} + S^{\mu\nu}_{\text{heat}}$$

$$\left\langle S^{\mu\nu}_{\text{vis}}(x) S^{\alpha\beta}_{\text{vis}}(y) \right\rangle = 2T\left[\eta\left(H^{\mu\alpha}H^{\nu\beta} + H^{\mu\beta}H^{\nu\alpha}\right) + \left(\zeta - \tfrac{2}{3}\eta\right)H^{\mu\nu}H^{\alpha\beta}\right]\delta^4(x-y)$$

$$\left\langle S^{\mu\nu}_{\text{heat}}(x) S^{\alpha\beta}_{\text{heat}}(y) \right\rangle = 2\chi T^2\left[H^{\mu\alpha}u^\nu u^\beta + H^{\nu\beta}u^\mu u^\alpha + H^{\mu\beta}u^\nu u^\alpha + H^{\nu\alpha}u^\mu u^\beta\right]\delta^4(x-y)$$

$$\left\langle S^{\mu\nu}_{\text{vis}}(x) S^{\alpha\beta}_{\text{heat}}(y) \right\rangle = 0$$

Similar expressions arise in the Eckart approach.

Procedure

- Solve equations of motion for arbitrary source function
- Perform averaging to obtain correlations/fluctuations
- Stochastic fluctuations need not be perturbative

Example: Boost Invariant Bjorken Model

$$u^\mu = \left(\cosh\eta_s, 0, 0, \sinh\eta_s\right) \quad T = T(\tau) \quad s(\tau) = \frac{s_0\tau_0}{\tau}$$

Fluctuation

$$X(\eta_s,\tau) = \frac{s_0\tau_0}{v_s^2(\tau)} \frac{\delta T(\eta_s,\tau)}{T(\tau)}$$

Solution

$$\tilde{X}(k,\tau) = -\int_{\tau_0}^{\tau} d\tau' \underbrace{\tilde{G}(k;\tau,\tau')}_{\text{response function}} \overbrace{\tilde{f}(k,\tau')}^{\text{noise}}$$

$$\left\langle X(\Delta\eta_s,\tau_f) X(0,\tau_f)\right\rangle = \frac{1}{\pi A}\int_{\tau_0}^{\tau_f} d\tau \frac{T(\tau)}{\tau}\left[\frac{4}{3}\eta(\tau) + \varsigma(\tau)\right]\int_{-\infty}^{\infty} dk\, e^{ik\Delta\eta_s} \underbrace{|\tilde{G}(k;\tau_f,\tau)|^2}_{\text{calculable}}$$

and similarly for fluctuations in the local flow velocity…

In the small viscosity limit

$$T(z)\tilde{G}(k;z',z) = \frac{1}{2\mu}\sqrt{\frac{z'}{z}}\,\mathrm{e}^{z'-z}\left[\left(1-2\mu^2+\mu+z\right)\left(\frac{z}{z'}\right)^{\mu} - \left(1-2\mu^2-\mu+z\right)\left(\frac{z}{z'}\right)^{-\mu}\right]$$

$$z = \frac{(\eta/s)_0}{T_0\tau_0}\left(\frac{\tau}{\tau_0}\right)^{2/3} k^2 \qquad \& \qquad \mu = \frac{1}{2}\sqrt{1-3k^2}$$

This leads to delta functions and their derivatives.

Cause: Space-time delta functions (white noise) in the original correlation functions.

Cure: Use finite range correlations (colored noise).

Why colored noise?

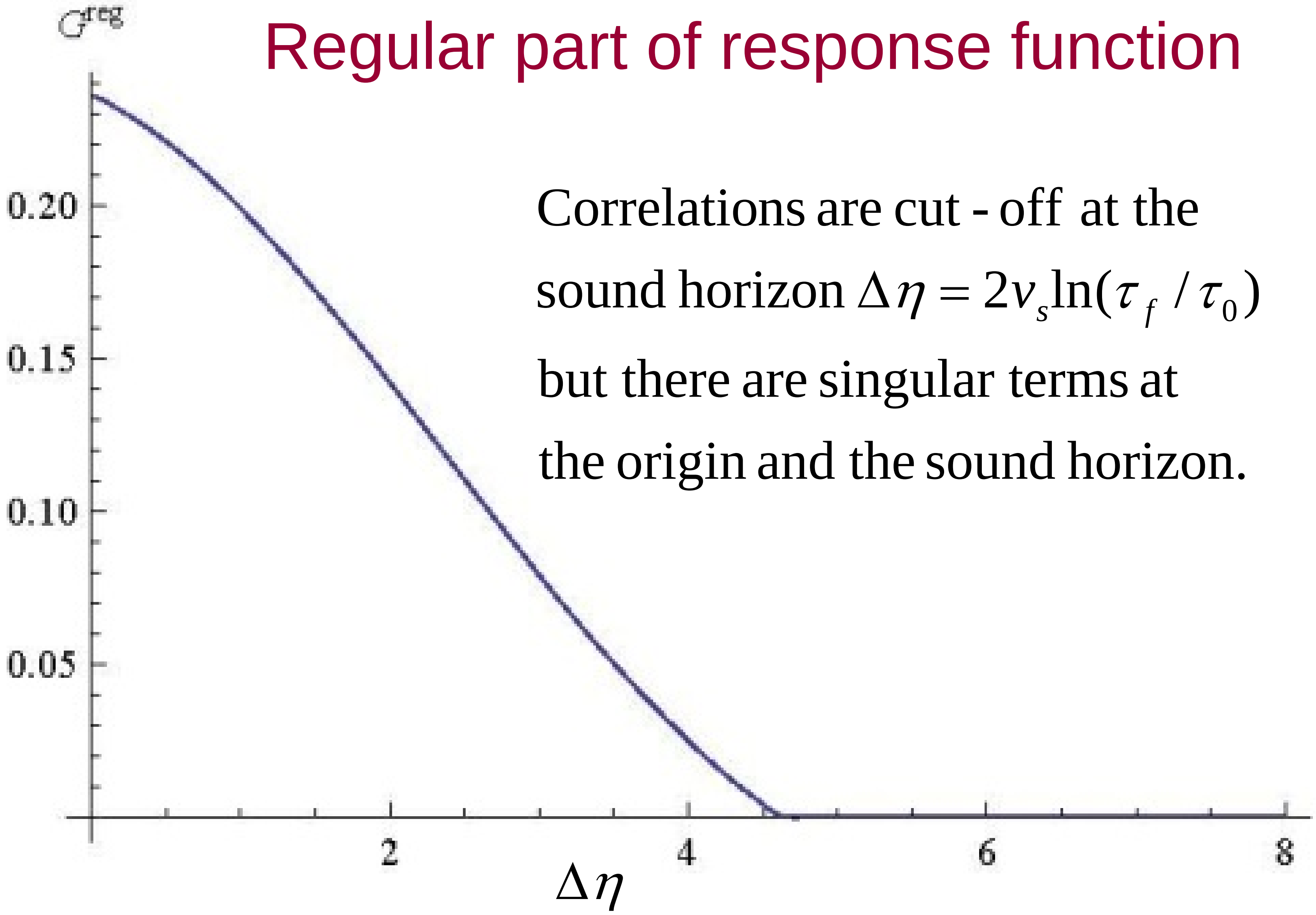
In the local rest frame:

$$\left\langle S^{\mu\nu}(\mathbf{x},t)\right\rangle = 0, \quad \left\langle S^{ij}S^{kl}(\mathbf{x},t)\right\rangle = 2\eta T\delta(\mathbf{x})\delta(t)M^{ijkl}$$

This is white noise (Fourier transform is a constant).
It is OK for hydrodynamics if noise is treated as a perturbation, but creates havoc if it is treated nonperturbatively as there will be a dependence on the coarse-grained cell size.

$$S_{\mathrm{rms}} \approx \sqrt{\frac{2\eta T}{\Delta V \Delta t}} = \sqrt{\frac{2(\eta / s)w}{\Delta V \Delta t}}$$

Regular part of response function



Correlations are cut - off at the sound horizon $\Delta\eta = 2v_s \ln(\tau_f / \tau_0)$ but there are singular terms at the origin and the sound horizon.

Fluctuations in the local temperature and flow velocity fields

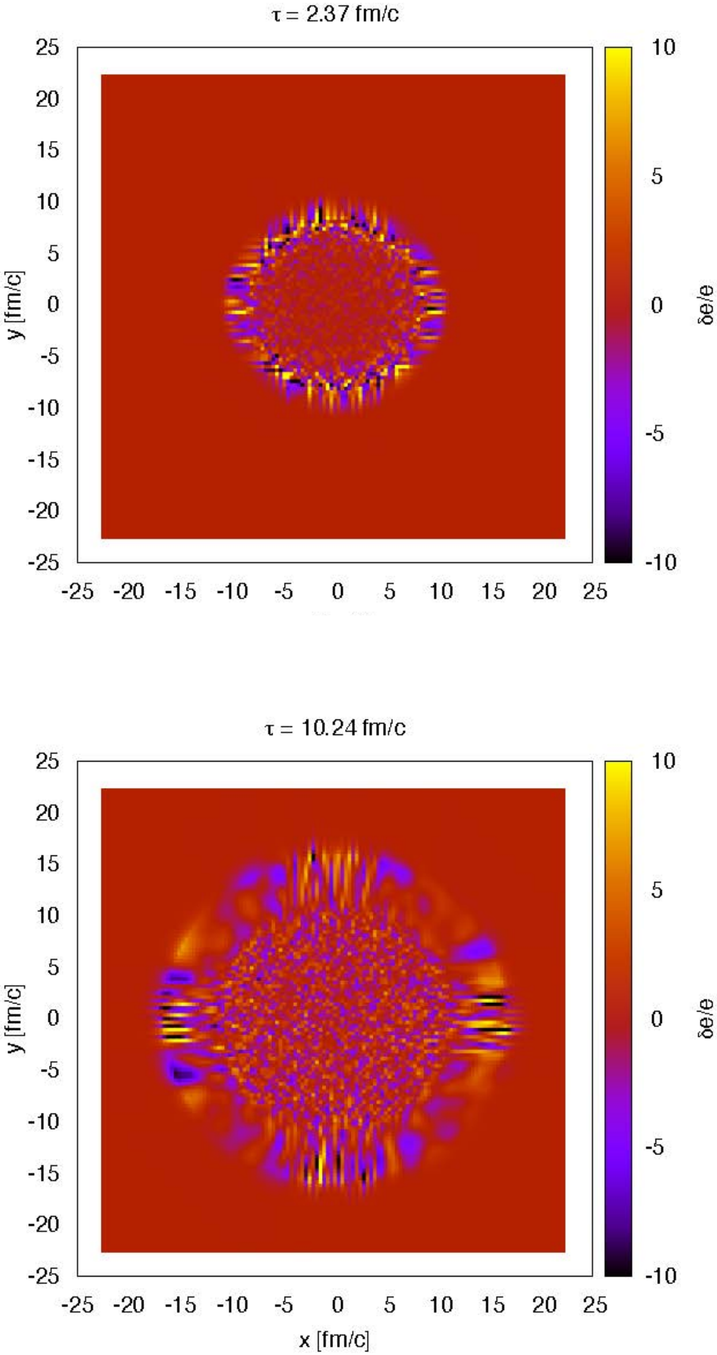
$$u_z = \sinh\left(\eta_s + \omega(\eta_s, \tau)\right)$$

give rise to a nontrivial 2-particle correlation function when the fluid elements freeze-out to free-streaming hadrons.

Noise in MUSIC

Numerical 3+1d hydrodynamics
Event by event
LHC energy – shear viscosity only
Noise treated perturbatively

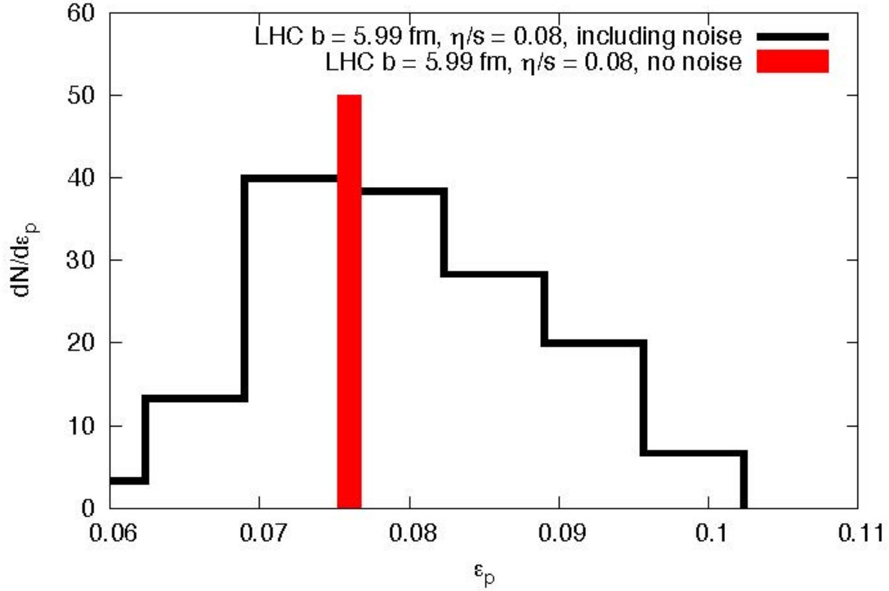
Young; Young, Kapusta, Gale, Jeon, Schenke



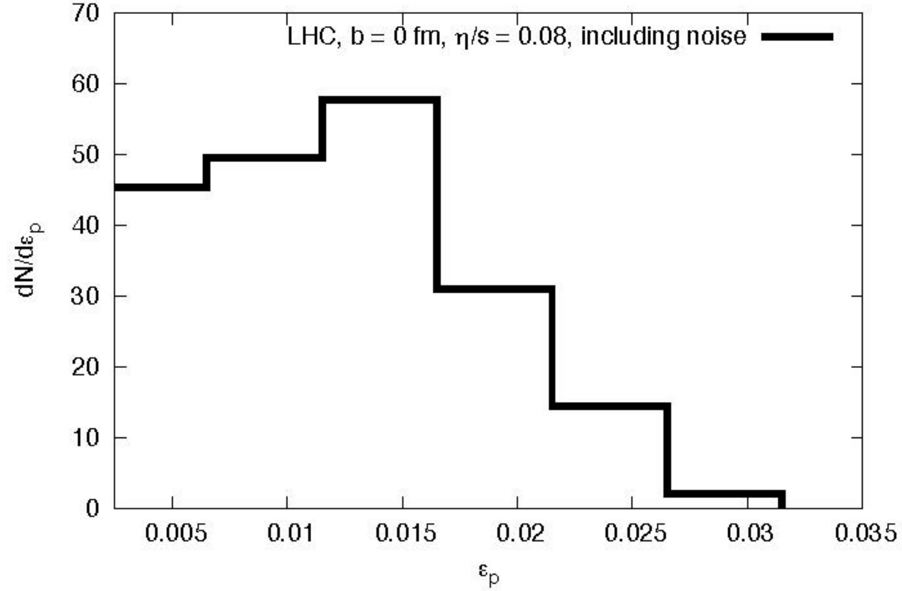
τ = 2.37 fm/c
τ = 6.30 fm/c
τ = 10.24 fm/c
τ = 14.18 fm/c
y [fm/c]
x [fm/c]
δe/e
25
20
15
10
5
0
-5
-10
-15
-20
-25

Eccentricity Fluctuations

b=6 fm



b=0



$$v_2 > 0 \quad \text{even for } \; b = 0$$

Simpler example: Baryon diffusion.

Kapusta & Young

The need for causality

The diffusion equation propagates information instantaneously. No good for hydrodynamic modeling of high energy nuclear collisions!

$$J^\mu = nu^\mu + \Delta J^\mu$$

$$\Delta J^\mu = \sigma T \Delta^\mu (\beta\mu), \quad \Delta_\mu = \partial_\mu - u^\mu (u \cdot \partial)$$

$$\left(\frac{\partial}{\partial t} - D\nabla^2 \right) n = 0$$

$$\sigma = D(\partial n / \partial \mu)$$

What should the current be modified to?

The need for colored noise

Add noise to the current; in the local rest frame:

$$\left\langle I^{\mu}(\mathbf{x},t)\right\rangle = 0, \quad \left\langle I^{i} I^{j}(\mathbf{x},t)\right\rangle = 2\sigma T \delta(\mathbf{x})\delta(t)\delta_{ij}$$

This is white noise (Fourier transform is a constant). It is OK for hydrodynamics if noise is treated as a perturbation, but creates havoc if it is treated nonperturbatively as there will be a dependence on the coarse-grained cell size.

Descriptions of heat conduction

Ordinary diffusion equation - 1st order

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right)n = 0$$

Cattaneo equation (1948) - 2nd order

$$\left(\frac{\partial}{\partial t} - D\nabla^2 + \tau_1 \frac{\partial^2}{\partial t^2}\right)n = 0$$

Gurtin - Pipkin equation (1968) - 3rd order

$$\left(\frac{\partial}{\partial t} - D\nabla^2 + \tau_1 \frac{\partial^2}{\partial t^2} + \tau_2^2 \frac{\partial^3}{\partial t^3} - \tau_3' D \frac{\partial}{\partial t}\nabla^2\right)n = 0$$

The Associated Baryon Current

$$\Delta J^{\mu} = \sigma T \Delta^{\mu} \frac{1 + \tau_4 (u \cdot \partial)}{1 + \tau_1 (u \cdot \partial) + \tau_2^2 (u \cdot \partial)^2 + \tau_3 D \Delta^2} \beta\mu$$

$$\text{Ordinary diffusion equation}: \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$$

$$\text{Cattaneo equation}: \tau_2 = \tau_3 = \tau_4 = 0$$

$$\text{Here } \tau_3' \equiv \tau_3 + \tau_4$$

Fluctuation-Dissipation Theorem

$$A(\mathbf{k},\omega) \equiv \omega + \frac{iDk^2(1-i\tau_4\omega)}{1-i\tau_1\omega-\tau_2^2\omega^2+\tau_3 Dk^2}$$

Response function $$G_R(\mathbf{k},\omega) = \left(\frac{\partial n}{\partial \mu}\right)\frac{\omega}{A}$$

Density correlator $$\left\langle \delta n \delta n(\mathbf{k},\omega)\right\rangle = iT\left(\frac{\partial n}{\partial \mu}\right)\left(\frac{1}{A}-\frac{1}{A^*}\right)$$

Noise $$\frac{1}{3}k^2\left\langle I^l I^l(\mathbf{k},\omega)\right\rangle = -iT\left(\frac{\partial n}{\partial \mu}\right)\left(A-A^*\right)$$

density correlator $\Leftrightarrow$ zeros of A

noise $\Leftrightarrow$ poles of A

Ordinary Diffusion Equation

$$\left\langle \delta n \delta n(\mathbf{x}, t) \right\rangle = T \left(\frac{\partial n}{\partial \mu} \right) \left(\frac{1}{4\pi D t} \right)^{3/2} \exp\left(-r^2 / 4Dt \right)$$

$$\left\langle I^i I^j(\mathbf{x}, t) \right\rangle = 2\sigma T \delta(\mathbf{x}) \delta(t) \delta_{ij}$$

Cattaneo Equation

For the density correlator there is a pair of imaginary poles for $k < k_c$, and a pair of complex poles for $k > k_c$ where $k_c^2 = 1/4\tau_1 D$.

Group velocity $v_g = \dfrac{v_0 k}{\sqrt{k^2 - k_c^2}}, \quad v_0 = \sqrt{\dfrac{D}{\tau_1}}$

(Infinite group velocity is not an issue; Brillouin.)

$$\left\langle I^i I^j(\mathbf{x},t)\right\rangle = \frac{\sigma T}{\tau_1}\delta(\mathbf{x})\exp(-|t|/\tau_1)\delta_{ij}$$

Gurtin-Pipkin Equation

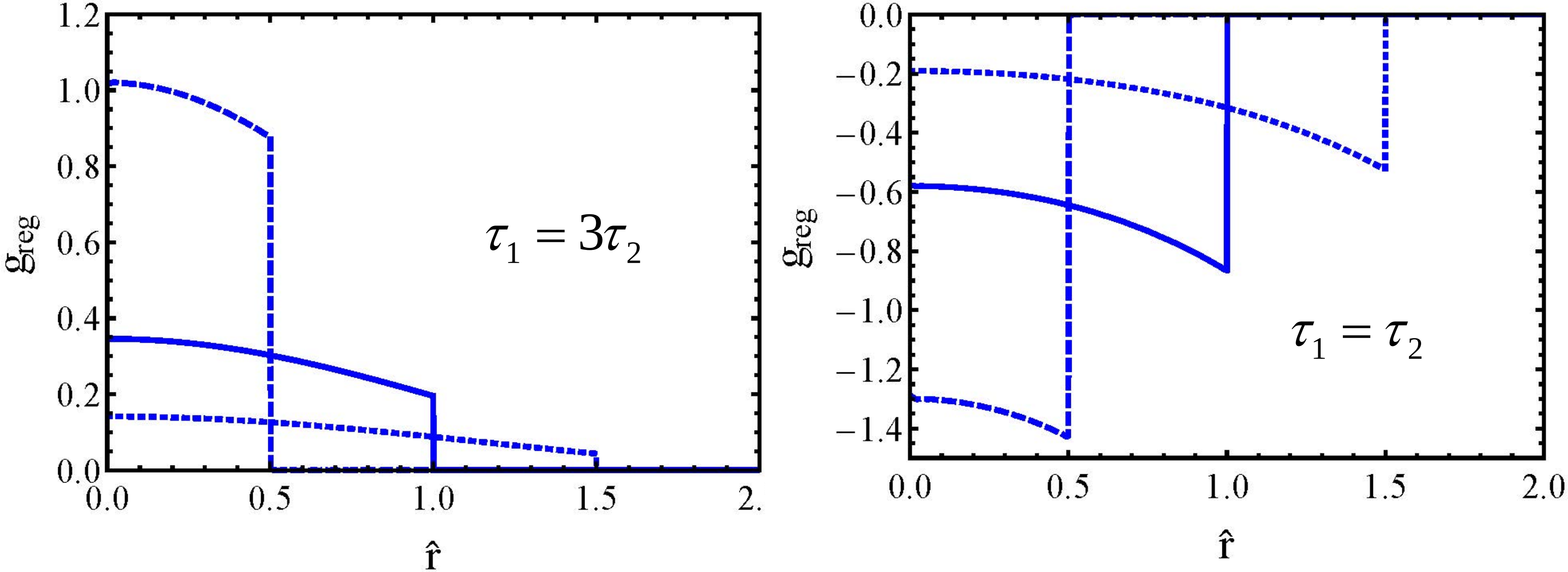
For the noise and for $\tau_1 > 2\tau_2$ there are a pair of imaginary poles for $k < k_c$ and a pair of complex poles for $k > k_c$ with $v_g = \dfrac{v_0 k}{\sqrt{k^2 - k_c^2}}$.

For $\tau_1 < 2\tau_2$ there are a pair of complex poles with $v_g = \dfrac{v_0 k}{\sqrt{k^2 + k_0^2}}$ where $v_0 = \sqrt{\tau_3 D / \tau_2^2}$.

Dimensionless noise

$$g(\widehat{r},\widehat{t}\,)=\frac{\pi}{2}\frac{\exp(-\tau_1\widehat{t}\,/\tau_2)}{\widehat{r}}\delta(\widehat{r}-\widehat{t}\,)+g_{reg}(\widehat{r},\widehat{t}\,)$$

with $\tau_4 = 0$ and $v_0^2 = 1/3.$



When $\tau_1 = 2\tau_2$ there is no wake.

Summary

- Fluctuations and dissipation are intimately related.
- Noise can be implemented perturbatively but non-perturbative implementation is a challenge.
- Colored noise seems to be needed which might require 3rd order fluid dynamics!

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