Geometric Aspects in Supersymmetry

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Outline

What is Supersymmetry?

Some References

- Yu. A. Gol'fand and E. P. Likhtman, "Extension of the Algebra of Poincaré Group and Violation of P Invariance", JETP Lett. 13: 323, (1971).
- P. Ramond, "Dual Theory for Free Fermions", Phys. Rev. D 3 (10): 2415, (1971).
- A. Neveu, J. H. Schwarz, "Factorizable dual model of pions", Nucl. Phys. B 31: 86–112, (1971).
- J.-L. Gervais, B. Sakita, "Field Theory Interpretation of Supergauges in Dual Models". Nucl. Phys. B 34 (2): 632, (1971).
- D.V. Volkov, V.P. Akulov, Pisma Zh. Eksp. Teor. Fiz. 16: 621, (1972).
- J. Wess, B Zumino, "Supergauge Transformations in Four Dimensions". Nucl. Phys. B 70: 39, (1974).

Coleman - Mandula Theorem

S. Coleman, J. Mandula, "All Possible Symmetries of the S Matrix" Phys. Rev. 159: 1251–1256, (1967).

Let G be a connected symmetry group of the S-matrix, *id est* a group whose generators commute with the S-matrix, and make the following assumptions:

1. Lorentz invariance (causality).
2. Particle finiteness and positivity of energy.
3. Weak elastic analyticity.
4. Occurrence of scattering at almost all energies.
5. Technical assumption on the generators' kernel of G.

Coleman - Mandula Theorem

Thus:
The group G is locally isomorphic to the direct product of a compact internal symmetry group and the Poincaré group.

Supersymmetry Algebra

R. Haag, J. T. Lopuszanski and M. Sohnius, "All Possible Generators of Supersymmetries of the S Matrix", Nucl. Phys. B 88: 257, (1975).

The *supersymmetry algebra* is a *graded Lie algebra* of grade one, namely called $\mathbb{Z}_2$ graded Lie algebra:

$$L = L_0 \oplus L_1$$

where L is a vector space in which we assign to any generator $X_\mu \in L$ a degree $g_\mu \equiv g(X_\mu) \in \{0, 1\}$ by defining

$$\begin{array}{lll} g_\mu = 0 & \Leftrightarrow X_\mu \in L_0 & \textit{even} \\ g_\mu = 1 & \Leftrightarrow X_\mu \in L_1 & \textit{odd} \end{array}$$

in order to define a product $\circ$ as follow

$$\circ : (X_\mu, X_\nu) \longrightarrow X_\mu \circ X_\nu \equiv X_\mu X_\nu - (-)^{g_\mu g_\nu} X_\nu X_\mu$$

for any $X_\mu, X_\nu \in L$.

Supersymmetry Algebra

We identify the spaces L_0 and L_1 with

$$\begin{aligned} L_0 &= Span\{P_\mu,\, M_{\mu\nu},\, B_l\} \\ L_1 &= Span\left\{Q^I_\alpha,\, \bar{Q}^I_{\dot\alpha}\right\} \qquad with \quad I = 1, \ldots, N \end{aligned}$$

where P_μ, $M_{\mu\nu}$ are the generators of Poincaré's transformations and B_l are the generators of some compact internal group. While Q^I_α and $\bar{Q}^I_{\dot\alpha}$ are a set of $N + N = 2N$ anticommuting fermionic generators transforming respectively in the Majorana spinor representation $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group, so that the indices α and $\dot\alpha$ denote the spinor component $1, 2$.

Supersymmetry Algebra

$$[P_\mu, P_\nu] = 0$$
$$[M_{\mu\nu}, P_\rho] = i\,(P_\mu \eta_{\nu\rho} - P_\nu \eta_{\mu\rho})$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = i\,(M_{\mu\sigma}\eta_{\nu\rho} - M_{\mu\rho}\eta_{\nu\sigma} - M_{\nu\sigma}\eta_{\mu\rho} + M_{\nu\rho}\eta_{\mu\sigma})$$

$$\left\{ Q^I_\alpha, \bar{Q}_{\dot\alpha\, J} \right\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^I_J$$
$$\left\{ Q^I_\alpha, Q^J_\beta \right\} = \varepsilon_{\alpha\beta} Z^{IJ}$$
$$\left\{ \bar{Q}_{\dot\alpha I}, \bar{Q}_{\dot\beta J} \right\} = \varepsilon_{\dot\alpha\dot\beta}\,(Z_{IJ})^*$$

$$\left[M_{\mu\nu}, Q^I_\alpha \right] = i\,(\sigma_{\mu\nu})_\alpha^{\ \beta}\, Q^I_\beta$$
$$\left[M_{\mu\nu}, \bar{Q}^{\dot\alpha}_I \right] = i\,(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\, \bar{Q}^{\dot\beta}_I$$
$$\left[P_\mu, Q^I_\alpha \right] = 0$$
$$\left[P_\mu, \bar{Q}^{\dot\alpha}_I \right] = 0$$

$$[B_l, B_m] = i f^n_{lm} B_n$$
$$[P_\mu, B_l] = 0$$
$$[M_{\mu\nu}, B_l] = 0$$

$$\left[Q^I_\alpha, B_l \right] = (b_l)^I{}_J\, Q^J_\alpha$$
$$\left[\bar{Q}_{\dot\alpha\, I}, B_l \right] = -\bar{Q}_{\dot\alpha\, J}\, (b^*_l)_I{}^J$$

Superspace

The superPoincaré Lie group mathematically is the contracted orthosymplectic group $\overline{Osp(4 \mid 1)}$. Given a generic group element of $\overline{Osp(4 \mid 1)}$ as

$$G\left(x,\theta,\bar{\theta},\omega\right)=e^{i\left(x_{\mu}P^{\mu}+\theta Q+\bar{\theta}\bar{Q}+\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}\right)},$$

the $\mathcal{N}=1$ superspace is defined as the eight dimensional coset space

$$\mathcal{M}_{4|1}=\frac{\overline{Osp(4 \mid 1)}}{SO(1,3)}.$$

Parametrized by the *local coordinates* $z^{M}\equiv\left(x^{\mu},\theta^{\alpha},\bar{\theta}_{\dot{\alpha}}\right)$. Each coset class, that correspond to a class of equivalent points on the superspace $\mathcal{M}_{4|1}$, is identified with a unique group representative element. This can be written as

$$\left(x^{\mu},\theta^{\alpha},\bar{\theta}_{\dot{\alpha}}\right)\longleftrightarrow e^{i\left(x_{\mu}P^{\mu}+\theta Q+\bar{\theta}\bar{Q}\right)}.$$

Superfields

The most general superfield $S = S\left(x,\theta,\bar{\theta}\right)$ that can be defined on $\mathcal{M}_{4|1}$ derives from the power series expansion of S in θ and $\bar{\theta}$

$$S\left(x,\theta,\bar{\theta}\right) = f\left(x\right) + \theta\psi\left(x\right) + \bar{\theta}\bar{\chi}\left(x\right) + \theta\theta m\left(x\right) + \bar{\theta}\bar{\theta}n\left(x\right) +$$

$$+\theta\sigma^{\mu}\bar{\theta}v_{\mu}\left(x\right) + \theta\theta\bar{\theta}\bar{\lambda}\left(x\right) + \bar{\theta}\bar{\theta}\theta\rho\left(x\right) + \theta\theta\bar{\theta}\bar{\theta}d\left(x\right)\ ,$$

where from the Grassman nature of the coordinates θ and $\bar{\theta}$, all higher powers vanish. Every element of the expansion is a usual field with some non-trivial tensor structure: scalar, spinor, vector *et cetera*.

Chiral and Vector Superfields

Irreducible representations of the supersymmetry algebra are realized by imposing some supersymmetric constrains on $S\left(x,\theta,\bar{\theta}\right)$. We have the two main following cases:

1. $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$ - Chiral superfield Φ

$$\begin{aligned}\Phi\left(x,\theta,\bar{\theta}\right) = \quad & \phi\left(x\right) + i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi\left(x\right) + \tfrac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi\left(x\right) \\ & +\sqrt{2}\theta\psi\left(x\right) - \tfrac{i}{\sqrt{2}}\theta\theta\partial_{\mu}\psi\left(x\right)\sigma^{\mu}\bar{\theta} + \theta\theta F\left(x\right)\end{aligned}$$

2. $V = \bar{V}$ - Vector superfield V

$$\begin{aligned}V\left(x,\theta,\bar{\theta}\right) = \quad & C\left(x\right) + i\theta\chi\left(x\right) - i\bar{\theta}\bar{\chi}\left(x\right) + \tfrac{1}{2}i\theta\theta\left[M\left(x\right) + iN\left(x\right)\right] \\ & -\tfrac{1}{2}i\theta\theta\left[M\left(x\right) - iN\left(x\right)\right] + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D - \tfrac{1}{2}\Box C\right] \\ & +\theta\sigma^{\mu}\bar{\theta}v_{\mu}\left(x\right) + i\theta\theta\bar{\theta}\left[\bar{\lambda}\left(x\right) + \tfrac{i}{2}\bar{\sigma}^{\mu}\partial_{\mu}\chi\left(x\right)\right] \\ & -i\bar{\theta}\bar{\theta}\theta\left[\lambda\left(x\right) + \tfrac{i}{2}\sigma^{\mu}\partial_{\mu}\bar{\chi}\left(x\right)\right]\end{aligned}$$

Supersymmetric Lagrangians

Supersymmetric Matter Lagrangian:

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta}\, K\left(\Phi, \bar{\Phi}\right) + \int d^2\theta\; W\left(\Phi\right) + \int d^2\bar{\theta}\; \bar{W}\left(\bar{\Phi}\right) \; .$$

Supersymmetric Gauge Lagrangians:

$$\mathcal{L}_{Abelian} = \frac{1}{32}\left[\int d^2\theta\; G^{\alpha} G_{\alpha} + \int d^2\bar{\theta}\; \bar{G}_{\dot{\alpha}} \bar{G}^{\dot{\alpha}}\right] .$$

Kahler potential:

$$K\left(\Phi, \bar{\Phi}\right) = \sum_{m,n=1}^{+\infty} a_{mn} \bar{\Phi}^m \Phi^n, \qquad \text{Renormalizable when } K_R\left(\Phi, \bar{\Phi}\right) = a\bar{\Phi}\Phi.$$

Superpotential:

$$W\left(\Phi\right) = \sum_{n=1}^{+\infty} c_n \Phi^n, \qquad \text{Renormalizable for } n \leq 3.$$

Field strength superfield: $G_{\alpha}\left(x, \theta, \bar{\theta}\right) = \bar{D}^2 D_{\alpha} V\left(x, \theta, \bar{\theta}\right) .$

Geometric Aspects

Non-linear σ-models

Consider the general action of a supersymmetric matter Lagrangian for a set of chiral superfields Φ^i and $\bar{\Phi}_i$

$$\mathcal{A} = \int d^4x \left[\int d^2\theta d^2\bar{\theta}\, K\left(\Phi^i, \bar{\Phi}_i\right) + \int d^2\theta\, W\left(\Phi^i\right) + \int d^2\bar{\theta}\, \bar{W}\left(\bar{\Phi}_i\right) \right].$$

We adopt the following notations to denote the derivatives of the Kähler potential:

$$K_i = \frac{\partial}{\partial\phi^i} K\left(\phi, \bar{\phi}\right), \qquad K^j = \frac{\partial}{\partial\bar{\phi}_j} K\left(\phi, \bar{\phi}\right), \qquad K_i^j = \frac{\partial^2}{\partial\phi^i \partial\bar{\phi}_j} K\left(\phi, \bar{\phi}\right),$$

et cetera. And similarly for the superpotential

$$W_i = \frac{\partial}{\partial\phi^i} W\left(\phi\right), \qquad W_{ij} = \frac{\partial^2}{\partial\phi^i \partial\phi^j} W\left(\phi\right).$$

Non-linear σ-models

A Taylor expansion of the above action leads to

$$
\begin{aligned}
\mathcal{A}_\sigma &= \int d^4x \left[\int d^2\theta d^2\bar\theta\, K\left(\Phi, \bar\Phi\right) + \int d^2\theta\; W\left(\Phi\right) + \int d^2\bar\theta\; \bar W\left(\bar\Phi\right) \right] \\
&= \int d^4x \left[K_i^j \left(\partial_\mu \phi^i \partial^\mu \bar\phi_j - \frac{i}{2} \psi^i \sigma^\mu D_\mu \bar\psi_j + \frac{i}{2} D_\mu \psi^i \sigma^\mu \bar\psi_j \right) - \left(K^{-1}\right)_j^i W_i W^j \right. \\
&\quad - \frac{1}{2}\left(W_{ij} - \Gamma_{ij}^k W_k \right) \psi^i \psi^j - \frac{1}{2}\left(W^{ij} - \Gamma_k^{ij} W^k \right) \bar\psi_i \bar\psi_j + \frac{1}{4} R_{ij}^{kl} \psi^i \psi^j \bar\psi_k \bar\psi_l .
\end{aligned}
$$

Where we have defined the *Levi-Civita-Kähler connection* and the *Riemann-Kähler tensor:*

$$
\Gamma_{ij}^l = \left(K^{-1}\right)_k^l K_{ij}^k , \qquad \Gamma_l^{ij} = \left(K^{-1}\right)_l^k K_k^{ij} ,
$$

$$
R_{ij}^{kl} = K_{ij}^{kl} - K_{ij}^m \left(K^{-1}\right)_m^n K_n^{kl} .
$$

As well as the *Kähler covariant derivative* of spinors:

$$
\begin{aligned}
D_\mu \psi^i &\equiv \partial_\mu \psi^i + \Gamma_{jk}^i \left(\partial_\mu \phi^j\right) \psi^k = \partial_\mu \psi^i + \left(K^{-1}\right)_l^i K_{jk}^l \left(\partial_\mu \phi^j\right) \psi^k , \\
D_\mu \bar\psi_j &\equiv \partial_\mu \bar\psi_j + \Gamma_j^{ki} \left(\partial_\mu \bar\phi_k\right) \bar\psi_i = \partial_\mu \bar\psi_j + \left(K^{-1}\right)_j^l K_l^{ki} \left(\partial_\mu \bar\phi_k\right) \bar\psi_i \; .
\end{aligned}
$$

Non-linear σ-models

1. The coefficient K_i^j on the first term in the action $\mathcal{A}_\sigma$

$$K_i^j = \frac{\partial^2}{\partial \phi^i \partial \bar{\phi}_j} K\left(\phi, \bar{\phi}\right) \equiv g_{i\underline{j}}\left(\phi, \bar{\phi}\right)$$

   is Hermitian since K is a real scalar function. Moreover it is positive definite and non-singular. Therefore we may interpret K_i^j as a metric called the *Kähler metric* $g_{i\underline{j}}$. In addition, the complex scalar fields ϕ^i and $\bar{\phi}_j$ may be interpreted as *local complex coordinates* on a target complex manifold.

2. A complex manifold is said to be Kähler if the holomorphic and antiholomorphic curl of the metric vanishes:

$$\partial_k g_{i\bar{j}} - \partial_i g_{k\bar{j}} = 0\,, \qquad \text{and} \qquad \partial_{\bar{k}} g_{i\bar{j}} - \partial_{\bar{j}} g_{i\bar{k}} = 0\,,$$

   where the new notation $\underline{i}\,, \underline{j}\,, \ldots$ indicates the antiholomorphic coordinates $\bar{\phi}_j$. The above Kähler conditions are naturally satisfied by the Kähler metric $g_{i\underline{j}}$ since the derivatives in respect to the scalar fields commute. *Id est, the target manifold of the supersymmetric sigma model is Kähler.*

The Sphere Rolls

Let us consider a rotation of a two dimensional sphere along the three axes (x, y, z) close to the identity, that is a rotation over the three infinitesimal angles (α, β, γ). The rotation along one axis may be described by an element R of the group $SO(3)$. Given two elements R_i, $R_j \in SO(3)$:

$$[R_i, R_j] = \varepsilon_{ij}^{\ k} R_k \qquad \text{for any} \quad i, j, k = x, y, z,$$

we read, for an infinitesimal rotation along the three axes:

$$R(\alpha, \beta, \gamma) = \begin{pmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{pmatrix}.$$

By projecting a rotated point of the sphere on a complex tangent plane we find the corresponding change in the complex so-called stereographic coordinates $(\phi, \bar{\phi})$. This transformation may then be cast as

$$\begin{aligned} \phi' &= \phi + \delta\phi = (1 + i\gamma)\phi + \frac{1}{2}\beta\left(1 + \phi^2\right) - \frac{i}{2}\alpha\left(1 - \phi^2\right), \\ \bar{\phi}' &= \bar{\phi} + \delta\bar{\phi} = (1 - i\gamma)\bar{\phi} - \frac{1}{2}\beta\left(1 + \bar{\phi}^2\right) + \frac{i}{2}\alpha\left(1 - \bar{\phi}^2\right). \end{aligned}$$

The Sphere Rolls

Under such infinitesimal rotation, the metric transforms as

$$g'_{\phi'\bar{\phi}'} = \frac{1}{\left[1+\bar{\phi}'\phi'\right]^2} = \frac{1}{\left[\left(1+\bar{\phi}\phi\right)\left(1+\beta\mathrm{Re}\phi-\alpha\mathrm{Im}\phi\right)\right]^2} \cong g_{\phi\bar{\phi}} \, .$$

And as a consequence the line element of the sphere is also invariant: $ds'^2 = ds^2$. Therefore, the above transformation is an *isometry* generated by the infinitesimal vector fields $\delta\phi$ and $\delta\bar{\phi}$. Hence, we may conclude that $\delta\phi \equiv \xi$ and $\delta\bar{\phi} \equiv \bar{\xi}$ are Killing vector fields, and they obey the *Killing equation*:

$$\frac{\partial\xi}{\partial\phi} + \frac{\partial\bar{\xi}}{\partial\bar{\phi}} = 0.$$

Moreover their linear combination as well as their Lie bracket $\left[\xi, \bar{\xi}\right]$ are also Killing vector fields. It follows that they satisfy the Lie algebra of the isometry group of transformations that may be identified by representing the sphere $\mathcal{S}^2$ as

$$\mathcal{S}^2 \simeq \frac{SU(2)}{U(1)}.$$

Killing vector fields are therefore generating a global $SU(2)$ transformation over the complex coordinates.

Supersymmetry on the Sphere

Given the Kähler potential as:

$$K\left(\Phi,\bar{\Phi}\right)=\ln\left(1+\bar{\Phi}\Phi\right), \qquad \frac{\partial^2 K}{\partial\phi\partial\bar{\phi}}=\frac{1}{\left(1+\bar{\phi}\phi\right)^2}=g_{\phi\bar{\phi}}.$$

The supersymmetric Lagrangian invariant under the above non-linear $SU(2)$ transformations reads

$$\mathcal{L}_{sphere-on}=\frac{1}{\left(1+\bar{\phi}\phi\right)^2}\left[\partial_\mu\bar{\phi}\partial^\mu\phi-\frac{i}{2}\psi\sigma^\mu D_\mu\bar{\psi}+\frac{i}{2}D_\mu\psi\sigma^\mu\bar{\psi}\right]+\frac{1}{4}R^{\phi\phi}_{\phi\phi}\bar{\psi}^2\psi^2,$$

where

$$D_\mu\psi=\partial_\mu\psi+\Gamma^{\phi}_{\phi\phi}\left(\partial_\mu\phi\right)\psi,$$
$$D_\mu\bar{\psi}=\partial_\mu\bar{\psi}+\Gamma^{\phi\phi}_{\phi}\left(\partial_\mu\bar{\phi}\right)\bar{\psi},$$

are the Kähler covariant derivatives of fermions.

Open Conclusions

- Homogeneous spaces of Kähler type have been widely studied and fully classified by mathematicians. Every one of such models offers a different particle spectra, based exclusively on the geometric features of the Kähler target space. The above scheme can potentially be extended to larger algebraic structures. Therefore, unified models can be thought, which may, include the gauge group of the Standard Model, like $\frac{SO(2n)}{U(n)}$, $\frac{SU(5)}{SU(3)\times SU(2)\times U(1)}$, $\frac{E_6}{SO(10)\times U(1)}$, *et cetera*. As yet, not all the physics and phenomenology of this class of models have been explored.

Afterword

- General criteria are still wanted in order to classify the particle spectra of the above class of supersymmetric models, for the purpose of devolving simplified models that can be easily tested at colliders.
- These constructions are plagued by anomalies (vector and spinor currents), for which further calculation techniques are required. It is hence an open challenge for mathematicins to understand the reasons why these anomalies arise, even in the case of "good" manifolds.
- New hints to simplify, and extend, this model building scheme are under study and bring the name of Linear Gauged Sigma Models, offering a vast field of investigation.

Thank you very much