
RENORMALIZED
HAMILTONIAN
TRUNCATION
J. Elias-Miró, M. Montull, M.R.

# An example in QM: Anharmonic oscillator

$$H = H_0 + V = \frac{1}{2}(p^2 + x^2) + \lambda x^4$$

The best approach to find the spectrum is the Rayleight-Ritz method:

Build the interacting Hamiltonian for states below a given energy ET and diagonalize it

$$H_0|n\rangle = E_n|n\rangle$$

$$(H_{Trunc})_{ij} = \langle i|H_0 + V|j\rangle\,, \quad i,j < N_{Trunc}$$

Converges exponentially in ET!

# Hamiltonian truncation for a QFT

Start in a well controlled enviroment:

Mass perturbation for a scalar field in 1+1 dimensions

$$S_0 = \frac{1}{2}\int_{-\infty}^{\infty} dt \int_0^L dx \; :(\partial\phi)^2 - m^2\phi^2: \; , \qquad V = g_2 \int_0^L dt \; :\phi^2:$$

The theory is studied in a cylinder of length L,

$$\phi(x) = \sum_k \frac{1}{\sqrt{2L\omega_k}} \left( a_k e^{ikx} + a_k^\dagger e^{-ikx} \right) \, ,$$

so the free eigenstates are

$$|\, E_i \,\rangle = \frac{a_{k_N}^{\dagger\, n_N}}{\sqrt{n_N!}} \cdots \frac{a_{k_2}^{\dagger\, n_2}}{\sqrt{n_2!}} \frac{a_{k_1}^{\dagger\, n_1}}{\sqrt{n_1!}} |\, 0 \,\rangle \qquad E_i = \sum_{s=1}^N n_s \sqrt{k_s^2 + m^2}$$

# Hamiltonian truncation for a QFT

Once the free theory is normal ordered, the vacuum energy including the mass perturbation can be found via Bogolyubov transformation:

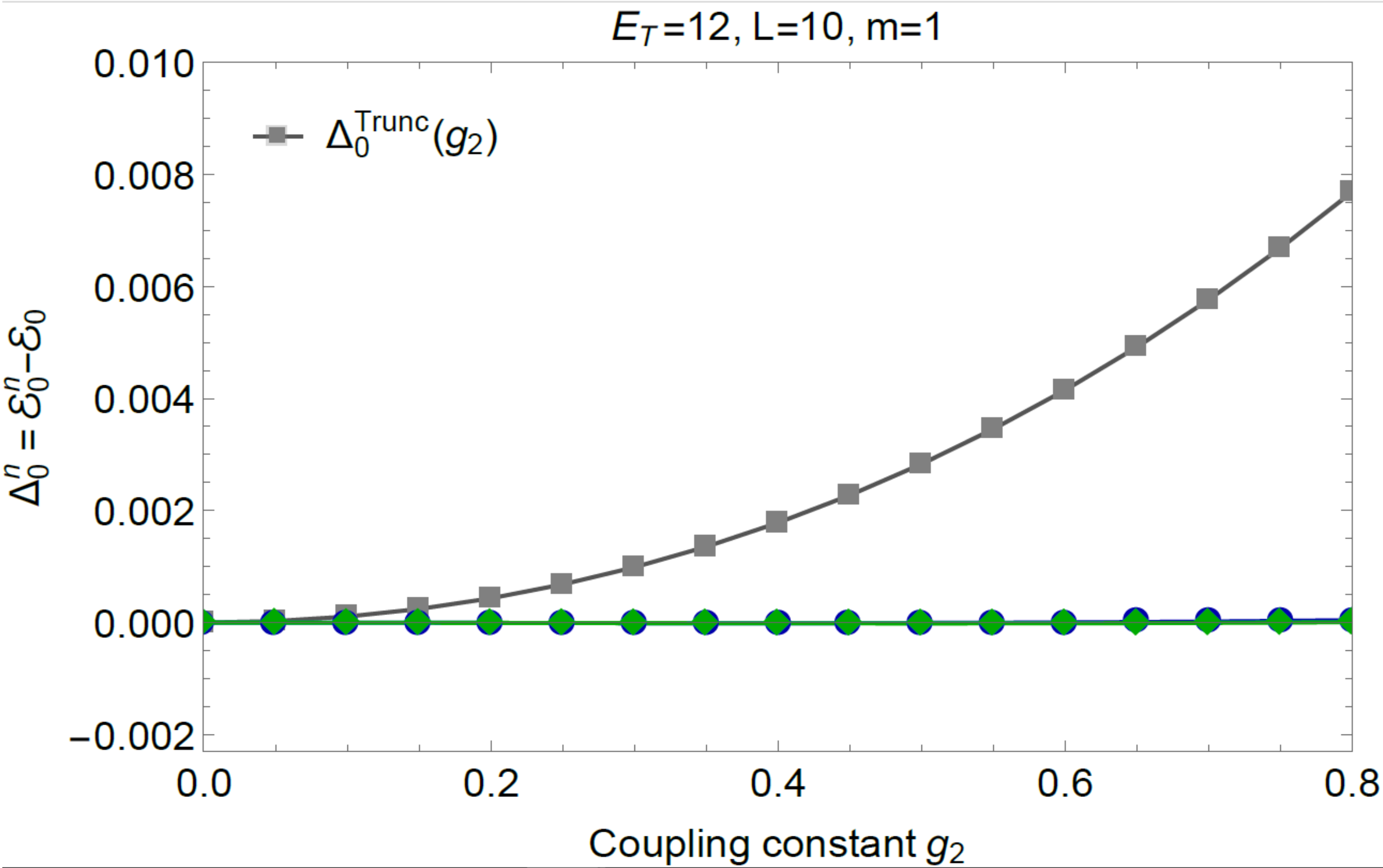
$$\mathcal{E}_0(g_2) = \frac{L\left(m^2 + 2g_2\right)}{8\pi}\left[\log\left(\frac{m^2}{m^2 + 2g_2}\right) + \frac{2g_2}{m^2 + 2g_2}\right],$$

Then the spectrum is just

$$\mathcal{E}_n = \mathcal{E}_0(g_2) + \sum_i \sqrt{k_i^2 + m^2 + 2g_2}$$

# Hamiltonian truncation for a QFT

Vacuum energy of the truncated Hamiltonian:




2% error
At g=0.8!

# **Renormalized** Hamiltonian Truncation

One can improve the method by taking into account the effect of the heavy states:

$$H = \begin{pmatrix} H_{ll} & H_{hl} \\ H_{lh} & H_{hh} \end{pmatrix}$$

$$|\,\mathcal{E}_h\,\rangle \;=\; (\mathcal{E} - H_{hh})^{-1}\, H_{hl}|\,\mathcal{E}_l\,\rangle$$

$$H_{eff} \equiv H_T + \Delta H(\mathcal{E}),$$

$$\Delta H(\mathcal{E}) = V_{lh}\frac{1}{\mathcal{E} - H_{0\,hh} - V_{hh}}V_{hl}\,,$$

# **Renormalized** Hamiltonian Truncation

Expand as a geometric series:

$$\Delta H(\mathcal{E},\, E_T) = \sum_{n=0}^{\infty} \Delta H_n(\mathcal{E}, E_T)$$

This is what we want

$$= V_{lh} \frac{1}{\mathcal{E} - E_{hh}} \left( V_{hh} \frac{1}{\mathcal{E} - E_{hh}} \right)^n V_{hl}$$

But this is what we will compute!

$$= \sum_{j_1, \cdots, j_n - 1 :\, E_{j_i} > E_T} V_{rj_1} \frac{1}{\mathcal{E} - E_{j_1}} V_{j_1 j_2} \frac{1}{\mathcal{E} - E_{j_2}} V_{j_2 j_3} \cdots V_{j_{n-2} j_{n-1}} \frac{1}{\mathcal{E} - E_{j_{n-1}}} V_{j_{n-1} s}\,,$$

**The difference is in the pole structure**

$$\Delta \widehat{H}_n(\mathcal{E})_{rs} = \sum_{j_1, \ldots, j_{n-1} = 1}^{\infty} V_{rj_1} \frac{1}{\mathcal{E} - E_{j_1}} V_{j_1 j_2} \frac{1}{\mathcal{E} - E_{j_2}} V_{j_2 j_3} \cdots V_{j_{n-2} j_{n-1}} \frac{1}{\mathcal{E} - E_{j_{n-1}}} V_{j_{n-1} s}$$

# **Renormalized** Hamiltonian Truncation

Can be rewritten as a half-Fourier transform of n potentials

$$\Delta\widehat{H}_n(\mathcal{E})_{rs} = \sum_{j_1,\ldots,j_{n-1}=1}^{\infty} V_{rj_1}\frac{1}{\mathcal{E}-E_{j_1}}V_{j_1j_2}\frac{1}{\mathcal{E}-E_{j_2}}V_{j_2j_3}\cdots V_{j_{n-2}j_{n-1}}\frac{1}{\mathcal{E}-E_{j_{n-1}}}V_{j_{n-1}s}$$

$$= \lim_{\epsilon\to 0}(-i)^{n-1}\int_0^\infty dt_1\cdots dt_{n-1}\; e^{i(\mathcal{E}-E_r+i\epsilon)(t_1+\cdots t_{n-1})}\,\mathcal{T}\left\{V\left(T_1\right)\cdots V\left(T_n\right)\right\}_{rs}$$

Now we know how to compute this!

The $\Delta\widehat{H}_n(\mathcal{E})_{rs}$ operator splits into **non-local** operators that look like even powers of $\phi$

## **Renormalized** Hamiltonian Truncation

Renormalization of the mass perturbation theory:

One of the generated operators is proportional to the identity,

$$\Delta\widehat{H}_2^{\mathbb{1}}(\mathcal{E})_{rs} = \lim_{\epsilon\to 0} -is_2 g_2^2 \int_0^\infty dt \int_{-L/2}^{L/2} dz\; e^{i(\mathcal{E}-E_r+i\epsilon)t_1} D_L^2(t,z)\,\mathbb{1}_{rs}$$

$$= \frac{s_2 g_2^2}{4L} \sum_k \frac{1}{\omega_k^2} \frac{1}{\mathcal{E}-E_r-2\omega_k}\,\mathbb{1}_{rs}$$

By the way, this equation is **EXACT**

Recall the pole identification!!

$$\Delta H_2^{\mathbb{1}}(\mathcal{E})_{rs} = \frac{s_2 g_2^2}{L} \sum_{k:\,E_r+2\omega_k>E_T} \frac{1}{4\,\omega_k^2} \frac{1}{\mathcal{E}-E_r-2\omega_k}\,\mathbb{1}_{rs}$$

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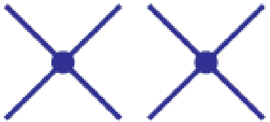
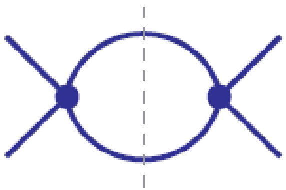
The computations **AT ANY ORDER** can get trivialized by using funny diagrams with some nice properties:

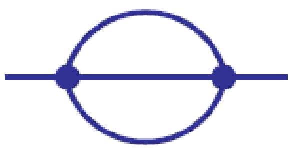


$$= s_0 \frac{L}{\mathcal{E} + i\epsilon}$$

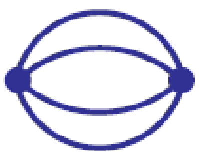
$$= \frac{s_1}{2} \frac{\lambda^2}{m} \frac{1}{\mathcal{E} - m + i\epsilon}$$



$$= \frac{s_2}{2^2} \frac{\lambda^2}{L} \sum_k \frac{1}{\omega_k^2} \frac{1}{\mathcal{E} - 2\omega_k + i\epsilon}$$

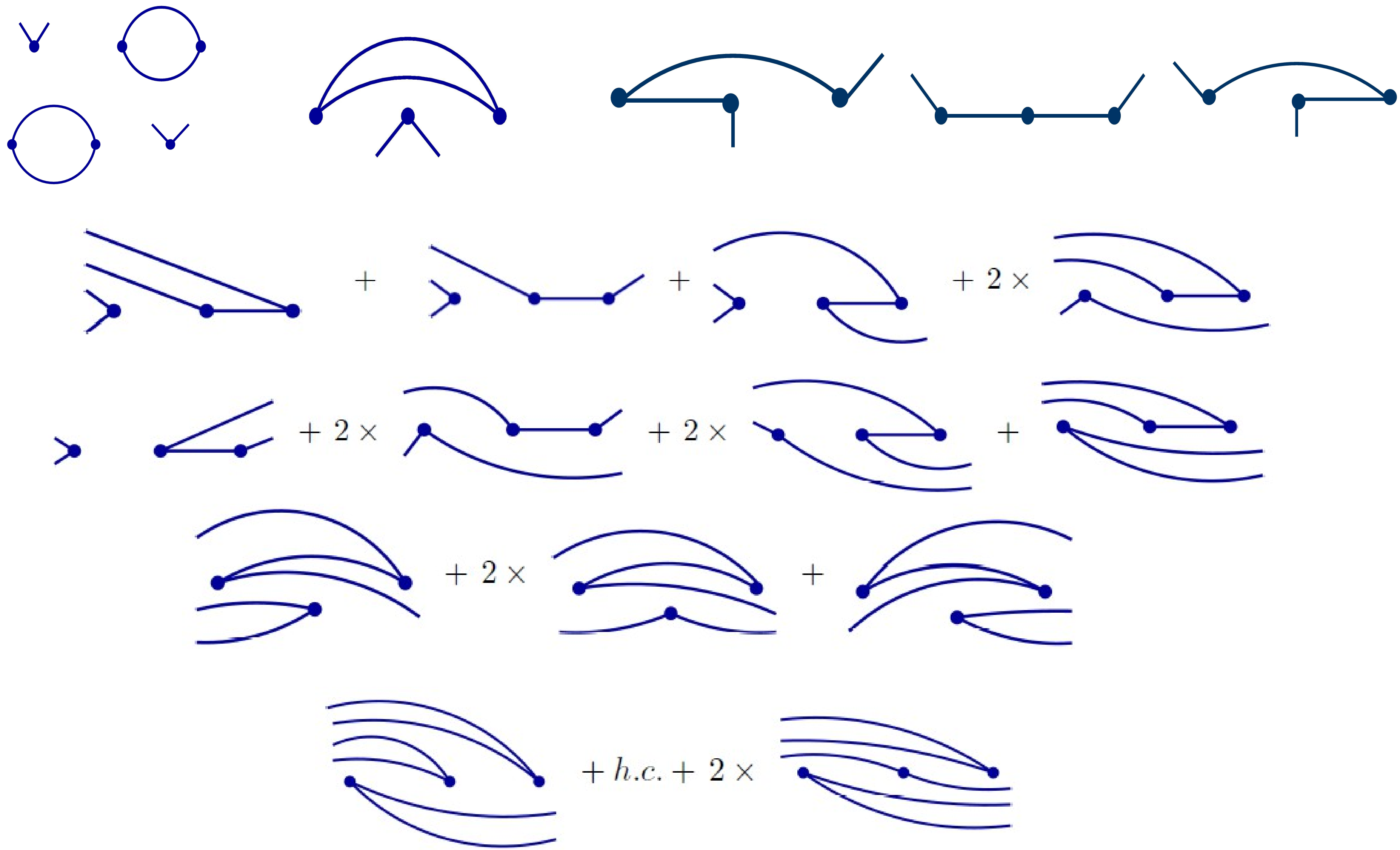


$$= \frac{s_3}{2^3} \frac{\lambda^2}{L^2} \sum_{k,q,p} \frac{1}{\omega_k} \frac{1}{\omega_q} \frac{1}{\omega_p} \frac{1}{\mathcal{E} - \omega_k - \omega_q - \omega_p + i\epsilon} \delta_{0,k+p+q}$$

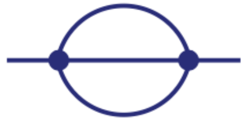


$$= \frac{s_4}{2^4} \frac{\lambda^2}{L^3} \sum_{k,q,p,t} \frac{1}{\omega_k} \frac{1}{\omega_q} \frac{1}{\omega_p} \frac{1}{\omega_r} \frac{1}{\mathcal{E} - \omega_k - \omega_q - \omega_p - \omega_r + i\epsilon} \delta_{0,k+p+q+r}$$

# The mass perturbation at 3rd order (VVV)



# The operators are related to the n-particle phase space



$$= s_3\lambda^2 \int \frac{dE}{2\pi} \frac{1}{\mathcal{E} - E + i\epsilon} \Phi_3(E)$$

$$\Phi_3(E) = \frac{g^2}{\pi} \frac{1}{(E-m)} \frac{1}{\sqrt{(E+m)^2 - 4m^2}} K(\alpha) \,,$$

$$\alpha = 1 - \frac{16Em^3}{(E-m)^3(E+3m)} \text{ and } K(\alpha) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-\alpha\sin^2(\varphi)}}$$

$$s_3/(2\pi)\Phi_3(E) = \frac{72}{E^2\pi^2} \log\left(\frac{E}{m}\right) + \mathcal{O}(m/E)$$

In agreement with Rychkov, Vitale '15

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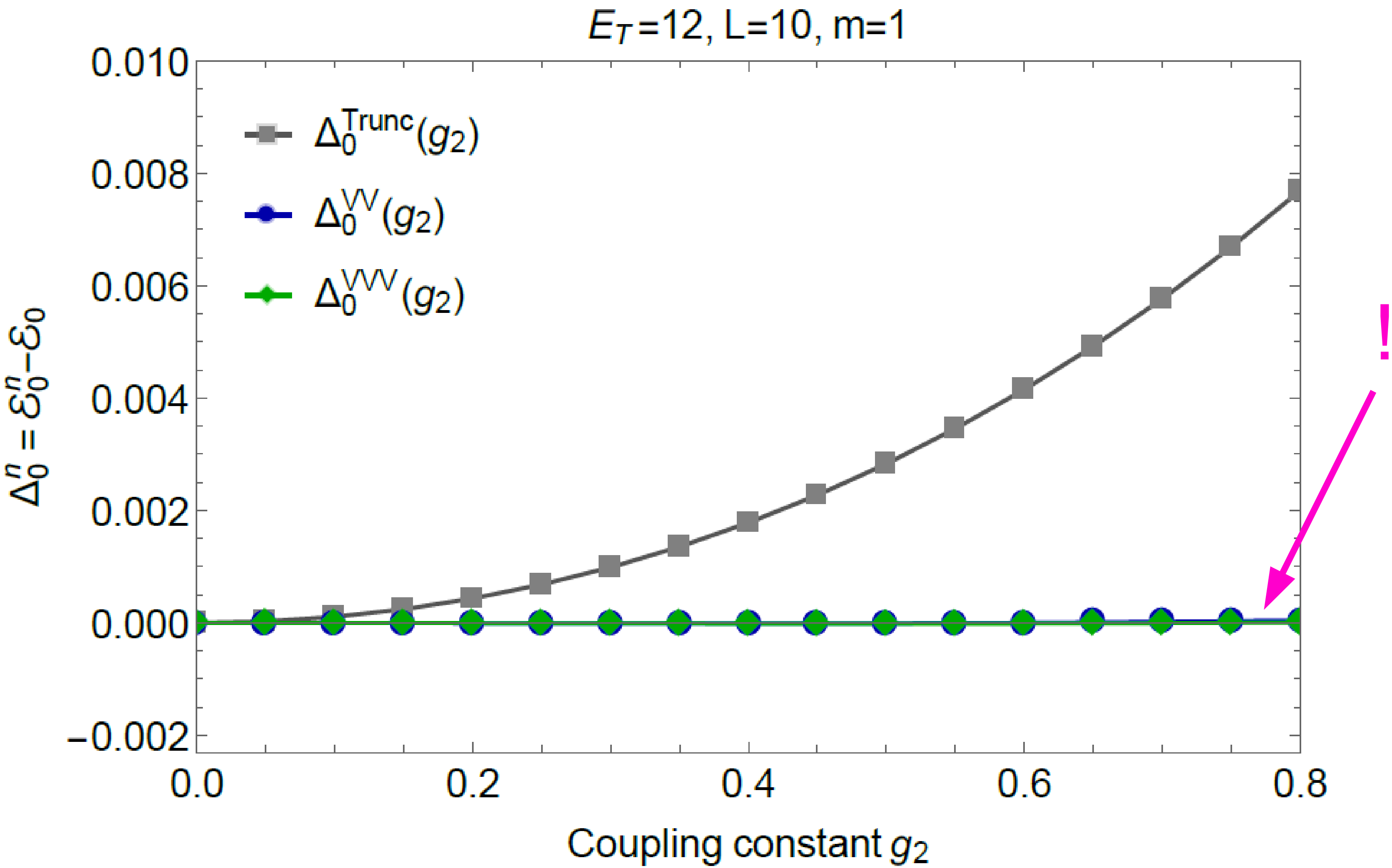
# Numerical results



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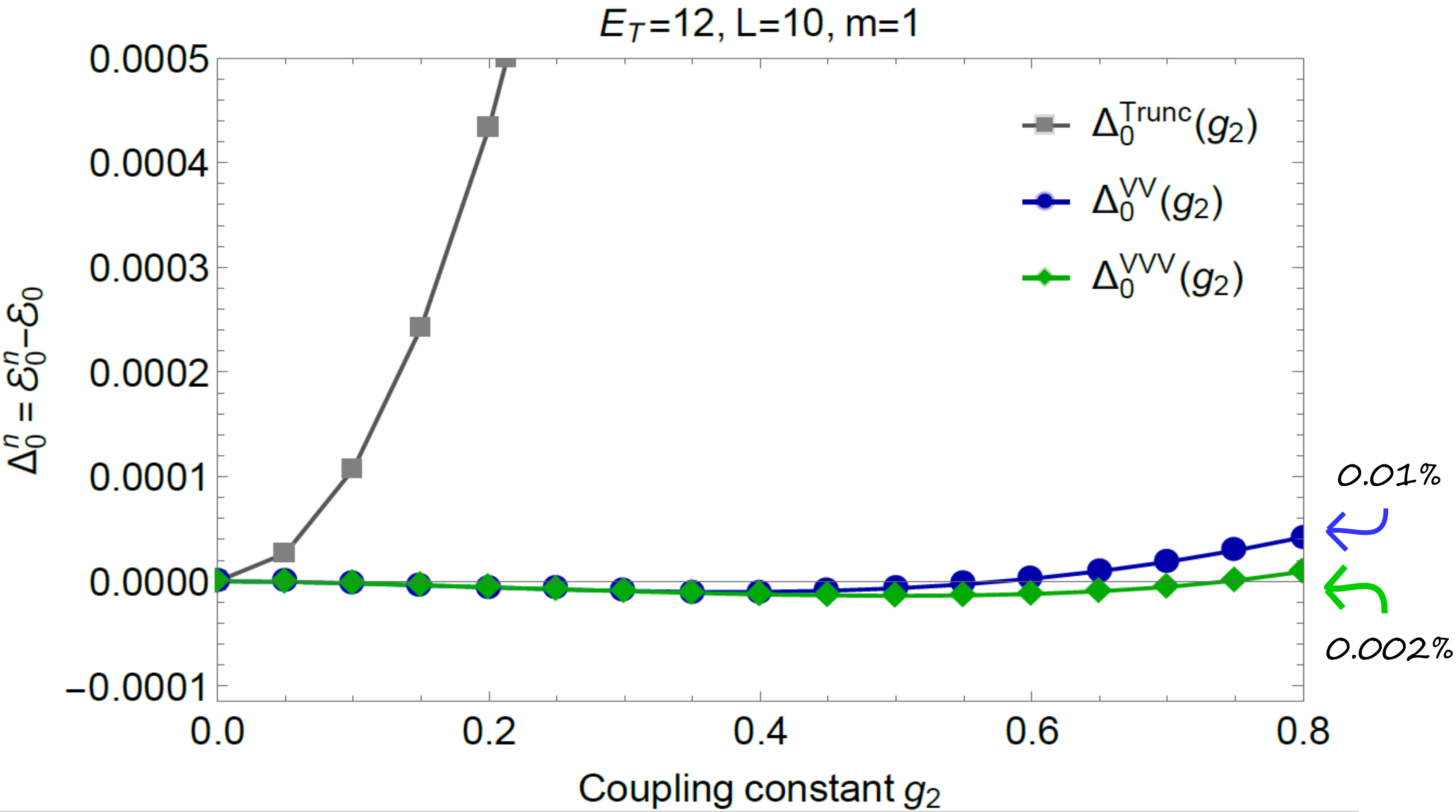

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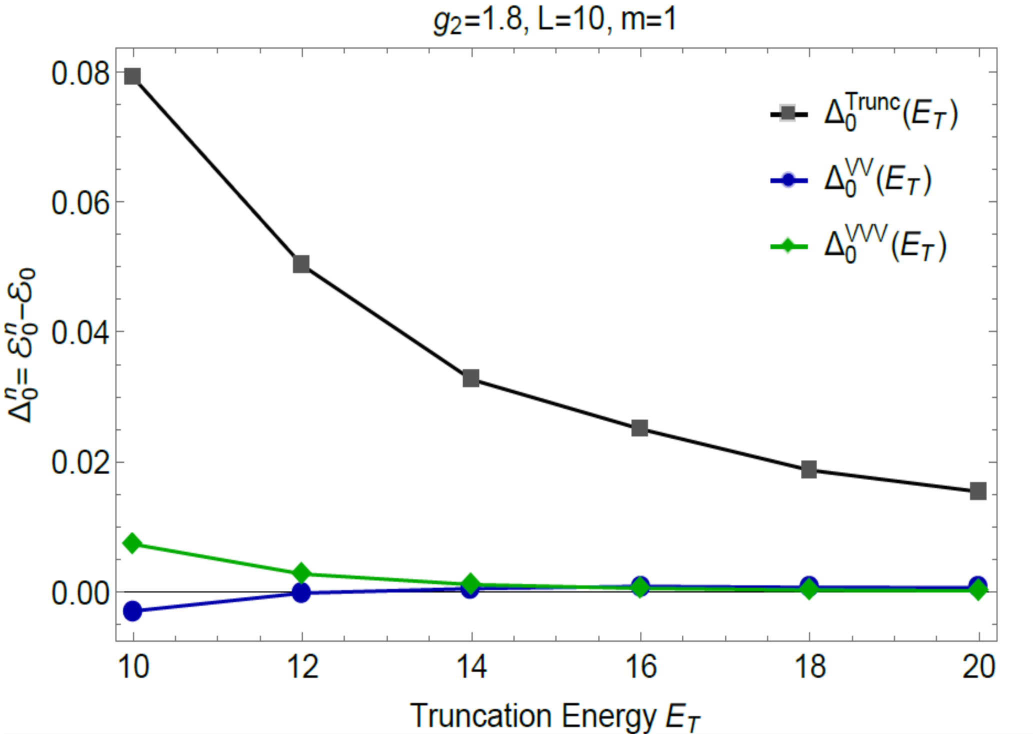


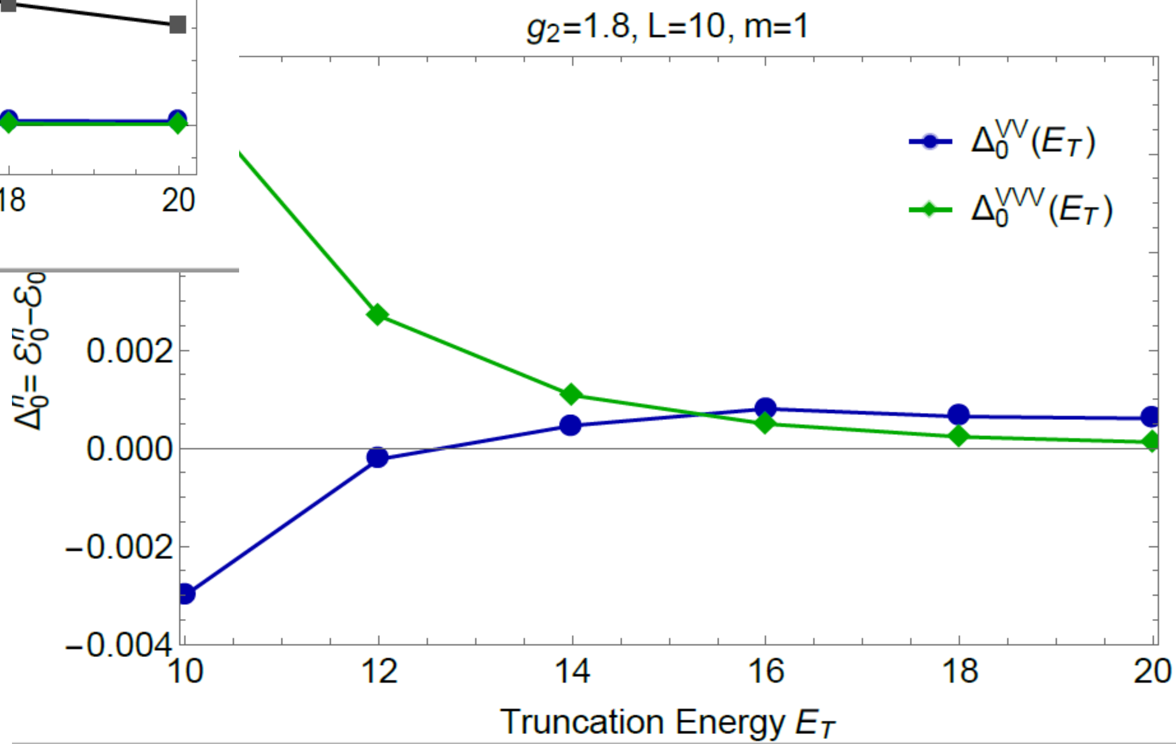

# Conclusions

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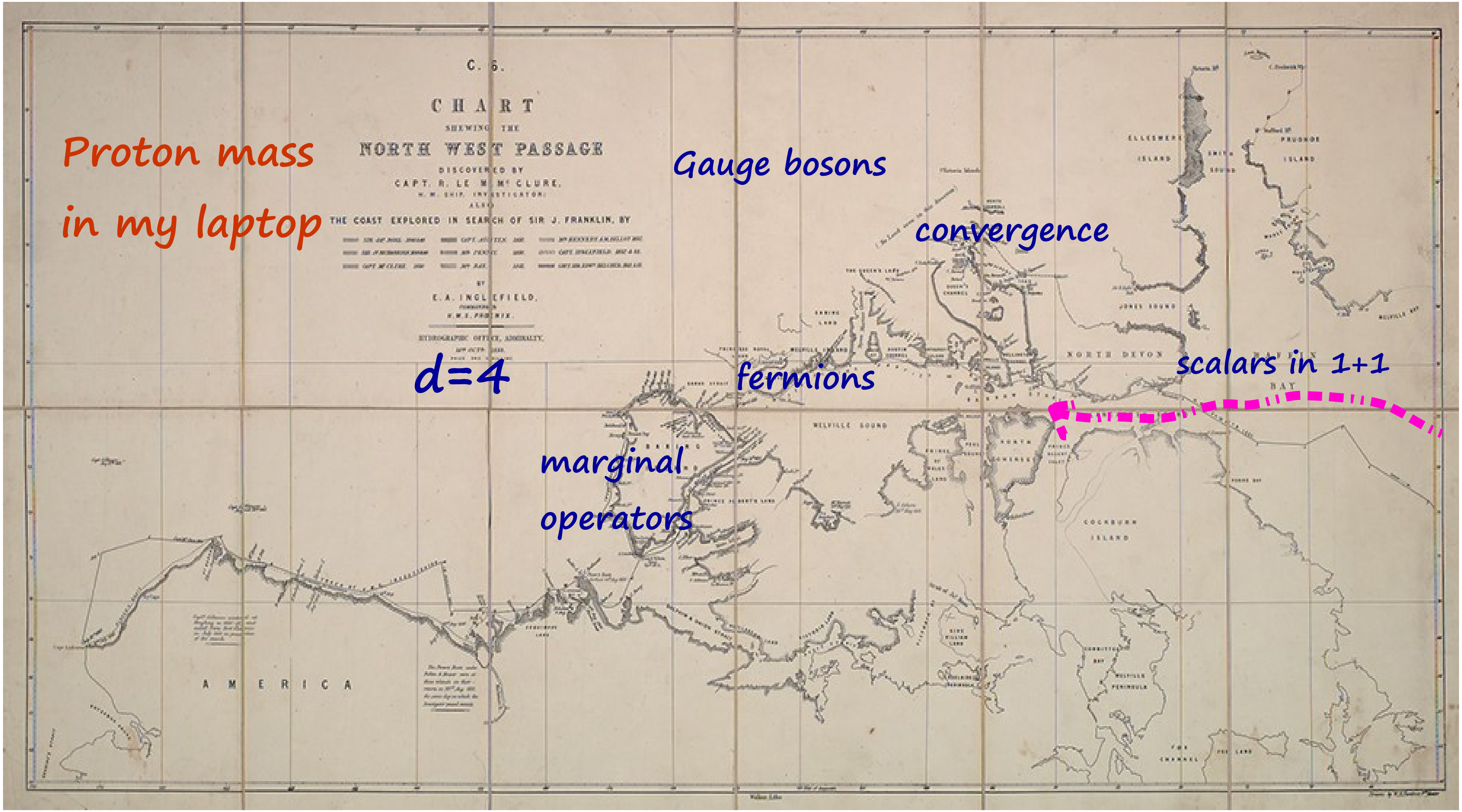


Thank you!