On the Frame Problem in Inflation

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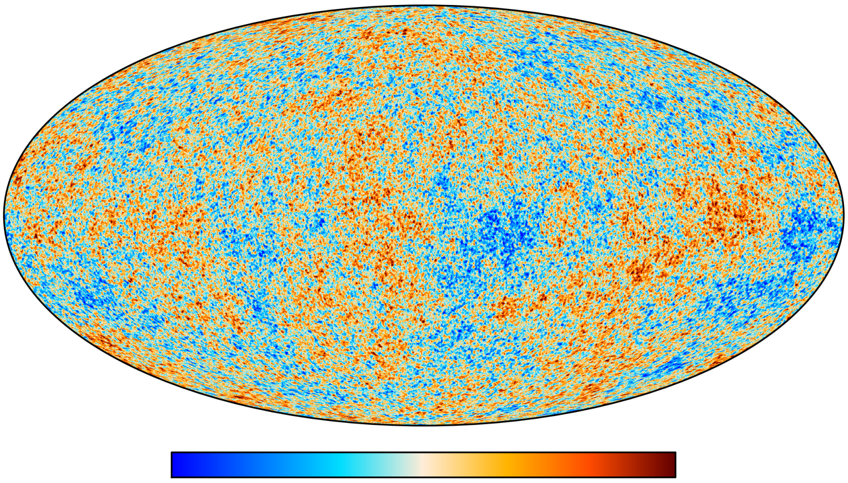
Outline

Introduction



- Inflation solves flatness, horizon, relic... problems
- Inflation is also generic explanation for origin of anisotropies in the CMB
- No definitive driving mechanism for inflation exists; increasing complexity of models poses a challenge in extracting analytical predictions

Scalar-Curvature Theories

- Action $S = S[g_{\mu\nu}, \varphi, f(\varphi), k(\varphi), V(\varphi)]$ for wide class of inflation models in the *Jordan frame* given by:

$$S = \int d^4x \sqrt{-g} \left[-\frac{f(\varphi)}{2} R + \frac{1}{2} k(\varphi) g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi) \right]$$

- *Conformal transformation* and *reparametrisation* of the inflaton (collectively *frame transformation*) link different scalar-curvature theories (*Einstein frame* defined by $f(\varphi) = M_P^2, k(\varphi) = 1$):

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu},$$
$$\varphi \to \tilde{\varphi} = \tilde{\varphi}(\varphi), \qquad (d\tilde{\varphi}/d\varphi)^2 = K(\varphi)$$

- The above transformation properties of model parameters imply

$$S[g_{\mu\nu}, \varphi, f(\varphi), k(\varphi), V(\varphi)] = S[\tilde{g}_{\mu\nu}, \tilde{\varphi}, \tilde{f}(\tilde{\varphi}), \tilde{k}(\tilde{\varphi}), \widetilde{V}(\tilde{\varphi})]$$

- *Frame problem*: are frame transformations physical or not? A given action may not necessarily be *a priori* frame invariant
- Aim to answer this by developing formalism applicable to scalar-curvature theories with general model functions

Classical Dynamics

- Einstein field equations modified due to non-minimal coupling $f(\varphi) \neq M_P^2$
- Assume homogeneous inflaton and Friedman-Walker-Robertson metric:

$$g_{\mu\nu} = \mathrm{diag}(N_L^2, -a^2, -a^2, -a^2)$$

- N_L is general lapse function, used to define the Hubble parameter

$$H \equiv \frac{\dot{a}}{a} \equiv \frac{1}{N_L}\frac{d\ln a}{dt}$$

- Energy density and pressure are replaced by their non-minimal (NM) extensions:

$$\frac{\rho^{(\mathrm{NM})}}{M_P^2} = \frac{\rho}{f} - \frac{3H\dot{f}}{f}, \qquad \frac{p^{(\mathrm{NM})}}{M_P^2} = \frac{p}{f} + \frac{2H\dot{f}}{f} - \frac{\ddot{f}}{f}$$

- Form of acceleration, Friedman, and continuity equations is unchanged

Cosmological perturbations

- Observable anisotropies seeded by primordial perturbations of the metric ($g_{\mu\nu}dx^\mu dx^\nu = (1+2\phi)N_L^2 dt^2 + \cdots$) and the inflaton field ($\varphi = \bar{\varphi} + \delta\varphi$)
- The only physically relevant inflationary quantity is the diffeomorphism-invariant *comoving curvature perturbation*

$$\mathcal{R} \equiv \phi - \frac{H}{\dot{\bar{\varphi}}}\delta\varphi$$

- "Freezes" outside the horizon (Weinberg, 2003)
- Scalar and tensor power spectra extracted via two-point functions of $\mathcal{R}$ evaluated at horizon crossing:

$$P_{\mathcal{R}} \equiv \frac{k^3}{4\pi^2}\frac{H^4}{Z_{\mathcal{R}}\dot{\varphi}^2}, \qquad P_T \equiv \frac{2k^3}{\pi^2}\frac{H^2}{Z_T}, \qquad r = 8\frac{Z_{\mathcal{R}}}{Z_T}\frac{\dot{\varphi}^2}{H^2}$$

Slow-roll inflation

- Define *scalar spectral index* and its running, evaluated at horizon crossing of largest cosmological scales ($k = 0.002$ Mpc^{-1}):

$$n_{\mathcal{R}} - 1 = \left.\frac{d\ln P_{\mathcal{R}}}{d\ln k}\right|_{k=aH}, \qquad \alpha_{\mathcal{R}} = \frac{dn_{\mathcal{R}}}{d\ln k}$$

- Slow-roll approximation $\ddot{\varphi} \ll H\dot{\varphi} \ll H^2\varphi$ implies that Hubble *slow-roll parameters* (HSRP) (Hwang, 1996) are *small* and *slowly varying*:

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2}, \qquad \delta_H \equiv -\frac{\ddot{\varphi}}{H\dot{\varphi}},$$

$$\kappa_H \equiv \frac{1}{2}\frac{\dot{f}}{Hf}, \qquad \sigma_H \equiv \frac{1}{2}\frac{\dot{E}}{HE}, \qquad E \equiv kf + \frac{3}{2}f_{,\varphi}^2$$

- We may express cosmological observables in terms of HSRP:

$$\begin{aligned} n_{\mathcal{R}} &= 1 - 4\epsilon_H - 2\delta_H + 2\kappa_H - 2\sigma_H, \\ r &= 16(\epsilon_H + \kappa_H) \end{aligned}$$

Slow-roll inflation

- HSRP expressions for $n_{\mathcal{R}}$ and r require solving the cosmological equations of motion (impractical in all but the simplest of cases)
- *Inflationary attractor*: class of solutions which all inflationary trajectories approach in phase space
- The equations of motion simplify in the limit of vanishing SRP:

$$H^2 \approx \frac{fU}{3}, \qquad -3EH\dot{\varphi} \approx f^3 U_{,\varphi}, \qquad \frac{H}{\dot{\varphi}} \approx -\frac{EU}{f^2 U_{,\varphi}}$$

- New *potential* slow-roll parameters (PSRP) reduce to the corresponding HSRP in the slow-roll limit:

$$\epsilon_U \equiv \frac{1}{2}\frac{fU_{,\varphi}(fU)_{,\varphi}}{EU^2}, \qquad \delta_U \equiv \frac{1}{2}\frac{fU_{,\varphi}(fU)_{,\varphi}}{EU^2} + \left(\frac{f^2 U_{,\varphi}}{EU}\right)_{,\varphi},$$

$$\kappa_U \equiv -\frac{f_{,\varphi}}{2}\frac{fU_{,\varphi}}{EU}, \qquad \sigma_U \equiv -\frac{1}{2}\frac{E_{,\varphi}}{E^2}\frac{f^2 U_{,\varphi}}{U}$$

Frame covariance

- Extract model functions f, k, and V from transformed action after a general frame trasformation
- PSRP transform as

$$\tilde{\epsilon}_U = \epsilon_U - 2\Delta_\Omega, \qquad \tilde{\delta}_U = \delta_U - 2\Delta_\Omega + \Delta_K,$$
$$\tilde{\kappa}_U = \kappa_U + 2\Delta_\Omega, \qquad \tilde{\sigma}_U = \sigma_U + 4\Delta_\Omega + \Delta_K,$$
$$\Delta_\Omega \equiv \frac{1}{2}\frac{f^2 U_{,\varphi}}{EU}\frac{\Omega_{,\varphi}}{\Omega}, \qquad \Delta_K \equiv \frac{1}{2}\frac{f^2 U_{,\varphi}}{EU}\frac{K_{,\varphi}}{K}$$

- Inflationary observables are frame-invariant to first (slow-roll) order
- Number of e-foldings ($dN \equiv HN_L\, dt$) not frame-invariant to all orders:

$$N \to \widetilde{N} = N + \ln[\Omega(t)/\Omega(t_{\rm end})]$$

- In potential formalism, N is frame-invariant as long as end-of-inflation condition $\max(\epsilon_U, |\epsilon_U + \delta_U|) = 1$ is extended to:

$$\max(\epsilon_U + \kappa_U, |\epsilon_U + \delta_U + 4\kappa_U - \sigma_U|) = 1$$

Specific models

- Induced gravity inflation: $f(\varphi) = \xi\varphi^2$, $V(\varphi) = \lambda(\varphi^2 - 1/\xi)^2$
- Higgs inflation: $f(\varphi) = M_P^2 + \xi\varphi^2$, $V(\varphi) = \lambda(\varphi^2 - v^2)^2$

	Induced gravity inflation	Higgs inflation
r	$\frac{128\,\xi\,(1+6\xi)}{[(8N-6)\xi-1]^2}$	$\frac{16(1+6\xi)}{8\xi N^2+N}$
$n_{\mathcal{R}}$	$\frac{4\left(16N^2-56N-15\right)\xi^2-4(4N+1)\xi+1}{[(8N-6)\xi-1]^2}$	$\frac{64\xi^2N^3+\left(1-40\xi-192\xi^2\right)N-16\xi(8\xi-1)N^2-3(1+6\xi)}{N(1+8\xi N)^2}$
$\alpha_{\mathcal{R}}$	$-\frac{1024N\xi^3(2(4N+9)\xi+3)}{[(6-8N)\xi+1]^4}$	$-\frac{1}{N^2(1+8\xi N)^4}\left[2048\xi^3N^2\left(4N^2+15N+9\right)\xi^4 +32N\left(160N^2+300N+81\right) + 12(8N + 3)\xi + 3 +4\left(272N^2+252N+27\right)\xi^2\right]$
α_T	$-\frac{2048N\xi^3(1+6\xi)}{[(6-8N)\xi+1]^4}$	$-\frac{2(1+6\xi)\left(32N(4N+3)\xi^2+6(4N+1)\xi+1\right)}{N^2(8\xi N+1)^3}$

- Possible to incorporate $F(R)$ models in the formalism via Legendre transformation $\chi = F(R)_{,R}$

Conclusion

- Frame problem resolved in a natural way; frame invariance of the action not imposed “by hand”
- Easy-to-use calculational tool for extracting cosmological observables from any scalar-curvature theory:
  - All higher-order runnings of cosmological observables may be calculated without defining additional slow-roll parameters
  - No further approximation beyond inflationary attractor is necessary (such as approximating the high-field potential in the Einstein frame)
- Possible extensions to the formalism:
  - Multi-field non-minimally coupled inflation
  - $F(\varphi, R)$ theories (recast in multi-field form via Legendre transformations)
- Form invariance of action is starting point for studying effect of frame transformations to radiative corrections in inflation

Cosmological perturbations: backup slides

- SVT decomposition leads to linearised modified Einstein field equations (where $T_{\mu\nu}^{(NM)}$ is non-minimal energy-momentum tensor):

$$\delta G_{\mu\nu} = \delta T_{\mu\nu}^{(NM)}$$

- Comoving curvature perturbation satisfies

$$\frac{1}{N_L^2 a^3 Q_{\mathcal{R}}} \frac{d}{dt}\left(N_L a^3 Q_{\mathcal{R}} \dot{\mathcal{R}}\right) + \frac{k^2 \mathcal{R}}{a^2} = 0, \quad Q_{\mathcal{R}} = \frac{k\dot{\varphi}^2 + \dfrac{3\dot{f}^2}{2f}}{\left(H + \dfrac{\dot{f}}{2f}\right)^2} \equiv \frac{\dot{\varphi}^2}{H^2} Z_{\mathcal{R}}$$

- Canonicalizing the primordial perturbation via $z_{\mathcal{R}} \equiv a\sqrt{Q_{\mathcal{R}}}$, $v_{\mathcal{R}} \equiv z_{\mathcal{R}}\mathcal{R}$, we derive the generalised *Sasaki-Mukhanov* (SM) equation (with $N_L dt \equiv a\, d\tau$)

$$\frac{d^2 v_{\mathcal{R},k}}{d\eta^2} + \left(k^2 - \frac{1}{z_{\mathcal{R}}} \frac{d^2 z_{\mathcal{R}}}{d\eta^2}\right) v_{\mathcal{R},k} = 0$$

Cosmological perturbations: backup slides

- Power spectrum of perturbations is quantum in origin
- Quantise SM equation by imposing usual commutation relations and the *Bunch-Davies* vacuum (fields behave as if free at very early times)
- Solving for the mode functions at large scale limit, we find the two-point function of the primordial perturbations via

$$\langle v_{\mathcal{R},\mathbf{k}_1} | v_{\mathcal{R},\mathbf{k}_2} \rangle = |v_{\mathcal{R},k}|^2 \, \delta(\mathbf{k}_1 + \mathbf{k}_2)$$

- Taking into account transfer functions which induce a multiplicative multipole contribution, we extract the observed scalar power spectrum and tensor-to-scalar ratio from the two-point function:

$$P_{\mathcal{R}} \equiv \frac{k^3}{4\pi^2} \frac{H^4}{Z_{\mathcal{R}} \dot{\varphi}^2}, \qquad P_T \equiv \frac{2k^3}{\pi^2} \frac{H^2}{Z_T}, \qquad r = 8 \frac{Z_{\mathcal{R}}}{Z_T} \frac{\dot{\varphi}^2}{H^2}$$