Dispersive Treatment of $K_{\ell 4}$ Decays

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Definition of the $K_{\ell 4}$ decay

Decay of a kaon into two pions and a lepton pair:

$$K^+(p) \to \pi^+(p_1)\pi^-(p_2)\ell^+(p_\ell)\nu_\ell(p_\nu)$$

$\ell \in \{e, \mu\}$ is either an electron or a muon.

(Other modes involving neutral pions can be related by isospin symmetry.)

Importance of the $K_{\ell 4}$ decay

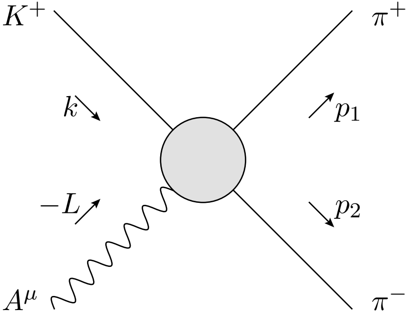
- provides information on $\pi\pi$-scattering lengths a_0^0, a_0^2
- K_{e4} very precisely measured $\Rightarrow$ test of χPT

  $\rightarrow$ Geneva-Saclay, E865, NA48/2
- best source of information on the χPT low-energy constants L_1^r, L_2^r and L_3^r
- happens at very low energy, where χPT is expected to converge best

Advantages of dispersion relations

- resummation of rescattering
- connect different energy regions
- based on analyticity and unitarity $\Rightarrow$ model independence

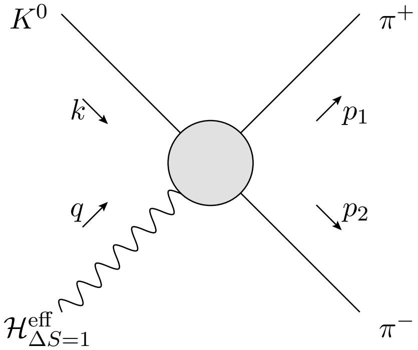
Hadronic part of $K_{\ell 4}$ as $2 \to 2$ scattering



Mandelstam variables:

$$s = (p_1 + p_2)^2, \quad t = (k - p_1)^2, \quad u = (k - p_2)^2$$

Similar to $K \to 2\pi$



$\to$ Büchler, Colangelo, Kambor, Orellana (2001)

Form factors

- Lorentz structure allows four form factors in the hadronic matrix element ($P = p_1 + p_2$, $Q = p_1 - p_2$):

$$\left\langle \pi^+(p_1)\pi^-(p_2)\middle|A_\mu(0)\middle|K^+(k)\right\rangle = -i\frac{1}{M_K}\left(P_\mu \boldsymbol{F} + Q_\mu \boldsymbol{G} + L_\mu \boldsymbol{R}\right)$$

$$\left\langle \pi^+(p_1)\pi^-(p_2)\middle|V_\mu(0)\middle|K^+(k)\right\rangle = -\frac{\boldsymbol{H}}{M_K^3}\epsilon_{\mu\nu\rho\sigma}L^\nu P^\rho Q^\sigma$$

- contribution of R invisible in the electron mode
- H chirally suppressed
- concentrate here on F and G
- form factors are functions of the Mandelstam variables s, t and u

Analytic properties

- $F(s,t,u)$ and $G(s,t,u)$ have a right-hand branch cut in the complex s-plane, starting at the $\pi\pi$-threshold
- left-hand cut present due to crossing
- analogous situation in t- and u-channel

Reconstruction theorem

→ Stern, Sazdjian, Fuchs (1993), Ananthanarayan, Buettiker (2001), . . .

- define a function that has just the right-hand cut of f_0, the first partial wave of F:

$$M_0(s) := P(s) + \frac{s^2}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{Im} f_0(s')}{(s' - s - i\epsilon){s'}^2} ds'$$

- similar functions take care of the right-hand cuts of all the other S- and P-waves (also crossed channels)
- all the discontinuities are split up into functions of a single variable
- neglect imaginary parts of D- and higher waves

Reconstruction theorem

Form factors decomposed into functions of one Mandelstam variable only:

$$F(s,t,u) = M_0(s) + \frac{u-t}{M_K^2} M_1(s) + (\text{functions of } t \text{ or } u),$$

$$G(s,t,u) = \tilde{M}_1(s) + (\text{functions of } t \text{ or } u).$$

Omnès representation

Function M_0 contains only right-hand cut of the partial wave f_0: difference is the 'inhomogeneity' $\hat{M}_0$:

$$f_0(s) = M_0(s) + \hat{M}_0(s)$$

Inhomogeneous Omnès problem:

$$\mathrm{Im} M_0(s) = (M_0(s) + \hat{M}_0(s)) e^{-i\delta_0^0(s)} \sin \delta_0^0(s)$$

Watson's theorem: δ_0^0 is elastic $\pi\pi$ phase shift

Omnès representation

Omnès function takes care of rescattering:

$$\Omega_l^I(s) := \exp\left\{\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\delta_l^I(s')}{(s'-s-i\epsilon)s'}ds'\right\}$$

δ_l^I: elastic $\pi\pi$ or $K\pi$ phase shifts

Write dispersion relation for $\dfrac{M_0(s)}{\Omega_0^0(s)}$

Omnès representation

Omnès solution for the functions $M_0(s)$, $M_1(s)$, $\tilde{M}_1(s)$, etc.:

$$M_0(s) = \Omega_0^0(s)\left\{P(s) + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\Lambda^2} \frac{\hat{M}_0(s')\sin\delta_0^0(s')}{|\Omega_0^0(s')|(s'-s-i\epsilon)s'^3}ds'\right\},$$

P: subtraction polynomial

$\hat{M}_i$: inhomogeneities, angular averages of all the functions M_i

Intermediate summary

- problem parametrised by 9 subtraction constants
- input: elastic $\pi\pi$- and $K\pi$-scattering phase shifts
- energy dependence fully determined by the dispersion relation

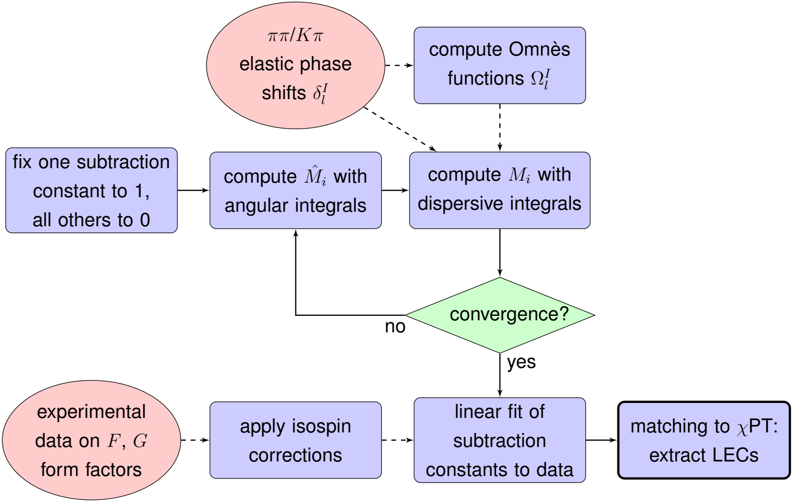
Intermediate summary

- set of coupled integral equations:
  $\Rightarrow M_0(s), M_1(s), \ldots$: DR involving $\hat{M}_0(s), \hat{M}_1(s), \ldots$
  $\Rightarrow \hat{M}_0(s), \hat{M}_1(s), \ldots$: angular integrals over $M_0(s), M_1(s), \ldots$
- system solved by iteration
- problem linear in the subtraction constants
  $\Rightarrow$ construct 9 basic solutions

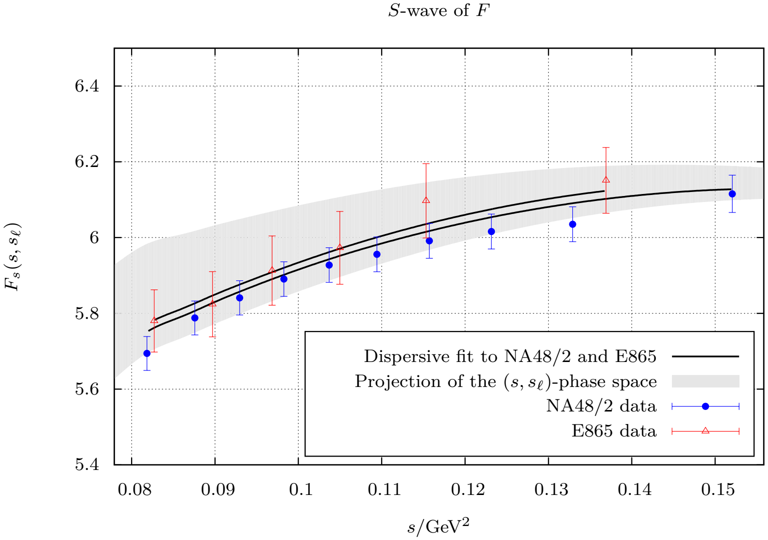
Determination of the subtraction constants

- fit to data of the high-statistics experiments E865 and NA48/2
- soft-pion theorems as additional constraints
- chiral input for the subtraction constants that are not well determined by data

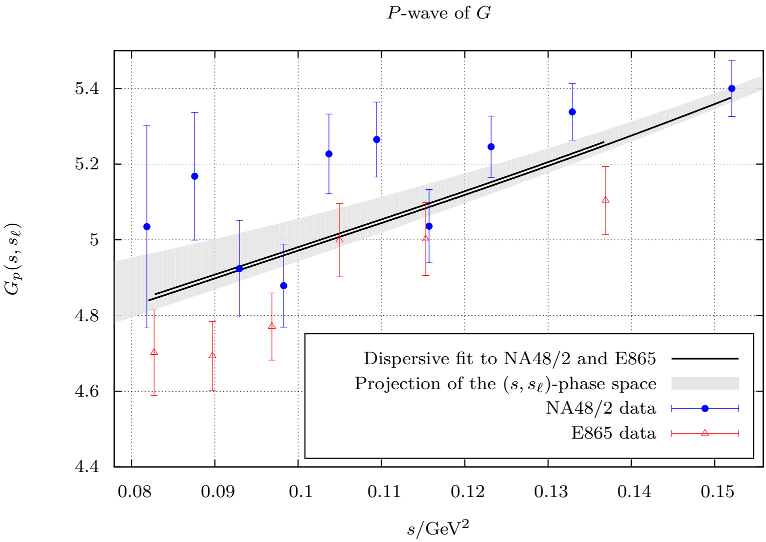
Numerical solution of the dispersion relation



Fit results for partial waves



Fit results for partial waves



Matching to χPT

- matching to χPT at the level of subtraction constants in Omnès form: separate rescattering effects
- fit to 2-dimensional data set of NA48/2
- L_9^r can be determined from dependence on s_ℓ

Matching at NNLO

- many poorly known LECs C_i^r at NNLO
- include additional constraints in the fit: require good chiral convergence
- input: C_i^r contribution to subtraction constants with $\pm 50\%$ uncertainty
- fit the C_i^r contribution
- not all sets of C_i^r input lead to a good chiral convergence: prefer BE14 $\rightarrow$ Bijnens, Ecker (2014)

Low-energy constants

Results for the LECs using χPT at NLO and NNLO.

	NLO	NNLO	Bijnens, Ecker (2014)
$10^3 \cdot L_1^r$	$0.51(2)(6)$	$0.69(16)(8)$	$0.53(6)$
$10^3 \cdot L_2^r$	$0.89(5)(7)$	$0.63(9)(10)$	$0.81(4)$
$10^3 \cdot L_3^r$	$-2.82(10)(7)$	$-2.63(39)(24)$	$-3.07(20)$
χ^2/dof	$141/116 = 1.2$	$124/122 = 1.0$	

What could be done in NA62?

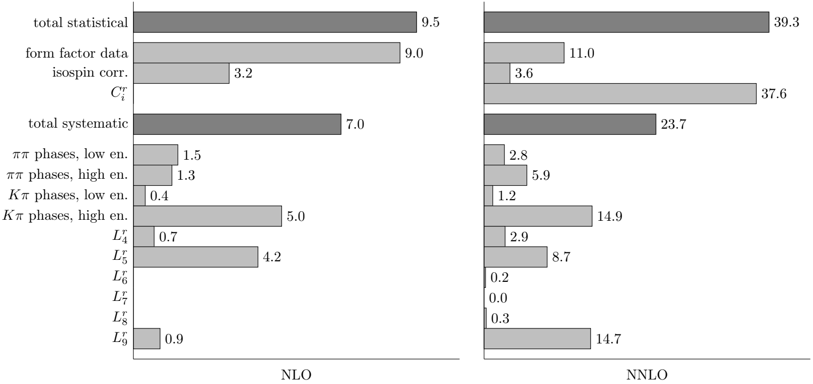
- $K_{\mu 4}$ (i.e. $K^+ \to \pi^+\pi^-\mu^+\nu_\mu$) may come through trigger
- K_{e4} does not, but is background for $K \to \pi\nu\bar{\nu}$
  $\to$ maybe a special run is planned?

Electron mode K_{e4}

What could be done with higher statistics?

- s_ℓ-dependence of F and G can be used to extract L_9^r
- determination of L_1^r, L_2^r, L_3^r with even higher precision
- (better) determination of linear combinations of C_i^r

Error budget: L_3^r



Radiative corrections for K_{e4}

→ EPJC **74** (2014) 2749

- full 1-loop calculation of photonic (and strong isospin-breaking) corrections in χPT$+\gamma+\ell$ (virtual and real corrections)
- NA48/2: PHOTOS Monte Carlo and Gamow-Sommerfeld factor
- a posteriori correction only possible for normalisation
- for NA62, $K_{e4}(\gamma)$ could be included in Monte Carlo

Muon mode $K_{\mu 4}$

- larger values of s_ℓ
- form factor R is accessible
- s-dependence of R contains L_4^r, L_5^r and L_9^r:

$$
\begin{aligned}
R &\propto \frac{Z}{s_\ell - M_K^2} + Q, \\
Z_L &= 32\big[\Sigma s - 4M_K^2 M_\pi^2\big] L_4^r \\
&\qquad + 4\big[\Sigma(s+t-u) - 8M_K^2 M_\pi^2\big] L_5^r + \ldots, \\
Q_L &= 2\big[(s+t-u) - (M_K^2 - s_\ell)\big] L_9^r + \ldots
\end{aligned}
$$

- information on $K\pi$ scattering

Summary

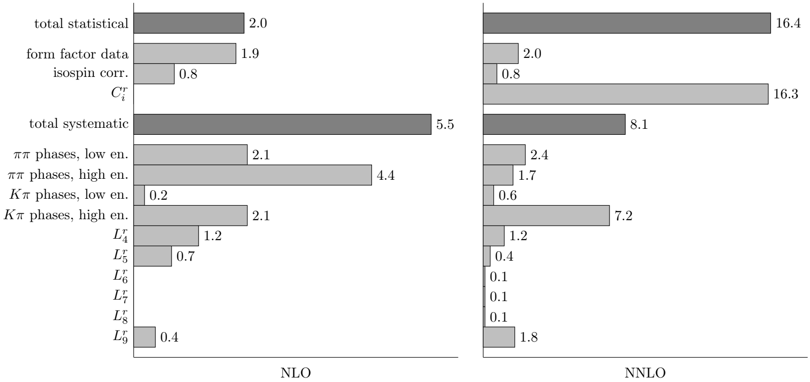
- parametrisation valid up to and including $\mathcal{O}(p^6)$
- model independence
- resummation of rescattering effects
- very precise data available
- determination of LECs from matching to χPT

Summary

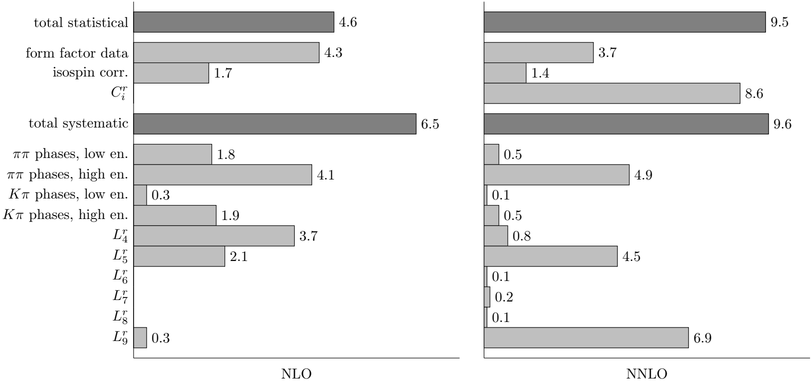
- even higher statistics could be useful for better determination of L_i^r and combinations of C_i^r
- better data on s_ℓ-dependence would enable independent determination of L_9^r
- radiative corrections should be included in Monte Carlo
- new form factor and further LECs accessible in $K_{\mu 4}$

Backup

Error budget: L_1^r



Error budget: L_2^r



Relation to $SU(2)$ LECs

$$l_1^r = 4L_1^r + 2L_3^r + x_1 + \mathcal{O}(p^2), \quad \bar{l}_1 = 96\pi^2 l_1^r - \ln\frac{M_\pi^2}{\mu^2},$$

$$l_2^r = 4L_2^r + x_2 + \mathcal{O}(p^2), \quad \bar{l}_2 = 48\pi^2 l_2^r - \ln\frac{M_\pi^2}{\mu^2},$$

	NLO matching, C_i^r BE14	→ CGL 2001
$\bar{l}_1$	-0.0 ± 0.3	-0.4 ± 0.6
$\bar{l}_2$	4.4 ± 0.2	4.3 ± 0.1

(only error due to L_1^r, L_2^r, L_3^r)