# Status and Prospects for Lattice Computations of Nonleptonic and Rare Kaon Decays

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*NA62 Kaon Physics Handbook*

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In June 2015, I gave a talk at a MIAPP workshop on

*New Directions in Lattice Flavour Physics*

with the following outline:

1. Introduction
2. Status of RBC-UKQCD Collaboration's calculations of $K \to \pi\pi$ decay amplitudes. *
3. Electromagnetic corrections to decay amplitudes.
4. Long-distance contributions to flavour changing processes

$$\int\!\!\int d^4x\, d^4y\, \langle f\,|\,T[Q_1(x)\, Q_2(y)]\,|\,i\rangle\,.$$

   (i) $K_L$-$K_S$ mass difference (and $\epsilon_K$)
   (ii) (Rare kaon decays)

* RBC=Riken Research Center, Brookhaven National Laboratory, Columbia University; UKQCD = Edinburgh + Southampton.

1. Introduction
2. Rare Kaon Decays $K \to \pi \ell^+ \ell^-$.
3. Rare Kaon Decays $K^+ \to \pi^+ \nu \bar{\nu}$.
4. Status of RBC-UKQCD Collaboration's calculations of $K \to \pi\pi$ decay amplitudes. *
5. Electromagnetic corrections to decay amplitudes $\Rightarrow$ Guido Martinelli's Talk.

* RBC=Riken Research Center, Brookhaven National Laboratory, Columbia University; UKQCD = Edinburgh + Southampton.

## 2. Rare Kaon Decays: $K_L \to \pi^0 \ell^+ \ell^-$

N.H.Christ, X.Feng, A.Portelli and C.T.Sachrajda, arXiv:1507.03094

Some comments from F.Mescia, C.Smith, S.Trine hep-ph/0606081:

- Rare kaon decays which are dominated by short-distance FCNC processes, $K \to \pi\nu\bar{\nu}$ in particular, provide a potentially valuable window on new physics at high-energy scales.
- The decays $K_L \to \pi^0 e^+ e^-$ and $K_L \to \pi^0 \mu^+ \mu^-$ are also considered promising because the long-distance effects are reasonably under control using ChPT.
  - They are sensitive to different combinations of short-distance FCNC effects and hence in principle provide additional discrimination to the neutrino modes.
  - A challenge for the lattice community is therefore to calculate the long-distance effects reliably (and to determine the Low Energy Constants of ChPT).
- We, the RBC-UKQCD collaboration, are attempting to meet this challenge but will need the help of the wider kaon physics community to do this as effectively as possible.

There are three main contributions to the amplitude:

1. Short distance contributions: F.Mescia, C,Smith, S.Trine hep-ph/0606081

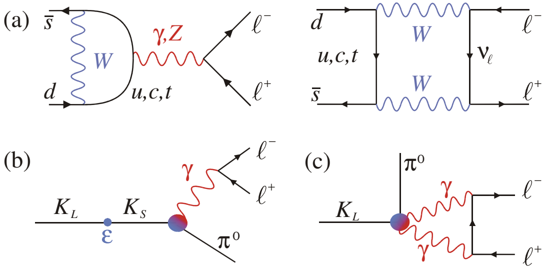
$$H_{\rm eff} = -\frac{G_F \alpha}{\sqrt{2}}\, V_{ts}^* V_{td} \{ y_{7V} (\bar{s}\gamma_\mu d)\, (\bar{\ell}\gamma^\mu \ell) + y_{7A} (\bar{s}\gamma_\mu d)\, (\bar{\ell}\gamma^\mu \gamma_5 \ell) \} + \text{h.c.}$$

   - Direct CP-violating contribution.
   - In BSM theories other effective interactions are possible.

2. Long-distance indirect CP-violating contribution

$$A_{ICPV}(K_L \to \pi^0 \ell^+ \ell^-) = \epsilon\, A(K_1 \to \pi^0 \ell^+ \ell^-)\,.$$

3. The two-photon CP-conserving contribution $K_L \to \pi^0 (\gamma^* \gamma^* \to \ell^+ \ell^-)\,.$



- The current phenomenological status for the SM predictions is nicely summarised by: V.Cirigliano et al., arXiv1107.6001

$$\mathrm{Br}(K_L\to\pi^0e^+e^-)_{\rm CPV} = 10^{-12}\times\left\{15.7|a_S|^2 \pm 6.2|a_S|\left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right)+2.4\left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right)^2\right\}$$

$$\mathrm{Br}(K_L\to\pi^0\mu^+\mu^-)_{\rm CPV} = 10^{-12}\times\left\{3.7|a_S|^2 \pm 1.6|a_S|\left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right)+1.0\left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right)^2\right\}$$

  - $\lambda_t = V_{td}V_{ts}^*$ and Im $\lambda_t \simeq 1.35\times 10^{-4}$.
  - $|a_S|$, the amplitude for $K_S\to\pi^0\ell^+\ell^-$ at $q^2=0$ as defined below, is expected to be $O(1)$ but the sign of $a_S$ is unknown. $|a_S| = 1.06^{+0.26}_{-0.21}$.
  - For $\ell = e$ the two-photon contribution is negligible.
  - Taking the positive sign (?) the prediction is

$$\begin{aligned}\mathrm{Br}(K_L\to\pi^0e^+e^-)_{\rm CPV} &= (3.1\pm0.9)\times10^{-11}\\ \mathrm{Br}(K_L\to\pi^0\mu^+\mu^-)_{\rm CPV} &= (1.4\pm0.5)\times10^{-11}\\ \mathrm{Br}(K_L\to\pi^0\mu^+\mu^-)_{\rm CPC} &= (5.2\pm1.6)\times10^{-12}\,.\end{aligned}$$

  - The current experimental limits (KTeV) are:

$$\mathrm{Br}(K_L\to\pi^0e^+e^-) < 2.8\times10^{-10}\quad\text{and}\quad \mathrm{Br}(K_L\to\pi^0\mu^+\mu^-) < 3.8\times10^{-10}\,.$$

G.Isidori, G.Martinelli and P.Turchetti, hep-lat/0506026

- We now turn to the CPC decays $K_S \to \pi^0\ell^+\ell^-$ and $K^+ \to \pi^+\ell^+\ell^-$ and consider

$$T_i^\mu = \int d^4x\, e^{-iq\cdot x}\, \langle \pi(p)\,|\,\mathrm{T}\{J^\mu_{\mathrm{em}}(x)\,Q_i(0)\,\}\,|\,K(k)\rangle\,,$$

  where $Q_i$ is an operator from the $\Delta S = 1$ effective weak Hamiltonian.

- EM gauge invariance implies that

$$T_i^\mu = \frac{\omega_i(q^2)}{(4\pi)^2}\,\left\{q^2(p+k)^\mu - (m_K^2 - m_\pi^2)\,q^\mu\right\}\,.$$

- Within ChPT the low energy constants $a_+$ and $a_S$ are defined by

$$a = \frac{1}{\sqrt{2}}\,V_{us}^*V_{ud}\left\{C_1\omega_1(0) + C_2\omega_2(0) + \frac{2N}{\sin^2\theta_W}f_+(0)C_{7V}\right\}$$

  where $Q_{1,2}$ are the two current-current GIM subtracted operators and the $C_i$ are the Wilson coefficients. ($C_{7V}$ is proportional to $y_{7V}$ above).

G.D'Ambrosio, G.Ecker, G.Isidori and J.Portoles, hep-ph/9808289

- Phenomenological values: $a_+ = -0.578 \pm 0.016$ and $|a_S| = 1.06^{+0.26}_{-0.21}$.

What can we achieve in lattice simulations?

- The generic non-local matrix elements which we wish to evaluate is

$$
\begin{aligned}
X &\equiv \int_{-\infty}^{\infty} dt_x\, d^3x\, \langle \pi(p)\,|\,\mathrm{T}\,[\,J(0)\,H(x)\,]\,|K(k)\rangle \\
&= i\sum_n \frac{\langle \pi(p)\,|\,J(0)\,|n\rangle\,\langle n\,|H(0)\,|\,K(k)\rangle}{E_K - E_n + i\epsilon} - i\sum_{n_s} \frac{\langle \pi(p)\,|\,H(0)\,|n_s\rangle\,\langle n_s\,|J(0)\,|\,K(k)\rangle}{E_{n_s} - E_\pi + i\epsilon}\,,
\end{aligned}
$$

  - $\{|n\rangle\}$ and $\{|n_s\rangle\}$ represent complete sets of non-strange and strange states.
- In Euclidean space we calculate correlation functions of the form

$$
C \equiv \int_{-T_a}^{T_b} dt_x\, \langle \phi_\pi(\vec{p}, t_\pi)\,\mathrm{T}\,[\,J(0)\,H(t_x)\,]\;\phi_K^\dagger(t_K)\,\rangle \equiv \sqrt{Z_K}\,\frac{e^{-E_K|t_K|}}{2m_K}\,X_E\,\sqrt{Z_\pi}\,\frac{e^{-E_\pi t_\pi}}{2E_\pi}\,,
$$

where $X_E = X_{E_-} + X_{E_+}$ and

$$
\begin{aligned}
X_{E_-} &= -\sum_n \frac{\langle \pi(p)\,|\,J(0)\,|n\rangle\,\langle n\,|H(0)\,|\,K(k)\rangle}{E_K - E_n}\left(1 - e^{(E_K - E_n)T_a}\right) \quad \text{and} \\
X_{E_+} &= \sum_{n_s} \frac{\langle \pi(p)\,|\,H(0)\,|n_s\rangle\,\langle n_s\,|J(0)\,|\,K(k)\rangle}{E_{n_s} - E_\pi}\left(1 - e^{-(E_{n_s} - E_\pi)T_b}\right)\,.
\end{aligned}
$$

# 4-pt Euclidean Correlation Functions


- In Euclidean space we calculate correlation functions of the form

$$C \equiv \int_{-T_a}^{T_b} dt_x \, \langle \phi_\pi(\vec{p}, t_\pi) \, \mathrm{T} \left[ J(0) \, H(t_x) \right] \, \phi_K^\dagger(t_K) \rangle \equiv \sqrt{Z_K} \, \frac{e^{-E_K |t_K|}}{2m_K} \, X_E \, \sqrt{Z_\pi} \, \frac{e^{-E_\pi t_\pi}}{2E_\pi} \,,$$

where $X_E = X_{E_-} + X_{E_+}$ and

$$X_{E_-} \;=\; -\sum_n \frac{\langle \pi(p) \,|\, J(0) \,|n\rangle \, \langle n \,|H(0) \,|\, K\rangle}{E_K - E_n} \left(1 - e^{(E_K - E_n)T_a}\right) \quad \text{and}$$

$$X_{E_+} \;=\; \sum_{n_s} \frac{\langle \pi(p) \,|\, H(0) \,|n_s\rangle \, \langle n_s \,|J(0) \,|\, K\rangle}{E_{n_s} - E_\pi} \left(1 - e^{-(E_{n_s} - E_\pi)T_b}\right) \,.$$

- In practice we may need to modify the above formulae to recognise the discrete nature of the lattice.
- For $E_K > E_n$ there are unphysical exponentially growing terms which need to be subtracted! This is a common feature in calculations of long-distance effects in Euclidean space. This requires the consideration of $\pi$, $\pi\pi$ and $\pi\pi\pi$ intermediate states.

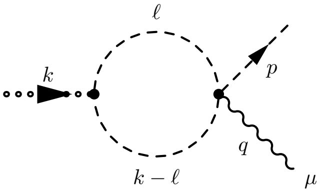
# Removal of single pion intermediate state

- For illustration, I consider the kaon to be at rest.
  - $X_{E_-} = -\sum_n \frac{\langle \pi(p) \mid J(0) \mid n\rangle \langle n \mid H(0) \mid K\rangle}{E_K - E_n}\left(1 - e^{(E_K - E_n)T_a}\right)$

- We use two methods to remove the contribution from the single pion state.
  1. We determine the matrix elements $\langle\pi|H|K\rangle$ and $\langle\pi|J|\pi\rangle$ and the energies from two and three-point correlations functions and then perform the subtraction directly.
  2. We add a term $c_S\,\bar{s}d$ to the effective Hamiltonian, with $c_S$ chosen for each momentum so that

$$\langle\pi\,|H - c_S\,\bar{s}d|K\rangle = 0\,.$$

  - The demonstration that the addition of a term proportional to $\bar{s}d$ does not change the physical amplitude can be found in our paper arXiv:1507.03094.

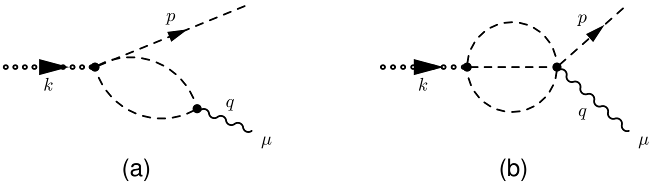



- In the continuum, space-time symmetries protect us from two-pion intermediate states:

$$\langle \pi(p_1)|J_\mu|\pi(p_2)\pi(p_3)\rangle = \epsilon_{\mu\nu\rho\sigma}p_1^\nu p_2^\rho p_3^\sigma F(s,t,u)$$

- After integrating over the momenta of the two intermediate pions, the only independent vectors are $k$, $p$ and $\epsilon_\gamma$ and so the indices of the Levi-Civita tensor cannot be saturated.
- This still leaves lattice artefacts two-pion contributions ($\propto a^2$) amplified by the growing exponential factors. While we expect these to be very small (as is the case for $\Delta m_K$), this will have to be confirmed numerically.

# The three pion contribution




- The finite-volume effects which vanish as powers of the volume are absent from diagram (a) for $q^2 < 4m_\pi^2$.
- The three-pion on-shell intermediate state contribution is heavily phase-space suppressed and is expected to be negligible (but in principle is also calculable as with method 1 for the single pion contribution).
- The suppression of finite-volume effects which only vanish as powers of the volume due to 2 or 3 particle on-shell intermediate states follows in a similar way.
- (It is only recently that the finite-volume corrections for three particle states have become understood theoretically, but the theory has not been applied in numerical calculations.) M.T.Hansen and S.R.Sharpe, arXiv:1504.04248

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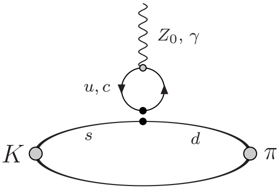
$$T_i^\mu = \int d^4x\, e^{-iq\cdot x}\, \langle \pi(p)\, |\, \mathrm{T}\{J^\mu(x)\, Q_i(0)\,\}\, |\, K(k)\rangle\,,$$

- Each of the two local $Q_i$ operators can be normalized in the standard way and for $J$ we imagine taking the conserved vector current.
- We must treat additional divergences as $x \to 0$.




- Quadratic divergence is absent by gauge invariance $\Rightarrow$ Logarithmic divergence.
  - Checked explicitly for Wilson and Clover at one-loop order.

    G.Isidori, G.Martinelli and P.Turchetti, hep-lat/0506026
  - Absence of power divergences does not require GIM.
  - Logarithmic divergence cancelled by GIM.

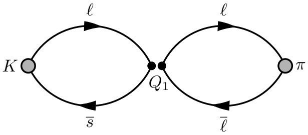
## Short Distance Effects - Postscript

- In the calculation described below we have followed the IMT approach, but the conserved vector current with DWF is a 5-D operator which adds considerably to the cost.
- We will now investigate whether it might not better to use a local vector current and non-perturbative renormalization for the residual logarithmic divergence.

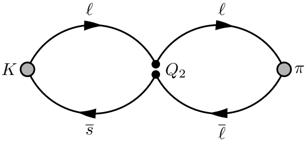
## Many diagrams to evaluate!



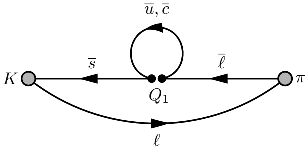
- For example for $K^+$ decays we need to evaluate the diagrams obtained by inserting the current at all possible locations in the three point function (and adding the disconnected diagrams):



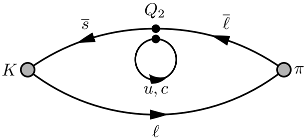

*W*




*C*




*S*




*E*

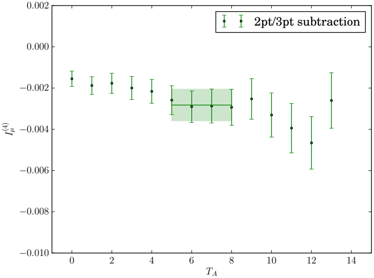
- $W$=Wing, $C$=Connected, $S$=Saucer, $E$=Eye.
- For $K_S$ decays there is an additional topology with a gluonic intermediate state.

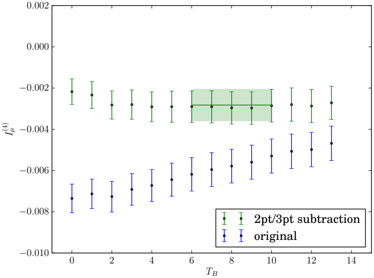
N.Christ, X.Feng, A.Jüttner, A.Lawson, A.Portelli and CTS

- The numerical study is performed on the $24^3 \times 64$ DWF+Iwasaki RBC-UKQCD ensembles with $am_l = 0.01$ ($m_\pi \simeq 420$ MeV), $am_s = 0.04$, $a^{-1} \simeq 1.73$ fm.
- 128 configurations were used with $\vec{k} = \vec{0}$ and $\vec{p}$ =(1,0,0), (1,1,0) and (1,1,1) in units of $2\pi/L$. (The (1,1,1) case is still being completed.)
- With this kinematics we are in the unphysical region, $q^2 < 0$.
- The charm quark is also lighter than physical $m_c^{\overline{\text{MS}}}(2\,\text{GeV}) \simeq 520$ MeV.
- The calculation is performed using the conserved vector current (5-dimensional), $J_{\text{em}}$.
- All results are preliminary.

# Method 1 for $\vec{p} = (1,0,0)$

Preliminary




$$A_0(q^2) = -0.0028(8).$$

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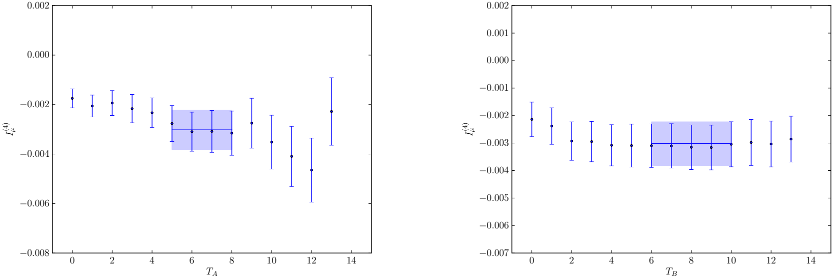
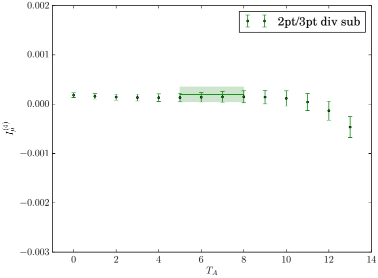
Preliminary

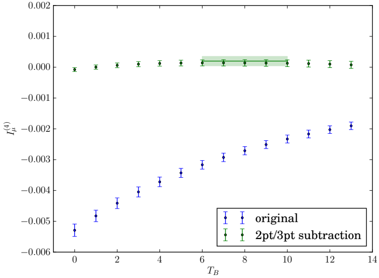



$$A_0(q^2) = -0.0030(8).$$

Numerical check that the matrix element with $H$ replaced by $\bar{s}d$ is consistent with zero.

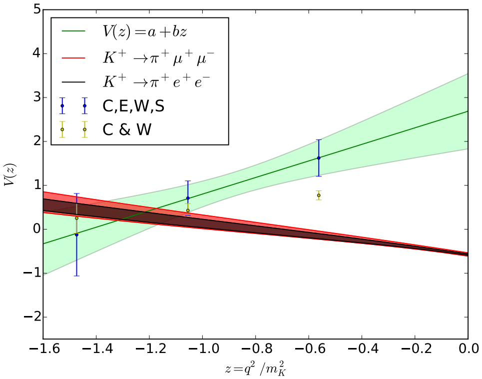
Preliminary




$$A_0^{\bar{s}d}(q^2) = 0.00020(15).$$

Working Plot



N.H.Christ, X.Feng, A.Portelli and CTS (in preparation)

- I don't need to mention at this meeting that these FCNC processes provide ideal probes for the observation of new physics effects.
- The dominant contributions from the top quark $\Rightarrow$ they are also very sensitive to $V_{ts}$ and $V_{td}$.
- Experimental results and bounds:

$$\mathrm{Br}(K^+ \to \pi^+\nu\bar{\nu})_{\mathsf{exp}} = 1.73^{+1.15}_{-1.05} \times 10^{-10}$$

A.Artamonov et al. (E949), arXiv:0808.2459

$$\mathrm{Br}(K_L \to \pi^0\nu\bar{\nu}) \le 2.6 \times 10^{-8} \text{ at } 90\% \text{ confidence level},$$

J.Ahn et al. (E291a), arXiv:0911.4789

- Sample recent theoretical predictions:

$$\mathrm{Br}(K^+ \to \pi^+\nu\bar{\nu})_{\mathsf{SM}} = (9.11 \pm 0.72) \times 10^{-11}$$
$$\mathrm{Br}(K_L \to \pi^0\nu\bar{\nu})_{\mathsf{SM}} = (3.00 \pm 0.30) \times 10^{-11},$$

A.Buras, D.Buttazzo, J.Girrbach-Noe, R.Knejgens, arXiv:1503.02693

- To what extent can lattice calculations reduce the theoretical uncertainty?

# Short and Long-Distance Contributions

- To what extent can lattice calculations reduce the theoretical uncertainty?
- $K \to \pi\nu\bar{\nu}$ decays are SD dominated and the hadronic effects can be determined from CC semileptonic decays such as $K^+ \to \pi^0 e^+ \nu$.
  - Lattice calculations of the $K_{\ell 3}$ form factors are well advanced,

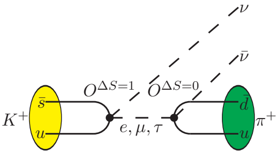
P.A.Boyle et al. (RBC-UKQCD), arXiv:1504.01692

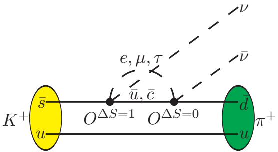
- LD contributions, i.e. contributions from distances greater than $1/m_c$ are negligible for $K_L$ decays and are expected to be $\leq 5\%$ for for $K^+$ decays.
  - $K_L$ decays are therefore one of the cleanest places to search for the effects of new physics.
  - The aim of our study is to compute the LD effects in $K^+$ decays. These provide a significant, if probably still subdominant, contribution to the theoretical uncertainty (which is dominated by the uncertainties in CKM matrix elements).
  - A phenomenological estimate of the long distance effects, estimated these to enhance the branching fraction by 6% with an uncertainty of 3%.

G.Isidori, F.Mescia and C.Smith, hep-ph/0503107

- Lattice QCD can provide a first-principles determination of the LD contribution with controlled errors.
  - Given the NA62 experiment, it is timely to perform a lattice QCD calculation of these effects.

- For this doubly weak decay there are a number of novel diagrams to evaluate:

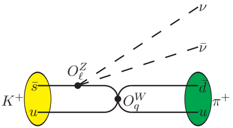



WW-diagrams

$$\mathcal{H}_{\rm eff}^{\rm LO} = -i\frac{G_F}{\sqrt{2}}\sum_{q,\ell}\left(V_{qs}^* O_{q\ell}^{\Delta S=1} + V_{qd} O_{q\ell}^{\Delta S=0}\right) - i\frac{G_F}{\sqrt{2}}\sum_q \lambda_q O_q^W - i\frac{G_F}{\sqrt{2}}\sum_\ell O_\ell^Z\,,$$

$$\begin{aligned} O_{q\ell}^{\Delta S=1} &= C_{\Delta S=1}^{\overline{\rm MS}}(\mu)\,\left[(\bar s q)_{V-A}\,(\bar\nu_\ell \ell)_{V-A}\right]^{\overline{\rm MS}}(\mu),\\ O_{q\ell}^{\Delta S=0} &= C_{\Delta S=0}^{\overline{\rm MS}}(\mu)\,\left[(\bar\ell \nu_\ell)_{V-A}\,(\bar q d)_{V-A}\right]^{\overline{\rm MS}}(\mu). \end{aligned}$$




Z-exchange diagrams

$$\mathcal{H}_{\rm eff}^{\rm LO} = -i\frac{G_F}{\sqrt{2}}\sum_{q,\ell}\left(V_{qs}^* O_{q\ell}^{\Delta S=1} + V_{qd} O_{q\ell}^{\Delta S=0}\right) - i\frac{G_F}{\sqrt{2}}\sum_q \lambda_q O_q^W - i\frac{G_F}{\sqrt{2}}\sum_\ell O_\ell^Z\,,$$

$$\begin{aligned} O_q^W &= C_1^{\overline{\rm MS}}(\mu)\, Q_{1,q}^{\overline{\rm MS}}(\mu) + C_2^{\overline{\rm MS}}(\mu)\, Q_{2,q}^{\overline{\rm MS}}(\mu), \\ O_\ell^Z &= C_Z^{\overline{\rm MS}}(\mu)\, \left[J_\mu^Z\, \bar{\nu}_\ell \gamma^\mu (1-\gamma_5)\nu_\ell\right]^{\overline{\rm MS}}(\mu) \end{aligned}$$

- The issues encountered in $K^+ \to \pi^+\ell^+\ell^-$ decays (additional ultra-violet divergences, subtraction or suppression of growing unphysical exponential terms and FV effects which fall as powers of the volume) must also be dealt with here.
- Theoretical paper almost complete. N.H.Christ, X.Feng, A.Portelli, CTS
- An exploratory study of $K^+ \to \pi^+\nu\bar{\nu}$ decays is also underway and the parameters and early results were presented at Lattice 2015 by Xu Feng.

X.Feng, https://indico2.riken.jp/indico/confSpeakerIndex.py?confId=1805

# Summary and Conclusions on Prospects for Rare Kaon Decays

- For $K^+ \to \pi^+\ell^+\ell^-$ or $K_S \to \pi^0\ell^+\ell^-$ decays we now have a “complete" theoretical framework with which to perform lattice computations of the amplitudes.

  N.H.Christ, X.Feng, A.Portelli and C.T.Sachrajda, arXiv:1507.03094

  - Exploratory numerical simulations are underway and the preliminary results are very encouraging.
  - To use this framework in a simulation with physical quark masses would require a major project.
  - This would undoubtedly happen if there was a strong prospect of the corresponding experimental programme and will probably happen as part of the $K^+ \to \pi^+\nu\bar{\nu}$ project.
- For the evaluation of the LD contributions to $K^+ \to \pi^+\nu\bar{\nu}$ decays we are very close to being at the same stage, with a theoretical paper to be released in the next few weeks.
  - The exploratory numerical results are surprisingly (to me) encouraging.

- In May RBC-UKQCD published our first result for $\epsilon'/\epsilon$ computed at physical quark masses and kinematics, albeit still with large errors:

$$\left.\frac{\epsilon'}{\epsilon}\right|_{\text{RBC-UKQCD}} = (1.38 \pm 5.15 \pm 4.59) \times 10^{-4}$$

to be compared with

$$\left.\frac{\epsilon'}{\epsilon}\right|_{\text{Exp}} = (16.6 \pm 2.3) \times 10^{-4}\,.$$

RBC-UKQCD, arXiv:1505.07863

- This is by far the most complicated project that I have ever been involved with.
- This single result hides much important (and much more precise) information which we have determined along the way.
- In this section I will review the main obstacles to computing $K \to \pi\pi$ decay amplitudes, the techniques used to overcome them and our main results.

1. $A_0$ and $A_2$ amplitudes with unphysical quark masses and with the pions at rest.

   "$K$ to $\pi\pi$ decay amplitudes from lattice QCD,"
   T.Blum, P.A.Boyle, N.H.Christ, N.Garron, E.Goode, T.Izubuchi, C.Lehner, Q.Liu, R.D. Mawhinney, C.T.S, A.Soni, C.Sturm, H.Yin and R. Zhou, Phys. Rev. D **84** (2011) 114503 [arXiv:1106.2714 [hep-lat]].

   "Kaon to two pions decay from lattice QCD, $\Delta I = 1/2$ rule and CP violation"
   Q.Liu, Ph.D. thesis, Columbia University (2010)

2. $A_2$ at physical kinematics and a single coarse lattice spacing.

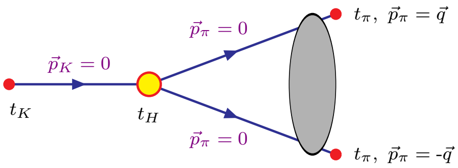
   "The $K \to (\pi\pi)_{I=2}$ Decay Amplitude from Lattice QCD,"
   T.Blum, P.A.Boyle, N.H.Christ, N.Garron, E.Goode, T.Izubuchi, C.Jung, C.Kelly, C.Lehner, M.Lightman, Q.Liu, A.T.Lytle, R.D.Mawhinney, C.T.S., A.Soni, and C.Sturm
   Phys. Rev. Lett. **108** (2012) 141601 [arXiv:1111.1699 [hep-lat]],

   "Lattice determination of the $K \to (\pi\pi)_{I=2}$ Decay Amplitude $A_2$"
   Phys. Rev. D **86** (2012) 074513 [arXiv:1206.5142 [hep-lat]]

   "Emerging understanding of the $\Delta I = 1/2$ Rule from Lattice QCD,"
   P.A. Boyle, N.H. Christ, N. Garron, E.J. Goode, T. Janowski, C. Lehner, Q. Liu, A.T. Lytle, C.T. Sachrajda, A. Soni, and D.Zhang, Phys. Rev. Lett. **110** (2013) 15, 152001 [arXiv:1212.1474 [hep-lat]].

3. $A_2$ at physical kinematics on two finer lattices $\Rightarrow$ continuum limit taken.
"$K \to \pi\pi$ $\Delta I = 3/2$ decay amplitude in the continuum limit,"
T.Blum, P.A.Boyle, N.H.Christ, J.Frison, N.Garron, T.Janowski, C.Jung, C.Kelly, C.Lehner, A.Lytle, R.D.Mawhinney, C.T.S., A.Soni, H.Yin, and D.Zhang
Phys. Rev. D **91** (2015) 7, 074502 [arXiv:1502.00263 [hep-lat]].

4. $A_0$ at physical kinematics and a single coarse lattice spacing.
"Standard-model prediction for direct CP violation in $K \to \pi\pi$ decay,"
Z.Bai, T.Blum, P.A.Boyle, N.H.Christ, J.Frison, N.Garron, T.Izubuchi, C.Jung, C.Kelly, C.Lehner, R.D.Mawhinney, C.T.S, A. Soni, and D. Zhang,
Phys. Rev. Lett. **115** (2015) 21, 212001 [arXiv:1505.07863 [hep-lat]].




- $K \to \pi\pi$ correlation function is dominated by lightest state, i.e. the state with two-pions at rest. Maiani and Testa, PL B245 (1990) 585

$$C(t_\pi) = A + B_1 e^{-2m_\pi t_\pi} + B_2 e^{-2E_\pi t_\pi} + \cdots$$

  - Solution 1: Study an excited state. Lellouch and Lüscher, hep-lat/0003023
  - Solution 2: Introduce suitable boundary conditions such that the $\pi\pi$ ground state is $|\pi(\vec{q})\pi(-\vec{q})\rangle$. RBC-UKQCD, C.h.Kim hep-lat/0311003

For $B$-decays, with so many intermediate states below threshold, this is the main obstacle to producing reliable calculations.

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- For $A_2$, there is no vacuum subtraction and we can use the Wigner-Eckart theorem to write

$$\underbrace{\langle(\pi\pi)^{I=2}_{I_3=1}\,|}_{\frac{1}{\sqrt{2}}(\langle\pi^+\pi^0|+\langle\pi^0\pi^+|)}\; Q^{\Delta I=3/2}_{\Delta I_3=1/2,i}\,|\,K^+\rangle = \frac{3}{2}\,\underbrace{\langle(\pi\pi)^{I=2}_{I_3=2}\,|}_{\langle\pi^+\pi^+|}\,Q^{\Delta I=3/2}_{\Delta I_3=3/2,i}\,|\,K^+\rangle\,,$$

  and impose anti-periodic conditions on the d-quark in one or more directions.
- If we impose the anti-periodic boundary conditions in all 3 directions then the ground state is

$$\Big|\pi\left(\frac{\pi}{L},\frac{\pi}{L},\frac{\pi}{L}\right)\,\pi\left(\frac{-\pi}{L},\frac{-\pi}{L},\frac{-\pi}{L}\right)\Big\rangle\,.$$

- With an appropriate choice of $L$ and the number of directions, we can arrange that $E_{\pi\pi}=m_K$.
- Isospin breaking by the boundary conditions is harmless here.

CTS & G.Villadoro, hep-lat/0411033

# Finite Volume Effects

- These are based on the Poisson summation formula:

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} f(p_n^2) = \int_{-\infty}^{\infty}\frac{dp}{2\pi} f(p^2) + \sum_{n\neq 0}\int_{-\infty}^{\infty}\frac{dp}{2\pi} f(p^2)e^{inpL}\,,$$

- For single-hadron states the finite-volume corrections decrease exponentially with the volume $\propto e^{-m_\pi L}$. For multi-hadron states, the finite-volume corrections generally fall as powers of the volume.
- For two-hadron states, there is a huge literature following the seminal work by Lüscher and the effects are generally understood.
  - The spectrum of two-pion states in a finite volume is given by the scattering phase-shifts. M. Luscher, Commun. Math. Phys. 105 (1986) 153, Nucl. Phys. B354 (1991) 531.
  - The $K \to \pi\pi$ amplitudes are obtained from the finite-volume matrix elements by the Lellouch-Lüscher factor which contains the derivative of the phase-shift. L.Lellouch & M.Lüscher, hep-lat/:0003023, C.h.Kim, CTS & S.R.Sharpe, hep-lat/0507006 $\cdots$
  - Recently we have also determined the finite-volume corrections for $\Delta m_K = m_{K_L} - m_{K_S}$. N.H.Christ, X.Feng, G.Martinelli & CTS, arXiv:1504.01170
- For three-hadron states, there has been a major effort by Hansen and Sharpe leading to much theoretical clarification.

M.Hansen & S.Sharpe, arXiv:1408.4933, 1409.7012, 1504.04248

- Since we cannot perform simulations with lattice spacings $< 1/M_W$ or $1/m_t$ we exploit the standard technique of the Operator Product Expansion and write schematically:

$$\text{Physics} = \sum_i C_i(\mu) \times \langle f|O_i(\mu)|i\rangle\,.$$

- Until recently, the (perturbative) Wilson coefficients $C_i(\mu)$ were typically calculated with much greater precision than our knowledge of the matrix elements.
  - The $C_i$ are typically calculated in schemes based on dimensional regularisation (such as $\overline{\text{MS}}$) which are intrinsically perturbative.
  - We can compute the matrix elements non-perturbatively, with the operators renormalised in schemes which have a non-perturbative definition (such as RI-MOM schemes) but not in purely perturbative schemes based on dim.reg.

G.Martinelli, C.Pittori, CTS, M.Testa and A.Vladikas, hep-lat/9411010

- Thus the determination of the $C_i$ in $\overline{\text{MS}}$-like schemes is not the complete perturbative calculation. Matching between $\overline{\text{MS}}$ and non-perturbatively defined schemes must also be performed.
  - This is beginning to be done.
  - We are now careful to present tables of matrix elements of operators renormalized in RI-MOM schemes, which can be used to gain better precision once improved perturbative calculations are performed.

# Error budgets in our calculation of $A_2$



RBC-UKQCD, T.Blum et al., arXiv:1502:00263

| Source | Re$A_2$ | Im$A_2$ |
|---|---|---|
| NPR (nonperturbative) | 0.1% | 0.1% |
| NPR (perturbative) | 2.9% | 7.0% |
| Finite volume corrections | 2.4% | 2.6% |
| Unphysical kinematics | 4.5% | 1.1% |
| Wilson coefficients | 6.8% | 10% |
| Derivative of the phase shift | 1.1% | 1.1% |
| Total | 9% | 12% |

- *Wilson Coefficients* and *NPR(perturbative)* errors are not from our lattice calculation.
- Step-scaling can be used to increase the scale at which the matching is performed.

- Our first results for $A_2$ at physical kinematics were obtained at a single, rather coarse, value of the lattice spacing ($a \simeq 0.14$ fm). Estimated discretization errors at 15%. arXiv:1111.1699, arXiv:1206.5142
- Our recent results were obtained on two new ensembles, $48^3$ with $a \simeq 0.11$ fm and $64^3$ with $a \simeq 0.084$ fm so that we can make a continuum extrapolation:

$$\begin{aligned} \mathrm{Re}(A_2) &= 1.50(4)_{\mathrm{stat}}(14)_{\mathrm{syst}} \times 10^{-8}\ \mathrm{GeV}. \\ \mathrm{Im}(A_2) &= -6.99(20)_{\mathrm{stat}}(84)_{\mathrm{syst}} \times 10^{-13}\ \mathrm{GeV}\,. \end{aligned}$$

arXiv:1502.00263

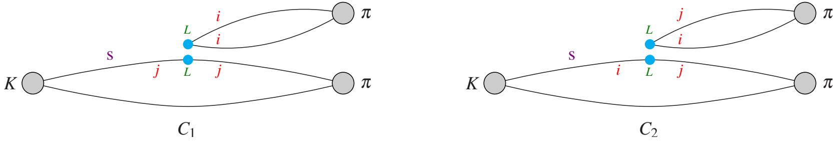
- Although the precision can still be significantly improved (partly by perturbative calculations), the calculation of $A_2$ at physical kinematics can now be considered as standard.

RBC-UKQCD Collaboration, arXiv:1212.1474

- Re $A_2$ is dominated by a simple operator:

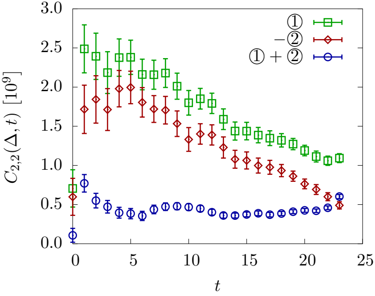
$$O^{3/2}_{(27,1)} = (\bar{s}^i d^i)_L \left\{ (\bar{u}^j u^j)_L - (\bar{d}^j d^j)_L \right\} + (\bar{s}^i u^i)_L \, (\bar{u}^j d^j)_L$$

  and two diagrams:

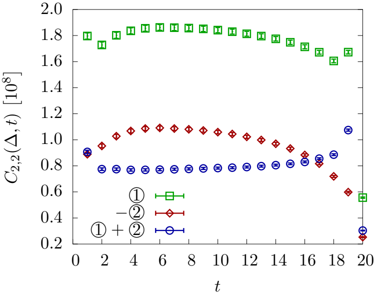



- Re $A_2$ is proportional to $C_1 + C_2$.
- The contribution to Re $A_0$ from $Q_2$ is proportional to $2C_1 - C_2$ and that from $Q_1$ is proportional to $C_1 - 2C_2$ with the same overall sign.
- Colour counting might suggest that $C_2 \simeq \frac{1}{3} C_1$.
- We find instead that $C_2 \approx -C_1$ so that $A_2$ is significantly suppressed!
- **We believe that the strong suppression of Re $A_2$ and the (less-strong) enhancement of Re $A_0$ is a major factor in the $\Delta I = 1/2$ rule.**

# Evidence for the Suppression of Re $A_2$



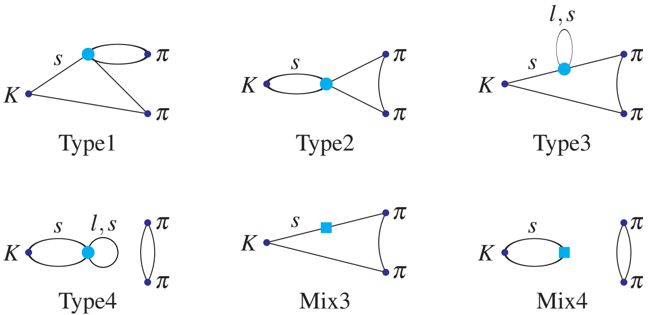

Physical Kinematics




$m_\pi \simeq$ 330 MeV at threshold.

- Notation ⓘ $\equiv C_i,\ i = 1, 2.$
- Of course before claiming a quantitative understanding of the $\Delta I = 1/2$ rule we needed to compute Re $A_0$ at physical kinematics and reproduce the experimental value of 22.5.
- Much early phenomenology was based on the vacuum insertion approach. although the qualitative picture we find had been suggested by Bardeen, Buras and Gerard in 1987.

- The calculation is much more difficult for the $K \to (\pi\pi)_{I=0}$ amplitude $A_0$:
  - The presence of disconnected diagrams, vacuum subtraction, ultra-violet power divergences, $\cdots$




  - $|\pi^+(\pi/L)\pi^-(\text{-}\pi/L)\rangle$ has a different energy from $|\pi^0(\vec{0})\pi^0(\vec{0})\rangle$.
  - We have developed the implementation of $G$-parity boundary conditions in which $(u,d) \to (\bar{d}, -\bar{u})$ at the boundary .

U. Wiese, Nucl.Phys. B375 (1992) 45 , RBC-UKQCD, C.h.Kim hep-lat/0311003

Slide shown at the annual UK Christmas Theory meeting, 2013

- RBC-UKQCD have computed $A_0$ with the two pions at rest and with unphysical masses, finding e.g. arXiv:1106.2714, Qi Liu Columbia Un.Thesis

$$\frac{\mathrm{Re}\,A_0}{\mathrm{Re}\,A_2} = 9.1 \pm 2.1 \qquad 877\,\mathrm{MeV}\text{ kaon decaying into two }422\,\mathrm{MeV}\text{ pions}$$

$$\frac{\mathrm{Re}\,A_0}{\mathrm{Re}\,A_2} = 12.0 \pm 1.7 \qquad 662\,\mathrm{MeV}\text{ kaon decaying into two }329\,\mathrm{MeV}\text{ pions}$$

- Whilst both these results are obtained at unphysical kinematics and are different from the physical value of 22.5, it is nevertheless interesting to understand the origin of these enhancements.
- 99% of the contribution to the real part of $A_0$ and $A_2$ come from the matrix elements of the current-current operators.
- For a calculation of $\epsilon'/\epsilon$ at physical kinematics, RBC-UKQCD are developing G-parity boundary conditions (estimate timescale $\sim$ 2 years).

- Computations were performed on a $32^3 \times 64$ lattice with the Iwasaki and DSDR gauge action and $N_f = 2+1$ flavours of Möbius Domain Wall Fermions)

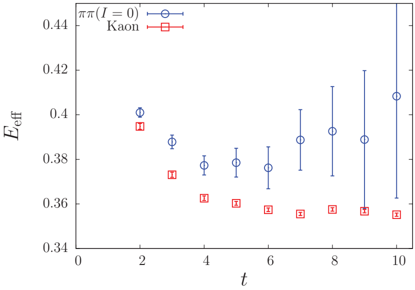
$$a^{-1} = 1.379(7)\,\text{GeV},\, m_\pi = 143.2(2.0)\,\text{MeV},\, (E_\pi = 274.8(1.4)\,\text{MeV})$$

- The $\pi\pi$ energies are

$$E_{\pi\pi}^{I=0} = (498 \pm 11)\,\text{MeV} \quad E_{\pi\pi}^{I=2} = (565.7 \pm 1.0)\,\text{MeV}$$

to be compared with $m_K = (490.6 \pm 2.4)$ MeV.

- Lüscher's quantisation condition $\Rightarrow E_{\pi\pi}^{I=0}$ corresponds to $\delta_0 = (23.8 \pm 4.9 \pm 1.2)^\circ$, which is somewhat smaller than phenomenological expectations.



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$$H_W = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_{i=1}^{10} \left[ z_i(\mu) + \tau y_i(\mu) \right] Q_i(\mu). \qquad \left( \tau = -\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} \right)$$

Wilson coefficients from Buchalla, Buras, Lautenbacher, hep-ph/9512380

| i | Re($A_0$)(GeV) | Im($A_0$)(GeV) |
|---|---|---|
| 1 | $1.02(0.20)(0.07) \times 10^{-7}$ | 0 |
| 2 | $3.60(0.90)(0.28) \times 10^{-7}$ | 0 |
| 3 | $-1.28(1.69)(1.20) \times 10^{-10}$ | $1.53(2.03)(1.44) \times 10^{-12}$ |
| 4 | $-2.01(0.69)(0.36) \times 10^{-9}$ | $1.80(0.61)(0.32) \times 10^{-11}$ |
| 5 | $-8.93(2.23)(1.84) \times 10^{-10}$ | $1.54(0.38)(0.32) \times 10^{-12}$ |
| 6 | $3.51(0.89)(0.23) \times 10^{-9}$ | $-3.56(0.90)(0.24) \times 10^{-11}$ |
| 7 | $2.38(0.40)(0.00) \times 10^{-11}$ | $8.49(1.44)(0.00) \times 10^{-14}$ |
| 8 | $-1.28(0.04)(0.00) \times 10^{-10}$ | $-1.71(0.05)(0.00) \times 10^{-12}$ |
| 9 | $-7.38(1.97)(0.48) \times 10^{-12}$ | $-2.41(0.64)(0.16) \times 10^{-12}$ |
| 10 | $7.29(2.62)(0.68) \times 10^{-12}$ | $-4.72(1.69)(0.44) \times 10^{-13}$ |
| Total (stat only) | $4.66(0.96)(0.27) \times 10^{-7}$ | $-1.90(1.19)(0.32) \times 10^{-11}$ |
| Final (incl. syst) | $4.66(1.00)(1.21) \times 10^{-7}$ | $-1.90(1.23)(1.04) \times 10^{-11}$ |

- Representative Errors

| Description | Error | Description | Error |
|---|---|---|---|
| Finite lattice spacing | 8% | Finite volume | 7% |
| Wilson coefficients | 12% | Excited states | $\leq 5\%$ |
| Parametric errors | 5% | Operator renormalization | 15% |
| Unphysical kinematics | $\leq 3\%$ | Lellouch-Lüscher factor | 11% |
| Total (added in quadrature) | | | 26% |

## Conclusions for $K \to \pi\pi$ decays

- As a results of our work, the computation of $A_2$ is now “standard".
- It appears that the explanation of the $\Delta I = 1/2$ rule has a number of components, of which the significant cancelation between the two dominant contributions to Re$A_2$ is a major one.
- We have completed the first calculation of $\epsilon'/\epsilon$ with controlled errors $\Rightarrow$ motivation for further refinement (systematic improvement by collecting more statistics, working on larger volumes, $\geq$2 lattice spacings etc.)
- $\epsilon'/\epsilon$ is now a quantity which is amenable to lattice computations.