$\pi \pi$ scattering length measurement

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Outline

Low energy theorems, chiral expansion

Dispersive methods
  Roy equations
  Chiral symmetry + dispersive methods

What have we learnt?

Concluding remarks
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Low-energy theorem for $\pi\pi$ scattering

Some notation

$$\langle \pi^i \pi^j \text{ out} | \pi^k \pi^l \text{ in} \rangle = \delta^{ij} \delta^{kl} A(s, t, u) + \delta^{ik} \delta^{jl} A(t, u, s) + \delta^{il} \delta^{jk} A(u, t, s)$$

All physical amplitudes can be expressed in terms of $A(s, t, u)$

$$T^{l=0}(s, t, u) = 3A(s, t, u) + A(t, s, u) + A(u, t, s)$$

Low energy theorem

Weinberg 1966

$$A(s, t, u) = \frac{s - M_\pi^2}{F_\pi^2} + \mathcal{O}(p^4) \quad \Rightarrow \quad T^{l=0} = \frac{2s - M_\pi^2}{F_\pi^2}$$

S wave projection

$$t_0^0(s) = \frac{2s - M_\pi^2}{32\pi F_\pi^2} \quad a_0^0 = t_0^0(4M_\pi^2) = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16$$
Low-energy theorem for $\pi\pi$ scattering

Some notation

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Low energy theorem

Weinberg 1966

$$A(s, t, u) = \frac{s - M^2}{F^2_\pi} + \mathcal{O}(p^4) \quad \Rightarrow \quad T^{I=0} = \frac{2s - M^2}{F^2_\pi}$$

S wave projection

(I=2)

$$t_0^2(s) = \frac{2M^2_\pi - s}{32\pi F^2_\pi} \quad a_0^2 = t_0^2(4M^2_\pi) = \frac{-M^2_\pi}{16\pi F^2_\pi} = -0.045$$
Higher orders

Higher order corrections are suppressed by $O(p^2/\Lambda^2)$
$\Lambda \sim 1$ GeV $\Rightarrow$ expected to be a few percent

$$a_0^0 = 0.200 + O(p^6) \quad a_0^2 = -0.0445 + O(p^6)$$

Gasser and Leutwyler (84)
Higher orders

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$$a_0^0 = 0.200 + O(p^6) \quad a_0^2 = -0.0445 + O(p^6)$$

The reason for the rather large correction in $a_0^0$ is a chiral log

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} \left[ 1 + \frac{9}{2} \ell_\chi + \ldots \right] \quad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} \left[ 1 - \frac{3}{2} \ell_\chi + \ldots \right]$$

$$\ell_\chi = \frac{M_\pi^2}{16\pi^2 F_\pi^2} \ln \frac{\mu^2}{M_\pi^2}$$

Gasser and Leutwyler (84)
Higher orders

\[ \text{Re}[t_0^0] \]

- \( O(p^2) \)
- \( O(p^2) + O(p^4) \)
- \( O(p^4) \)

\[ E(\text{GeV}) \]
Higher orders

\[ Re[t_0^2] \]

\[ O(p^2) \]
\[ O(p^2) + O(p^4) \]
\[ O(p^4) \]
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Roy equations

Unitarity effects can be calculated *exactly* using dispersive methods

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

**Input:** imaginary parts above 0.8 GeV

two subtraction constants, *e.g.* $a_0^0$ and $a_0^2$

**Output:** the full $\pi\pi$ scattering amplitude below 0.8 GeV

**Note:** if $a_0^0, a_0^2$ are chosen within the universal band

the solution exist and is unique

**Numerical solutions of the Roy equations:**

Pennington-Protopopescu, Basdevant-Froggatt-Petersen (70s)

Ananthanarayan, GC, Gasser and Leutwyler (00)

Descotes-Genon, Fuchs, Girlanda and Stern (01)

Garcia-Martin, Kaminski, Pelaez, Ruiz de Elvira, Yndurain (08,11)
Numerical solutions

![Graph showing numerical solutions with points S_0, S_1, S_2, and S_3 on a 2D plot. The axes are labeled a_0^2 and a_0. The values range from -0.06 to 0.05 for a_0^2 and 0.15 to 0.3 for a_0. The points S_0, S_1, S_2, and S_3 are marked on the graph.]
Numerical solutions

![Graph showing numerical solutions with labels for I=0, I=1, and I=2. The graph includes the real part of the Roy equations and input for E>0.8 GeV.]
Numerical solutions
Numerical solutions

![Graph showing numerical solutions for Roy equations with different values of I (0, 1, 2). The x-axis represents E (GeV) ranging from 0.2 to 1.2, and the y-axis represents the real part ranging from -0.4 to 0.6. The graph includes a line for Roy's equation and another for input for E > 0.8 GeV.]
Numerical solutions

Example of a fit to data

![Graph showing a fit to data with various solutions and data points.](image)
Combining CHPT and dispersive methods

In CHPT the two subtraction constants are predicted

Subtracting the amplitude at threshold \((a_0^0, a_0^2)\) is not mandatory

The freedom in the choice of the subtraction point can be exploited to use the chiral expansion where it converges best, i.e. below threshold
Combining CHPT and dispersive methods
Combining CHPT and dispersive methods

The convergence of the series at threshold is greatly improved if CHPT is used only below threshold

**CHPT at threshold**

\[
\begin{align*}
a_0^0 &= 0.159 \rightarrow 0.200 \rightarrow 0.216 \\
10 \cdot a_0^2 &= -0.454 \rightarrow -0.445 \rightarrow -0.445 \\
p^2 & \quad p^4 & \quad p^6
\end{align*}
\]

GC, Gasser and Leutwyler (01)
Combining CHPT and dispersive methods

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\]

\[
\begin{array}{c c c}
    p^2 & p^4 & p^6 \\
\end{array}
\]

**CHPT below threshold + Roy**

\[
\begin{align*}
    a_0^0 & = 0.197 \rightarrow 0.2195 \rightarrow 0.220 \\
    10 \cdot a_0^2 & = -0.402 \rightarrow -0.446 \rightarrow -0.444
\end{align*}
\]

GC, Gasser and Leutwyler (01)
Roy+ChPT: final results

GC, Gasser and Leutwyler (01)

Scattering lengths

\[ a_0^0 = 0.220 \pm 0.001 + 0.009 \Delta \ell_4 - 0.002 \Delta \ell_3 \]
\[ 10 \cdot a_0^2 = -0.444 \pm 0.003 - 0.01 \Delta \ell_4 - 0.004 \Delta \ell_3 \]

where \( \bar{\ell}_4 = 4.4 + \Delta \ell_4 \)
\( \bar{\ell}_3 = 2.9 + \Delta \ell_3 \)

Adding errors in quadrature

\[ \Delta \ell_4 = 0.2, \ \Delta \ell_3 = 2.4 \]

\[ a_0^0 = 0.220 \pm 0.005 \]
\[ 10 \cdot a_0^2 = -0.444 \pm 0.01 \]
\[ a_0^0 - a_0^2 = 0.265 \pm 0.004 \]
Roy+ChPT: final results

GC, Gasser and Leutwyler (01)

Universal Band
- tree, one loop, two loops
- $\tilde{l}_4 = 4.4 \pm 0.2$, $\tilde{l}_3$ free
- $\tilde{l}_3 = 2.9 \pm 2.4$, $\tilde{l}_4$ free
- Colangelo, Gasser & Leutwyler 2001
Final result for the phase shifts

Phase shifts:

$\delta_1$ (degrees)

E(GeV)

Hyams et al.
Protopopescu et al.

phase of the form factor
Final result for the phase shifts

Phase shifts:

\[ \delta_0(\text{degrees}) \]

\[ E(\text{GeV}) \]

- ACM (A) data
- ACM (B) data
- Losty et al. data
- RBC/UKQCD
Final result for the phase shifts

Phase shifts:

\[ \delta_0(\text{degrees}) \]

\[ E(\text{GeV}) \]

Hyams et al.
Protopopescu et al.
RBC/UKQCD
Final result for the phase shifts

Determination by RBC/UKQCD:

\[
\begin{align*}
\delta_0^0(MK) &= 23.8(4.9)(1.2) \\
\delta_0^2(MK) &= -11.6(2.5)(1.2)
\end{align*}
\]

Our determination (at \(M_k = 0.4976\) GeV)

\[
\begin{align*}
\delta_0^0(MK) &= 39.2(1.5) \\
\delta_0^2(MK) &= -8.5(0.15)
\end{align*}
\]

or (at \(M_k = 0.4906\) GeV value used by RBC/UKQCD):

\[
\begin{align*}
\delta_0^0(MK) &= 38.0(1.3) \\
\delta_0^2(MK) &= -8.3(0.15)
\end{align*}
\]
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Sensitivity to the quark condensate

The constant $\ell_3$ appears in the chiral expansion of the pion mass

$$M_{\pi}^2 = 2B\hat{m}\left[1 + \frac{2B\hat{m}}{16\pi F_{\pi}^2}\ell_3 + \mathcal{O}(\hat{m}^2)\right]$$

$$\hat{m} = \frac{m_u + m_d}{2} \quad B = -\frac{1}{F^2}\langle 0|\bar{q}q|0\rangle$$
Sensitivity to the quark condensate

The constant $\bar{\ell}_3$ appears in the chiral expansion of the pion mass

$$M^2_\pi = 2B\hat{m} \left[ 1 + \frac{2B\hat{m}}{16\pi F^2_\pi} \bar{\ell}_3 + O(\hat{m}^2) \right]$$

$$\hat{m} = \frac{m_u + m_d}{2} \quad B = -\frac{1}{F^2} \langle 0 | \bar{q} q | 0 \rangle$$

Its size tells us what fraction of the pion mass is given by the Gell-Mann–Oakes–Renner term

$$M^2_{\text{GMOR}} \equiv 2B\hat{m}$$
Sensitivity to the quark condensate

The E865 data on $K_{\ell 4}$ imply that

$M_{\text{GMOR}} > 94\% M_\pi$

GC, Gasser and Leutwyler PRL (01)
Experimental tests

- Universal band
- Tree (66), one loop (83), two loops (96)
- Prediction (ChPT + dispersion theory, 2001)
- DIRAC (2005)
- NA48 K -> 3 π (2005)
- E865
- NA48
Experimental tests
Experimental tests

E865 corrected their data
Experimental tests

E865 corrected their data
isospin breaking corrections recently calculated for $K_{e4}$ are essential at this level of precision

GC, Gasser, Rusetsky (09)
Experimental tests

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GC, Gasser, Rusetsky (09)
Experimental tests

Figure from NA48/2 Eur.Phys.J.C64:589,2009
Lattice determinations of $\bar{\ell}_{3,4}$

Figure courtesy of H. Leutwyler
Lattice determinations of $\bar{\ell}_{3,4}$

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Remarks on a possible new measurement of $a_0^0$

- the burning question about the mechanism of chiral symmetry breaking has been answered experimentally by past measurements of $a_0^0$
- the precision of the theoretical prediction for $a_0^0$ has *de facto* already been improved by lattice determinations of $\bar{l}_3$
- increasing the precision of the experimental measurement of $a_0^0$ will require a better handling of radiative corrections (→ talks by P. Stoffer and M. Knecht)
- this could lead to a precise determination of $\bar{l}_3$, *i.e.* of the quadratic dependence of $M_\pi$ on $\hat{m}$
- in itself this is a remarkable achievement, but not a qualitative change in our knowledge of nonperturbative QCD
Remarks on the relevance of the measurement of $a_0^0$

- the burning question about the mechanism of chiral symmetry breaking has been answered experimentally by past measurements of $a_0^0$
- an accurate measurement of the $S$ wave scattering lengths implies also a precise knowledge of the $\pi\pi$ phase shifts below $\sim 1 \text{ GeV}$
- which makes a dispersive treatment of several other low-energy matrix elements – in particular $K$ decays – meaningful and potentially very precise
- e.g. $K_{\ell 4}$, $K \rightarrow \pi\pi$, $K \rightarrow 3\pi$, $K_S \rightarrow \gamma(\ast)\gamma(\ast)$, $K_S \rightarrow \ell^+\ell^-$, $K \rightarrow \pi\gamma\gamma$, $K \rightarrow \pi\ell^+\ell^-$, $\Delta M_K$, $\epsilon_K$

(→ talks by P. Stoffer and R. Stucki)
analyticity properties of Green’s functions (and form factors and scattering amplitudes) can be rigorously established

the absence of singularities for complex (unphysical) values of kinematic variables\(^1\) follows from causality

the presence of singularities is related to dynamical phenomena (exchange of particles) and can be understood in terms of the underlying dynamics

analytic functions are determined by their singularities: dispersion relations provide an explicit representation of this mathematical property

QFTs satisfy these properties automatically. *Weinberg*: QFT emerges by imposing analyticity and unitarity (and other properties)

\(^1\)Exceptions known: anomalous thresholds.
Dispersion relations: basics II

- dispersion relations are exact
- their usefulness is directly related to our knowledge of the singularities of the function of interest
- depending on where one wants to calculate the function, some singularities (or regions thereof) may be more important than others: approximation schemes may be successfully applied
- singularities at infinity = subtraction constants, if present are essential input
- use of dispersion relations in combination with QFT calculations (whether perturbative or not) is always possible