Efficient Eccentric EOB Dynamics via Near-Identity Averaging Transformations

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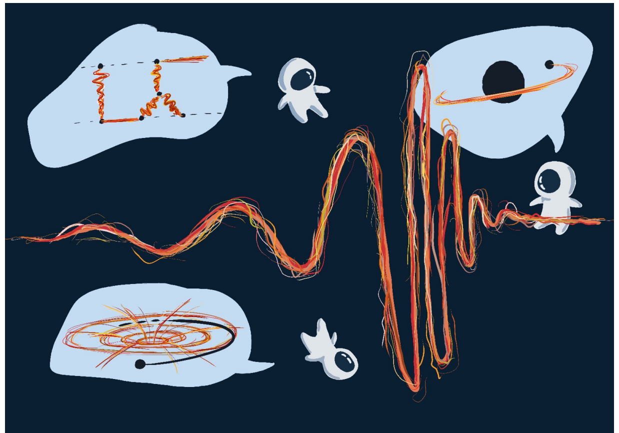

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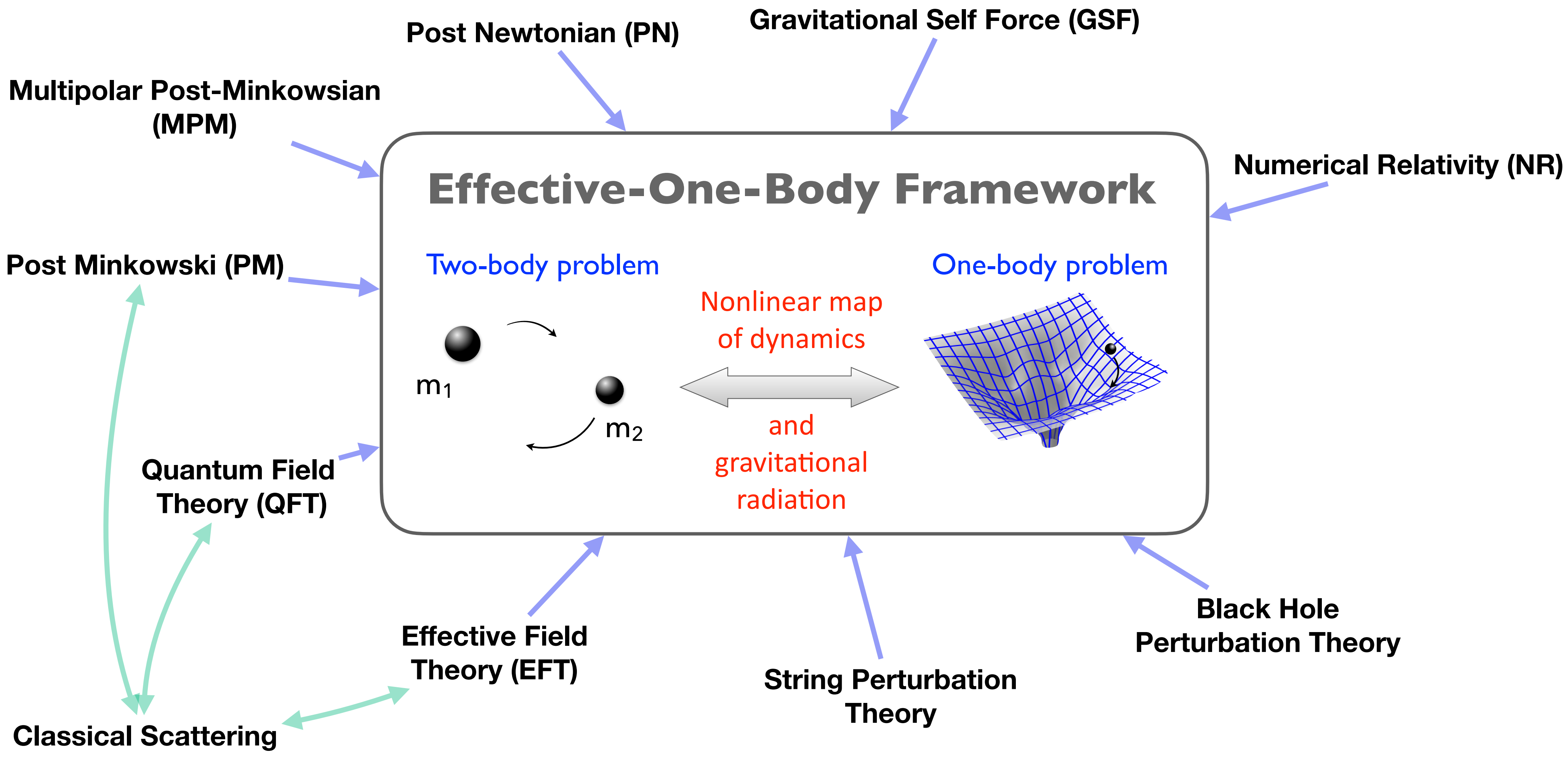
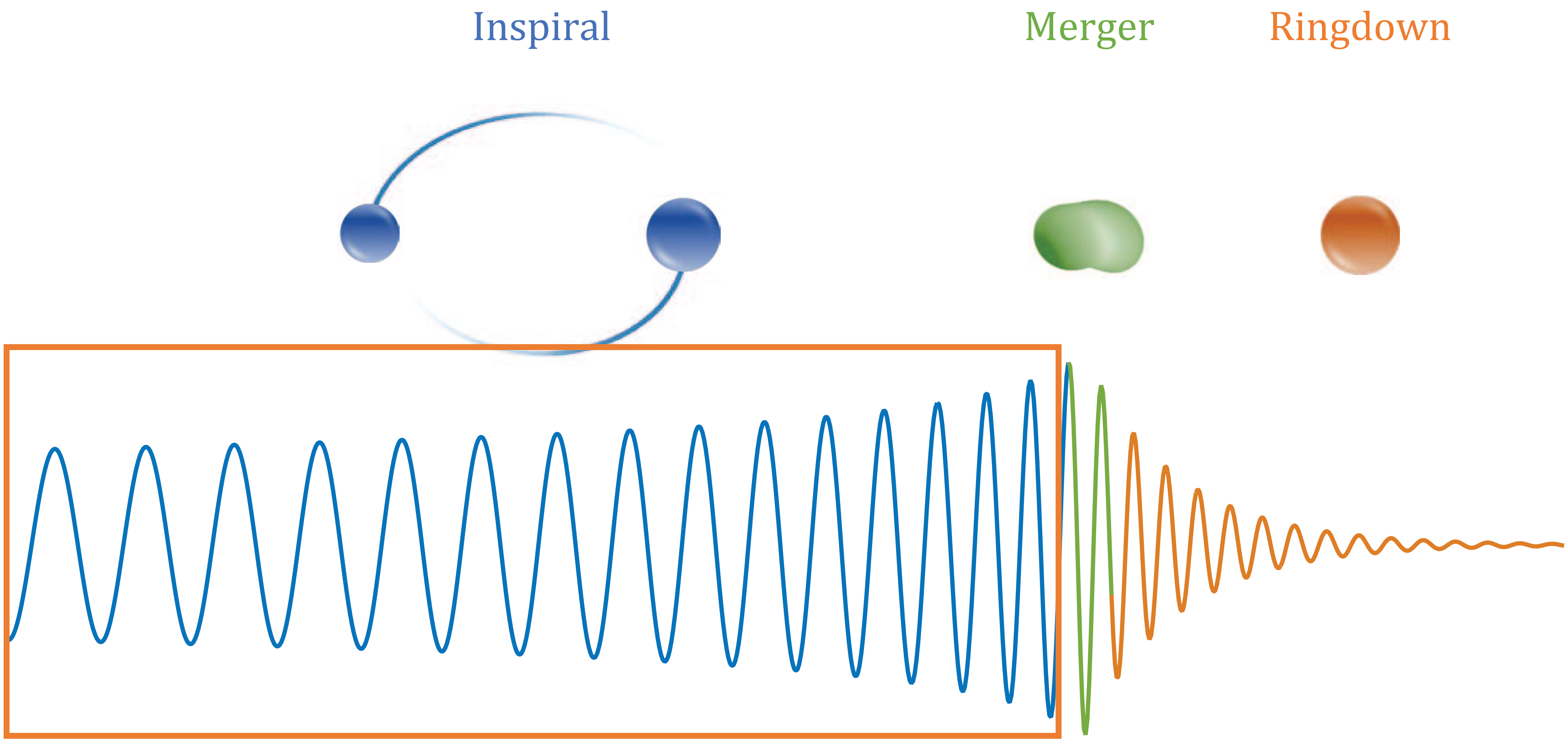


[LISA WF Whitepaper: arXiv: 2311.01300]



[Antelis & Moreno 1610.03567]

Motivation

- Need to model long eccentric **MBHB** signals for **LISA**
- **Quasicircular** dynamics are fast w/ **"Post-Adiabatic expansion"**
  - [Nagar & Rettengo 18],[Rettengo +19] [Reimenschneider+21], [Mihaylov+21]
- **Eccentric dynamics** are slow
- Borrow techniques from **EMRI** modelling:
  - Speed up the **inspiral** section
  - **Osculating Orbital Elements** (OOEs)
  - **Near Identity Averaging Transformations** (NITs)
- Proof of concept:
  - h_{22} for *non-spinning eccentric* binaries

EOB Inspiral Dynamics

$$H_{\rm EOB} = M\sqrt{1+2\nu(\hat{H}_{\rm eff}-1)}$$

$$\hat{H}_{\rm eff} = \sqrt{p_{r_*}^2 + A(\nu,r)\left(1+\frac{p_\phi^2}{r^2}+Q(\nu,r,p_{r_*})\right)}$$

$$p_{r_*} = A(\nu,r)\sqrt{\bar{D}(\nu,r)}p_r$$

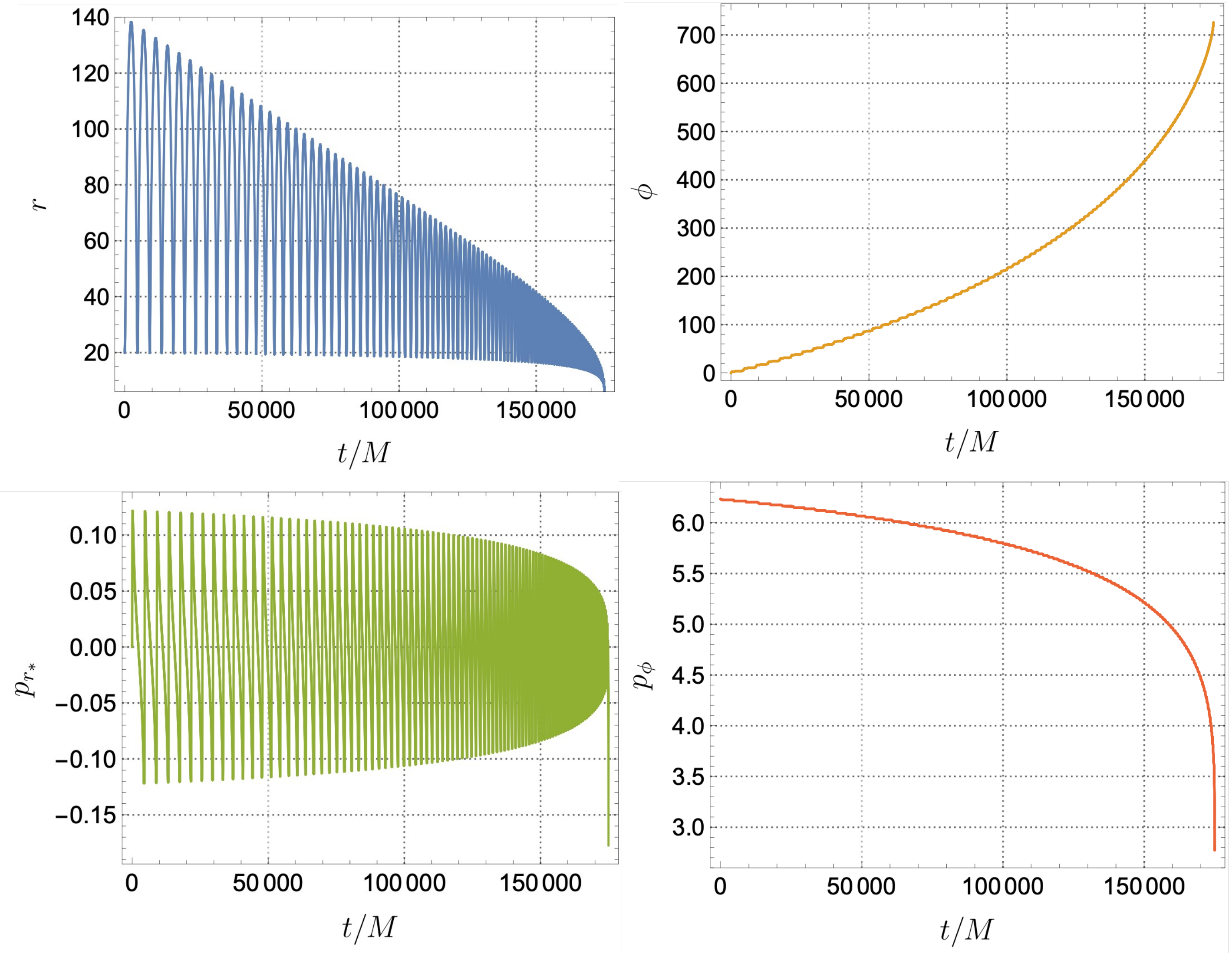
$$\frac{dr}{dt} = \frac{\partial p_{r_*}}{\partial p_r}\frac{\partial H_{\rm EOB}}{\partial p_{r_*}} \qquad \frac{d\phi}{dt} = \frac{\partial H_{\rm EOB}}{\partial p_\phi}$$

$$\frac{dp_{r_*}}{dt} = -\frac{\partial p_{r*}}{\partial p_r}\frac{\partial H_{\rm EOB}}{\partial r} + \mathcal{F}_r(\nu,r,p_r,p_\phi)$$

$$\frac{dp_\phi}{dt} = \mathcal{F}_\phi(\nu,r,p_r,p_\phi)$$

$$h(t) = \frac{1}{d_L}\sum_{\ell=2}^{\ell_{\rm max}}\sum_{m=-\ell}^{\ell} h_{\ell m}(\nu,r,\phi,p_r,p_\phi)\,{}_{-2}Y_{\ell m}(\Theta,\Phi)$$

- Potentials: SEOBNRv5
- RR. Forces: 3PN Eccentric
- WF Modes: 3PN Eccentric



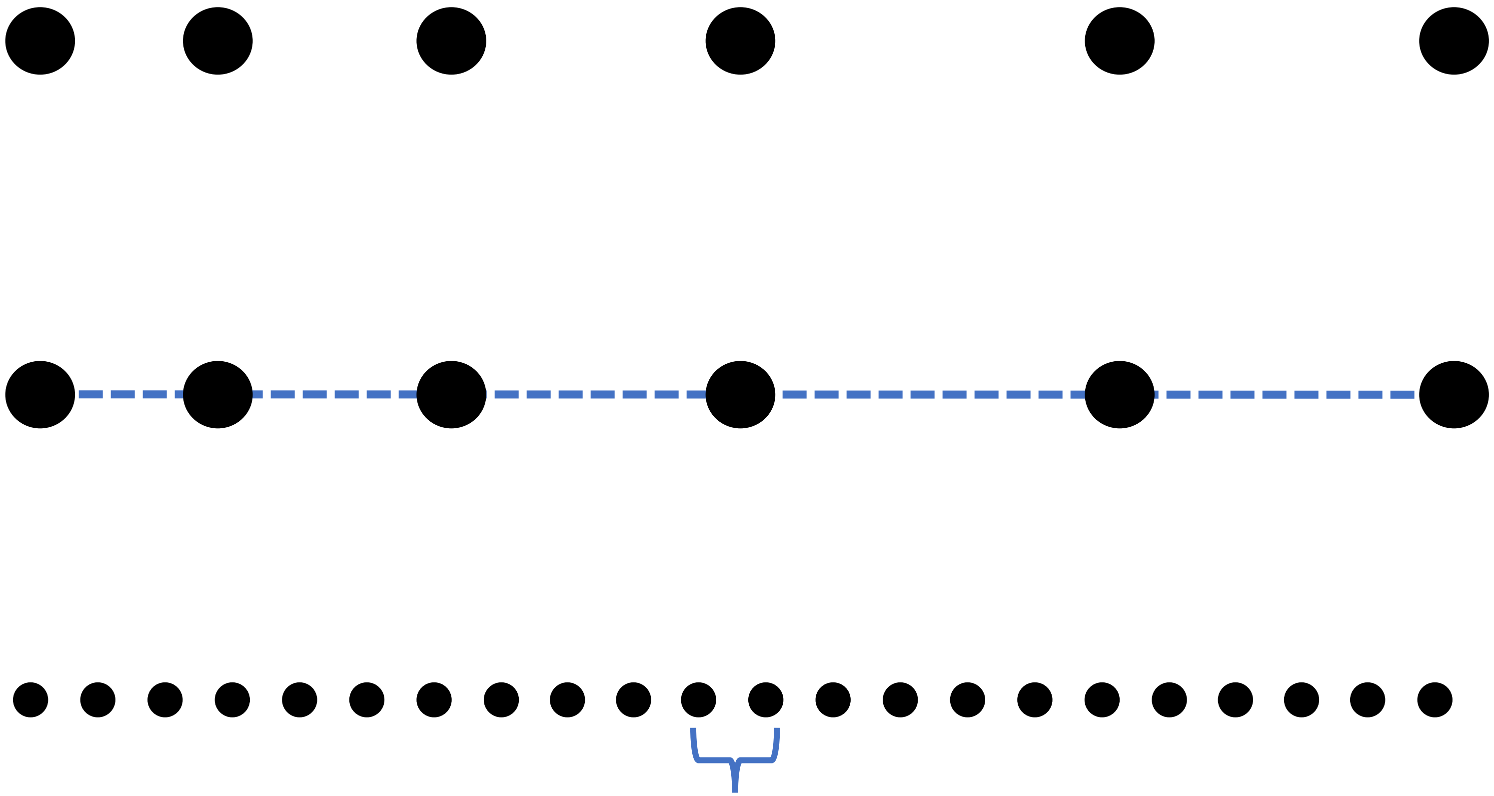
- $m_1 = m_2 = 30\ M_\odot @\ 10Hz$
- $(\nu, p_0, e_0 \xi_0) = (0.25, 34.94, 0.75, 0)$
- Using RK4 adaptive time stepper
- Steps taken: 86,567
- Dynamics time: 3.34s

$(r(t), \phi(t), p_r(t), p_\phi(t))$

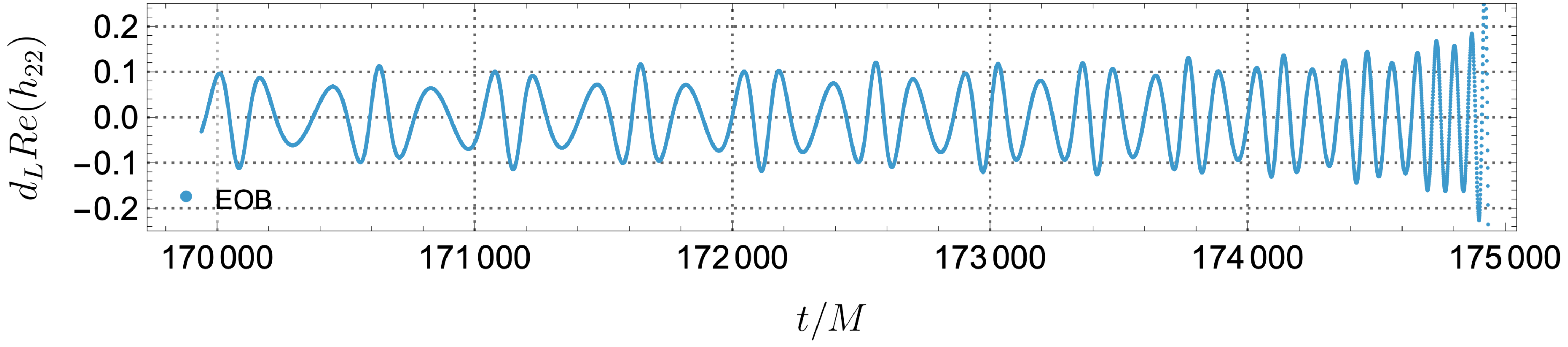
$h_{22}(t)$

$h_{22}(N\,\Delta t)$

$\hat{h}_{22}(f)$

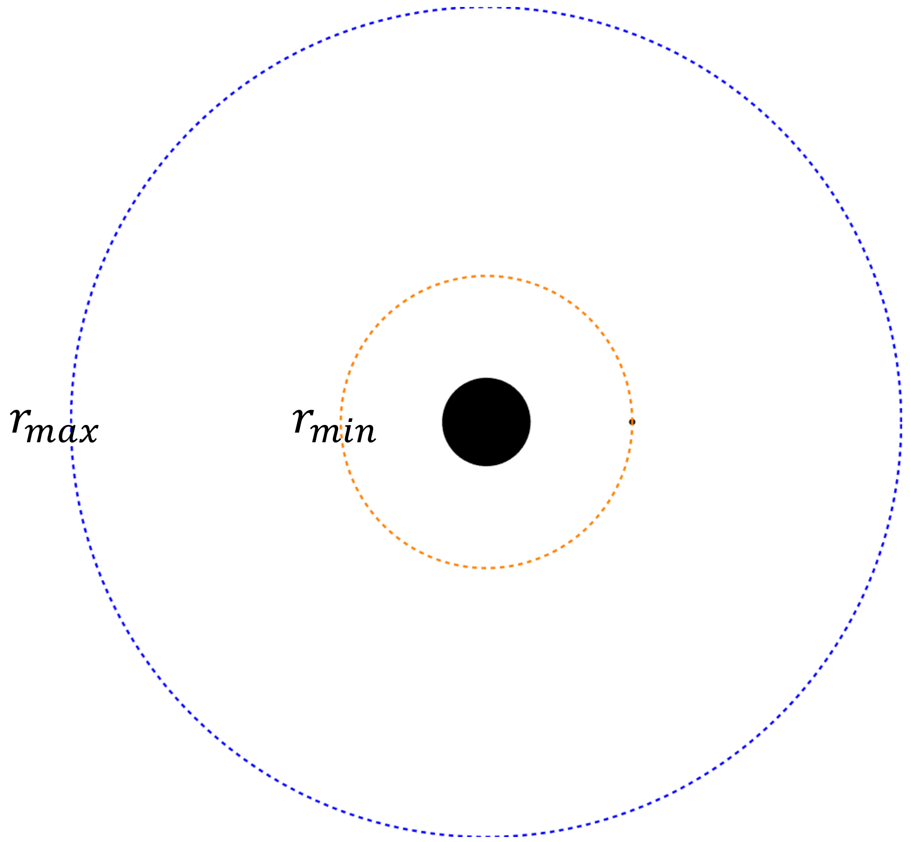


$$\Delta t \leq \frac{1}{2\,f_{max}}$$



- WF sampled $\Delta t = 0.5\ M$
- WF Generation time: $\sim$ 2.68s
- Time to beat: 6.02s

Quasi-Keplerian Parametrization



- Semilatus Rectum: $p = \frac{2 r_{max} r_{min}}{r_{max} + r_{min}}$
- Eccentricity: $e = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}$
- Relativistic anomaly: $\xi = \chi - \chi_0$
- $r = \frac{p}{1 + e \cos \xi}$
- $H_{eff}(p, e), p_\phi(p, e), p_r(p, e, \xi)$
  - [Hinderer & Babak 17]

Osculating Orbital elements

- Constants of motion p & e initial phase χ_0

- Orbital Elements: $I^A = \{p, e, \chi_0\} \ \rightarrow I(t)^A$

- Osculating Conditions:

  1. $x^\alpha(t) = x^\alpha_{cons}(I^A(t), t)$

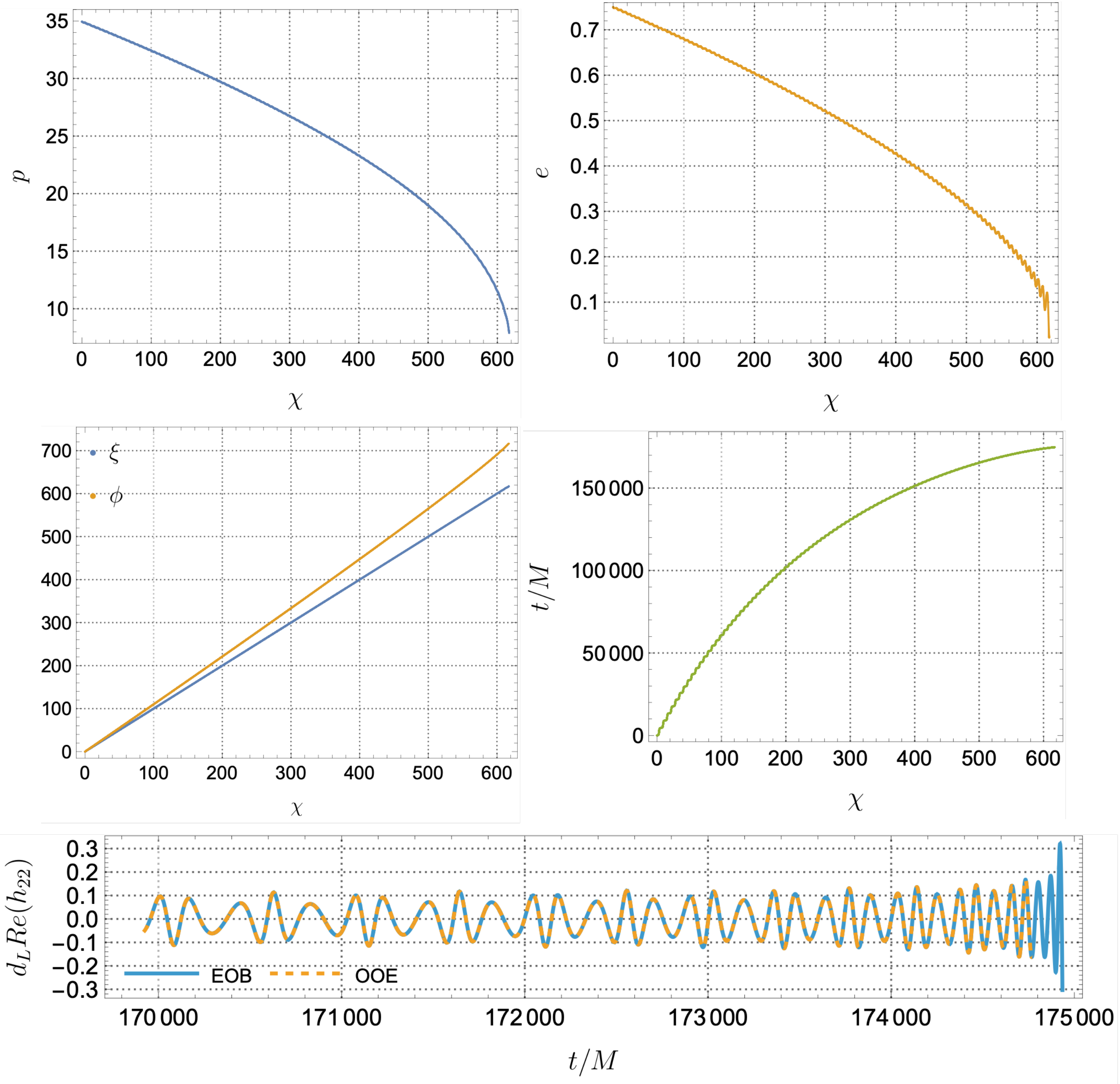
  2. $u^\alpha(t) = u^\alpha_{cons}(I^A(t), t)$

$$\frac{dp}{d\chi} = \epsilon F_p \qquad \frac{de}{d\chi} = \epsilon F_e$$

$$\frac{d\xi}{d\chi} = 1 + \epsilon f_\xi^{(1)}$$

$$\frac{d\phi}{d\chi} = f_\phi \qquad \frac{dt}{d\chi} = f_t$$

- Bookkeeping parameter: ϵ
- Functions are **interpolants** of (ν, p, e)



- $(\nu, p_0, e_0 \xi_0) = (0.25, 34.94, 0.75, 0)$
- Dynamics time: 218.9s
- Steps taken: 3,202
- WF Generation time: ~ 1.13s
- Mismatch: 8.01×10^{-6}

Near-Identity (Averaging) Transformations

Transform to new variables in series exp. in ϵ

$$\tilde{p} = p + \epsilon Y_p^{(1)} + \epsilon^2 Y_p^{(2)} + \mathcal{O}(\epsilon^3)$$

$$\tilde{e} = e + \epsilon Y_e^{(1)} + \epsilon^2 Y_e^{(2)} + \mathcal{O}(\epsilon^3)$$

$$\tilde{\xi} = \xi + \epsilon X_\xi^{(1)} + \mathcal{O}(\epsilon^2)$$

$$\tilde{\phi} = \phi + Z_\phi^{(0)} + \epsilon Z_\phi^{(1)} + \mathcal{O}(\epsilon^2)$$

$$\tilde{t} = t + Z_t^{(0)} + \epsilon Z_t^{(1)} + \mathcal{O}(\epsilon^2)$$

To obtain equations of motion independent of ξ

$$\frac{d\tilde{p}}{d\chi} = \epsilon \tilde{F}_p^{(1)} + \epsilon^2 \tilde{F}_p^{(2)} + \epsilon^3 \tilde{F}_p^{(3)} + \mathcal{O}(\epsilon^4)$$

$$\frac{d\tilde{e}}{d\chi} = \epsilon \tilde{F}_e^{(1)} + \epsilon^2 \tilde{F}_e^{(2)} + \epsilon^3 \tilde{F}_e^{(3)} + \mathcal{O}(\epsilon^4)$$

$$\frac{d\tilde{\xi}}{d\chi} = 1 + \epsilon f_\xi^{(1)} + \epsilon^2 f_\xi^{(2)} + \mathcal{O}(\epsilon^3)$$

$$\frac{d\tilde{\phi}}{d\chi} = \tilde{f}_\phi^{(0)} + \epsilon \tilde{f}_\phi^{(1)} + \epsilon^2 \tilde{f}_\phi^{(2)} + \mathcal{O}(\epsilon^3)$$

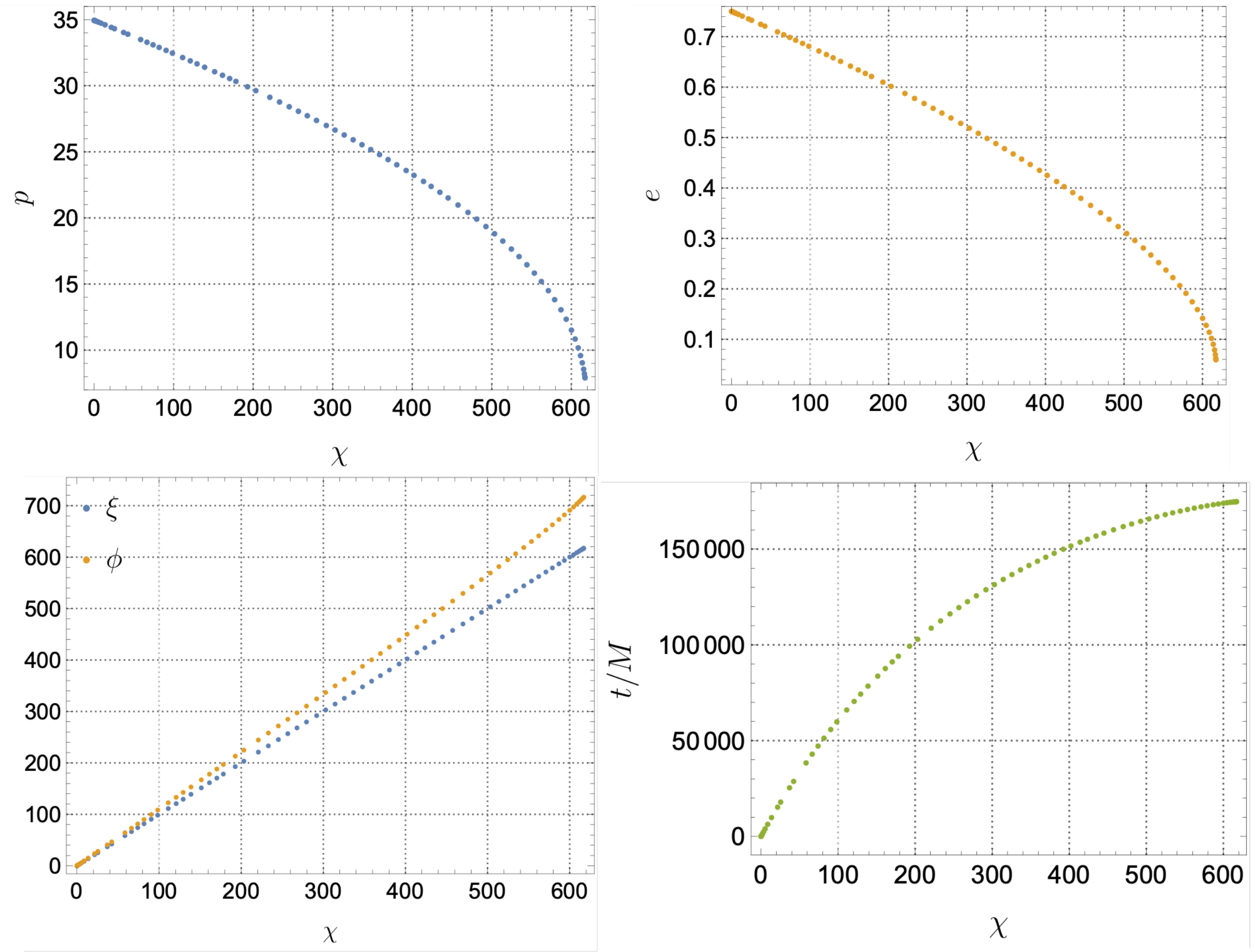
$$\frac{d\tilde{t}}{d\chi} = \tilde{f}_t^{(0)} + \epsilon \tilde{f}_t^{(1)} + \epsilon^2 \tilde{f}_t^{(2)} + \mathcal{O}(\epsilon^3)$$

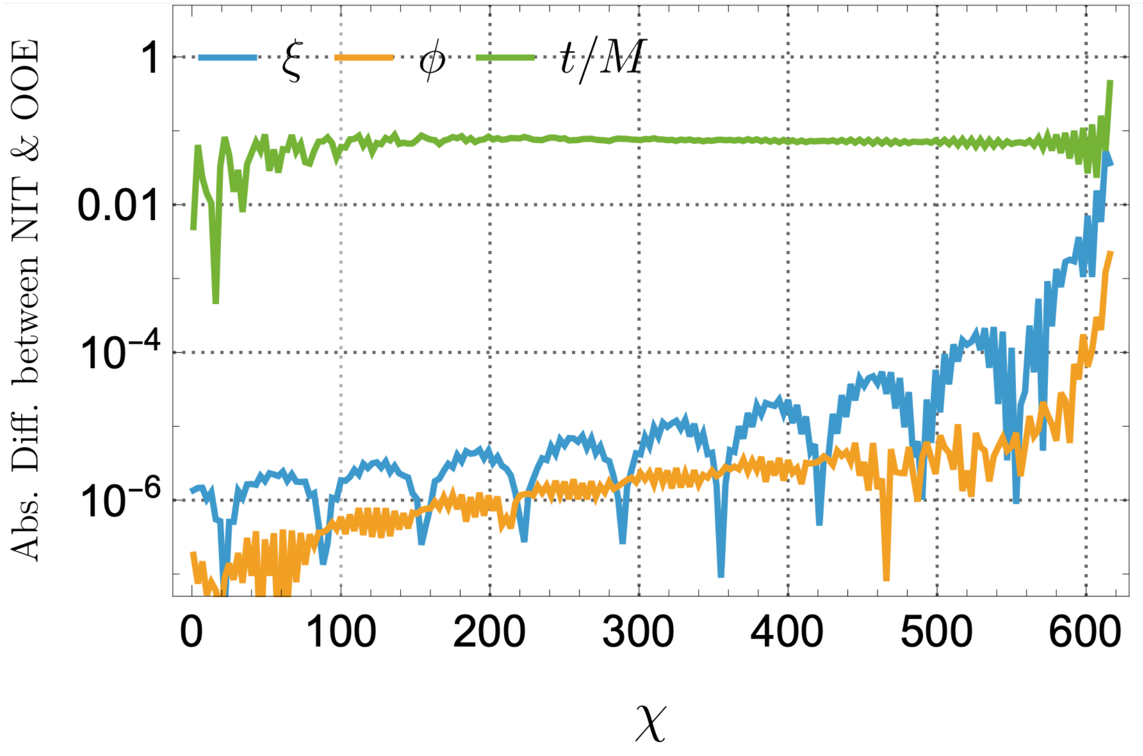
0PA

1PA

2PA

Functions are interpolants of $(\nu, \tilde{p}, \tilde{e})$

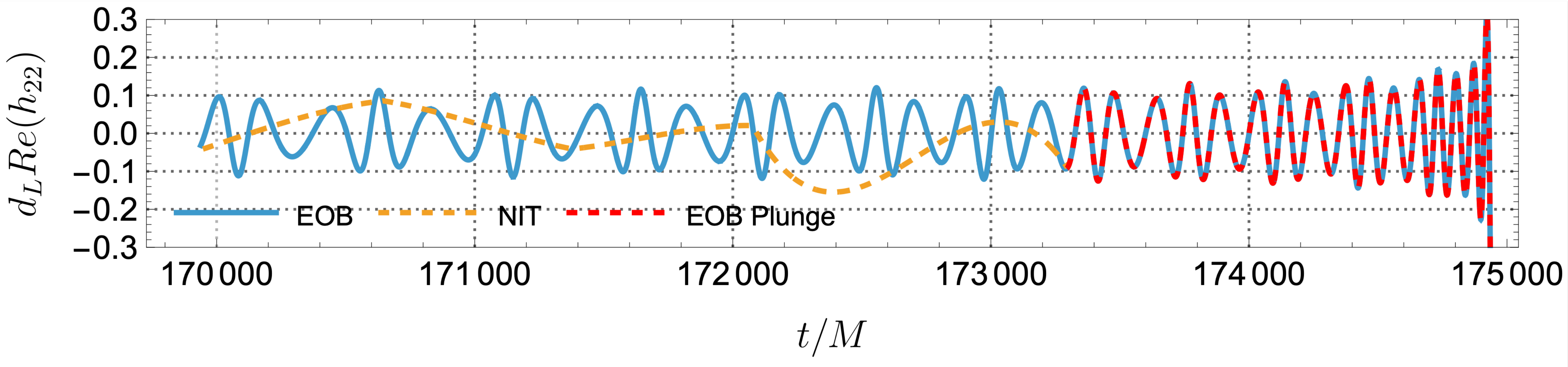




- Steps taken: 75
- Dynamics time: ~ 0.058s
- 2 orders of magnitude of speed up

- NIT breaks down close to LSO
- Stop NIT when $Q = \frac{\Omega^2}{\dot{\Omega}} < 100$
- Use EOB for Transition to Plunge

Generating Waveforms with NIT dynamics



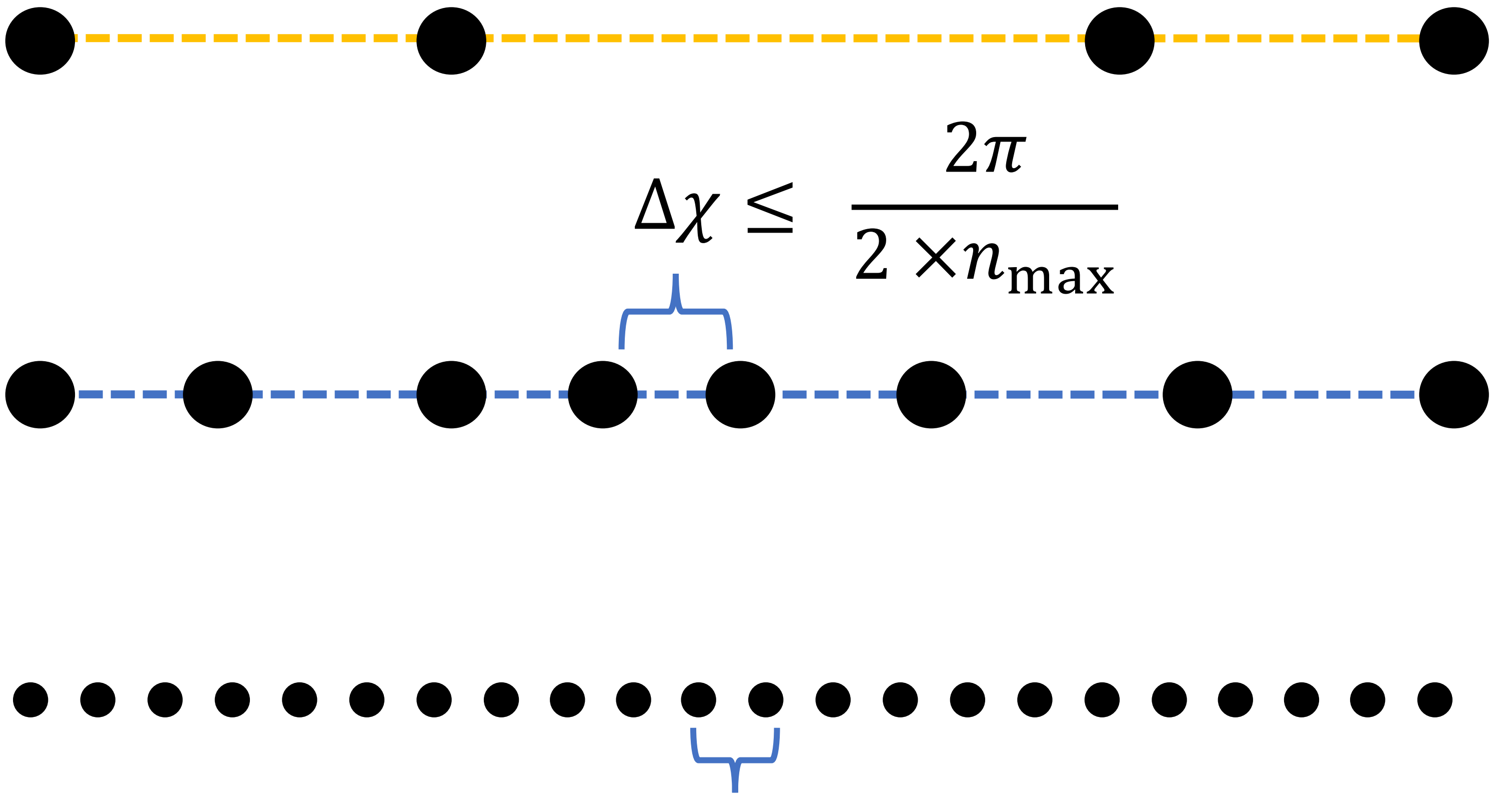
- Steps: 66

$$(\tilde{p}(\chi), \tilde{e}(\chi), \tilde{\chi}(\chi), \tilde{\phi}(\chi))$$

$$h_{22}(\tilde{t}(\chi) - Z_t^{(0)} - \epsilon Z_t^{(1)})$$

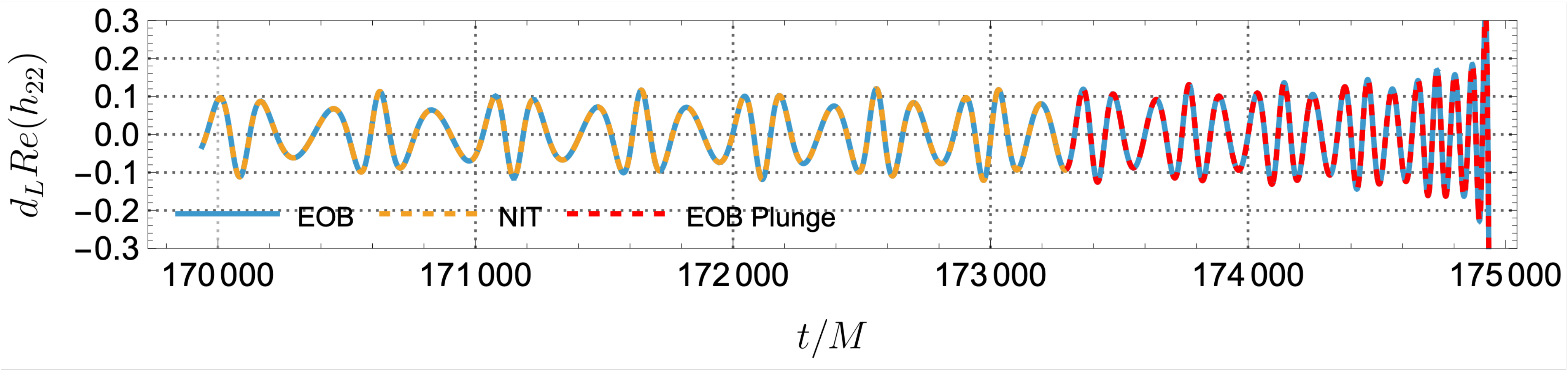
$$h_{22}(N\,\Delta t)$$

$$\hat{h}_{22}(f)$$



$$\Delta\chi \leq \frac{2\pi}{2 \times n_{\max}}$$

$$\Delta t \leq \frac{1}{2\, f_{max}}$$



WF Evaluations

- NIT: 6,406
- EOB: 86,567

Total time (Dynamics + WF)

- NIT 2.2s
- EOB: 6.02s
- Speed-up: $\sim\times 2.74$

Mismatch

- 2.48×10^{-6}

What order PA Order do we Need?

Transform to new variables in series exp. in ϵ

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0PA

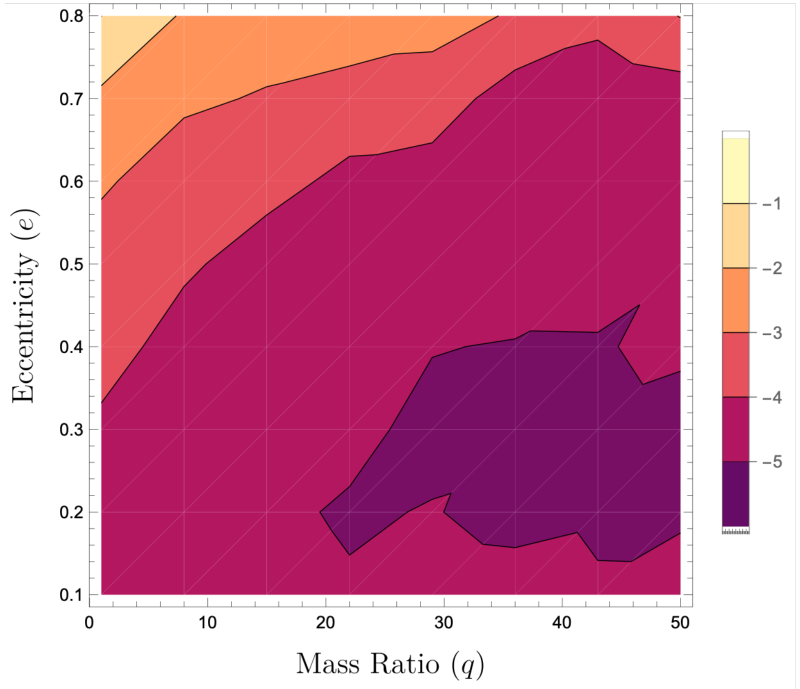
1PA

2PA

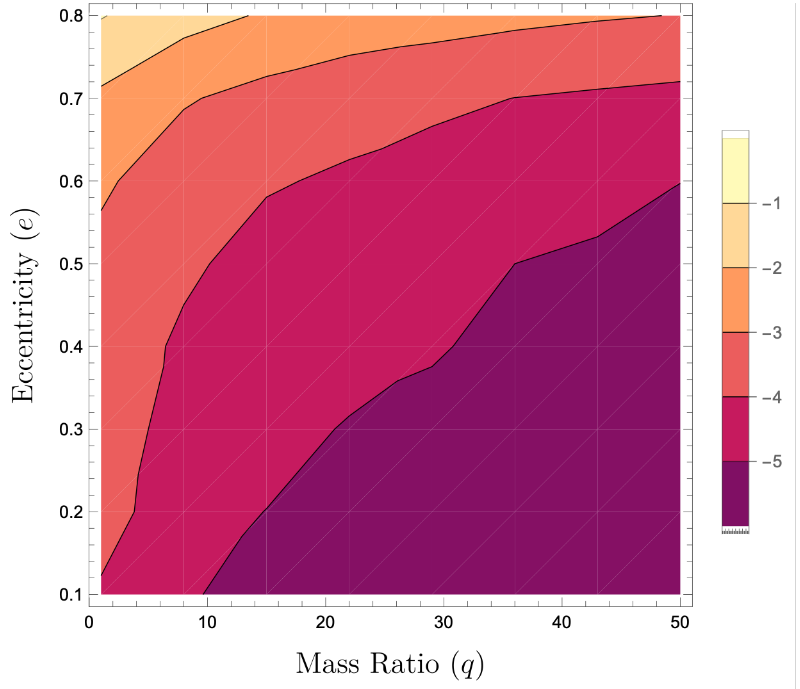
Functions are interpolants of $(\nu, \tilde{p}, \tilde{e})$

What order PA Order do we Need?

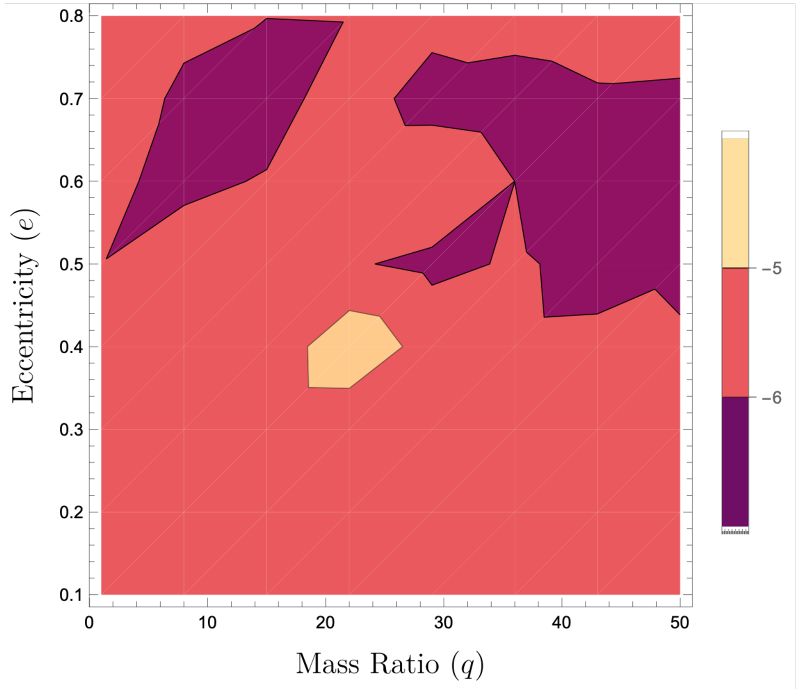
0PA



1PA



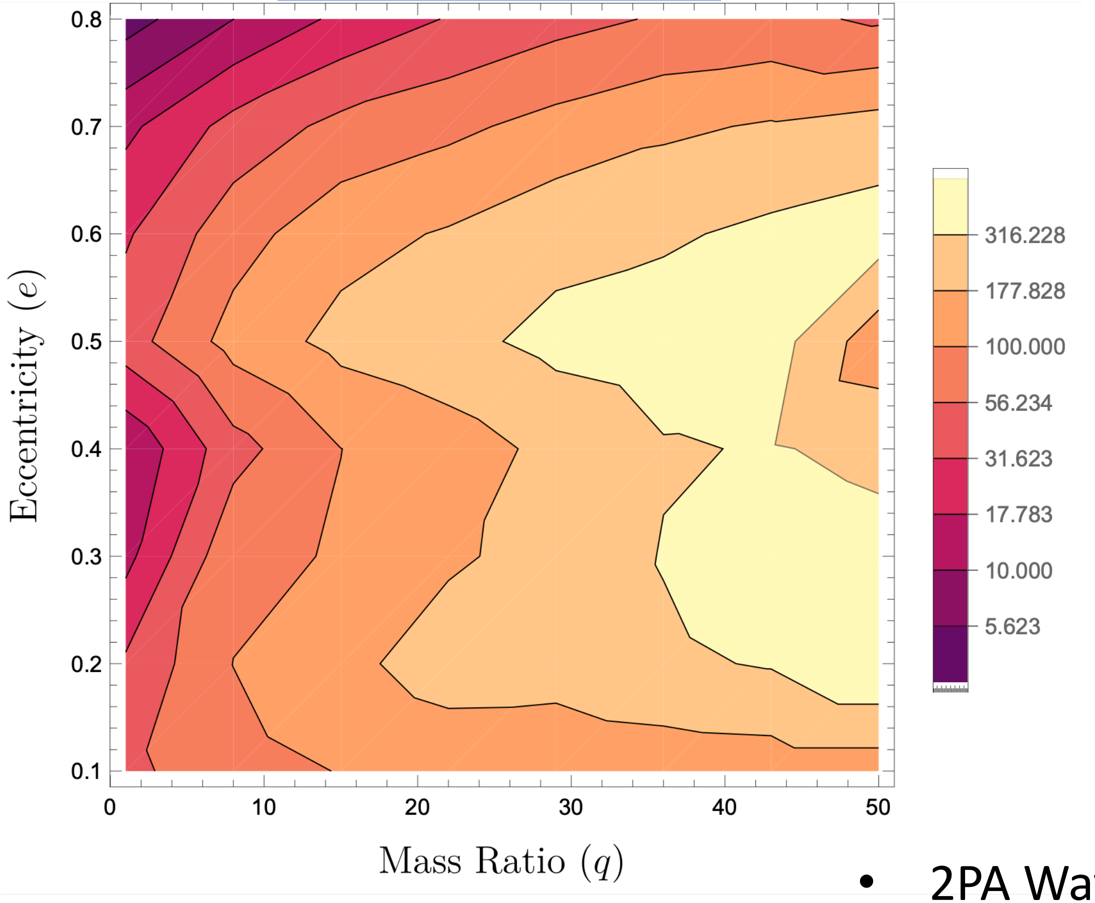
2PA



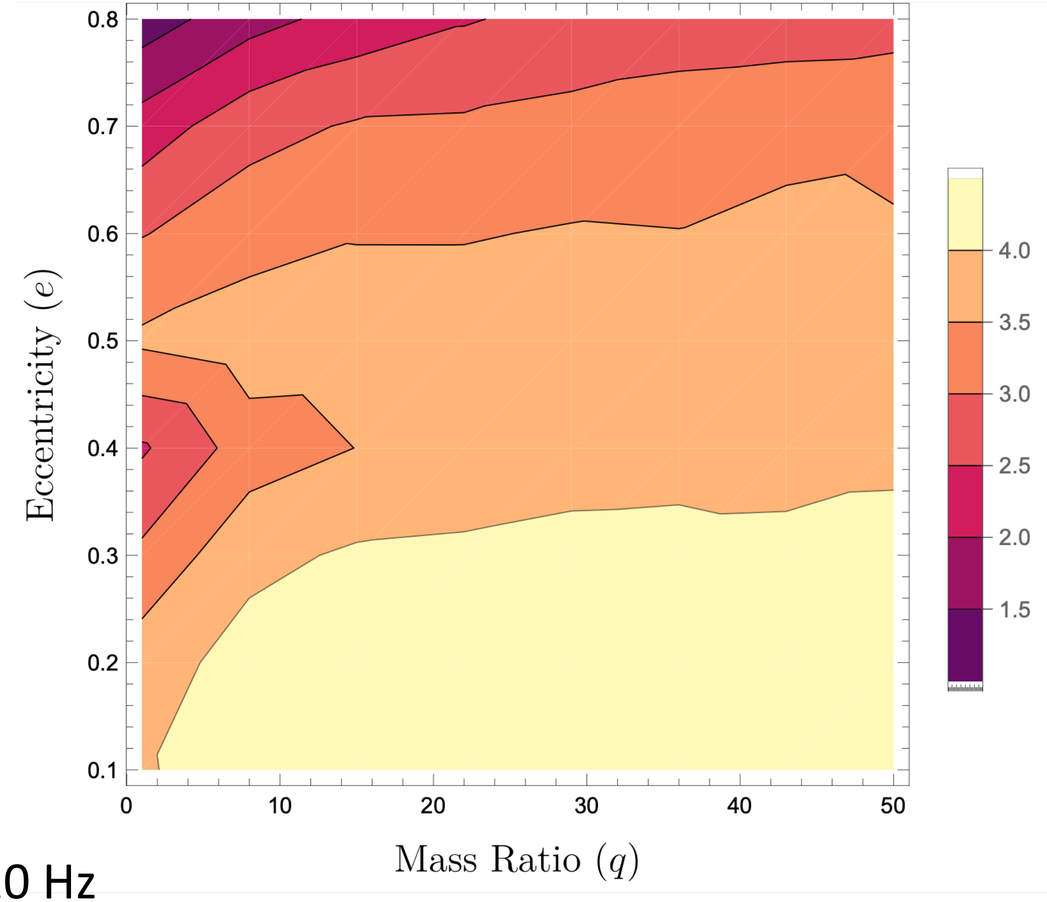
M = 60 @ 10 Hz

Speedup?

Dynamics



Dynamics + Waveform Generation



- 2PA Waveforms 60M @ 10 Hz
- Dynamics: 4- 540 times speedup
- Walltime: 1.2 – 5 times speedup

Future Work

Near term

- Add **higher modes**
- **Small e** needs different orbital elements
- More rigorous testing

Medium-term

- Adding **alligned spins**
- Need **fast multidimensional interpolation** method
- Copy tech. from FastEMRIWaveforms Package

Long-term

- Combine NIT with ν expansion
- **Hybrid EOB-GSF 1PA** model for EMRIs
- **Calibrate EOB fluxes** to ecc. 0PA fluxes

Summary

- Using **OOEs and NITs** speeds up the inspiral **dynamics** by **a factor of q**
- WF generation sped up by $\sim 1.2 - 5$
- 0PA and 1PA give not accurate enough, but 2PA gives $\mathcal{M} \sim 10^{-5}$
- Opens up new **synergies** between **EOB and EMRI WF** modelling