Lorenz Gauge Metric Reconstruction

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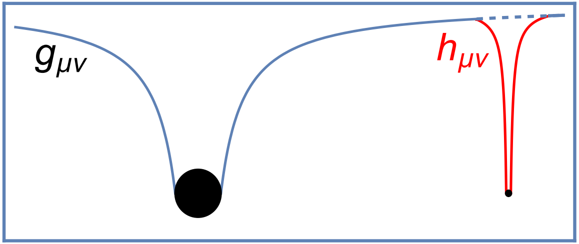
Perturbation Theory

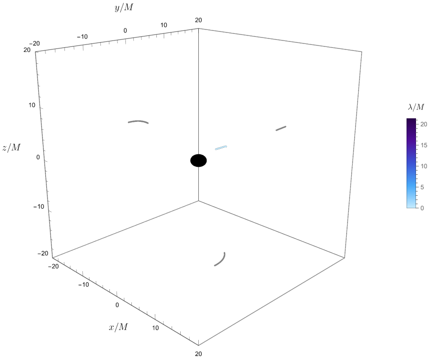
$$g_{\mu\nu}^{\text{EMRI}} = g_{\mu\nu}^{\text{Kerr}} + \epsilon h_{\mu\nu}^{(1)} + \epsilon^2 h_{\mu\nu}^{(2)} + \mathcal{O}(\epsilon^3),$$
$$T_{\mu\nu} = T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)} + \mathcal{O}(\epsilon^3),$$

$$G[g^{\text{EMRI}}]_{\mu\nu} = 8\pi T_{\mu\nu},$$
$$\mathcal{E}[h^{(1)}]_{\mu\nu} = 8\pi T_{\mu\nu}^{(1)},$$
$$\mathcal{E}[h^{(2)}]_{\mu\nu} = 8\pi T_{\mu\nu}^{(2)} + \delta^2 G[h^{(1)}, h^{(1)}]_{\mu\nu},$$



- Second order effects will be sourced by the first order metric perturbation (MP).
- We need first order data in a suitable gauge, Lorenz gauge is the current best option.

Stress-Energy Source: Geodesic Orbits



- $T^{(1)}_{\mu\nu}(x^\alpha) = \mu \frac{u_\mu u_\nu}{u^t\sqrt{-g}} \delta^{(3)}(\mathbf{x} - \mathbf{x}_{\text{Geo}}(\mathbf{t}))$
- Starting point is bound Kerr geodesics.
- Kerr spin parameter, $0 \leq a \leq M$,
- semi-latus rectum, $r_{\text{min}} \leq p \leq r_{\text{max}}$,
- eccentricity, $0 \leq e = \frac{r_{\text{min}} r_{\text{max}}}{r_{\text{max}} - r_{\text{min}}} \leq 1$,
- inclination $0 \leq x \leq 1$,
- EMRI trajectory transitions slowly through a series of geodesics, driven by a radiation reaction.

First Order Schwarzschild

In Schwarzschild spacetime the Linearised EFEs allow metric perturbations that can decompose onto a harmonic basis. This gives us options in how we solve for the MP:

- **Time Domain:** Solve the EFE's as PDEs for $h^{\ell m}_{\mu\nu}(t,r)$.
  - Flexible: deals with any orbital configuration
  - Suffers from mode instabilities
- **Frequency Domain:** Decompose to frequency domain, Solve ODEs for $h^{\ell m\omega}_{\mu\nu}(r)$.
  - ODEs are faster to solve.
  - More complicated configurations $\rightarrow$ more work.
- **Regge-Wheeler:** Solve separable scalar equations, do metric reconstruction in RW gauge (or Lorenz gauge).
  - decouples MP components.
  - More work to do reconstruction.

First Order Kerr

In Kerr spacetime, trying to decompose the MP onto a harmonic basis leads to nasty mode coupling. This changes the option we have for solutions:

- **m-modes:** Separate out the $e^{i(m\phi-\omega t)}$, solve PDEs for $h^{m\omega}_{\mu\nu}(r,\theta)$.
  - No summation over ℓ-modes.
  - Use tools from NR (WIP).
- **Teukolsky:** Solve the separable Teukolsky Equation, use metric reconstruction to get a radiation (or Lorenz) gauge MP.
  - Separable equations, MP components decouple.
  - Radiation gauge reconstruction is extra work.
  - Lorenz gauge reconstruction is a LOT of extra work.

Teukolsky, 1973

$$\mathcal{O}\Psi = \mathcal{S}T$$

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}$$
$$- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi}$$
$$- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 4\pi\Sigma T. \quad (4.7)$$

The Teukolsky Formalism

Linearised EFEs: 10 coupled PDEs for $h^{(1)}_{\mu\nu}$ — Projection onto null tetrad, Separation of Variables, Harmonic Basis → **Teukolsky Equation**: 5 complex ODEs for Weyl Scalars

E.g.

$$\psi_4 = \frac{1}{(r - ia\cos\theta)^4} \sum_{lm} \int \mathrm{d}\omega \; {}_{-2}\psi_{lm\omega}(r) \, {}_{-2}S^{a\omega}_{lm}(\theta,\phi) e^{-i\omega t}.$$

$$\left[\frac{1}{\Delta^2(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{\Delta(r)}\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{K^2(r) + 4i(r-M)K(r)}{\Delta(r)} - 8i\omega r + \lambda\right] {}_{-2}\psi_{lm\omega}(r) = {}_{-2}T_{lm\omega}(r),$$

$$K(r) = (r^2 + a^2)\omega - ma, \quad \Delta(r) = r^2 - 2Mr + a^2$$

Lots of techniques and tools to solve this equation: `Black hole perturbation Toolkit`, `pyBHPT`, spectral methods.

Sources

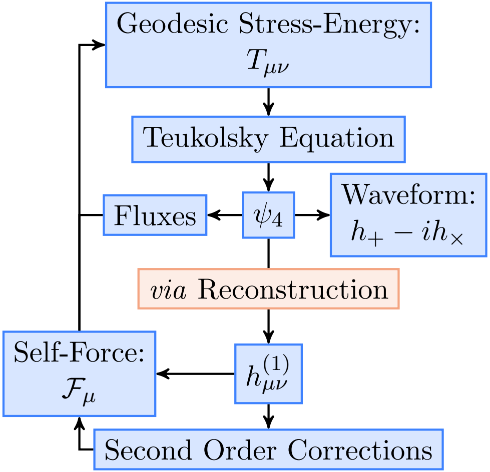
$$T_{\mu\nu} = \frac{u_\mu u_\nu}{u^t \sqrt{-g}} \delta^{(3)}(\mathbf{x} - \mathbf{x}_p(t))$$

Because the source is periodic and bounded, the integrals above evaluate to constants outside the libration region, with a discrete spectrum:

$${}_{-2}\psi_{lmnk}(r) = {}_{-2}Z^{\mathrm{Up}}_{lmnk}\ {}_{-2}R^{\mathrm{Up}}_{lmnk}(r)\Theta(r - r_{\mathrm{max}}) + {}_{-2}Z^{\mathrm{In}}_{lmnk}\ {}_{-2}R^{\mathrm{In}}_{lmnk}(r)\Theta(r_{\mathrm{min}} - r)$$

Uses and Limits of Teukolsky

- Weyl scalars are useful for quantities like fluxes, and extracting waveforms, but we do need $h^{(1)}_{\mu\nu}$.
- CCK reconstruction from Hertz potentials: Chrzanowski (1975), Wald (1978), Kegeles and Cohen (1979).
- Typically the metric perturbation is reconstructed in a radiation gauge.



Metric Reconstruction

Given $h_{\mu\nu}$ satisfying the linearised EFEs, and some Ψ_s satisfying a Teukolsky equation:

$$\mathcal{E}[h] = T, \qquad \mathcal{O}\Psi = \mathcal{S}T,$$

where Ψ_s and $h_{\mu\nu}$ are related by:

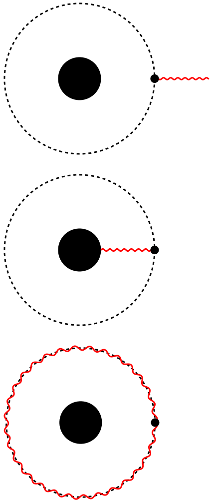
$$\Psi = \mathcal{T}h.$$

Then we use adjoint rules to show:

$$\mathcal{O}\mathcal{T}[h] = \mathcal{S}\mathcal{E}[h] \quad \rightarrow \quad \mathcal{E}[\mathcal{S}^\dagger\Phi] = \mathcal{T}^\dagger\mathcal{O}^\dagger\Phi,$$

which tells us that $\mathcal{S}^\dagger\Phi$ is a vacuum solution of the linearised EFEs, **if** $\mathcal{O}^\dagger\Phi = 0$. Getting a non-vacuum solution is extra work.

WKD Lorenz Gauge *vs* Radiation Gauge



- Radiation gauge has been used for most Kerr calculations to date.
- Radiation gauge includes extended singularities that make it unsuitable for second order source calculations.
- Bourg *et al.*: GHZ Corrector tensor approach.
- Neef and Musaeus: "Eyeball method".
- Lorenz gauge has no extended singularities, and is well understood by the GSF community.

WKD framework

Our starting point is the Aksteiner, Andersson, Bäckdahl (AAB) gauge:

$$h^{\mathrm{AAB}}_{\alpha\beta} = \frac{4}{3}\left(\mathcal{S}_4^\dagger \zeta^4 \psi_0 - \mathcal{S}_0^\dagger \zeta^4 \psi_4\right)_{\alpha\beta} + 8\pi(\mathcal{N}T)_{\alpha\beta} + h^{\partial M}_{\alpha\beta} + h^{\partial a}_{\alpha\beta},$$

where $(\mathcal{N}T)_{\mu\nu}$ is a corrector tensor.
This is related to Lorenz gauge by a transformation found by Wardell, Kavanagh and Dolan:

$$\pounds_T h^{\mathrm{L}}_{\alpha\beta} = h^{\mathrm{AAB}}_{\alpha\beta} - 2\xi_{(\alpha;\beta)}$$

where

$$\pounds_T \nabla^\alpha \bar{h}^{\mathrm{L}}_{\alpha\beta} = 0 \implies \Box\xi_\alpha = \nabla^\beta h^{\mathrm{AAB}}_{\alpha\beta}$$

Auxiliary Fields

ξ_α is written in terms of the following solutions to the Teukolsky Equations and their derivatives:

Field	Spin Weight	Compact Source	Implementation
ψ_4	-2	✓	Generic
ψ_0	$+2$	✓	Generic
ϕ_2	-1	✓	$\sim$ Generic
ϕ_0	$+1$	✓	$\sim$ Generic
χ	0	✓	$\sim$ Generic
h^L	0	✓	Generic
κ	0	✗, $\Box\kappa = \frac{1}{2}\pounds_T h^L$	$\sim$ Generic

κ calculation

The compact source fields can be tackled using existing variation of parameters techniques. This leaves κ, which is sourced everywhere by the trace of the metric perturbation.

$$\Box\kappa = \frac{1}{2}\pounds_t h^L$$

$$\Downarrow$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta(r)\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{K(r)^2}{\Delta(r)} + \lambda_{lm\omega}\right]\kappa_{j;lm\omega}(r) = -\frac{i\omega}{2}(\delta_{jl}r^2 + a^2\gamma_{jl})h^L_{jm\omega}(r).$$

l=j: Boundary Conditions

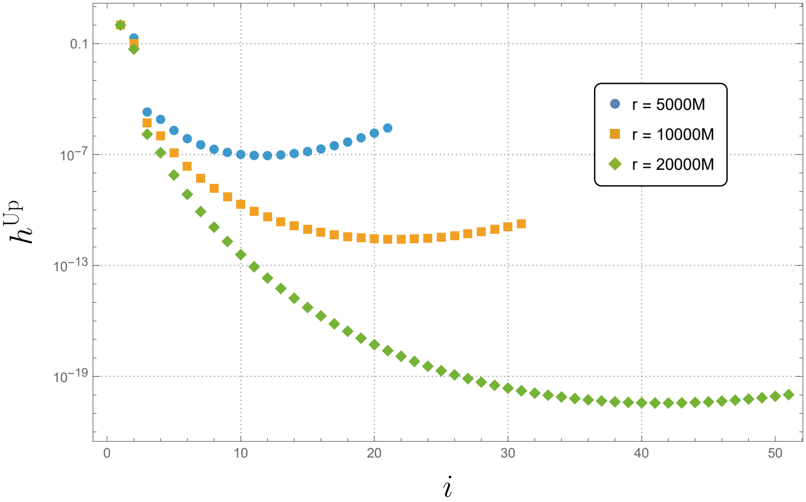
Construct asymptotic series to act as boundary conditions. Coefficients are given by recursion relations obtained from the field equation.

$$\kappa^{\mathrm{Up}}(r) = e^{-i\omega r_*}\left[\sum_{i=0}^{i_{\max}} \frac{\kappa_i^{\mathrm{Up}}}{r^i} + \mathcal{O}(r^{-i_{\max}-1})\right], \quad h^{\mathrm{Up}}(r) = \frac{e^{-i\omega r_*}}{r}\left[\sum_{i=0}^{i_{\max}} \frac{h_i^{\mathrm{Up}}}{r^i} + \mathcal{O}(r^{-i_{\max}-1})\right],$$

$$\kappa^{\mathrm{In}}(r) = (r-r_+)e^{i\omega r_*}\left[\sum_{i=0}^{i_{\max}} \kappa_i^{\mathrm{In}}(r-r_+)^i + \mathcal{O}(r-r_+)^{i_{\max}+1}\right],$$

$$h^{\mathrm{In}}(r) = e^{i\omega r_*}\left[\sum_{i=0}^{i_{\max}} h_i^{\mathrm{In}}(r-r_+)^i + \mathcal{O}(r-r_+)^{i_{\max}+1}\right].$$

Convergeance



κ Calculation: Circular (and spherical) orbits

Construct Up and In solutions above and below the particle.

$$\kappa_{j;lm}^{\text{Up/In}} = \kappa_{j;lm}^{\text{Particular, Up/In}} + \beta_{j;lm}^{\text{Up/In}} h_{lm}^{L,\text{Up/In}}$$

where the β-coefficients are determined by enforcing continuity at the particle location, and

$$\kappa_{j;lm}^{\text{Particular, Up/In}} = \begin{cases} \frac{a^2\gamma_{j;lm}}{\lambda_j - \lambda_l} h_{jm}^{L,\text{Up/In}}, & l \neq j \\ \kappa_{l;lm}^{\text{Num, Up/In}}, & l = j \end{cases}$$

where

$$\gamma_{j;lm} = \oint {}_0S_{\ell m}^{a\omega}(\theta,\phi)\ \cos^2\theta\ {}_0S_{jm}^{a\omega}(\theta,\phi)\mathrm{d}\Omega.$$

κ continuity: Circular (and spherical) orbits

We can get the β coefficients by enforcing continuity in κ and it's radial derivative:

$$\beta^{\text{Up}} h^{\text{Up, P}}(r_0) - \beta^{\text{In}} h^{\text{In, P}}(r_0) = -(\kappa^{\text{Up, P}}(r_0) - \kappa^{\text{In, P}}(r_0)),$$
$$\beta^{\text{Up}} h^{\text{Up, P}\prime}(r_0) - \beta^{\text{In}} h^{\text{In, P}\prime}(r_0) = -(\kappa^{\text{Up, P}\prime}(r_0) - \kappa^{\text{In, P}\prime}(r_0))$$

Set up simultaneous equations to solve for β^{Up} and β^{In}.

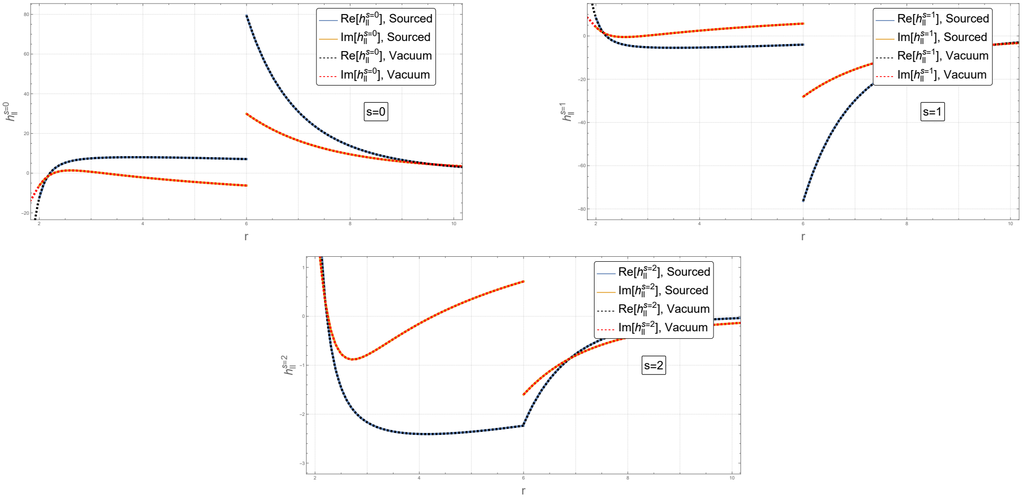
Reconstruction & Projection to Spheroidal Harmonics

Use $\psi_{0/4}$ to reconstruct the AAB gauge MP, and $\phi_{0/2}$, χ, h^L, and κ to construct the gauge transformation to Lorenz gauge. [Wardell et al. 2025]
Project each component of the metric perturbation onto spin-weighted spherical harmonics [Dolan et al. 2024]:

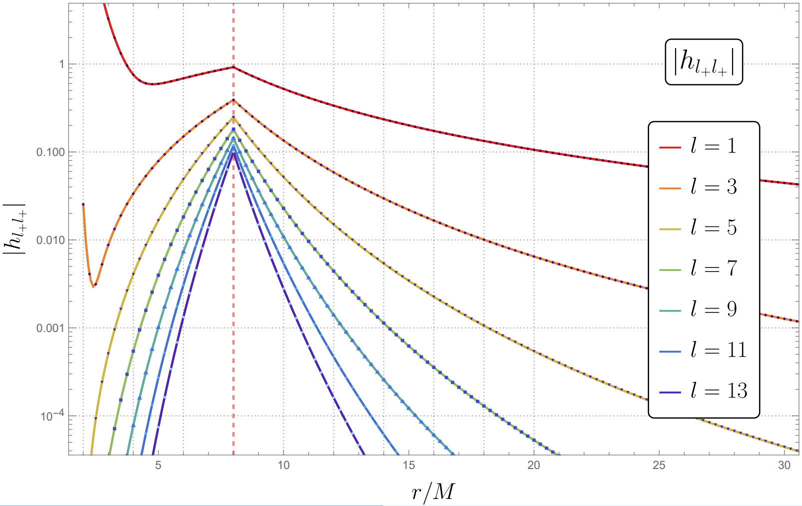
$$h_{(a)(b)}(t,r,\theta,\phi) = \sum_{lm} h^{lm}_{(a)(b)}(t,r)\,{}_sY_{lm}(\theta,\phi),$$

where $(a)(b)$ represent the components of the MP in tetrad form: $h_{ll} = h_{\mu\nu}l^\mu l^\nu$, the spin weight is given by counting the spin weight of the tetrad legs.

Circular orbits: Fine Cancelations

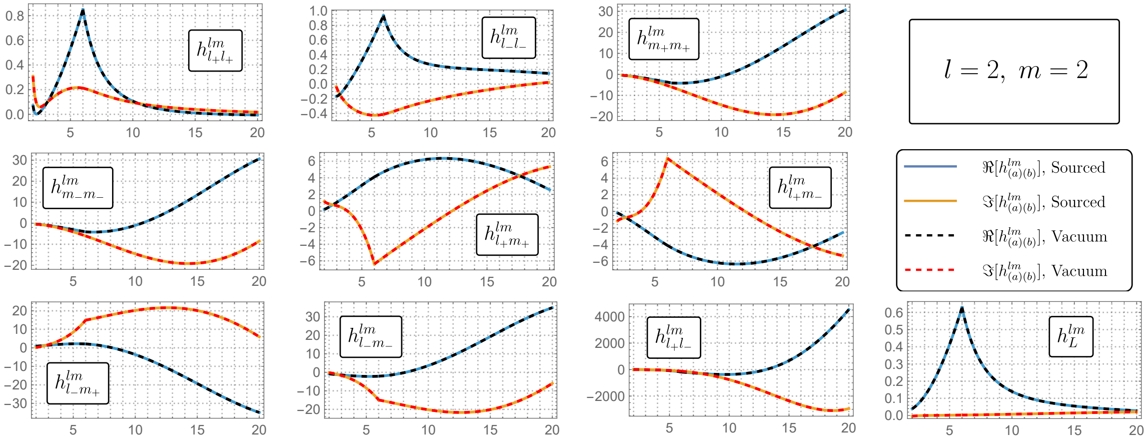


Circular orbits: $h^{\text{Lorenz}}_{l_+ l_+}$

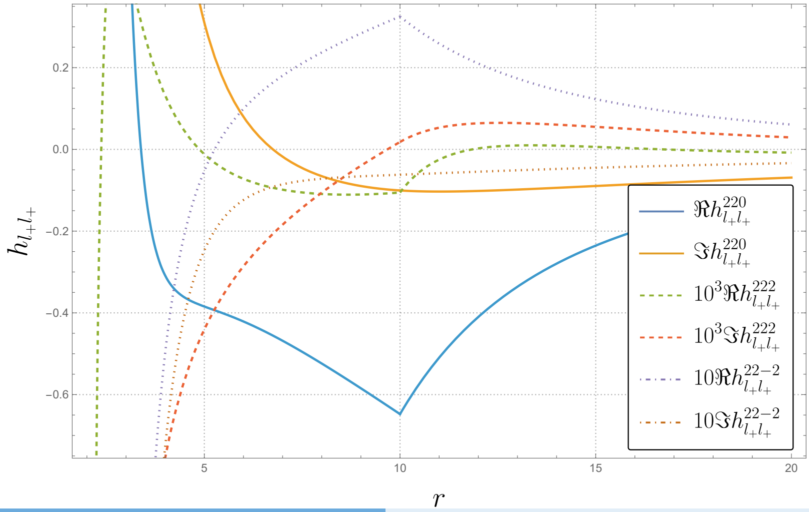


- $m = 1$
- $a = 0.6M, \ r_0 = 8M$
- Lines are my data, points are Dolan's
- exponential fall-off with l, except near the particle.

Circular Orbits: All components



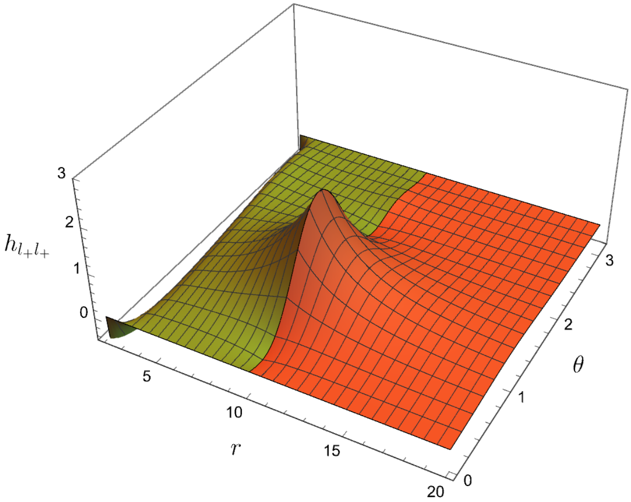
Spherical orbit results



- Similar to the circular case, but now with more modes.
- each k-mode now involves an integral over the polar motion.
- Recover continuity mode by mode.

Spherical Orbit Results

$$\phi = \phi_p(\lambda),$$
$$a = 0.99M, \quad p = 10M,$$
$$e = 0, \quad x = .9,$$
$$l_{\max} = 5 = m_{\max},$$
$$k_{\max} = 10, \quad k_{\min} = -10$$



κ Calculation: Eccentric orbits

New Problem: The radial motion of the particle smears our source across the libration region $\implies$ No continuity on the level of $j; lmn$ modes , but we still need to fix our βs somehow.

Solution: The **spherical** modes of the field must be continuous at the particle worldline in the **time domain**. By reconstructing κ in the time domain, and sampling the fields at enough points in time over a libration, we can construct a system of equations for the βs.

$$\sum_{\ell n} b_{\ell jm}^{a\omega_{mn}} \left[\beta_{\ell mn}^{\text{Up}} h_{\ell mn}^{\text{Up}}\left(r_p(t)\right) - \beta_{\ell mn}^{\text{In}} h_{\ell mn}^{\text{In}}\left(r_p(t)\right)\right] e^{-i\omega_{mn}t} = - \sum_{\ell n} b_{\ell jm}^{a\omega_{mn}} \left[\kappa_{\ell mn}^{\text{P, Up}}\left(r_p(t)\right) - \kappa_{\ell mn}^{\text{P, In}}\left(r_p(t)\right)\right] e^{-i\omega_{mn}t},$$

$$\sum_{\ell n} b_{\ell jm}^{a\omega_{mn}} \left[\beta_{\ell mn}^{\text{Up}} h_{\ell mn}^{\text{Up}}{}'\left(r_p(t)\right) - \beta_{\ell mn}^{\text{In}} h_{\ell mn}^{\text{In}}{}'\left(r_p(t)\right)\right] e^{-i\omega_{mn}t} = - \sum_{\ell n} b_{\ell jm}^{a\omega_{mn}} \left[\kappa_{\ell mn}^{\text{P, Up}\prime}\left(r_p(t)\right) - \kappa_{\ell mn}^{\text{P, In}\prime}\left(r_p(t)\right)\right] e^{-i\omega_{mn}t},$$

$$\sum_{\ell n} b_{\ell jm}^{a\omega_{mn}} (-i\omega_{mn}) \left[\beta_{\ell mn}^{\text{Up}} h_{\ell mn}^{\text{Up}}\left(r_p(t)\right) - \beta_{\ell mn}^{\text{In}} h_{\ell mn}^{\text{In}}\left(r_p(t)\right)\right] e^{-i\omega_{mn}t} = - \sum_{\ell n} b_{\ell jm}^{a\omega_{mn}} (-i\omega_{mn}) \left[\kappa_{\ell mn}^{\text{P, Up}}\left(r_p(t)\right) - \kappa_{\ell mn}^{\text{P, In}}\left(r_p(t)\right)\right] e^{-i\omega_{mn}t}$$

Constructing matrices

We now have jump conditions for the spherical modes, each of which depends on **all** spheroidal modes.
We can still set up a matrix system, by combining all the spherical jumps. Schematically each of my submatrices from earlier looks like

$$\begin{matrix} [(j_{\min},\ t_1),\ (\ell_{min},\ n_{\min})] & \cdots & [(j_{\min},\ t_1),\ (\ell_{min},\ n_{\max})] & \cdots & [(j_{\min},\ t_1),\ (\ell_{max},\ n_{\max})] \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ [(j_{\min},\ t_{\tilde{N}}),\ (\ell_{min},\ n_{\min})] & \cdots & [(j_{\min},\ t_{\tilde{N}}),\ (\ell_{min},\ n_{\max})] & \cdots & [(j_{\min},\ t_{\tilde{N}}),\ (\ell_{max},\ n_{\max})] \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ [(j_{\max},\ t_{\tilde{N}}),\ (\ell_{min},\ n_{\min})] & \cdots & [(j_{\max},\ t_{\tilde{N}}),\ (\ell_{min},\ n_{\max})] & \cdots & [(j_{\max},\ t_{\tilde{N}}),\ (\ell_{max},\ n_{\max})] \end{matrix}$$

Constructing matrices

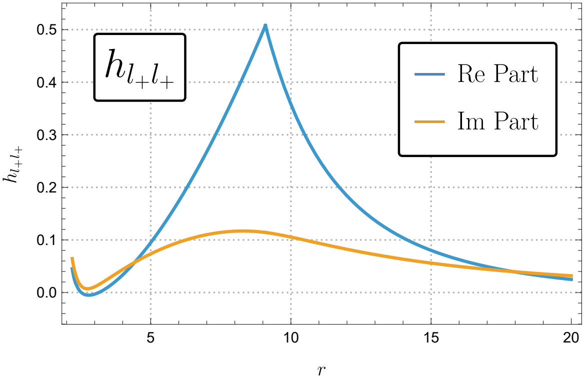
The matrices are quite big, but still numerically tractable.

$$3\tilde{N} \underset{\downarrow}{\overset{\uparrow}{\times}} L \left\{ \underbrace{\begin{pmatrix} H_m^{\mathrm{Up}} & -H_m^{\mathrm{In}} \\ \partial_r H_m^{\mathrm{Up}} & -\partial_r H_m^{\mathrm{In}} \\ \partial_t H_m^{\mathrm{Up}} & -\partial_t H_m^{\mathrm{In}} \end{pmatrix}}_{\leftarrow 2N \times L \rightarrow} \right. \begin{pmatrix} \beta_m^{\mathrm{Up}} \\ \beta_m^{\mathrm{In}} \end{pmatrix} = \begin{pmatrix} T_m \\ \partial_r T_m \\ \partial_t T_m \end{pmatrix} \tag{1}$$

We can choose $\tilde{N}$ to make the matrix square, or use it to overdetermine the system, and use least squares to solve for the βs.

Eccentric Orbit Results, $l = m = 2$

$$\theta = \theta_p = \frac{\pi}{2}, \quad \phi = \phi_p,$$
$$a = 0.99M, \quad p = 10M,$$
$$e = 0.1, \quad x = 1,$$
$$n_{\max} = 10, \quad n_{\min} = -5$$



Eccentric Orbit Results

$$\theta = \theta_p = \tfrac{\pi}{2},$$
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$$l_{\max} = 5 = m_{\max},$$
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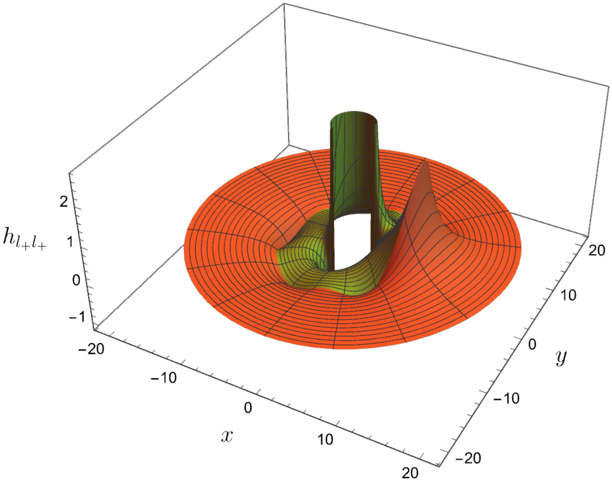

Eccentric Orbit Results

$$\phi = \phi_p(\lambda),$$
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$$e = 0.1, \quad x = 1,$$
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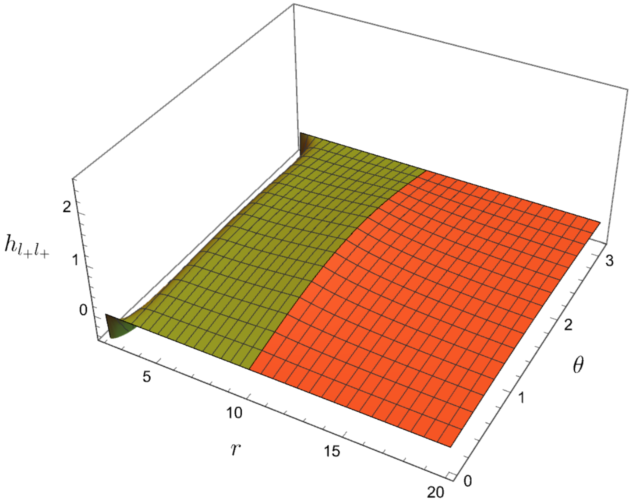
$$n_{\max} = 10, \quad n_{\min} = -5$$



Asymptotics

In the weak field ($\omega r \gg 1$) our Teukolsky solutions admit asymptotic expansions, e.g.

$$h^{L}_{lm\omega}(r) = Z^{\text{h, Up}}_{lm\omega} \frac{e^{i\omega r_*}}{r} \sum_{i=0}^{\infty} \frac{c^{h}_{lm\omega i}}{r^i},$$

where the expansion coefficients $c^{h}_{lm\omega i}$ depend on quantities like $\{l, m, \omega, a, \lambda\}$. This enables asymptotic expansions for the metric perturbation, e.g.

$$\begin{aligned} h^{lm\omega}_{l_+ l_+} e^{-i\omega r_*} = &\frac{i Z^{\text{h,Up}} S_0}{\omega r^2} \\ &+ \frac{1}{r^3}\left[Z^{\text{h,Up}} \frac{am\omega - 2\lambda}{\omega^2} - 8\left(Z^{\chi,\text{Up}} + Z^{\kappa,\text{Up}} + \beta^{\text{Up}}_{ll} Z^{\text{h,Up}}\right)\right] S_0 + \mathcal{O}\left(\frac{1}{r^4}\right) \qquad (2) \end{aligned}$$

Future Work

- **Generic Orbits:**
  - Calculations for inclined, eccentric sources.
  - Scattering trajectories.
- **Regularisation, Static Sector & Completion Pieces:**
  - Apply puncture techniques to speed up convergeance and fix cusping.
  - The reconstruction algorithm has not been written down yet for $\omega = 0$.
  - Need corrections corresponding to corrections to the primary mass.
- **Accuracy and Speed:**
  - Calculate the Teukolsky solutions using compactified, hyperboloidal co-ordinates and a spectral solver for the circular case.
- **Further Calculations:**
  - Calculate the first-order self-force directly.
  - Calculate gauge invariants.
  - Feed into the source for the second calculations.

Questions?

Thanks for Listening!
Any questions?

Adjoint Definition

The adjoint is defined in the sense that

$$\int \phi \, \mathcal{P}\psi \, \sqrt{-g} \, \mathrm{d}^4x = \int \psi \, \mathcal{P}^\dagger\phi \, \sqrt{-g} \, \mathrm{d}^4x$$

Bibliography I

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