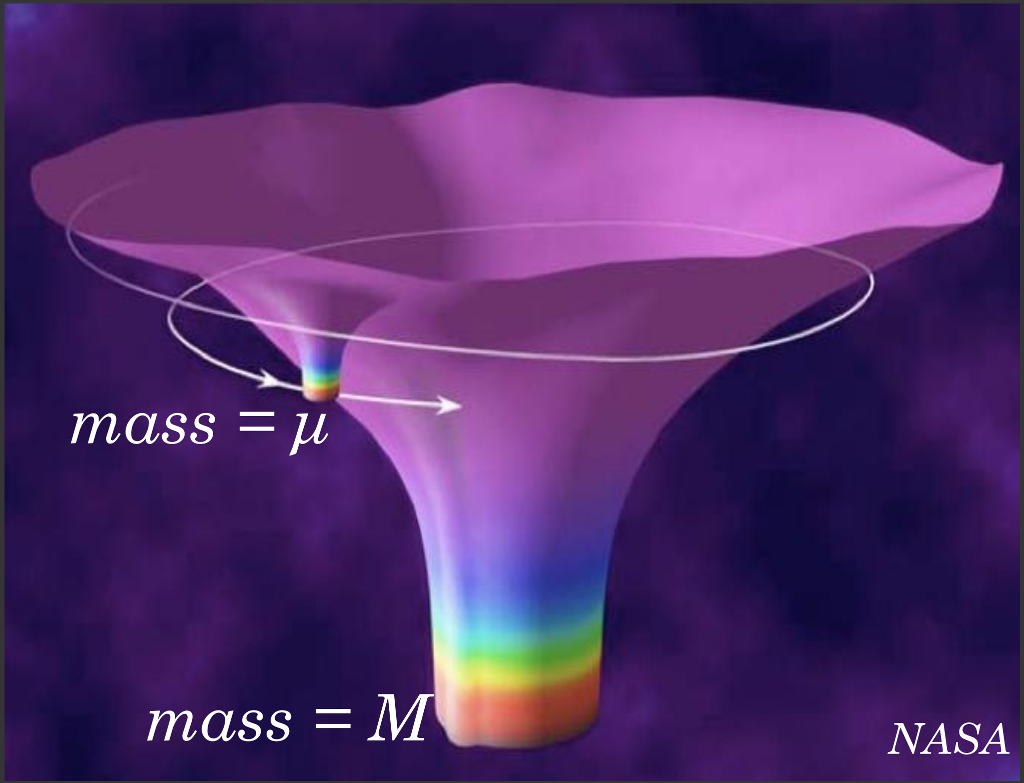
Introduction to black hole perturbation theory and self-force methods

$$g_{\alpha\beta} = g_{\alpha\beta}^{\text{Kerr}} + \left(\frac{\mu}{M}\right) h_{\alpha\beta}^{(1)} + \left(\frac{\mu}{M}\right)^2 h_{\alpha\beta}^{(2)} + \ldots$$



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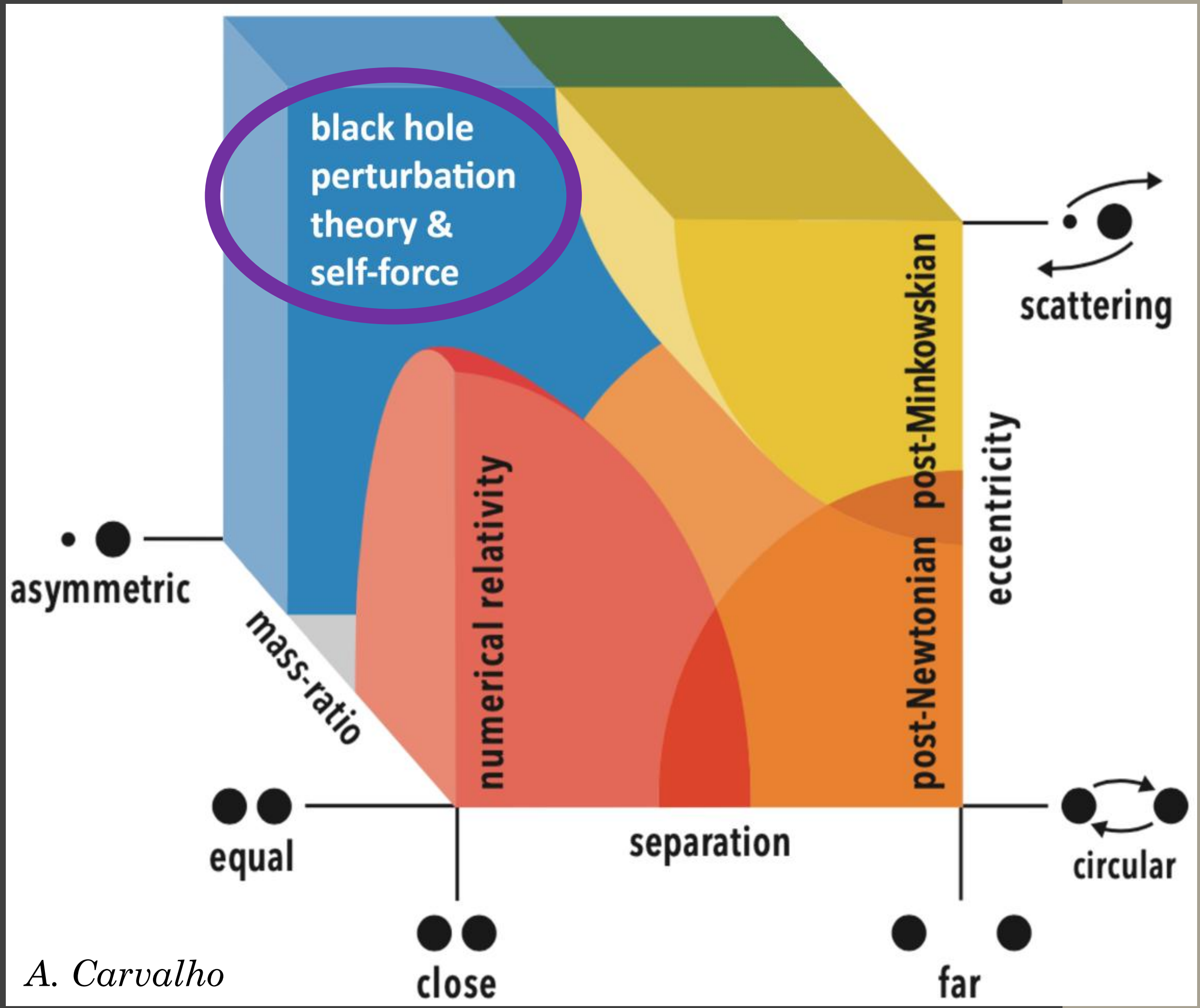
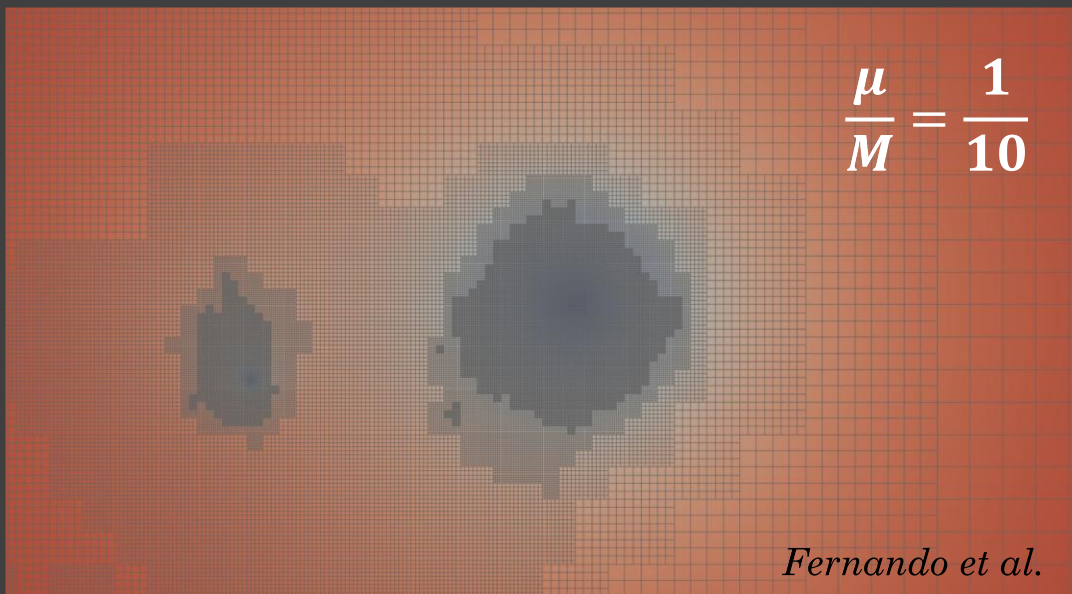
The overall premise of this talk

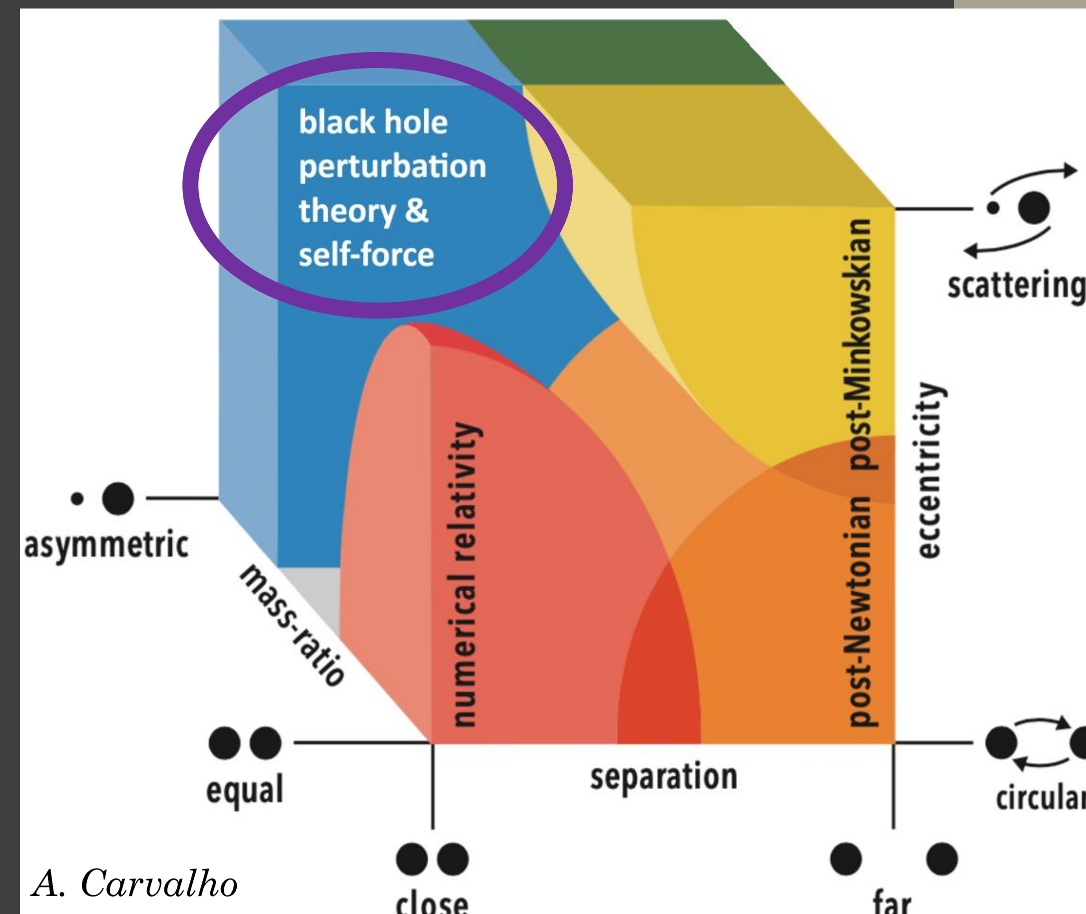
- Gravitational wave models involve the relativistic two-body problem
- Let's assume GR is the correct theory of gravity
- Don't have exact solutions
- Different methods are valid in different scenarios
- Need **black hole perturbation theory (BHPT) and self-force** for asymmetric mass-ratios



Challenges of asymmetric mass-ratios

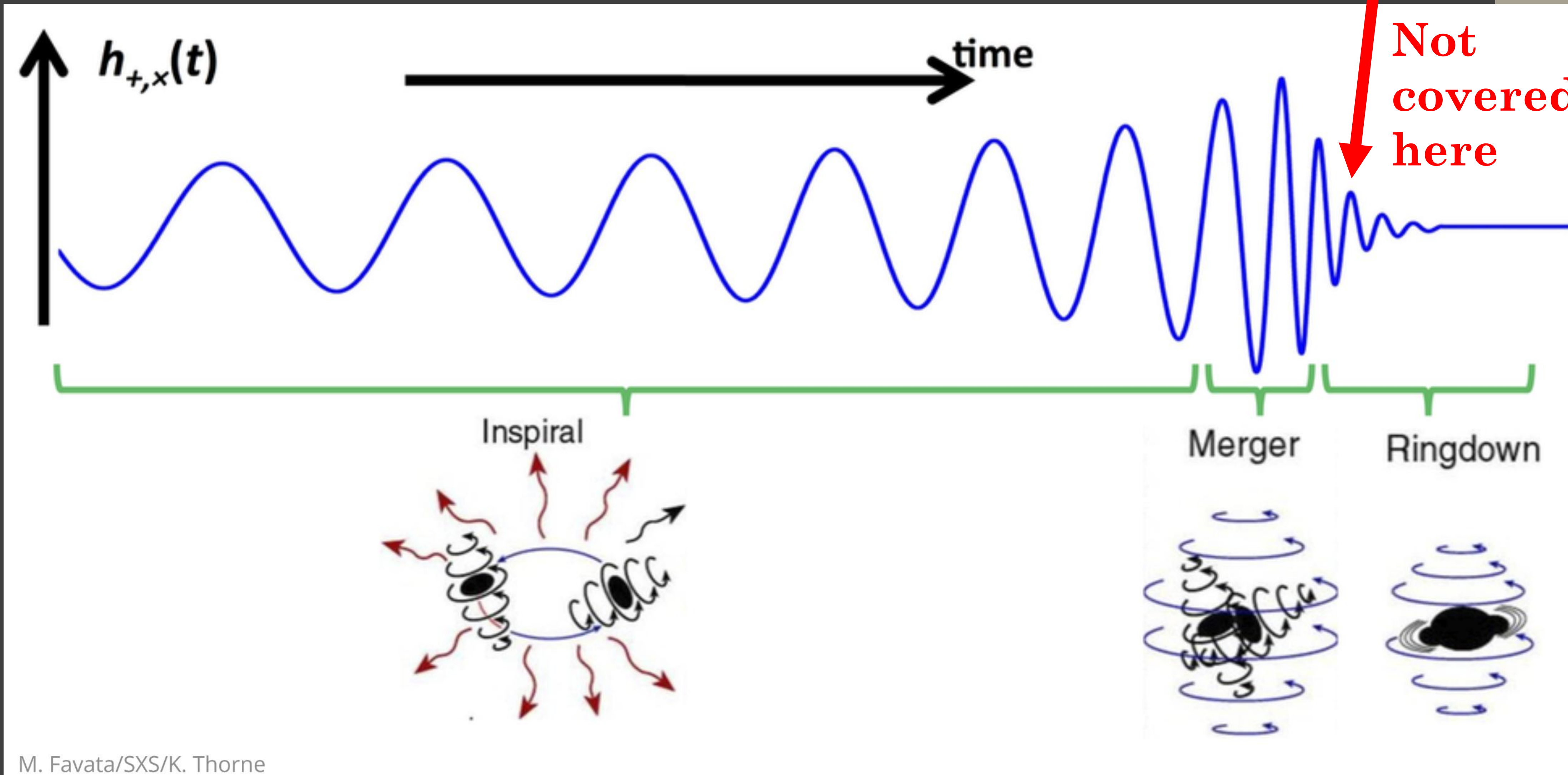
- There are scale disparities that worsen with mass-ratio





- BHPT avoids length disparities via zero-size limit of small body
- BHPT avoids resolving each cycle in time via the frequency domain

Interesting topic unrelated to this talk: *using BHPT for* ***quasi-normal modes***



Quick GR review

- Fields: $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right)$$

- Test body motion follows geodesics: $$\frac{d^2 x_p^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx_p^\beta}{d\tau} \frac{dx_p^\gamma}{d\tau} = 0$$

- Action: $$S = \int \left(\tfrac{1}{16\pi} \sqrt{-g}\, R + \mathcal{L}_{\mathrm{matter}} \right) \mathrm{d}^4 x$$

- Field equations:

$$R^\alpha{}_{\beta\mu\nu} = \frac{\partial \Gamma^\alpha_{\nu\beta}}{\partial x^\mu} - \frac{\partial \Gamma^\alpha_{\mu\beta}}{\partial x^\nu} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\beta}$$

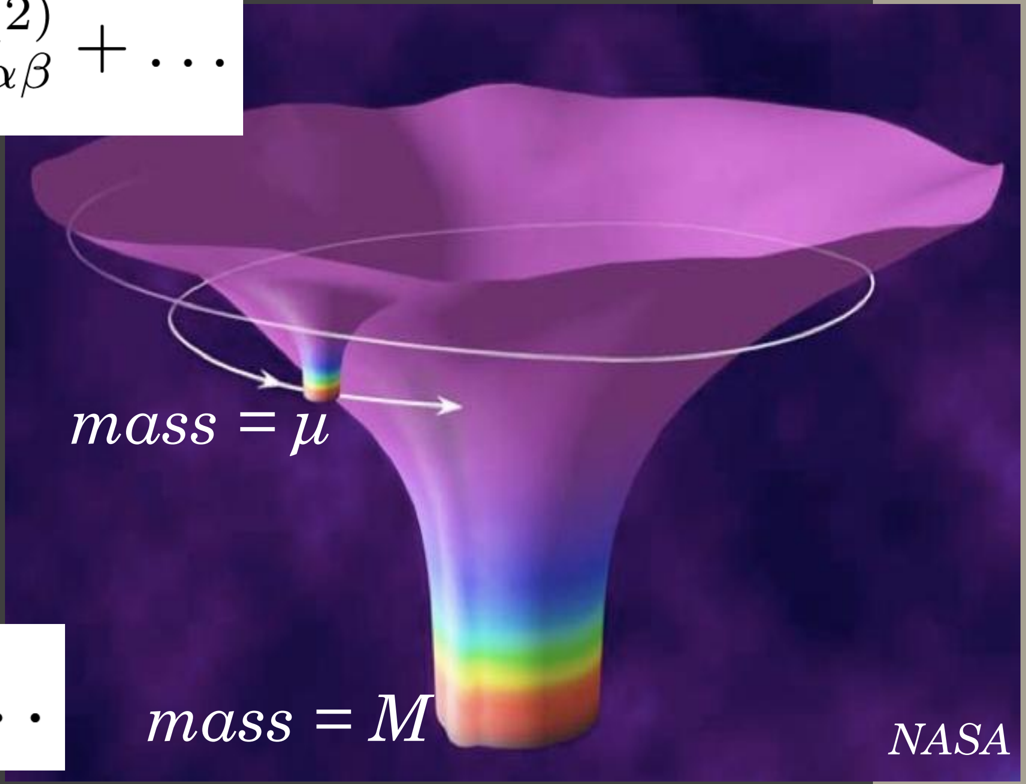
$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = G_{\mu\nu}$$

- Einstein field equation: $$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

This talk involves solving these PDEs for $g_{\alpha\beta}$

Black Hole Perturbation Theory (BHPT)

$$g^{\text{Total}}_{\alpha\beta} = g^{\text{Kerr}}_{\alpha\beta} + \left(\frac{\mu}{M}\right) h^{(1)}_{\alpha\beta} + \left(\frac{\mu}{M}\right)^2 h^{(2)}_{\alpha\beta} + \ldots$$



- Simplify notation: ϵ for mass-ratio

$$\epsilon = \frac{\mu}{M}$$

$$g_{\alpha\beta} = g^{\text{Kerr}}_{\alpha\beta}$$

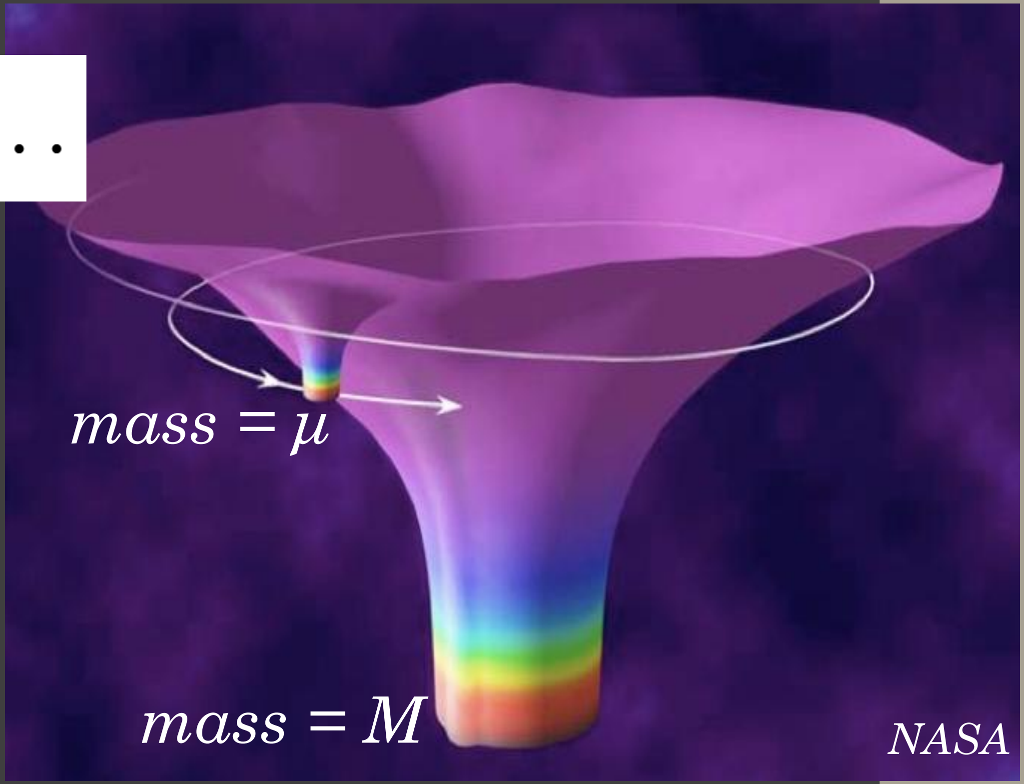
$$g^{\text{Total}}_{\alpha\beta} = g_{\alpha\beta} + \epsilon\, h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \ldots$$

- Solve Einstein field equation?

$$G_{\mu\nu}\left(g_{\alpha\beta} + \epsilon\, h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \ldots\right) = 8\pi T_{\mu\nu}$$

How to use the small body as a source?

$$g^{\text{Total}}_{\alpha\beta} = g_{\alpha\beta} + \epsilon\, h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \ldots$$



- If small body is a neutron star vs. black hole, is the method different?
- Recall that, although BBH $T_{\mu\nu} = 0$, $\epsilon \ll 1$ scale disparities prohibit numerically resolving the structure of the small body
- We need to extract the zero-size limit of the small body
- For $h^{(1)}_{\alpha\beta}$, the field equation is linear and we can use a point source

First-order metric perturbations

$$g^{\text{Total}}_{\alpha\beta} = g_{\alpha\beta} + \epsilon\, h^{(1)}_{\alpha\beta} + \ldots$$

$$G_{\mu\nu}\left(g_{\alpha\beta} + \epsilon\, h^{(1)}_{\alpha\beta} + \mathcal{O}(\epsilon^2)\right) = 8\pi T_{\mu\nu} + \mathcal{O}(\epsilon^2)$$

- Simplifying ingredients:

$$\bar{h}^{(1)}_{\mu\nu} = h^{(1)}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} h^{(1)}$$

$$Z^\alpha = \nabla_\alpha \bar{h}^{(1)\alpha\beta}$$

- Einstein tensor perturbation

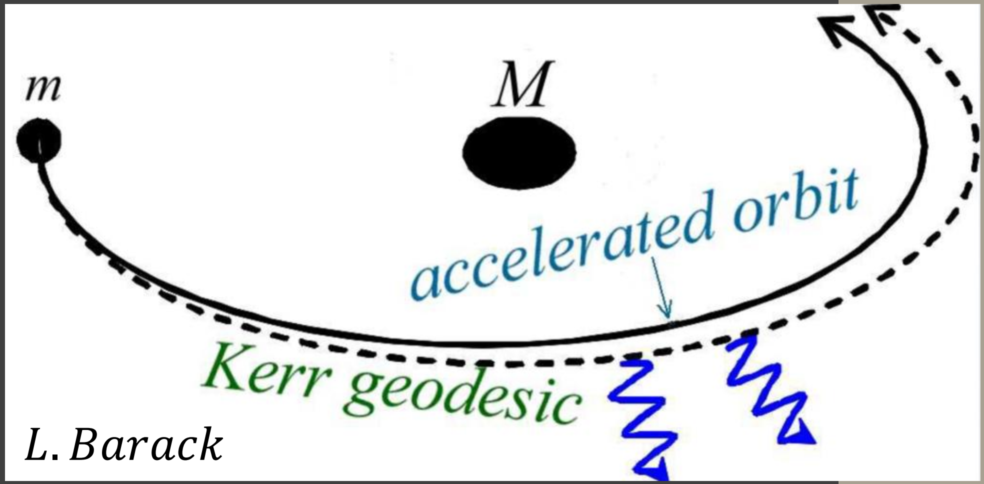
$$G_{\mu\nu}\left(g_{\alpha\beta} + \epsilon\, h^{(1)}_{\alpha\beta} + \mathcal{O}(\epsilon^2)\right) = -\frac{\epsilon}{2}\left(g^{\alpha\beta}\nabla_\alpha\nabla_\beta \bar{h}^{(1)}_{\mu\nu} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\, \bar{h}^{(1)}_{\alpha\beta} + g_{\mu\nu}\nabla_\alpha Z^\alpha - 2\nabla_{(\mu} Z_{\nu)}\right) + \mathcal{O}(\epsilon^2)$$

- 10 coupled linear PDEs
- How to handle motion of source? $\bar{h}^{(1)}_{\alpha\beta}$ affects the motion, but we need to know the motion for $T_{\mu\nu}$ before we can solve for $\bar{h}^{(1)}_{\alpha\beta}$

Perturbative expansion of worldline

- **How to handle motion of source?**

$\bar{h}^{(1)}_{\alpha\beta}$ affects the motion, but we need to know the motion for $T_{\mu\nu}$ before we can solve for $\bar{h}^{(1)}_{\alpha\beta}$



- Answer: expand small body worldline in powers of mass-ratio

$$x^\alpha_p = x^\alpha_{\text{geodesic}} + \epsilon\, x^\alpha_{(1)} + \ldots$$

$$g^{\text{Total}}_{\alpha\beta} = g_{\alpha\beta} + \epsilon\, h^{(1)}_{\alpha\beta} + \ldots$$

- Use $x^\alpha_{\text{geodesic}}$ to calculate $h^{(1)}_{\alpha\beta}$, then worry about $x^\alpha_{(1)}$ afterwards

$$G_{\mu\nu}\left(g_{\alpha\beta} + \epsilon\, h^{(1)}_{\alpha\beta} + \mathcal{O}(\epsilon^2)\right) = 8\pi T_{\mu\nu}\left(x^\alpha_{\text{geodesic}} + \mathcal{O}(\epsilon)\right) + \mathcal{O}(\epsilon^2)$$

Metric perturbations in Lorenz gauge

$$G_{\mu\nu}\big(g_{\alpha\beta} + \epsilon\, h^{(1)}_{\alpha\beta} + \mathcal{O}(\epsilon^2)\big) = -\frac{\epsilon}{2}\Big(g^{\alpha\beta}\nabla_\alpha\nabla_\beta\bar{h}^{(1)}_{\mu\nu} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\,\bar{h}^{(1)}_{\alpha\beta} + g_{\mu\nu}\nabla_\alpha Z^\alpha - 2\nabla_{(\mu}Z_{\nu)}\Big) + \mathcal{O}(\epsilon^2)$$

- Using gauge freedom we can simplify the equations
- Impose Lorenz gauge condition: $Z^\alpha = \nabla_\alpha \bar{h}^{(1)\alpha\beta} = 0$
- Lorenz gauge field equations:

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\bar{h}^{(1)}_{\mu\nu} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\,\bar{h}^{(1)}_{\alpha\beta} = -16\pi T^{(1)}_{\mu\nu}$$

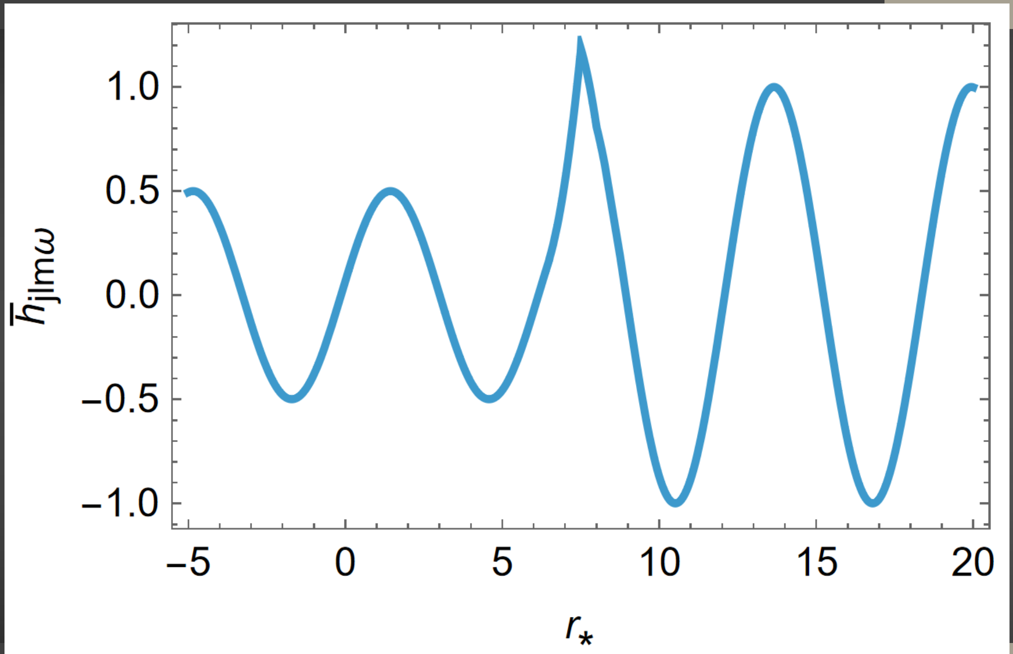
- 10 coupled hyperbolic PDEs with point source

Example method for simple case: Schwarzschild perturbations

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta \bar{h}^{(1)}_{\mu\nu} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\,\bar{h}^{(1)}_{\alpha\beta} = -16\pi T^{(1)}_{\mu\nu}$$

- Schwarzschild is highly symmetric: separation of variables
- The Schwarzschild operator has tensor spherical harmonic eigenfunctions (***lm* mode method**):

$$\bar{h}^{(1)}_{\mu\nu} = \sum_{lm\omega}\sum_{j=1}^{10} \bar{h}_{jlm\omega}(r)\,Y^{jlm}_{\mu\nu}(\theta,\phi)\,e^{-i\omega t}$$



- For a given *l*,*m*,*ω*, each radial function satisfies ODEs (simpler)

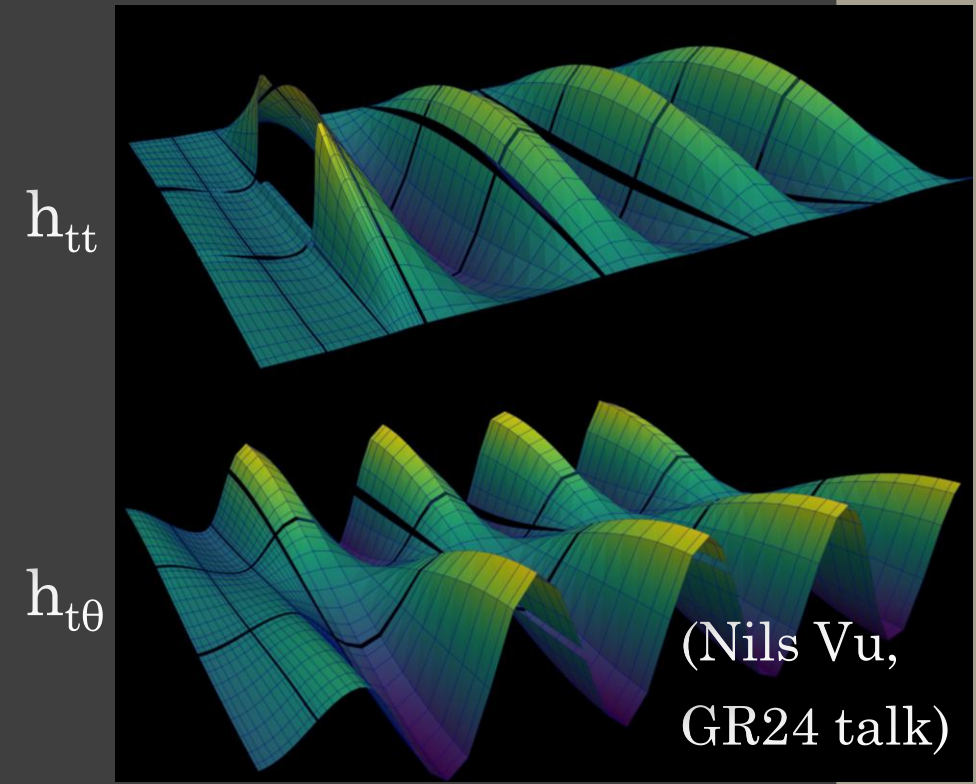
More complicated case: Kerr perturbations

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta \bar{h}^{(1)}_{\mu\nu} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\,\bar{h}^{(1)}_{\alpha\beta} = -16\pi T^{(1)}_{\mu\nu}$$

- Kerr only has 2 killing vectors: partial separation of variables

$$\bar{h}^{(1)}_{\mu\nu} = \sum_{m\omega} \bar{h}^{m\omega}_{\mu\nu}(r,\theta)\, e^{im\phi}\, e^{-i\omega t}$$

- Each ***m* mode** is infinite at source
- Each mode satisfies PDEs (tougher), but big advantages for second-order
- For first-order, would be nice to have full separation of variables for Kerr…



The path to separability: Newman Penrose

- The Newman-Penrose approach to GR is based on a null tetrad

$$g_{ab} = -\ell_a n_b - n_a \ell_b + m_a \bar{m}_b + \bar{m}_a m_b$$

- Formulation also based on “spin coefficients” (omitted here)
- Weyl scalars have important info about gravitational waves

$$\Psi_4 := C_{abcd} n^a \bar{m}^b n^c \bar{m}^d$$

- Ψ_4 vanishes for Kerr, so $\Psi_4^{(1)}$ is the leading order equation:

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}$$
$$- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi}$$
$$- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 4\pi\Sigma T\,.$$

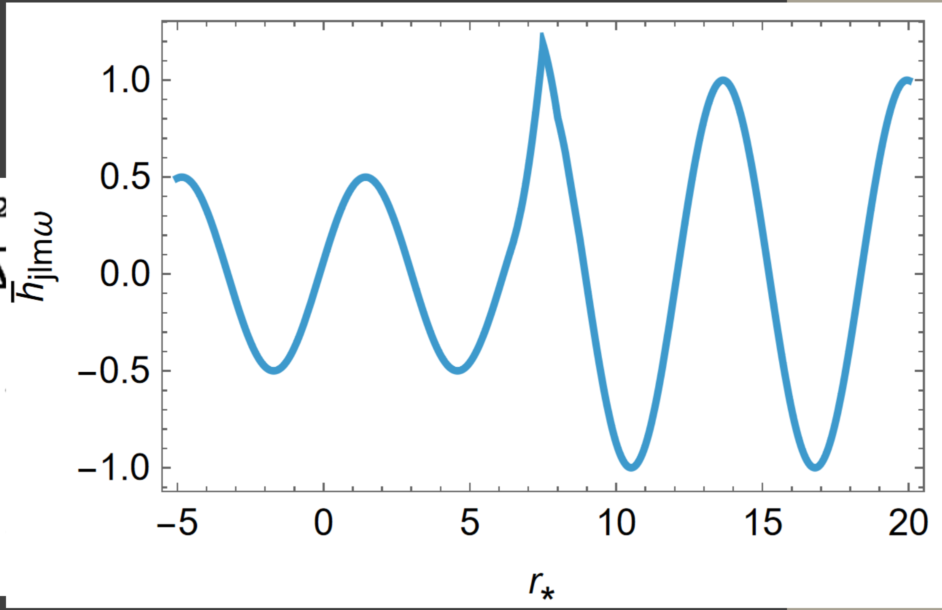
S. Teukolsky

The Teukolsky equation

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta}\right.$$

$$- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)$$

$$- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t}$$



- Even with only two symmetries, the field equation for $\Psi_4^{(1)}$ is fully separable in Kerr: (see talk by Zachary Nasipak later)

$$\Psi_4^{(1)}(t,r,\theta,\phi) = \frac{1}{(r-ia\cos\theta)^4}\sum_{\ell m\omega}\left(R^{(1)}_{\ell m\omega}(r)\,{}_{-2}S_{\ell m\omega}(\theta)\right)e^{i(m\phi-\omega t)}$$

- **Huge advantage**: $R^{(1)}_{lm\omega}$ satisfies an ODE for each mode
- **Disadvantage**: $\Psi_4^{(1)}$ is missing information for $h^{(1)}_{\mu\nu}$

Metric reconstruction

$$\Psi_4^{(1)}(t,r,\theta,\phi) = \frac{1}{(r - ia\cos\theta)^4} \sum_{\ell m\omega} \left(R^{(1)}_{\ell m\omega}(r)\ {}_{-2}S_{\ell m\omega}(\theta) \right) e^{i(m\phi - \omega t)}$$

- Extra step: extract as much info about $h^{(1)}_{\mu\nu}$ from $\Psi_4^{(1)}$ as we can

$$\rho^{-4}\psi_4 = \frac{1}{8}\left[\tilde{\eth}^4 \bar{\Psi}^{\mathrm{IRG}} - 12M\partial_t \Psi^{\mathrm{IRG}}\right]$$

Merlin et al.

- $\Psi_4^{(1)}$ sources a 4th order diff eq producing the "Hertz potential" Ψ^{IRG}, which is used to construct an incomplete $h^{(1)}_{\mu\nu}$ (CCK method)

$$\begin{aligned} h^{\mathrm{IRG}}_{\alpha\beta} = \{ & -\ell_\alpha \ell_\beta (\boldsymbol{\delta} + 2\beta)(\boldsymbol{\delta} + 4\beta) - m_\alpha m_\beta (\boldsymbol{D} - \varrho)(\boldsymbol{D} + 3\varrho) \\ & - \ell_{(\alpha} m_{\beta)} [\boldsymbol{D}(\boldsymbol{\delta} + 4\beta) + (\boldsymbol{\delta} + 4\beta)(\boldsymbol{D} + 3\rho)] \} \Psi^{\mathrm{IRG}} \pm \text{c.c.} \end{aligned}$$

- Completion adds the missing info: $h^{\mathrm{monopole}}_{\mu\nu} + h^{\mathrm{dipole}}_{\mu\nu} + h^{\mathrm{IRG}}_{\mu\nu}$

- **Alternate new path**: reconstruct $h^{(1)}_{\mu\nu}$ directly in Lorenz gauge

(Dolan et al., see talk by Kevin Cunningham later)

First-order self-force (1SF) calculation

- Assume you have calculated $h^{(1)}_{\mu\nu}$ in Lorenz gauge with a point source following $x_\mu^{\text{geodesic}}(\tau)$
- How does $h^{(1)}_{\mu\nu}$ affect the motion of the small body?
- Does the geodesic motion principle apply to $g_{\mu\nu} + \epsilon\, h^{(1)}_{\mu\nu}$???

$$u^\beta \nabla_\beta u^\alpha = -\epsilon \left(g^{\alpha\beta} + u^\alpha u^\beta\right) \left(\nabla_\nu h^{(1)}_{\beta\mu} - \frac{1}{2} \nabla_\beta h^{(1)}_{\mu\nu}\right) u^\mu u^\nu + \mathcal{O}(\epsilon^2)$$

- **No**, this expanded geodesic equation can't be right because $h^{(1)}_{\mu\nu}(x^\alpha \to x^\alpha_p) \sim \infty$ (~1/distance, like Coulomb potential)
- Any validity of the above would require careful regularization

Electromagnetic self-force analogy

- The EM self-force has same Coulomb regularization problem!
- How does EM handle it? Minkowski EM equations of motion:

$$\mu \frac{du^\alpha}{d\tau} = q \left(\nabla^\alpha A_\beta - \nabla_\beta A^\alpha \right) u^\beta$$

- Dirac's approach: split A^β into **singular** and **regular** pieces

finite everywhere

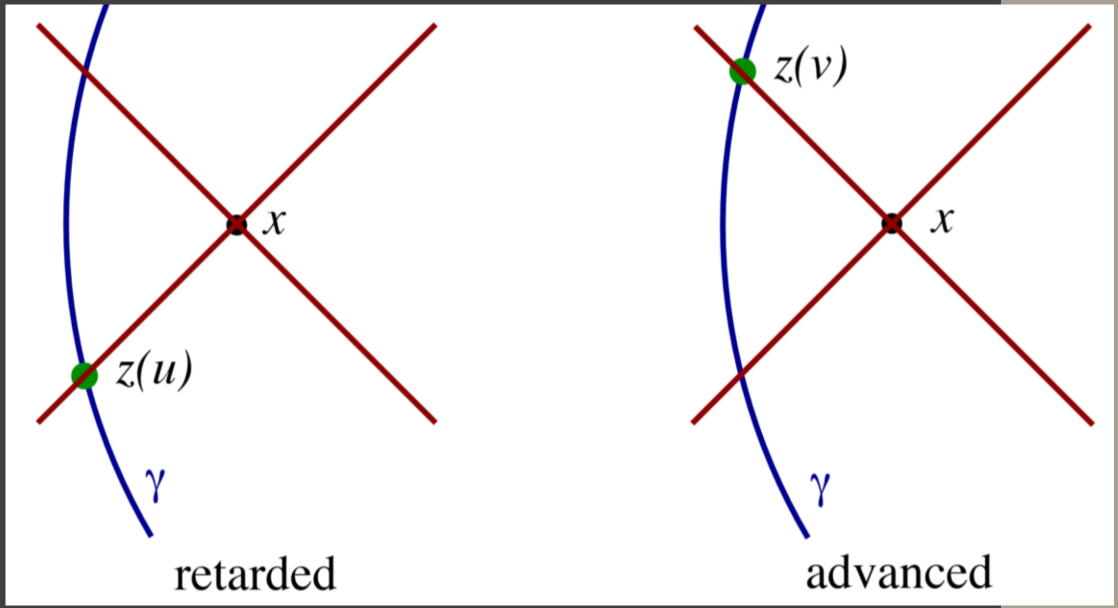
$$A^\beta = A^\beta_S + A^\beta_R$$

$$A^\beta_R = \frac{1}{2}\left(A^\beta_{\text{ret}} - A^\beta_{\text{adv}} \right)$$

$$A^\beta_S = \frac{1}{2}\left(A^\beta_{\text{ret}} + A^\beta_{\text{adv}} \right)$$

bad, but no force

E. Poisson



Regularizing the gravitational self-force

- Similar approach for $h_{\mu\nu}^{(1)}$? Tricky:
- More complicated than (*ret-adv*)/2
- Start from singular field $h_{\mu\nu}^{(1)S}$, what has correct divergence with zero force?
- Then: $\bar{h}_{\mu\nu}^{(1)R} = \bar{h}_{\mu\nu}^{(1)} - \bar{h}_{\mu\nu}^{(1)S}$

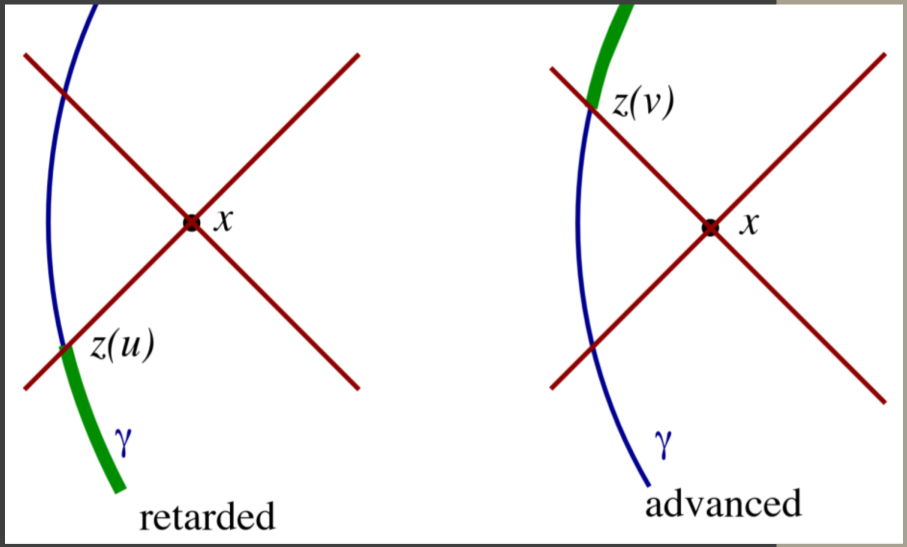
Detweiler et al.

- **Features**:

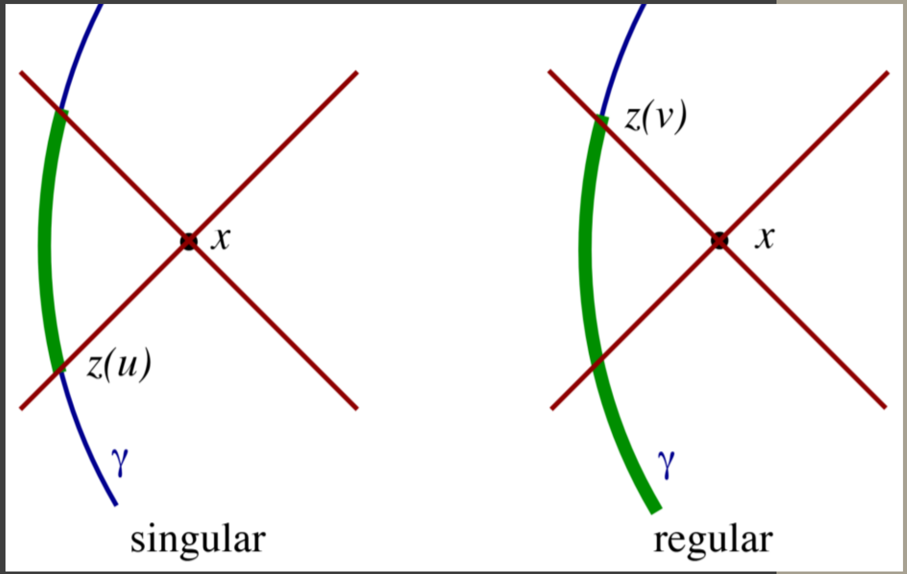
-Motion is geodesic in $g_{\mu\nu} + \epsilon\, h_{\mu\nu}^{(1)R}$

-$h_{\mu\nu}^{(1)S}$ is an inhomogeneous solution accessible from a local expansion

-$h_{\mu\nu}^{(1)R}$ is a homogeneous solution



E. Poisson



Implementing 1SF

$$\bar{h}^{(1)R}_{\mu\nu} = \bar{h}^{(1)}_{\mu\nu} - \bar{h}^{(1)S}_{\mu\nu}$$

1SF

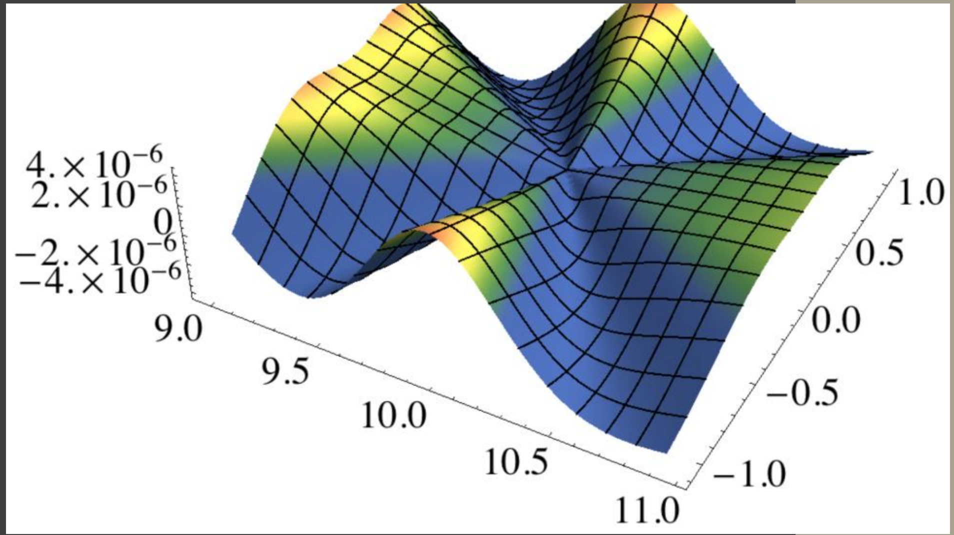
$$u^\beta \nabla_\beta u^\alpha = -\epsilon \left(g^{\alpha\beta} + u^\alpha u^\beta\right) \left(\nabla_\nu h^{(1)R}_{\beta\mu} - \frac{1}{2}\nabla_\beta h^{(1)R}_{\mu\nu}\right) u^\mu u^\nu + \mathcal{O}(\epsilon^2)$$

- **Issue**: Both $h^{(1)}_{\mu\nu}$ and $h^{(1)S}_{\mu\nu}$ are infinite at source position
- **Solution**: each lm mode is finite→ **Mode-sum regularization**

$$\bar{h}^{(1)S}_{\mu\nu} = \sum_{lm\omega} \sum_{j=1}^{10} \bar{h}^S_{jlm\omega}(r)\, Y^{jlm}_{\mu\nu}(\theta, \phi)\, e^{-i\omega t}$$

$$\bar{h}^{(1)R}_{\mu\nu} = \sum_{lm\omega} \sum_{j=1}^{10} \left(\bar{h}_{jlm\omega}(r) - \bar{h}^S_{jlm\omega}(r)\right) Y^{jlm}_{\mu\nu}(\theta, \phi)\, e^{-i\omega t}$$

Wardell et al.



- **Another solution**: effective source method

1SF Features

$$\bar{h}^{(1)R}_{\mu\nu} = \bar{h}^{(1)}_{\mu\nu} - \bar{h}^{(1)S}_{\mu\nu}$$

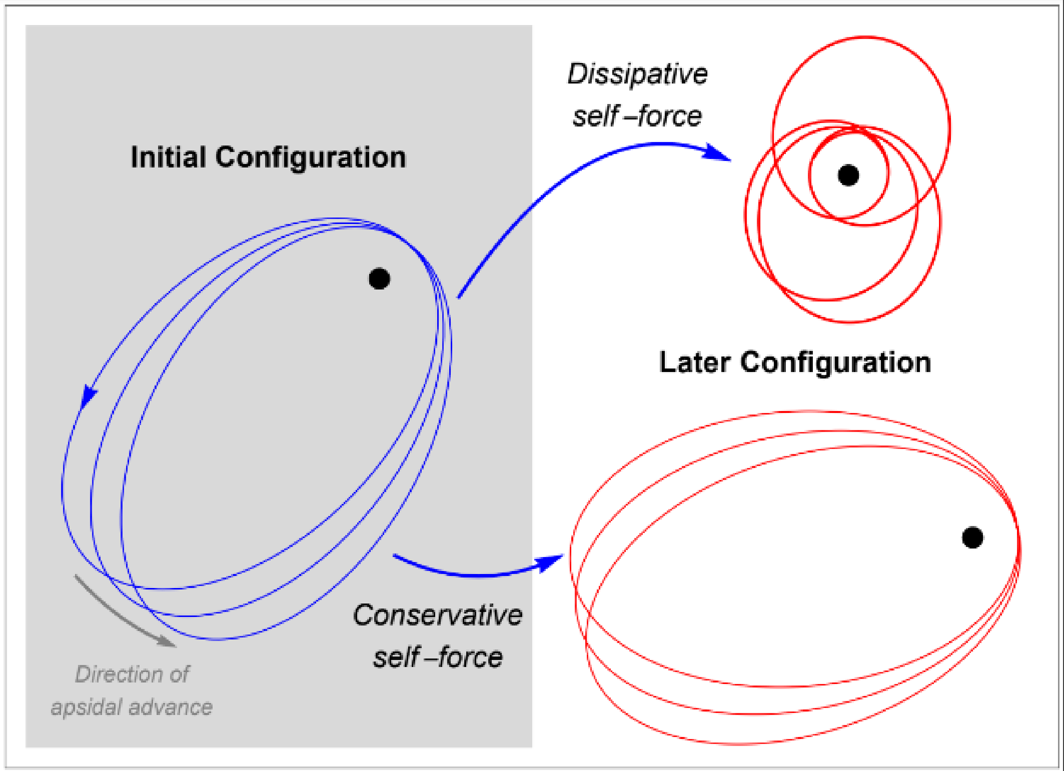
1SF

$$u^\beta \nabla_\beta u^\alpha = -\epsilon \left(g^{\alpha\beta} + u^\alpha u^\beta\right)\left(\nabla_\nu h^{(1)R}_{\beta\mu} - \frac{1}{2}\nabla_\beta h^{(1)R}_{\mu\nu}\right) u^\mu u^\nu + \mathcal{O}(\epsilon^2)$$

1SF has both **dissipative** and **conservative** pieces

- The biggest effect is the dissipative self-force (causes the constants of motion to drift over time)
- The conservative self-force perturbs the shape of the orbit

N. Warburton



1SF Trajectories

$$\bar{h}^{(1)R}_{\mu\nu} = \bar{h}^{(1)}_{\mu\nu} - \bar{h}^{(1)S}_{\mu\nu}$$

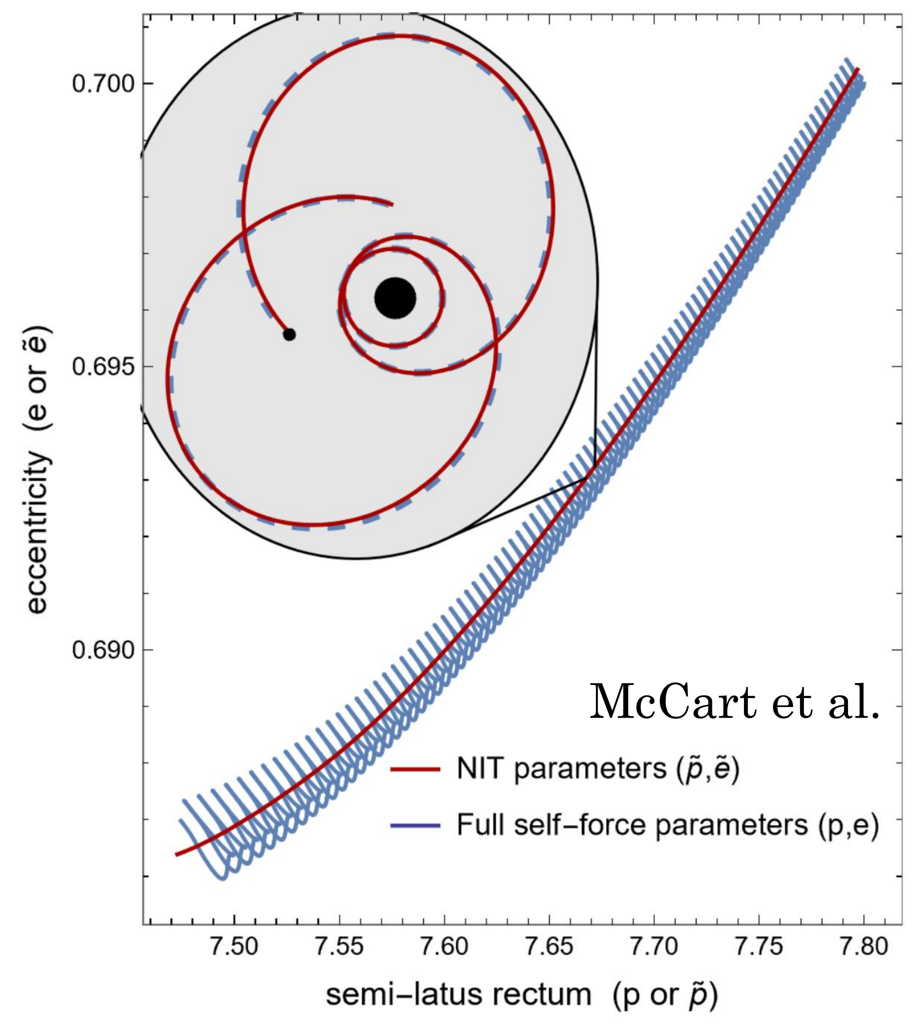
1SF

$$u^\beta \nabla_\beta u^\alpha = -\epsilon \left(g^{\alpha\beta} + u^\alpha u^\beta\right) \left(\nabla_\nu h^{(1)R}_{\beta\mu} - \frac{1}{2}\nabla_\beta h^{(1)R}_{\mu\nu}\right) u^\mu u^\nu + \mathcal{O}(\epsilon^2)$$

- **Osculating geodesics method**: Parameterize inspiral with evolving orbital parameters that describe the geodesic tangent to the worldline at that time

  -Issue: tangent geodesic parameters oscillate rapidly

- **Near-identity transform method** (see talk by Phillip Lynch later): Time-average the oscillating in the geodesic parameters in a way that is approximately invertible



1SF Waveforms

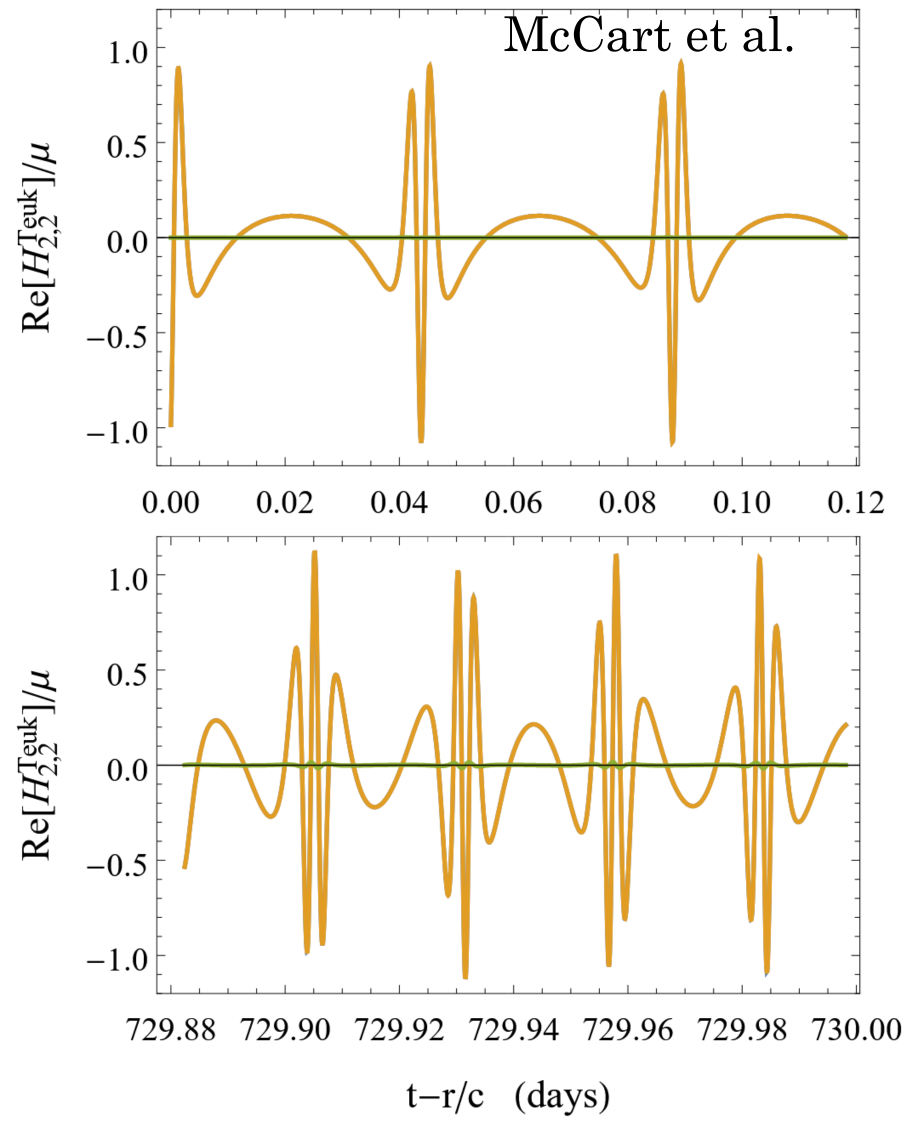
$$h_+ - i\,h_\times = \frac{1}{r}\sum_{lm} H_{lm}(t_{\text{ret}})\ {}_{-2}Y_{lm}(\theta,\phi)$$

$$\Phi_{mn}(t) = m\int_0^t \Omega_\phi\,dt + n\int_0^t \Omega_r\,dt$$

$$H_{lm}^{\text{Teuk}}(t) = \sum_n A_{lmn}(p,e)\ e^{-i\Phi_{mn}}$$

- **Example**: Constructing waveforms in the time domain (see later talk by Zachary Nasipak)

- It is also possible to construct waveforms in directly in the frequency domain, see Hughes et al. (this plus NIT successfully avoids the separation of timescales)



Towards 2SF

- Let's define the residual error from truncating at 1SF as $\delta^2 G_{\mu\nu}$

$$G_{\mu\nu}\left(g_{\alpha\beta} + \epsilon\, h^{(1)}_{\alpha\beta}\right) = 8\pi T_{\mu\nu}\left(x^\alpha_{\text{geodesic}}\right) + \delta^2 G_{\mu\nu} + \mathcal{O}(\epsilon^3)$$

- $\delta^2 G_{\mu\nu}$ will help us approach the second-order field equation

$$G_{\mu\nu}\left(g_{\alpha\beta} + \epsilon\, h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \mathcal{O}(\epsilon^3)\right) = 8\pi T_{\mu\nu}\left(x^\alpha_{\text{geodesic}} + \epsilon\, x^\alpha_{(1)} + \mathcal{O}(\epsilon^2)\right) + \mathcal{O}(\epsilon^3)$$

- Do difference between equations

$$G_{\mu\nu}\left(g_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \mathcal{O}(\epsilon^3)\right) = 8\pi\left(T_{\mu\nu}\left(x^\alpha_{\text{geo}} + \epsilon\, x^\alpha_{(1)}\right) - T_{\mu\nu}\left(x^\alpha_{\text{geo}}\right)\right) - \delta^2 G_{\mu\nu} + \mathcal{O}(\epsilon^3)$$

linear PDEs for $h^{(2)}_{\mu\nu}$!

extended non-linear source known: $\delta^2 G \sim (\nabla h^{(1)})(\nabla h^{(1)})$

2SF obstacles (see next talk by Adam Pound)

$$G_{\mu\nu}\left(g_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \mathcal{O}(\epsilon^3)\right) = 8\pi\left(T_{\mu\nu}\left(x^\alpha_{\rm geo} + \epsilon\, x^\alpha_{(1)}\right) - T_{\mu\nu}\left(x^\alpha_{\rm geo}\right)\right) - \delta^2 G_{\mu\nu} + \mathcal{O}(\epsilon^3)$$

- We have an extended source that isn't nice: $\delta^2 G \sim (\nabla h^{(1)})(\nabla h^{(1)})$
- Near small body, $\delta^2 G \sim 1/\text{distance}^4$!
- PDE solution, $h^{(2)} \sim 1/\text{distance}^2$!

$$\bar{h}^{(2)}_{\mu\nu} = \sum_{lm\omega}\sum_{j=1}^{10} \bar{h}^{(2)}_{jlm\omega}(r)\, Y^{jlm}_{\mu\nu}(\theta,\phi)\, e^{-i\omega t}$$

- Each lm mode is better, but still infinite
- Also, using $x^\mu_{\rm geo}$ for $h^{(1)}_{\mu\nu}$ emits waves throughout infinite past
- This causes more bad behavior of $\delta^2 G$ to mitigate
- Also, a 2SF Teukolsy lm approach is tougher, and for Kerr is much messier without analytic eigenfunctions
- Perhaps better to abandon full separation of variables (m-modes)

Thanks for having me!

Questions?