MAX-PLANCK-GESELLSCHAFT

# The eccentric radiation-reaction force within the effective-one-body formalism

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pySEOBNR

Lost in Translation: The languages of Gravitational Waves

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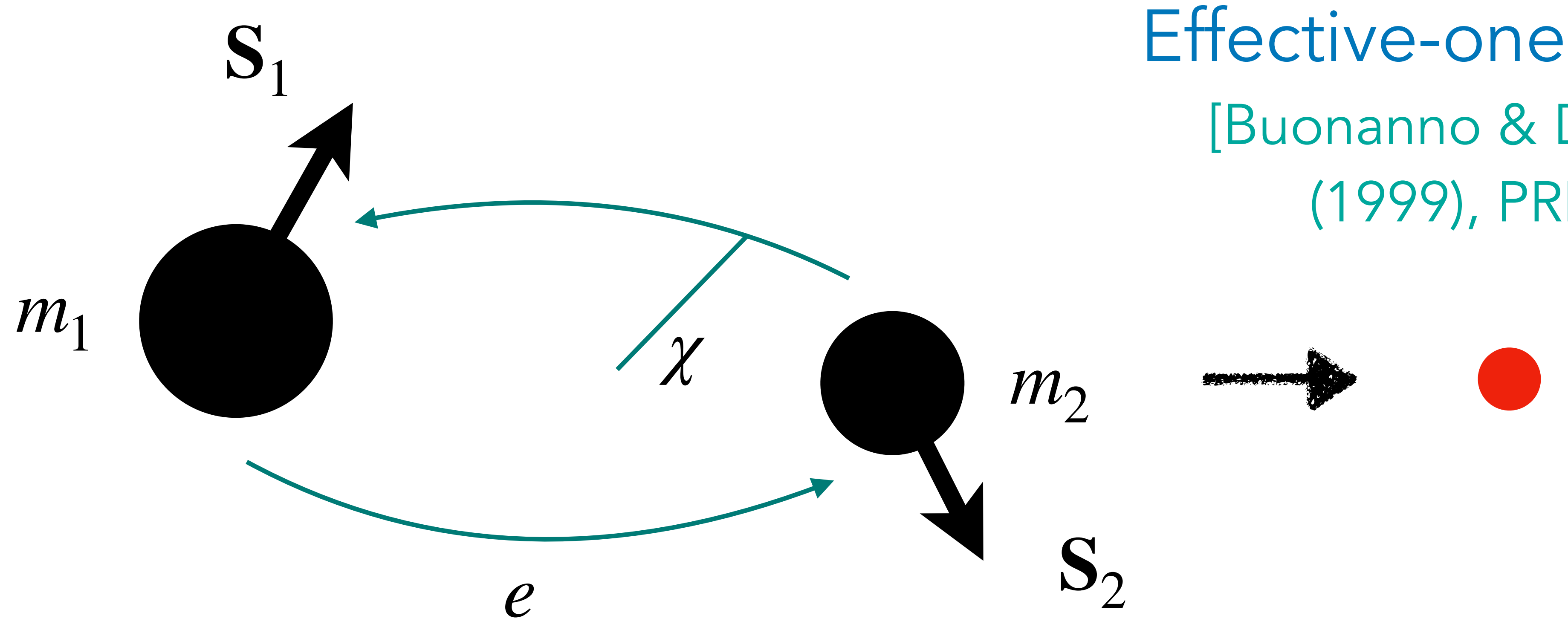
# Content

- EOB summary and the flux balance approach
- Resummations of the radiation-reaction force
- Comparing eccentric waveform models

# EOB waveform models in a nutshell




Effective-one-body (EOB) problem

[Buonanno & Damour, PRD 59, 084006 (1999), PRD 62, 064015 (2000)]

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

- EOB Hamiltonian

$$H_{\rm EOB} = M\sqrt{1+2\nu\left(\frac{H_{\rm eff}}{\mu}-1\right)}$$

***Resummations*** of analytical information

- Radiation-reaction (RR) force $\mathcal{F}$

Losses of energy and angular momentum due to the emission of GWs

- Binary dynamics:

$$\dot{r} = \frac{\partial H_{\rm EOB}}{\partial p_r} \qquad \dot{p}_r = -\frac{\partial H_{\rm EOB}}{\partial r} + \mathcal{F}_r$$

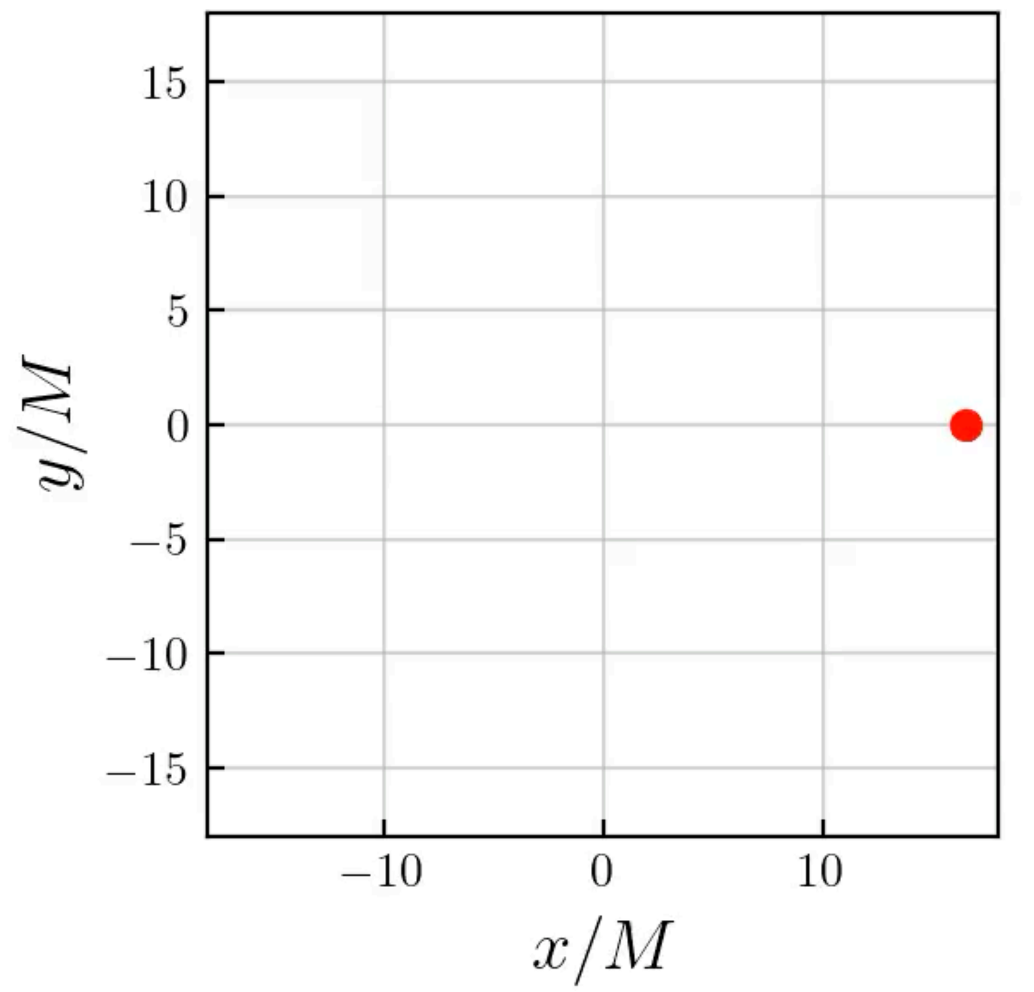
$$\Omega = \frac{\partial H_{\rm EOB}}{\partial p_\phi} \qquad \dot{p}_\phi = -\frac{\partial H_{\rm EOB}}{\partial \phi} + \mathcal{F}_\phi$$

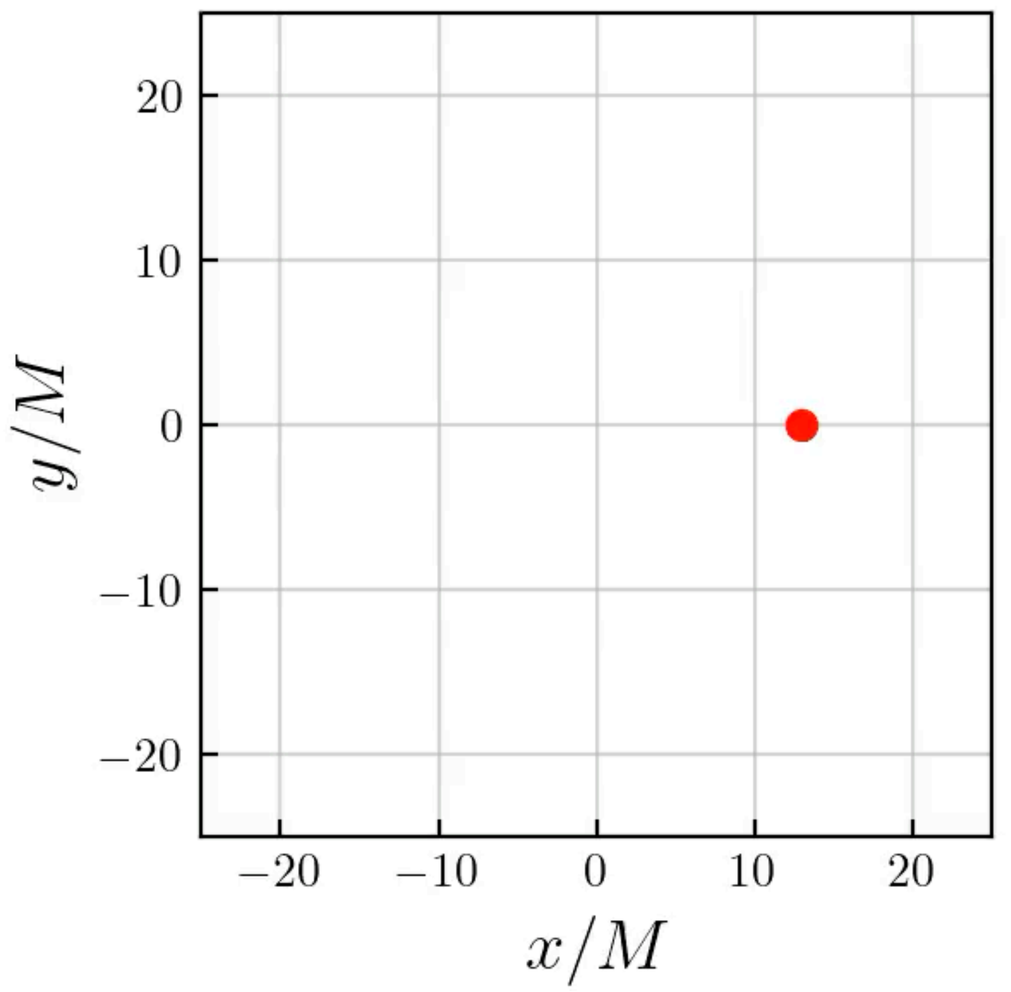
- GWs are decomposed in terms of waveform modes $h_{\ell m}$

$$h_+ - i\,h_\times = \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{m=\ell} {}_{-2}Y_{\ell m}(\iota,\varphi)\, h_{\ell m}(\Theta;t)$$

- The waveform modes $h_{\ell m}$ are obtained from factorized formulas and physically-motivated ansatz, and are further improved with calibrations to numerical relativity (NR) and BH perturbation theory (BHPT).

- SEOBNR $\equiv$ EOB dynamics + NR & BHPT info. $\longrightarrow$ GWs



# The flux-balance approach

- The RR force can be derived from first principles [e.g., Chandrasekhar & Esposito, '70; Burke & Thorne, '70; Damour & Deruelle, '81; …; Blanchet+, 2025].

- It can also be derived from a phenomenological method known as the **flux-balance approach** [Iyer & Will, '93, '95; Gopakumar+, '98], where the local losses of energy and angular momentum are related to the far-zone fluxes:

$$\dot{E}_{\rm sys} \sim \Phi_E$$
$$\dot{J}_{\rm sys} \sim \Phi_J$$

- To make the connection, one needs ***Schott terms*** (additional contributions to the system due to its interaction with the radiation field) [Schott, 1915; Bini & Damour, 2012]:

**Local (gauge-dependent)** → $\dot{E}_{\rm sys}$, $\dot{J}_{\rm sys}$, $\dot{E}_{\rm Schott}$, $\dot{J}_{\rm Schott}$

$$\dot{E}_{\rm sys} + \dot{E}_{\rm Schott} = \Phi_E$$
$$\dot{J}_{\rm sys} + \dot{J}_{\rm Schott} = \Phi_J$$

**Wave-zone (gauge-independent)**

# The flux-balance approach

- From the EOB equations of motion,

$$\dot{J}_{\rm sys} = \frac{dp_\phi}{dt} = \mathscr{F}_\phi \qquad\qquad \mathscr{F}_\phi + \dot{J}_{\rm Schott} = \Phi_J$$
$$\dot{E}_{\rm sys} = \frac{dH_{\rm EOB}}{dt} = \dot{r}\mathscr{F}_r + \dot{\phi}\mathscr{F}_\phi \quad \rightarrow \quad \dot{r}\mathscr{F}_r + \dot{\phi}\mathscr{F}_\phi + \dot{E}_{\rm Schott} = \Phi_E$$

- These equations can be solved for the components of the RR force:

$$\mathscr{F}_\phi = -\dot{J}_{\rm Schott} - \Phi_J$$
$$\mathscr{F}_r = \frac{-\Phi_E - \dot{E}_{\rm Schott} + \dot{\phi}\left(\Phi_J + \dot{J}_{\rm Schott}\right)}{\dot{r}}$$

- Thus, given the **post-Newtonian (PN) results for the fluxes** $\Phi_E(r, p_r, p_\phi)$ and $\Phi_J(r, p_r, p_\phi)$ [Arun+, 2008a, 2008b, 2009; Cho+, 2021; Henry+, 2023] (**expressed in EOB coordinates** — *canonical transformation*), and **ansatz for the Schott terms**, one can determine a prescription for the RR force.

# The flux-balance approach at leading order

- Example at **leading order**: we start from the expressions for the fluxes for generic orbits, together with ansatz for the Schott terms, following the approach from [Bini & Damour, 2012; Khalil+, 2021; AG+ 2025]:

$$\Phi_J^{\rm LO} = \frac{8\nu\, p_\phi}{5\, r^3}\left(2\,\frac{p_\phi^2}{r^2} - p_r^2 + \frac{2}{r}\right) \qquad J_{\rm Schott}^{\rm LO} = \frac{\nu\, p_r p_\phi}{r}\left(\alpha_1\, p_r^2 + \alpha_2\,\frac{p_\phi^2}{r^2} + \frac{\alpha_3}{r}\right)$$

$$\Phi_E^{\rm LO} = \frac{8\nu}{15\, r^4}\left(12\,\frac{p_\phi^2}{r^2} + p_r^2\right) \qquad E_{\rm Schott}^{\rm LO} = \frac{\nu\, p_r}{r^2}\left(\beta_1\, p_r^2 + \beta_2\,\frac{p_\phi^2}{r^2} + \frac{\beta_3}{r}\right)$$

where $\{\alpha_i, \beta_i\}$ are **gauge constants**, associated with a freedom in the choice of coordinates [Iyer & Will, '93, '95].

- Plugging this into the formulas before, gives

$$\mathscr{F}_\phi^{\rm LO} = \frac{8\nu\, p_\phi}{5\, r^3}\left[(2\alpha+1)\, p_r^2 - \frac{\alpha+2}{r^2} p_\phi^2 + \frac{\alpha-2}{r}\right],$$

$$\mathscr{F}_r^{\rm LO} = \frac{8\nu\, p_r}{15\, r^3}\left[6(\alpha-\beta+2)\, p_r^2 - \frac{3(\alpha-3\beta-1)}{r^2} p_\phi^2 + \frac{9\alpha-9\beta+17}{r}\right]$$

- At higher PN orders, tail and post-adiabatic effects need to be considered, and much more gauge constants start to appear. Current results are available up to 3PN order [AG+, 2025], and the approach has been extended, e.g., to scalar-tensor theories up to 1.5PN order [Jain & Rettegno, 2025].

# The need for a resummation

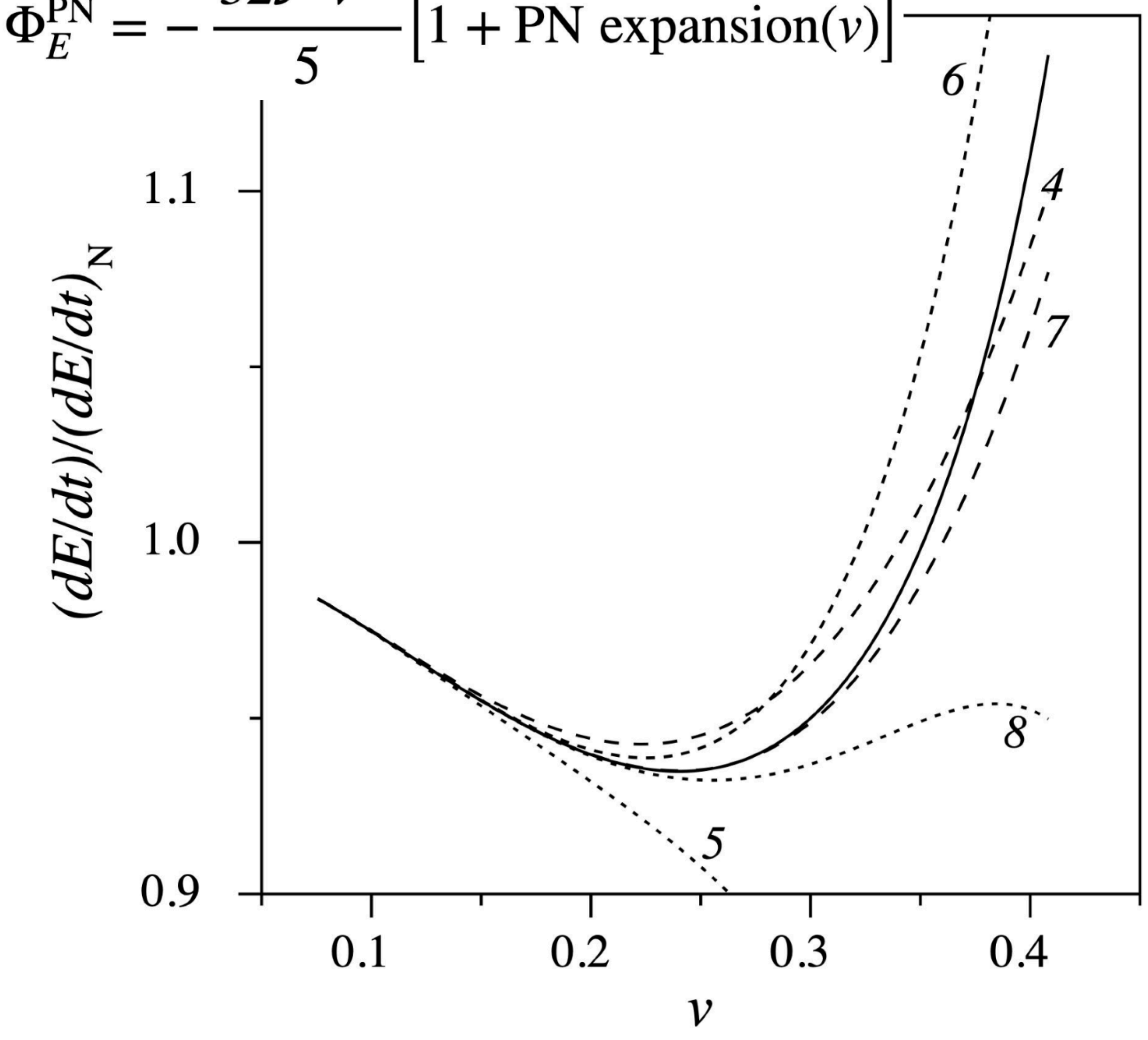
- **Using the PN expanded expressions** $\mathscr{F}(r, p_r, p_\phi, \{\alpha_i\}, \{\beta_i\})$ in the EOB equations of motion **leads to robustness problems** towards the end of the inspiral phase.

- Example: non-convergent behavior of the flux for circular orbits in the test-mass limit [e.g., Poisson, 1997].

- Resummation strategies are needed! (Resummation: rewriting a perturbative expansion to improve some property, e.g., its accuracy or robustness.)

- How to motivate a resummation?

$$\Phi_E^{\rm PN} = -\frac{32\nu^2 v^{10}}{5}\left[1 + \text{PN expansion}(v)\right]$$



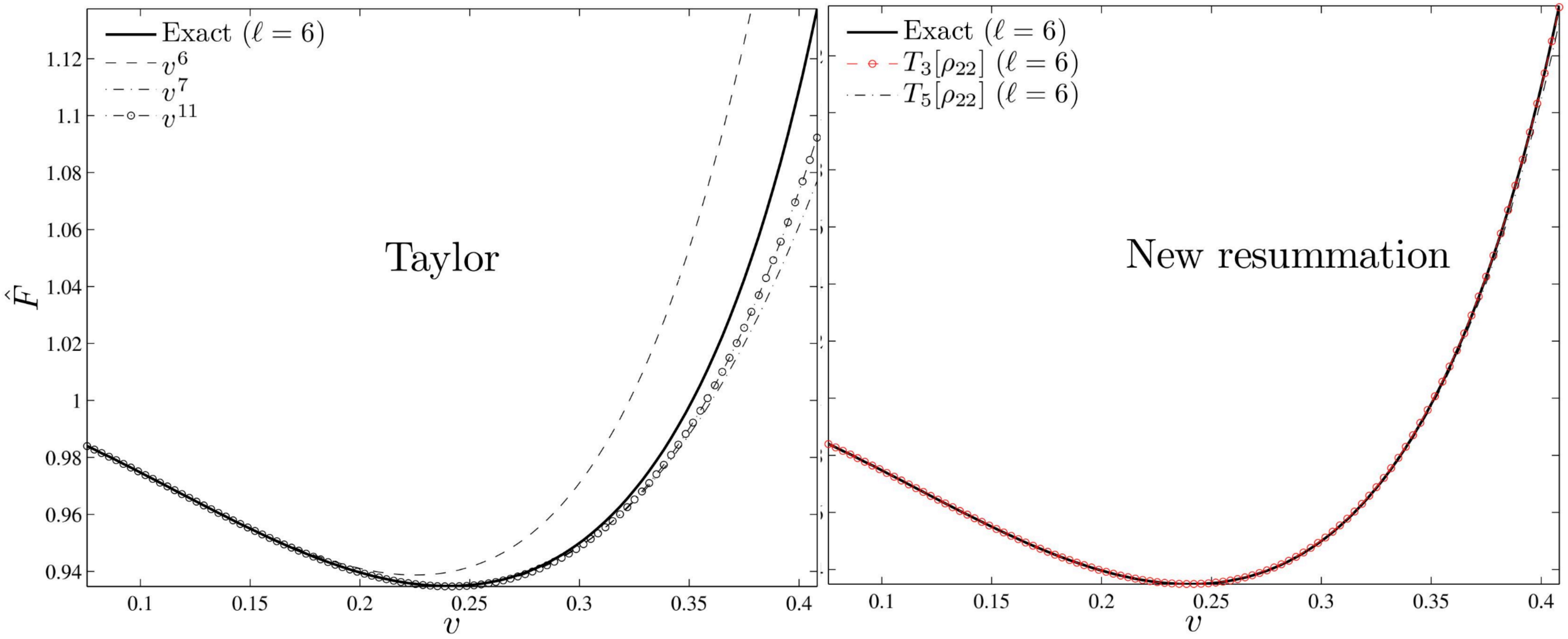

Poisson (1997)

# Resummations of the RR force

- From the generic expressions for the fluxes:

$$\Phi_E^{\rm gw} = \frac{1}{16\pi}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} |\dot{h}_{\ell m}|^2 \to_{QC} \quad \Phi_E^{\rm gw} \approx \frac{\Omega^2}{16\pi}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} m^2 |h_{\ell m}|^2$$

- [Damour, Nagar, Iyer, (2007, 2009)] proposed the use of **resummed expressions for the waveform modes** based on physical and numerical expectations in the test-mass limit.




Damour, Iyer, Nagar (2009)

$$h^{\rm F}_{\ell m} = h^{\rm N}_{\ell m}\, \hat{S}_{\rm eff}\, T_{\ell m}\, f_{\ell m}\, {\rm e}^{i\delta_{\ell m}}$$

- $h^{\rm N}_{\ell m}$ is a Newtonian prefactor.
- $S_{\ell m}$ is a source term motivated by the Regge-Wheeler-Zerilli equation.
- $T_{\ell m}$ relates far- and near-zone GW amplitudes, and it is a resummation of an infinite number of leading logarithms entering the tail contributions to the waveform. (Recently generalized in [Ivanov, Li, Parra-Martinez, & Zhou, 2025]).
- $f_{\ell m}$ and $\delta_{\ell m}$ are residual amplitude and phase terms, which can be exploited to introduce calibration parameters.

# The eccentric RR force

- The previous **resummation** works marvelously in the QC orbit limit: the RR force can be employed to generate the dynamics even in strong-field regions. **How to generalize this to the eccentric case?**

- Research has focused on factorizations of the type: $\mathscr{F} = \mathscr{F}^{\mathrm{QC}}\,\mathscr{F}^{\mathrm{ecc}}$:
  - **TEOBResumS-Dalí** [Chiaramello & Nagar, 2020; Nagar+, 2025]: generic Newtonian prefactor $\mathscr{F}^{\mathrm{ecc}}(r^{(n)}, \Omega^{(m)})$.
  - **SEOBNRv4EHM** [Ramos-Buades+, 2022]: $\mathscr{F}^{\mathrm{ecc}} = 1 \rightarrow$ incorrect but robust.
  - **SEOBNRv5EHM** [AG+, 2025]: $\mathscr{F}^{\mathrm{ecc}}(x = \langle M\Omega\rangle^{2/3}, e, \zeta) \rightarrow$ smooth deviation from QC limit, but requires extra equations, namely, $\{\dot{e}, \dot{\zeta}, x(\Omega)\}$, where $r = p/(1 + e\cos\zeta)$.

- **Requirements and complications:**
  - The RR force should be consistent with the known PN expressions for the fluxes.
  - It should be accurate and robust all the way towards plunge-merger.
  - Factorizing a QC part (e.g., the QC expressions for $T_{\ell m}$, $f_{\ell m}$, or $\delta_{\ell m}$) introduces residual terms in the eccentricity corrections.

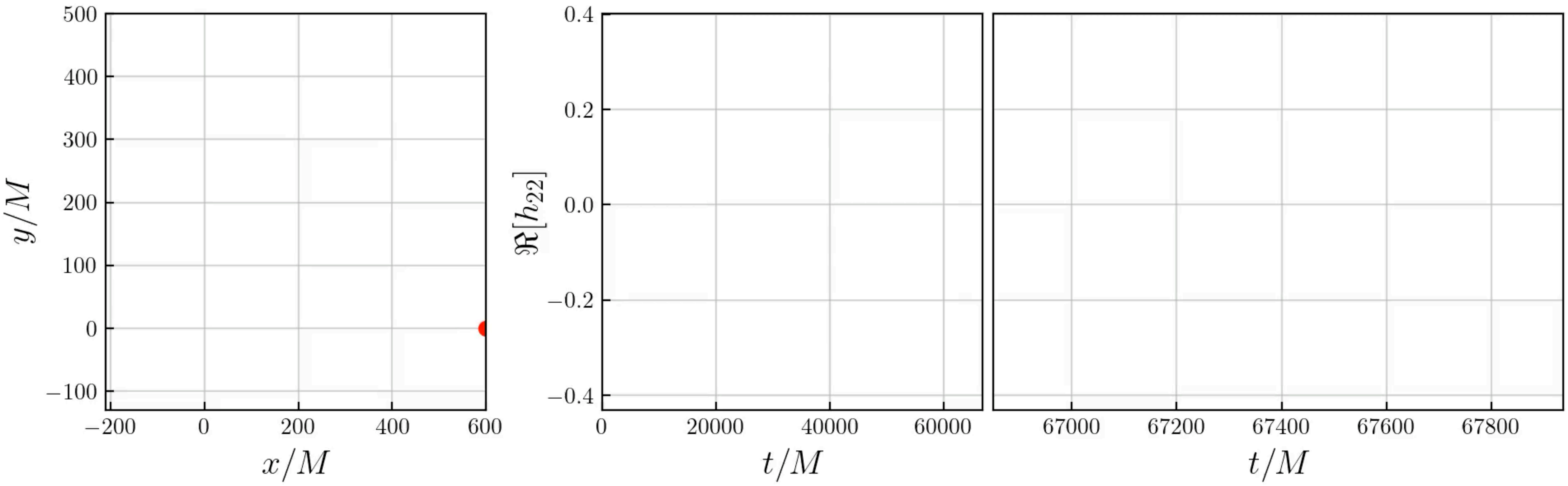
- **Extra:** A very similar situation appears for the waveform modes, $h_{\ell m} = h^{\mathrm{QC}}_{\ell m}\,h^{\mathrm{ecc}}_{\ell m}$.

# EOB in action

Despite all these complications, good prescriptions for the RR force can be obtained and plugged into the EOB equations of motion!

$$\dot{r} = \frac{\partial H_{\rm EOB}}{\partial p_r} \qquad \dot{p}_r = -\frac{\partial H_{\rm EOB}}{\partial r} + \mathscr{F}_r$$

$$\Omega = \frac{\partial H_{\rm EOB}}{\partial p_\phi} \qquad \dot{p}_\phi = -\frac{\partial H_{\rm EOB}}{\partial \phi} + \mathscr{F}_\phi$$



# How to compare eccentric waveform models?

- Comparing QC waveform models is facilitated by the gauge-invariant property of the angular frequency. This **simplicity is lost** for eccentric systems due to the **gauge-dependence of orbital eccentricity in general relativity**.

- **Gauge-dependence mixes with modeling systematics** (models are not built in terms of only gauge-invariant quantities).

- Data perspective: the only real observable is the waveform. **How to compare different eccentric waveforms?**

[Shaikh+, 2023]

- An eccentricity defined from the *dynamics* is *gauge-dependent*:

  e.g., $e_{\Omega_{\rm orb}} = \dfrac{\sqrt{\Omega^{\rm p}_{\rm orb}} - \sqrt{\Omega^{\rm a}_{\rm orb}}}{\sqrt{\Omega^{\rm p}_{\rm orb}} + \sqrt{\Omega^{\rm a}_{\rm orb}}}$ [Mora & Will, 2002].

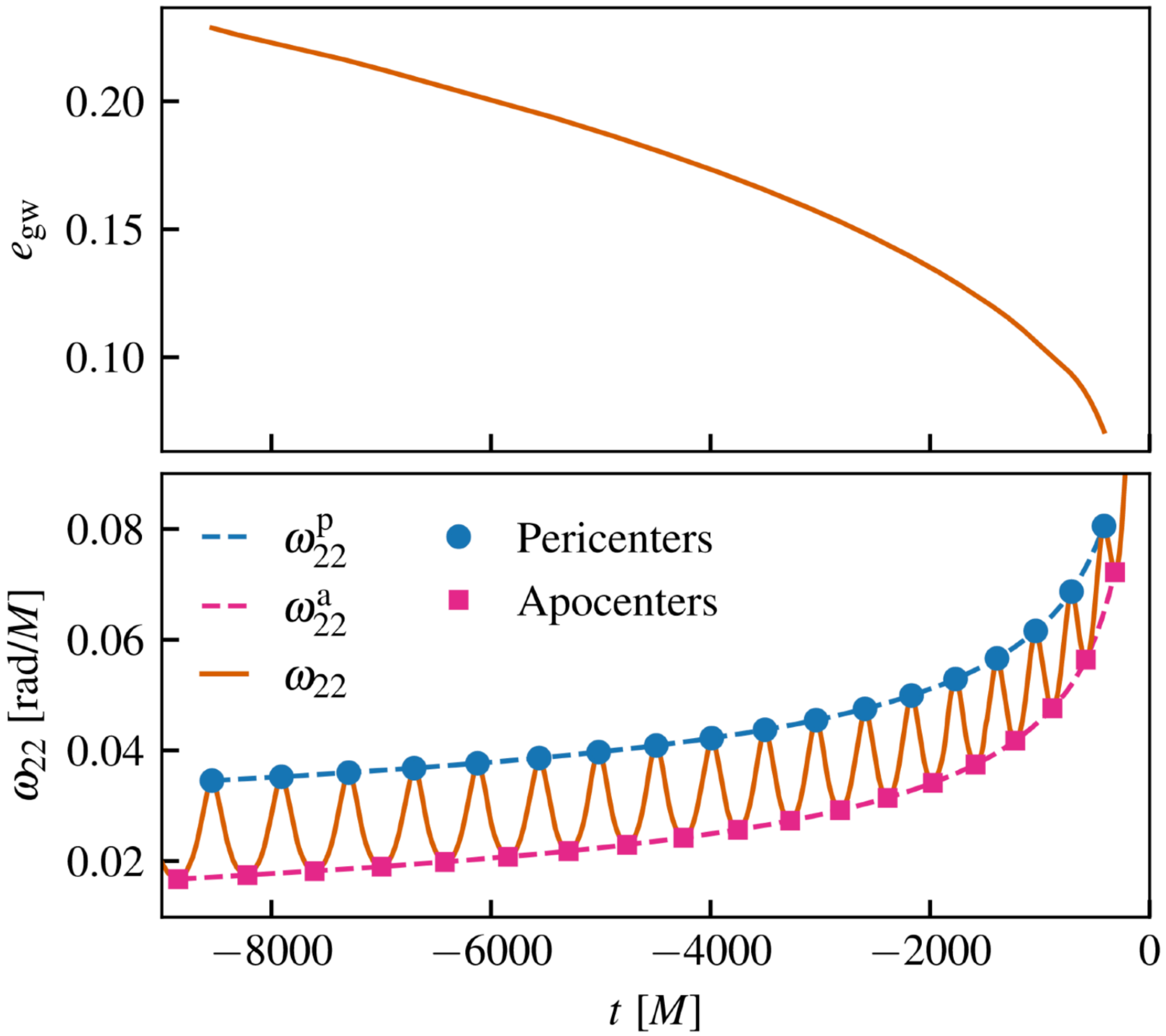
- An eccentricity defined from the *waveform* is *gauge-independent*:

  e.g., $e_{\omega_{22}} = \dfrac{\sqrt{\omega^{\rm p}_{22}} - \sqrt{\omega^{\rm a}_{22}}}{\sqrt{\omega^{\rm p}_{22}} + \sqrt{\omega^{\rm a}_{22}}}$ [Ramos-Buades+, 2020], but definitions like this one do not satisfy the correct Newtonian limit.

- [Ramos-Buades+, 2022] and [Shaikh+, 2023] have proposed to use

  $e_{\rm gw} = \cos(\Psi/3) - \sqrt{3}\sin(\Psi/3)$ where $\Psi = \arctan\left(\dfrac{1-e^2_{\omega_{22}}}{2e_{\omega_{22}}}\right)$

  which matches the correct Newtonian limit and approximately matches the geodesic eccentricity in the extreme mass ratio limit.



# How to compare eccentric waveform models?

- Given a NR eccentric waveform, what is the appropriate EOB waveform? **[state-of-the-art question!]**

- For example, [Bonino+, 2024] put forward a pipeline for measuring the eccentricity evolution of a given NR waveform and translate this to EOB initial conditions. Similarly, one can minimize the difference in the $e_{\rm gw}$ evolution [Chartier+, 2025]. However, these methods are very sensible to NR noise and the number of cycles.

- [Ramos-Buades+, 2022] developed a method based on **brute force optimization over the eccentric parameters** of an EOB model to get the **best-fitting EOB waveform to the NR data**:

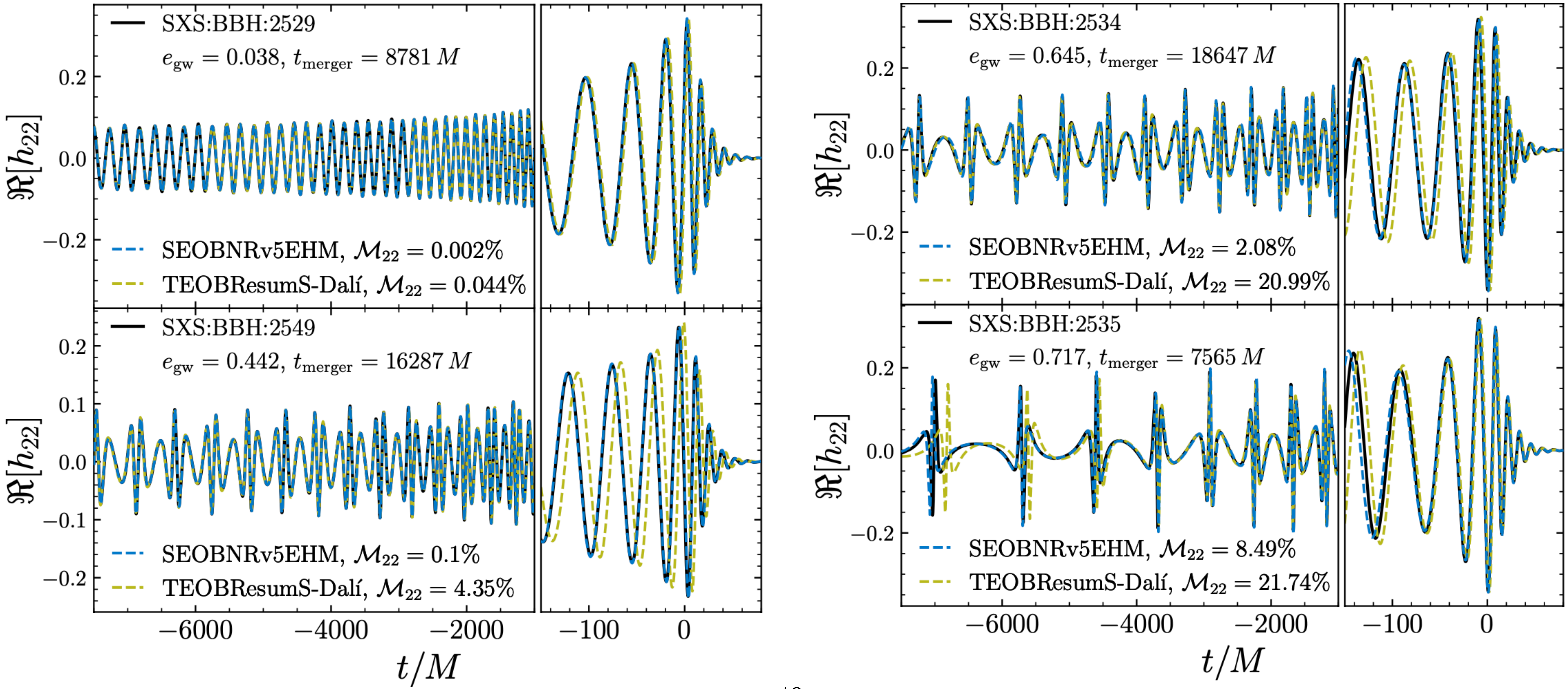
$$\mathcal{M}(M) = 1 - \max_{t_c, \varphi_t, e_t, \langle\Omega\rangle_t} \left[\langle \tilde{h}_{\rm NR} | \tilde{h}_{\rm EOB} \rangle\right] \quad \text{where} \quad \langle h_1 | h_2 \rangle \equiv 4\,\Re \int_{f_{\min}}^{f_{\max}} \frac{\tilde{h}_1(f)\,\tilde{h}_2^*(f)}{S_n(f)}\,df$$

**Note:** brute-force (and similar) techniques are needed due to the eccentric likelihood space complexity.

- Different studies have applied these optimizations **to assess the accuracy of eccentric waveform models against NR simulations** [e.g., Planas+, 2025; Paul+, 2024; AG+, 2025].

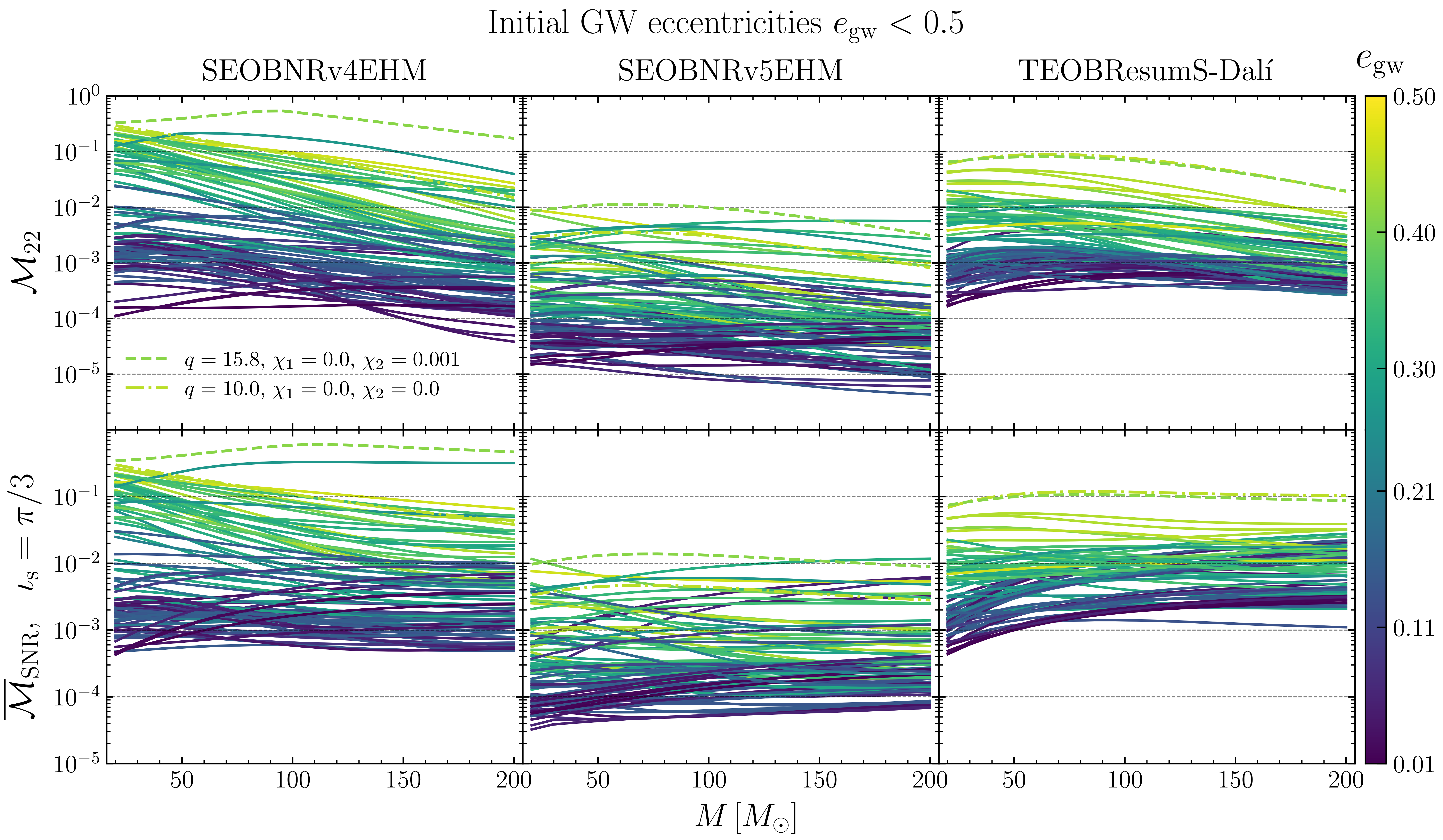
# Comparing models against eccentric NR simulations

- Particular examples: validation of the state-of-the-art **SEOBNRv5EHM** [AG+, 2025] and **TEOBResumS-Dalí** [e.g., Nagar+, 2024] waveform models.



# Comparing models against eccentric NR simulations

- **Mismatch** between **NR** and **EOB** waveforms for **99 eccentric** SXS NR **simulations** [Scheel+, 2025]
  - 84 with $e_{\rm gw} < 0.5$
  - 15 with $e_{\rm gw} > 0.5$
- **SEOBNRv5EHM** is about **one order of magnitude more accurate** than SEOBNRv4EHM and **TEOBResumS-Dalí**.




[AG+, 2025]

# Comparing models against eccentric NR simulations

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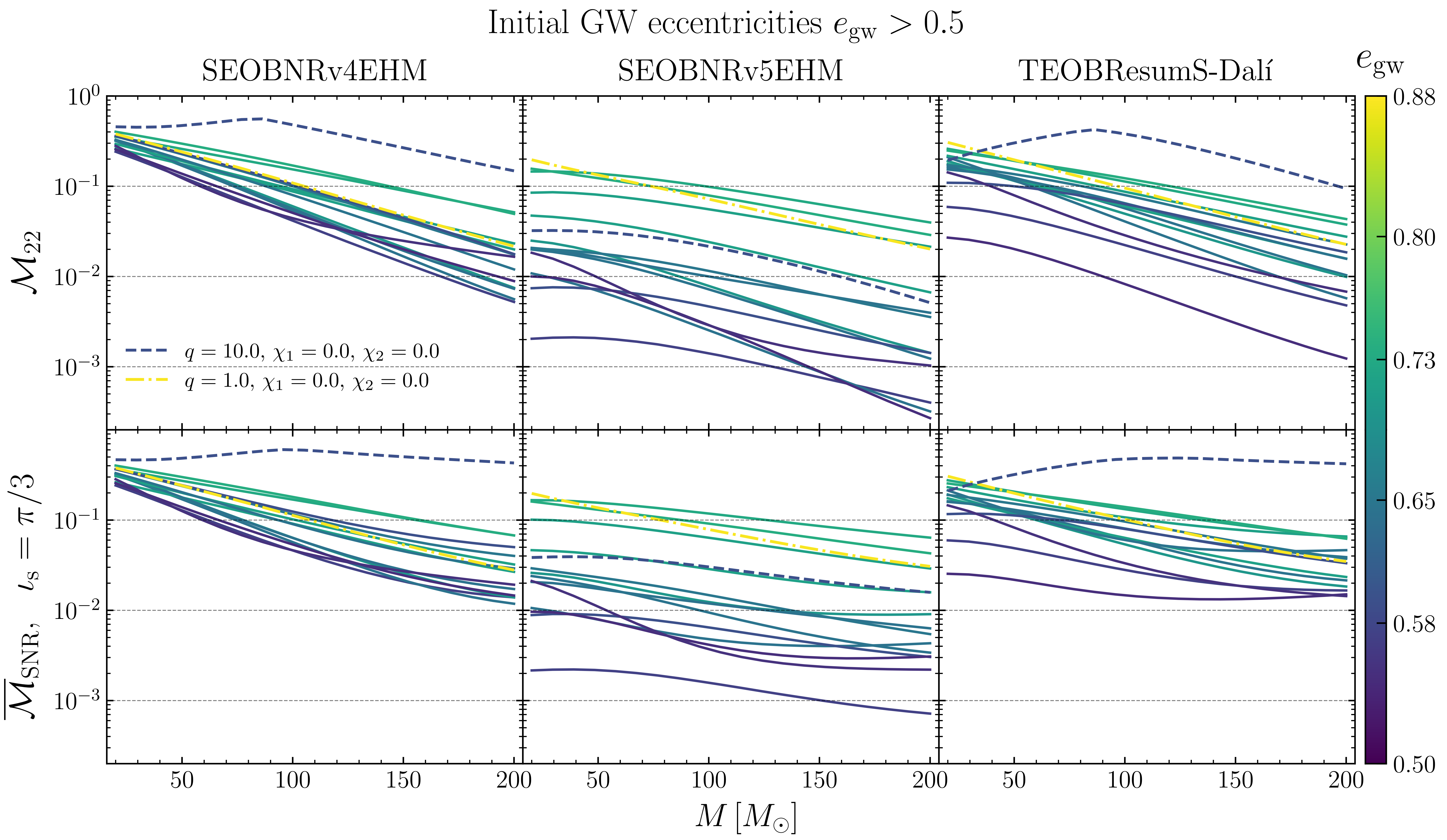



[AG+, 2025]

# Summary

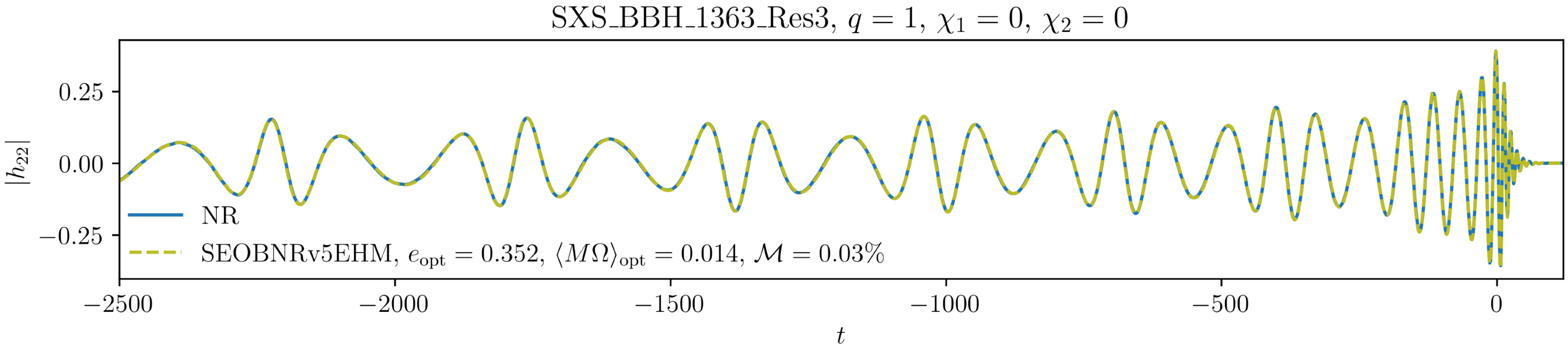
- The **flux-balance approach** can be employed to obtain a **prescription for the EOB RR force**.

- **Resummations are needed** to improve the **faithfulness** and **robustness** of the RR force, allowing for an accurate evolution of the binary *even in strong-field configurations*.

- **Comparing eccentric waveforms is not trivial** due to the gauge dependence of orbital eccentricity.

- **State-of-the-art eccentric waveform models still struggle with high eccentricities**.

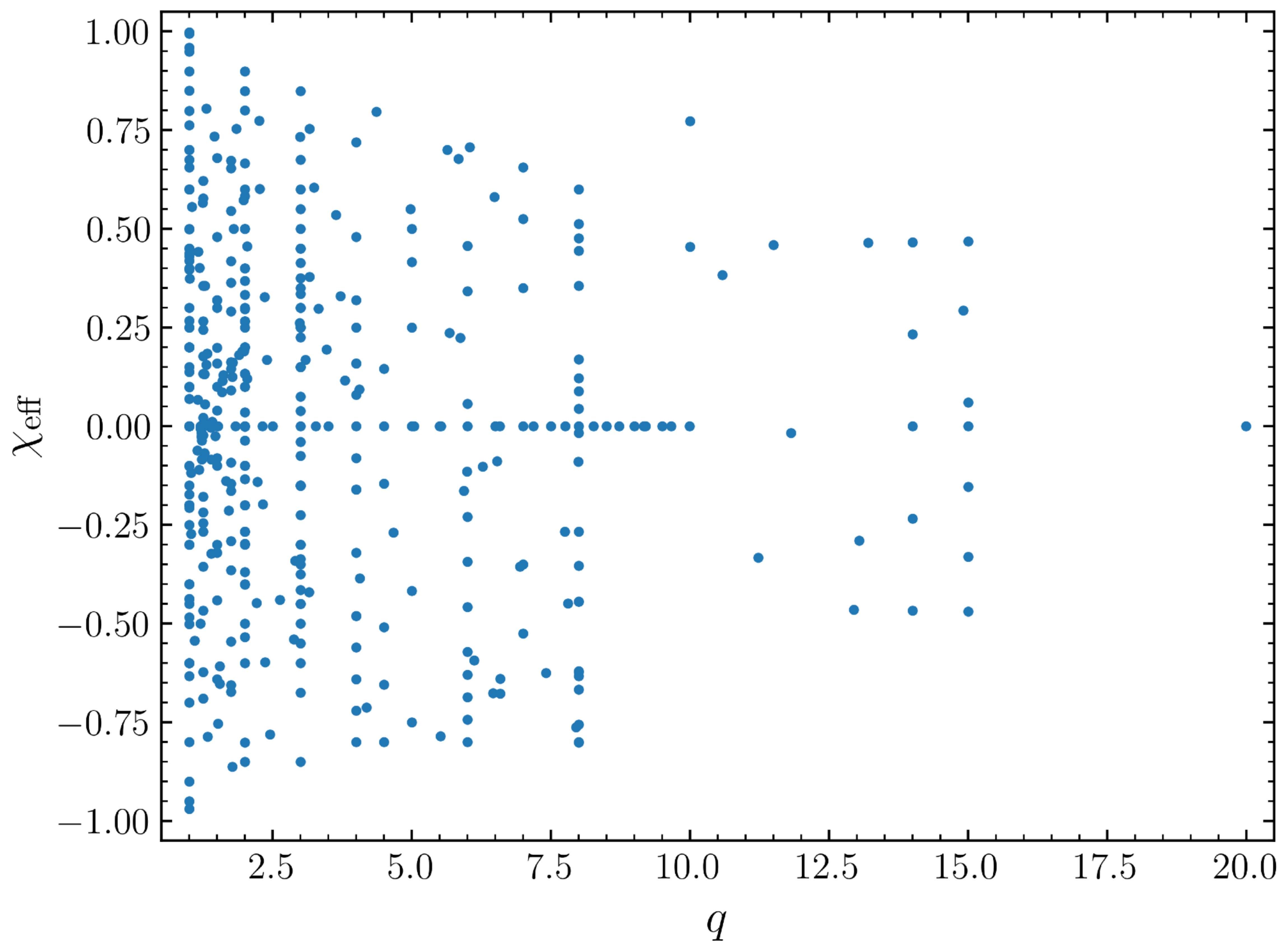
# Thanks for your attention!



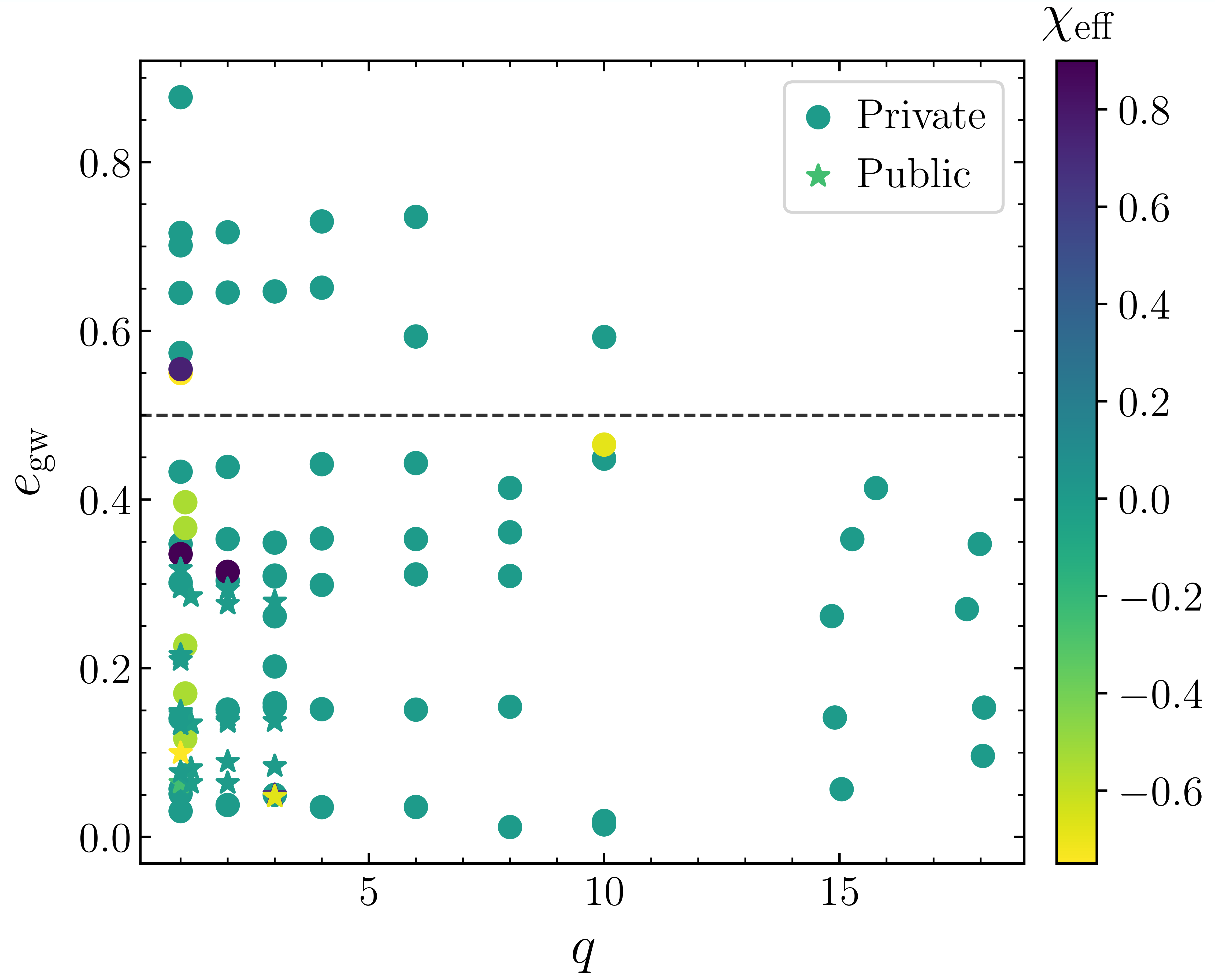
# Back-up slides

# Validation against Numerical Relativity waveforms

**441** public + private QC NR simulations from the SXS collaboration (see [Pompili+, PRD 108, 124035 (2023)] and references therein)




**99** eccentric NR simulations from the SXS collaboration [Hinder+, PRD 98, 044015 (2018), Ramos-Buades+, PRD 106, 124040 (2022)]



# SEOBNRv5EHM equations of motion

- We employ the eccentricity $e$ defined by the Keplerian parametrization:

$$r = \frac{p}{(1 + e\cos\zeta)} \qquad p = p(e,x) \qquad x = \langle M\Omega \rangle^{2/3}$$

where $p$ is the semi-latus rectum, $\zeta$ is the relativistic anomaly and $x$ is the orbit-averaged orbital frequency.

- Equations of motion for $(r, \phi, p_r, p_\phi, e, \zeta)$:

$$\dot{r} = \frac{\partial H_{\rm EOB}}{\partial p_r}, \qquad \Omega = \dot{\phi} = \frac{\partial H_{\rm EOB}}{\partial p_\phi}, \qquad \dot{p}_r = -\frac{\partial H_{\rm EOB}}{\partial r} + \mathscr{F}_r(x, e, \zeta), \qquad \dot{p}_\phi = \mathscr{F}_\phi(x, e, \zeta),$$

$$\dot{e} = \frac{\nu e x^4}{M}\left[-\frac{(121e^2 + 304)}{15\left(1 - e^2\right)^{5/2}} + \text{3PN terms}\right], \quad \dot{\zeta} = \frac{x^{3/2}}{M}\left[\frac{(1 + e\cos\zeta)^2}{\left(1 - e^2\right)^{3/2}} + \text{3PN terms}\right], \quad x = (M\Omega)^{2/3}\left[\frac{1 - e^2}{\left(1 + e\cos\zeta\right)^{4/3}} + \text{3PN terms}\right]$$

- RR force $\mathscr{F}$ and waveform modes $h_{\ell m}$ with eccentricity corrections up to 3PN order.

- This construction is ideal for recovering the underlying QC model in the $e \to 0$ limit.

# SEOBNRv5EHM radiation-reaction (RR) force and waveform modes

- Eccentric effects are included as multiplicative corrections to the modes and RR force:

$$\mathcal{F}_\phi = \mathcal{F}_{\text{modes}} \, \mathcal{F}_{\phi,\text{corr}}(x, e, \zeta) \qquad \mathcal{F}_r = \frac{p_r}{p_\phi} \mathcal{F}_{\text{modes}} \, \mathcal{F}_{r,\text{corr}}(x, e, \zeta)$$

$$\mathcal{F}_{\text{modes}} = -\frac{M\Omega}{8\pi} \sum_{\ell=2}^{8} \sum_{m=1}^{\ell} m^2 \left| d_L \, h^{\text{F}}_{\ell m} \right|^2$$

$$h^{\text{F}}_{\ell m} = h^{\text{N,qc}}_{\ell m} \, S_{\text{eff}} \, T^{\text{qc}}_{\ell m} \, f^{\text{qc}}_{\ell m} \, e^{i\delta^{\text{qc}}_{\ell m}} \, h^{\text{ecc,corr}}_{\ell m}(x, e, \zeta)$$

- This choice is ideal for recovering the underlying QC model in the $e \to 0$ limit:

$$\mathcal{F}^{\text{qc}}_{\text{modes}} = -\frac{\Phi_E}{\Omega} = -\Phi_J\,, \qquad \mathcal{F}_{\phi,\text{corr}} = 1, \qquad \mathcal{F}_{r,\text{corr}} = 1, \qquad \text{and} \qquad h^{\text{ecc,corr}}_{\ell m}(x, e, \zeta) = 1$$