MITP Lost in translation

JANUARY 2026

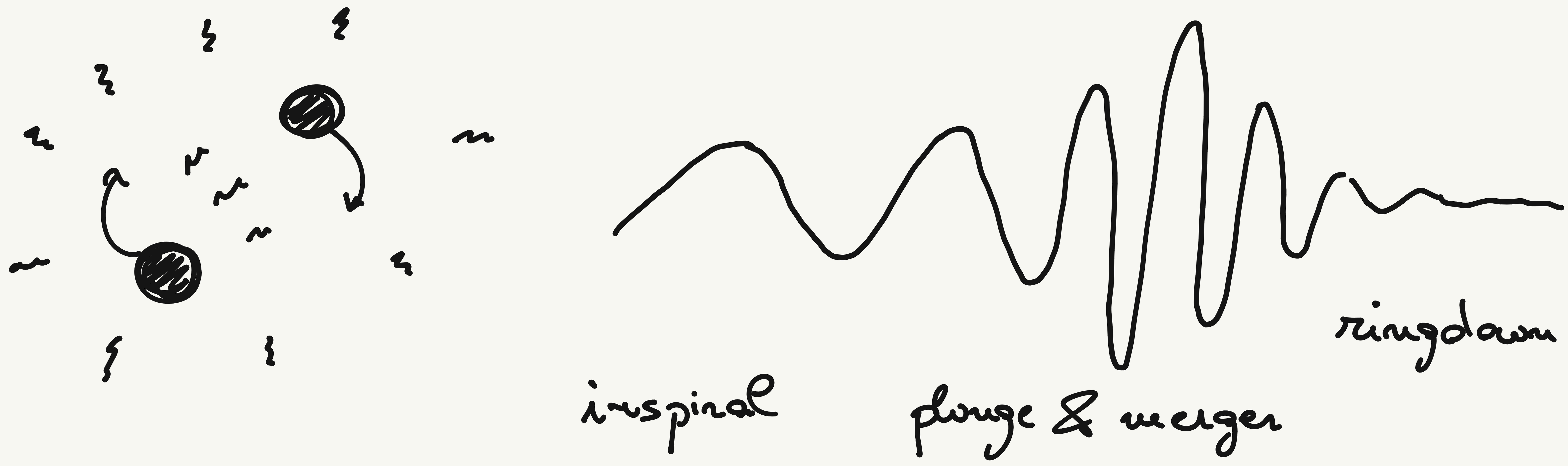
INTRODUCTION TO EOB

1) EOB mapping
   - Hamiltonians
   - Radiation reaction

2) Full GW models
   - Resummations
   - NR-calibration

GW basics

We want to estimate the GWs emitted by a compact binary coalescence.



→ solve Einstein's equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

BUT these are not solvable in the most generic case

analytical approximations (PN, PM, GSF) — numerical solutions (NR)

PN: small velocities $\left(\frac{v}{c}\right)^2 \ll 1$
weak fields $\left(\frac{GM}{Rc^2}\right) \ll 1$

PM: only weak fields

GSF: large mass ratios $q = \frac{m_1}{m_2} \gg 1$

Historically, people focused on PN results

→ good for inspiral
break down near merger

and quasi-circular orbits

→ easier & physically motivated

We will also consider a few simplifications

- BHs (treated as point particles)
- no spins
- circular orbits

From PN theory we get

Blanchet review (1310.1528)

$$- \quad E = -\frac{\mu c^2 x}{2}\left(1+\left(-\frac{3}{4}-\frac{\nu}{12}\right)x+\dots\right)$$

$$- \quad \mathcal{F}_E = \frac{32c^5}{5G}\nu^2 x^5\left(1+\left(-\frac{1247}{336}-\frac{35}{12}\nu\right)x+\dots\right)$$

$M = m_1 + m_2$

$\mu = \frac{m_1 m_2}{M}$

$\nu = \frac{\mu}{M}$

$x = \left(\frac{GM\Omega}{c^3}\right)^{2/3}$ expansion parameter

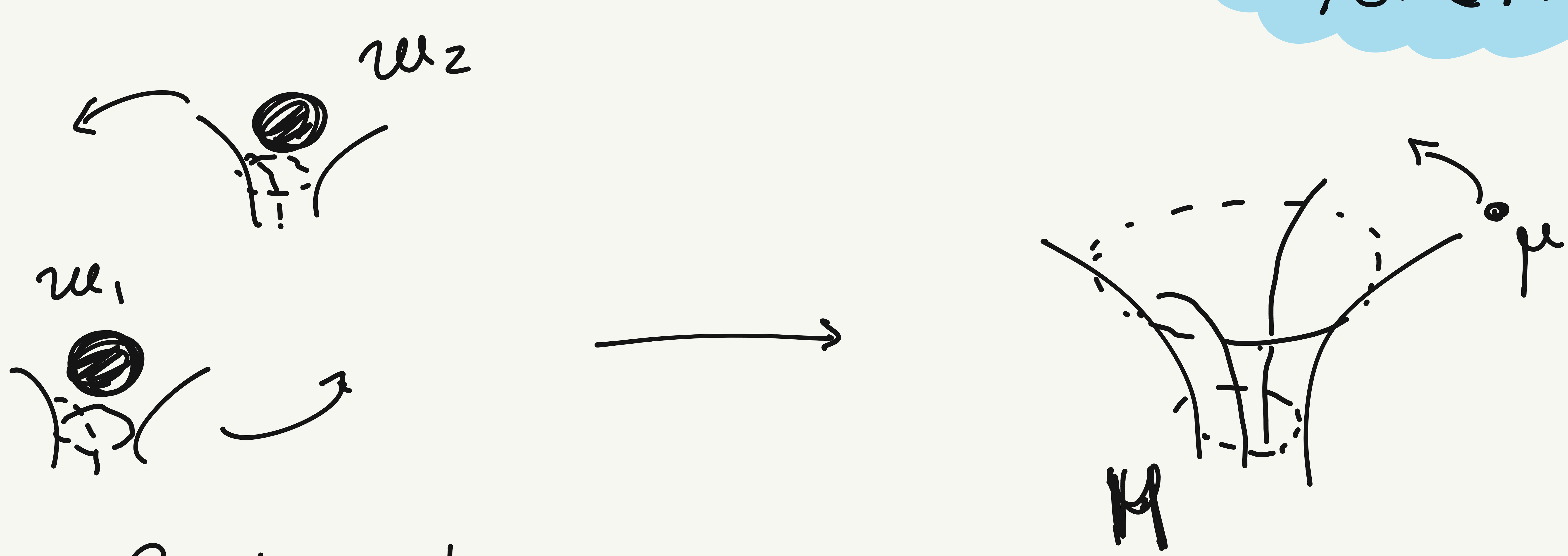
Ω orbital frequency

Note that for circular orbits

$$\left(\frac{v}{c}\right)^2 \ll 1 \iff \left(\frac{GM}{Rc^2}\right) \ll 1 \iff x \ll 1$$

EOB Hamiltonian

BD 9811091



real two-body system

effective one-body system

Similar to Newtonian mechanics

It does not solve the problem, but it improves PN performance

contains **exact** Schwarzschild solution

resums PN series

How to change coordinates?

gauge-inv. quantities

canonical transformation

The effective system is described by an effective metric

$$ds^2 = g_{\mu\nu}^{eff}\, dx^\mu dx^\nu = -A_{eff}\, dt^2 + B_{eff}\, dr^2 + r^2 d\Omega$$

$$\left(r = \frac{Rc^2}{GM} \right)$$

I want to impose a continuous Schwarzschild limit

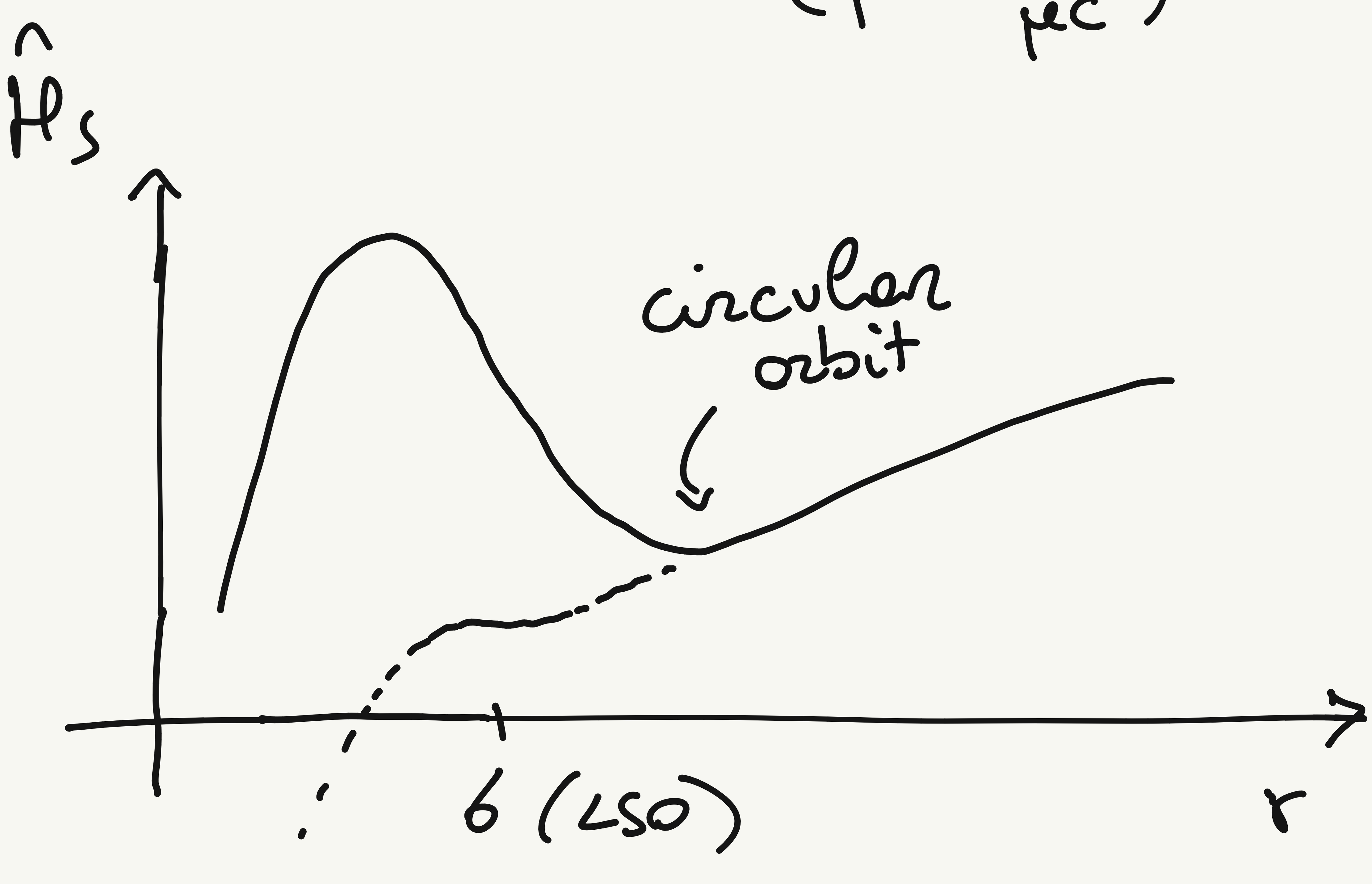
$$A_{eff} = 1 - \frac{2}{r} + O(\nu)$$
$$B_{eff} = 1 + \frac{2}{r} + O(\nu)$$

Remember how

$$\left(u = \frac{1}{r} = \frac{GM}{rc^2} \right)$$
$$\left(j = \frac{cJ}{GM\mu} \right)$$
$$\left(p = \frac{P}{\mu c} \right)$$

$$\hat{H}_S = \sqrt{A_S\left(1 + j^2u^2 + \frac{p_r^2}{B_S}\right)}$$

$$A_S = 1 - 2u$$
$$B_S = A_S^{-1}$$



Then we want

$$\hat{H}_{eff} = \sqrt{A_{eff}\left(1 + j^2u^2 + \frac{p_r^2}{B_{eff}}\right)}$$

$$A_{eff} = 1 - 2u + O(u^2)$$

$$B_{eff} = 1 + 2u + O(u^2)$$

The main issue is that

$$E_{real} \neq E_{eff}$$

In order to find the EOB map Buonanno and Damour used action / angle (Delauney) variables

$$J, \qquad I_r = \frac{1}{2\pi} \oint dr\, p_r$$

1) From PN computations we get the center-of-mass Hamiltonian

$$\hat{H}^{NR}(r,p) = \frac{1}{2}p^2 - \frac{1}{r} + \frac{1}{c^2}\left(-\frac{1}{8}(1-3\nu)p^4 - \frac{1}{2r}\left[(3+\nu)p^2 + \nu p_r\right] + \frac{1}{2r^2}\right) + O\left(\frac{1}{c^4}\right)$$

$$p^2 = p_r^2 + j^2 u^2$$

$$E_{real} = \mathcal{E}^{NR} + Mc^2$$

$$\hat{\mathcal{E}}^{NR} = \frac{\mathcal{E}^{NR}}{\mu} \longleftrightarrow \hat{H}^{NR}(r,p)$$

We can invert the relation to get p_r and then compute I_r

$$H(r, p_r, j) \longrightarrow p_r(r, H, j) \longrightarrow I_r(H, j)$$

At this point I can write the energy as a function of gauge-invariant quantities

Introducing $N = I_r + j$

$$E_{real}(N, j) = Mc^2 - \frac{1}{2}\frac{\mu\alpha^2}{N^2}\left[1 + \frac{\alpha^2}{c^2}\left(\frac{6}{Nj} - \frac{1}{4}\frac{15-\nu}{N^2}\right) + O\left(\frac{1}{c^4}\right)\right]$$

$$(\alpha = G m_1 m_2)$$

2) We repeat the same steps in the effective system

We parametrize the effective metric potentials as

$$A(u) = 1 - 2u + a_2 u^2 + a_3 u^3 + \ldots$$
$$B(u) = 1 + 2u + b_2 u^2 + \ldots$$

(u and p are different coordinates I should write u_{EOB} and u_{ADM})

From the EOB mass-shell condition

$$g_{eff}^{\mu\nu} p_\mu p_\nu + 1 = 0 \qquad \left(g^{\mu\nu} p_\mu p_\nu = \mu^2 c^2 \right)$$

$$-\frac{\mathcal{E}_{eff}^2}{A} + \frac{p_r^2}{B} + j^2 u^2 + 1 = 0 \qquad \left(\mathcal{E}_{eff} = \mu c^2 + \mathcal{E}_{eff}^{NR} \right)$$

$$\longrightarrow \; p_r(r, \mathcal{E}_{eff}, J) \longrightarrow I_r(\mathcal{E}_{eff}, J_{eff})$$

$$\rightsquigarrow \mathcal{E}_{eff}(N_{eff}, J_{eff}) = \mu c^2 - \frac{1}{2}\frac{\alpha^2}{N_{eff}^2}\left[1 + \frac{\alpha^2}{c^2}\left(\frac{6-a_2}{N_{eff} J_{eff}} - \frac{15}{4 N_{eff}^2}\right) + O\left(\frac{1}{c^4}\right)\right]$$

and again $\alpha = G M \mu$

3) Compare the two systems

We cannot equate $\mathcal{E}_{eff}$ to $\mathcal{E}_{real}$

We need

$$\mathcal{E}_{eff}(N_{eff}, J_{eff}) = f\left[\mathcal{E}_{real}(N, J)\right]$$

(Inspired by electromagnetic computations BD thought of N and J as quantized and equated the energy levels)

Requiring the same energy spectrum means imposing $\quad N = N_{eff}$ and $J = J_{eff}$

They wrote the energy map as a PN expansion:

$$\mathcal{E}_{eff}^{NR} = \mathcal{E}_{real}^{NR}\left(1 + \alpha_1\left(\frac{\mathcal{E}_{real}^{NR}}{\mu c^2}\right) + \alpha_2\left(\frac{\mathcal{E}_{real}^{NR}}{\mu c^2}\right)^2 + \dots\right)$$

Solving order by order they obtained:

1) $\alpha_1 = \frac{\nu}{2}$ $\rightsquigarrow$ $\frac{\mathcal{E}_{eff}}{\mu c^2} = \frac{\mathcal{E}_{real}^2 - m_1^2 c^4 - m_2^2 c^4}{2 m_1 m_2 c^4}$

2) $\alpha_2 = 0$ (and later $\alpha_3 = \alpha_4 = 0$)

EOB map was conjectured to be quadratic *

$$\frac{\mathcal{E}_{real}}{M} = \sqrt{1 + 2\nu\left(\frac{\mathcal{E}_{eff}}{\mu} - 1\right)}$$

3) $a_2 = 0$

1PN corrections vanish

This will continue at higher orders, with EOB potentials being far simpler than PN quantities.

This suggests the "real" answer probably has a similar structure to Schwarzschild

Damour
1609.00354

* Damour demonstrated the quadratic map using PM results

This time equating scattering angles

$$\vartheta = \pi + \chi = -\int_{-\infty}^{+\infty} dr \frac{\partial p_r}{\partial J}$$

$$\chi_{eff}(\mathcal{E}_{eff}, J) = \chi_{real}(\mathcal{E}_{real}, J)$$

$$\mathcal{E}_{eff} = f(\mathcal{E}_{real})$$

$\longrightarrow$ again $\dfrac{\mathcal{E}_{eff}}{\mu} = \dfrac{(\mathcal{E}_{real})^2 - m_1^2 - m_2^2}{2 m_1 m_2}$

<u>exact mapping</u>

Also, again 1PM terms disappear as they are contained in Schwarzschild

(at higher orders, less evident)

EOB radiation reaction

BD 0001013
Bini-D 1210.2834

$$\begin{cases} \dfrac{dr}{dt} = \dfrac{\partial H}{\partial p_r} \\ \dfrac{d\varphi}{dt} = \dfrac{\partial H}{\partial p_\varphi} \\ \dfrac{dp_r}{dt} = -\dfrac{\partial H}{\partial r} + \mathcal{F}_r \\ \dfrac{dp_\varphi}{dt} = \mathcal{F}_\varphi \end{cases}$$

radiation reaction forces

$$\dot{E} = \dot{r}\mathcal{F}_r + \dot{\varphi}\mathcal{F}_\varphi$$

Here we don't need an energy map

Φ_J and Φ_E are gauge-invariant

BUT we need to take into account so-called Schott terms

$$\begin{cases} \dot{E} + \dot{E}_{(\text{schott})} + \Phi_E = 0 \\ \dot{J} + \dot{J}_{(\text{schott})} + \Phi_J = 0 \end{cases}$$

For circular orbits, the computations are easy. The first EOB models choose

$$\mathcal{F}_\varphi = -\Phi_J \qquad \mathcal{F}_r = 0$$

Factorization and resummation

Why are EOB models more accurate than PN ones?

PN results are polynomial series in the expansion parameter.

They are asymptotic series and are unreliable when x is no longer $\ll 1$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

The EOB map imposes a particular resummation of PN results which has proven fruitful

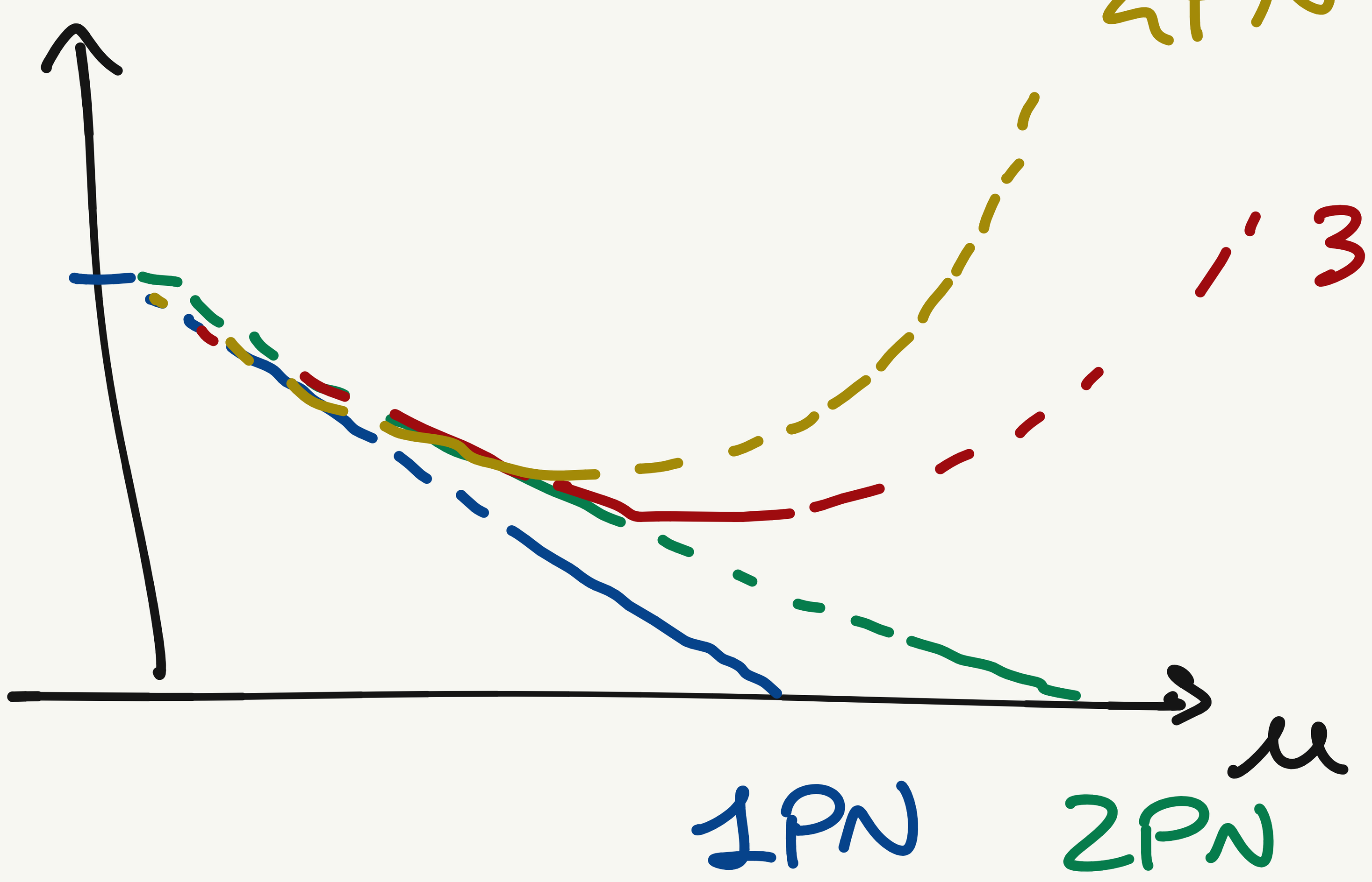
$$\hat{H}^{PN} = \frac{1}{2}p^2 - \frac{1}{r} + \dots \longrightarrow \hat{H}_{EOB} = \sqrt{1 + 2\nu(\hat{H}_{eff} - 1)}$$

$$\hat{H}_{eff} = \sqrt{(1 - 2u + \dots) \times \times (1 + j^2 u^2 + \dots)}$$

The symplicity of the EOB metric potentials is a "hint"

$$A = 1 - 2u + 2\nu u^3 + \nu\left(\frac{94}{3} - \frac{41}{32}\pi^2\right)u^4 + \ldots$$

On top of the inherent EOB resumm. we can impose other



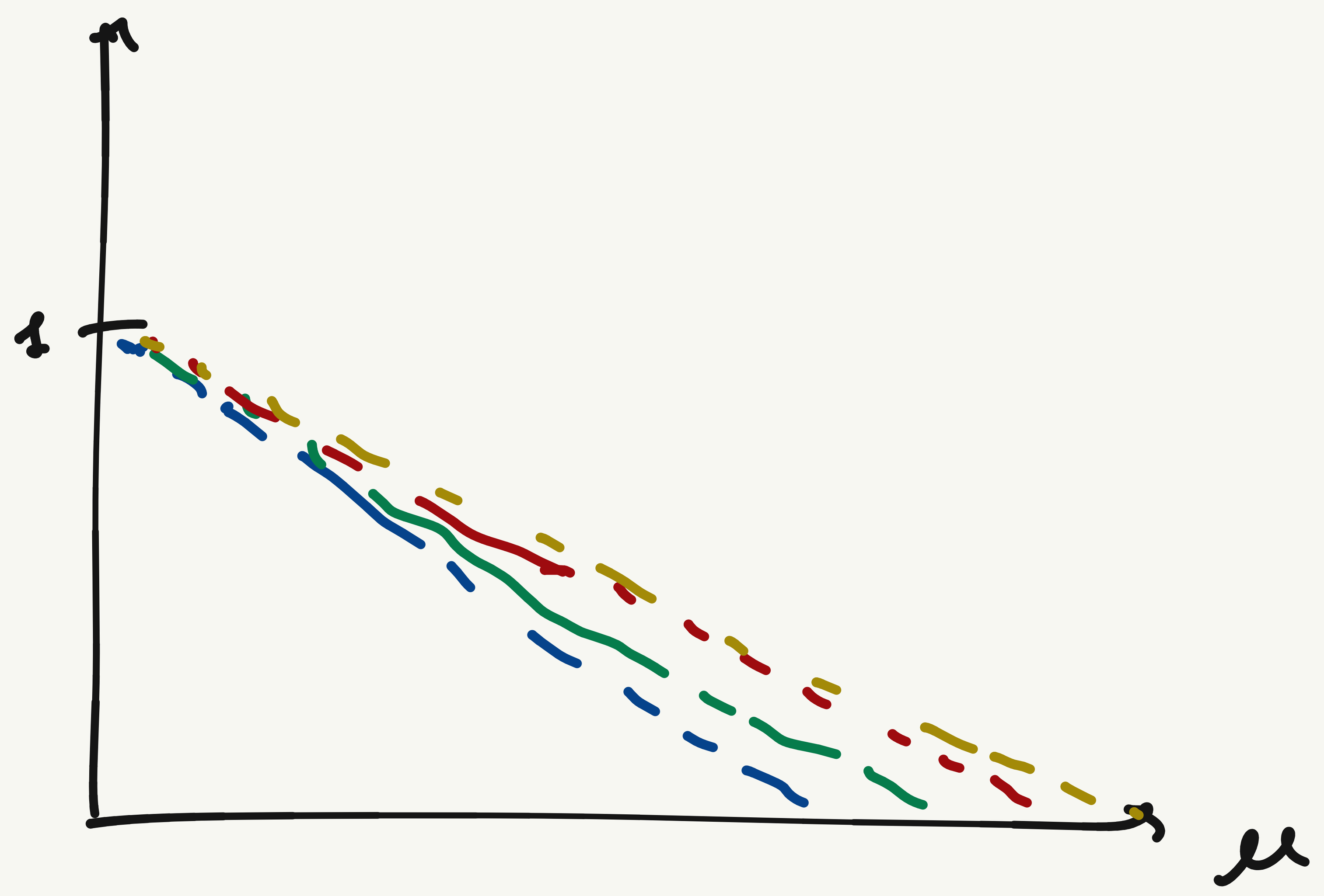
region of validity $x \ll 1$

behaviour completely determined by functional form

Let us stretch the region of validity of PN up to $x \to \infty$ (in reality $x \sim 0.3$)

$\longrightarrow$ Taylor of order $n \sim x^n$

We can tame this behaviour by taking rational functions



$$P^p_n = \frac{1 + \ldots + x^p}{1 + \ldots + x^n}$$

These choices are arbitrary, but they affect the result when dealing with perturbative expansions

(+ choice of coordinates, ...)

How to choose?

- physical intuition
- avoid pathological behaviour
- compare to exact (numerical) results

In general, you should:

- impose what you know
- guess what you don't

An example is the treatment of the radiation-reaction fluxes

$$\mathcal{F}_a = -\Phi_J = -\frac{1}{8\pi}\sum_{\ell m} m^2\,\Omega\,|R\,h_{\ell m}|^2$$

distance from the source

~ waveform

The waveform is factorized as

$$h_{\ell m} = h^{N}_{\ell m} \, h^{\text{tail}}_{\ell m} \, \hat{S}_{\text{eff}} \, \rho^{\ell}_{\ell m}$$

DIN 064004

Newtonian prefactor

tail contribution

source term

Imposing known behaviours, extracted from physical insights, vastly improves NR agreement

PN results + EOB mapping + resummations

already agrees with NR almost up to merger

One could attach a ringdown obtained from BHPT and have a fully analytical model

$\longrightarrow$ not good enough for LVK

NR information

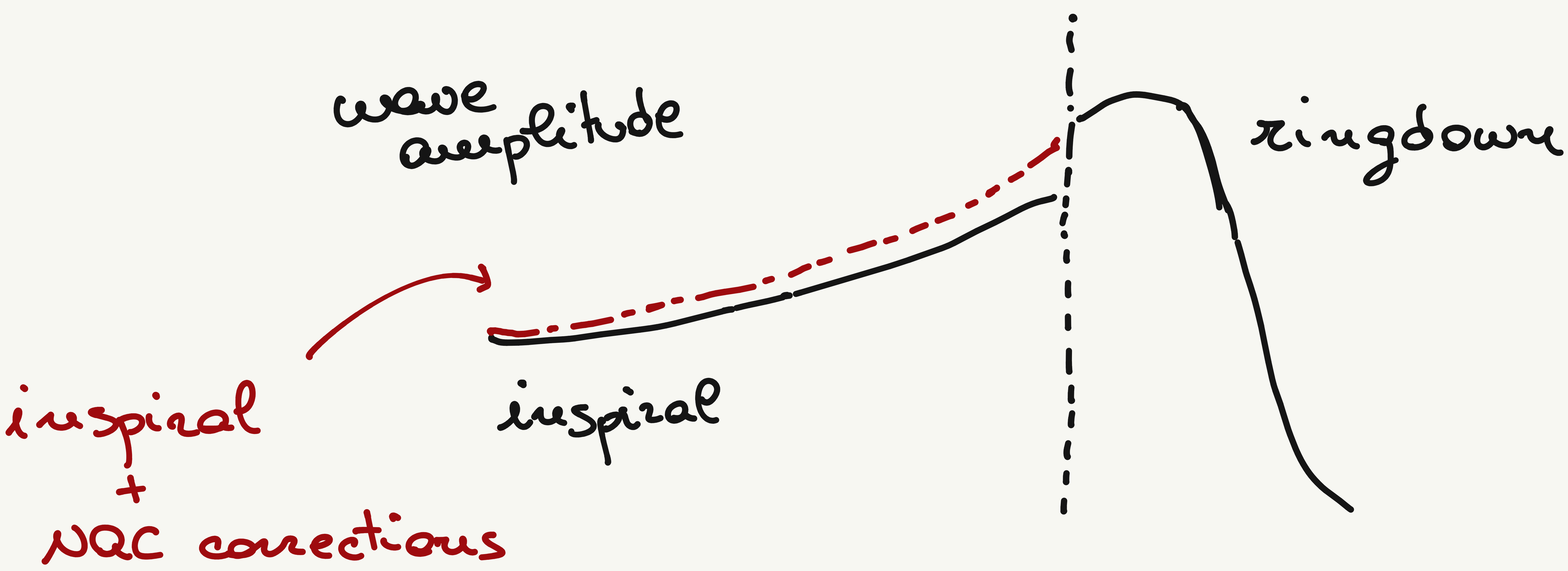
NR simulations inform models in many ways:

1) guide for resumm. choices
2) effective parameters

$$A = 1 - 2u + \dots + a_6^{NR} u^6$$

3) NR-informed ringdown
4) NQC parameters that impose a smooth transition between inspiral and ringdown

$$h_{lm}^{NR} = h_{\ell m} \times h_{\ell m}^{NQC}$$



EOB models are much more complicated than what I showed

1) Physical effects:

- tides
- spins and spin precession
- eccentricity
- beyond - GR terms

...

2) Not only PN

$$\downarrow$$

PN + PM + GSF

FIN