Post-Minkowskian Scattering of Black Holes

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Lost in Translation: The languages of Gravitational Waves

PM black hole scattering

- Compute black hole scattering observables analytically at high precision (in the weak field post-Minkowskian expansion)
- Use computational methods from collider physics and QFT
- Model black holes as point-like particles using worldlines

Many QFT-inspired frameworks: WQFT, PM-EFT, KMOC, HEFT, Eikonal, …

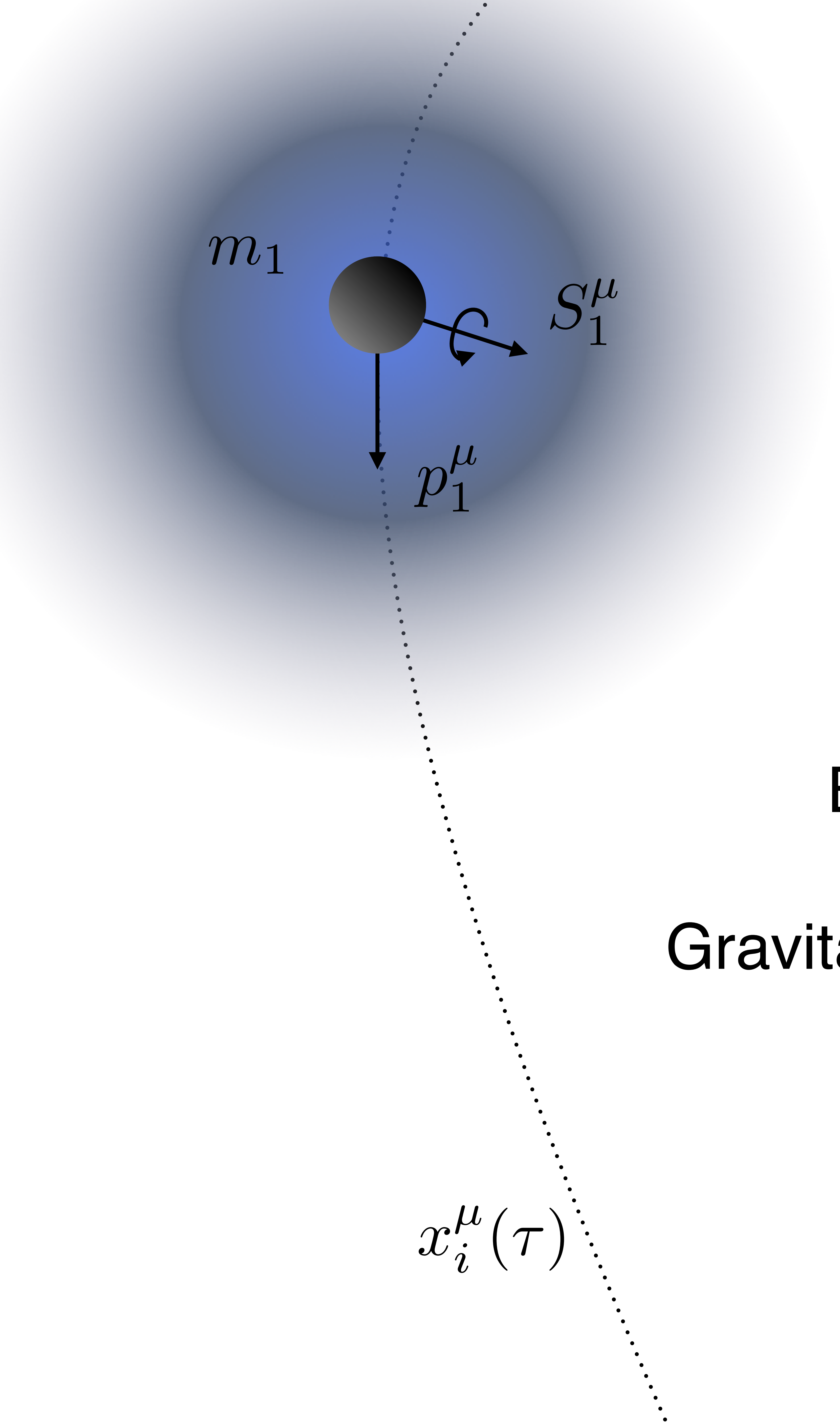
Part I:

- Set up equations of motion for PM black hole scattering with worldlines
- Introduce QFT language and diagrammatic representations

Part II:

- Compute impulse and total radiation of four momentum at 3PM order
- QFT methods for simplifying integrands and evaluating integrals


$m_1(-\infty)$
$S_1^\mu(-\infty)$
$p_1^\mu(-\infty)$
b
Before
$x_1^\mu(\tau)$
Interactions
After
+ radiation
$S_1^\mu(\infty)$
$m_1(\infty)$
$p_1^\mu(\infty)$



$$(Gm_i)^2 R[x_i(\tau)] \ll 1$$

$$\frac{Gm_i}{|\boldsymbol{x}_i|} \ll 1$$

Effective field theory: $S_i = -m_i \int \mathrm{d}\tau \sqrt{g_{\mu\nu}\dot{x}_i^\mu \dot{x}_i^\nu} + \ldots$

Black holes: $x_i^\mu(\tau) = b_i^\mu + v_i^\mu \tau + z_i^\mu(\tau)$ (+ spin)

Gravitational field: $g_{\mu\nu}(x) = \eta_{\mu\nu} + \sqrt{G}\, h_{\mu\nu}(x)$

General relativity: $S_{\mathrm{EH}} = -\frac{1}{16\pi G}\int \mathrm{d}^4x \sqrt{-g} R$

Why do point particles describe black holes?

Geodesic equation: $$\frac{\mathrm{d}^2 x_i^\mu(\tau)}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}[x_i^\mu(\tau)]\frac{\mathrm{d}x_i^\alpha(\tau)}{\mathrm{d}\tau}\frac{\mathrm{d}x_i^\beta(\tau)}{\mathrm{d}\tau} = 0$$

Einstein field equations: $$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = 8\pi G\left(T_1^{\mu\nu} + T_2^{\mu\nu}\right)$$

Point-particle energy-momentum: $$T_i^{\mu\nu}(x) = \frac{m_i}{\sqrt{-g}}\int_{-\infty}^{\infty}\mathrm{d}\tau\,\delta^d\big(x - x_i(\tau)\big)\dot{x}_i^\mu(\tau)\dot{x}_i^\nu(\tau)$$

Black holes as point particles is inconsistent without regulator!

Regulators are fundamental ingredients of EFT.

Dimensional regularization: mysteriously simple

What happens to self-interactions and self-force?

Post-Minkowskian expansion:

$$x_i^\mu(\tau) = b_i^\mu + v_i^\mu \tau + z_i^\mu(\tau)$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sqrt{G}\, h_{\mu\nu}(x)$$

$$\frac{\delta^{n+m} X}{\delta x^n \delta g^m}$$

$$X \in \{\Gamma^\mu_{\alpha\beta}\,,\, T^{\mu\nu}\,,\, R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu}\}$$

Interaction vertices

$$\frac{\delta^{n+m} S}{\delta x^n \delta g^m}$$

Equations of motion:

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = 8\pi G \left(T_1^{\mu\nu} + T_2^{\mu\nu}\right)$$

$$\frac{\mathrm{d}^2 x_i^\mu(\tau)}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}[x_i^\mu(\tau)] \frac{\mathrm{d}x_i^\alpha(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}x_i^\beta(\tau)}{\mathrm{d}\tau} = 0$$

Post-Minkowskian expansion:

$$x_i^\mu(\tau) = b_i^\mu + v_i^\mu \tau + z_i^\mu(\tau)$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sqrt{G}\, h_{\mu\nu}(x)$$

$$\partial_\mu \partial^\mu h = \sqrt{G} \sum_i \left[T_i \Big|_0 + \frac{\delta T_i}{\delta z_i}\bigg|_0 z_i + \frac{1}{2} \frac{\delta^2 T_i}{\delta z_i^2}\bigg|_0 z_i^2 + \dots \right]$$

$$+ \frac{\sqrt{G}}{2} \frac{\delta^2 \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}\right)}{\delta h^2}\bigg|_0 h^2 + \frac{G}{6} \frac{\delta^3 \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}\right)}{\delta h^3}\bigg|_0 h^3 + \dots$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^2 z_i = \sqrt{G} \left(\frac{\delta \Gamma \dot{x}_i \dot{x}_i}{\delta h}\bigg|_0 h + \frac{\delta^2 \Gamma \dot{x}_i \dot{x}_i}{\delta h \delta z_i}\bigg|_0 hz + \frac{1}{2} \frac{\delta^3 \Gamma \dot{x}_i \dot{x}_i}{\delta h \delta z_i^2}\bigg|_0 hz^2 + \dots \right)$$

Post-Minkowskian expansion:

$$x_i^\mu(\tau) = b_i^\mu + v_i^\mu \tau + z_i^\mu(\tau)$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sqrt{G}\, h_{\mu\nu}(x)$$

[shaded box] = interaction vertices

$$\partial_\mu \partial^\mu h = \sqrt{G} \sum_i \left[T_i \Big|_0 + \frac{\delta T_i}{\delta z_i}\Big|_0 z_i + \frac{1}{2}\frac{\delta^2 T_i}{\delta z_i^2}\Big|_0 z_i^2 + \dots \right] + \frac{\sqrt{G}}{2} \frac{\delta^2 \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}\right)}{\delta h^2}\Big|_0 h^2 + \frac{G}{6}\frac{\delta^3 \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}\right)}{\delta h^3}\Big|_0 h^3 + \dots$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^2 z_i = \sqrt{G} \left(\frac{\delta \Gamma \dot{x}_i \dot{x}_i}{\delta h}\Big|_0 h + \frac{\delta^2 \Gamma \dot{x}_i \dot{x}_i}{\delta h \delta z_i}\Big|_0 hz + \frac{1}{2}\frac{\delta^3 \Gamma \dot{x}_i \dot{x}_i}{\delta h \delta z_i^2}\Big|_0 hz^2 + \dots \right)$$

Post-Minkowskian expansion:

$$x_i^\mu(\tau) = b_i^\mu + v_i^\mu \tau + z_i^\mu(\tau)$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sqrt{G}\, h_{\mu\nu}(x)$$

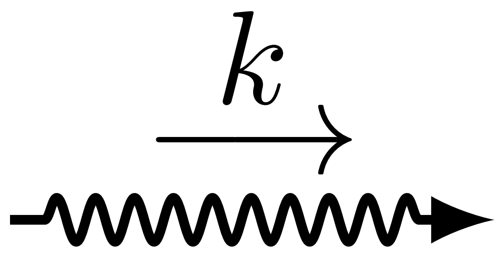
= interaction vertices

= (inverse) propagators

$$\partial_\mu \partial^\mu h = \sqrt{G} \sum_i \left[T_i\Big|_0 + \frac{\delta T_i}{\delta z_i}\Big|_0 z_i + \frac{1}{2}\frac{\delta^2 T_i}{\delta z_i^2}\Big|_0 z_i^2 + \dots \right]$$

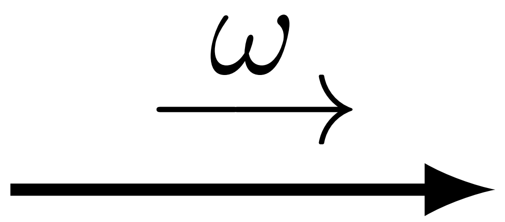
$$+ \frac{\sqrt{G}}{2} \frac{\delta^2\left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}\right)}{\delta h^2}\Big|_0 h^2 + \frac{G}{6}\frac{\delta^3\left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}\right)}{\delta h^3}\Big|_0 h^3 + \dots$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^2 z_i = \sqrt{G}\left(\frac{\delta \Gamma \dot{x}_i \dot{x}_i}{\delta h}\Big|_0 h + \frac{\delta^2 \Gamma \dot{x}_i \dot{x}_i}{\delta h \delta z_i}\Big|_0 hz + \frac{1}{2}\frac{\delta^3 \Gamma \dot{x}_i \dot{x}_i}{\delta h \delta z_i^2}\Big|_0 hz^2 + \dots \right)$$

$\mu\nu$ 

$\alpha\beta$

$$= i\left[\eta_{\mu(\alpha}\eta_{\beta)\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\right]\frac{1}{(k_0 + i\epsilon)^2 - \boldsymbol{k}^2} \qquad = \mathrm{F.T.}\left[(\partial_\mu\partial^\mu)^{-1}\right]$$

σ 

ρ

$$= -i\frac{\eta^{\sigma\rho}}{m_i}\frac{1}{(\omega + i\epsilon)^2} \qquad = \mathrm{F.T.}\left[\left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\right)^{-1}\right]$$

$$\partial_\mu\partial^\mu h = \sqrt{G}\sum_i \left[T_i\Big|_0 + \frac{\delta T_i}{\delta z_i}\Big|_0 z_i + \frac{1}{2}\frac{\delta^2 T_i}{\delta z_i^2}\Big|_0 z_i^2 + \ldots\right]$$

$$+ \frac{\sqrt{G}}{2}\frac{\delta^2\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)}{\delta h^2}\Big|_0 h^2 + \frac{G}{6}\frac{\delta^3\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)}{\delta h^3}\Big|_0 h^3 + \ldots$$

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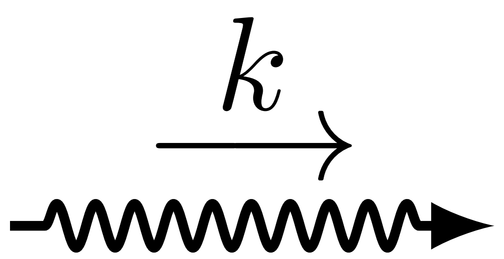
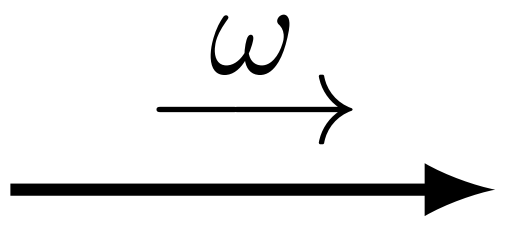
$\mu\nu$  $\alpha\beta$ $= i\left[\eta_{\mu(\alpha}\eta_{\beta)\nu} - \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\right]\frac{1}{(k_0 + i\epsilon)^2 - \boldsymbol{k}^2}$ $= \text{F.T.}\left[(\partial_\mu\partial^\mu)^{-1}\right]$

$\theta(t - t')\delta\big((x - x')^2\big)$

σ  ρ $= -i\frac{\eta^{\sigma\rho}}{m_i}\frac{1}{(\omega + i\epsilon)^2}$ $= \text{F.T.}\left[\left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\right)^{-1}\right]$

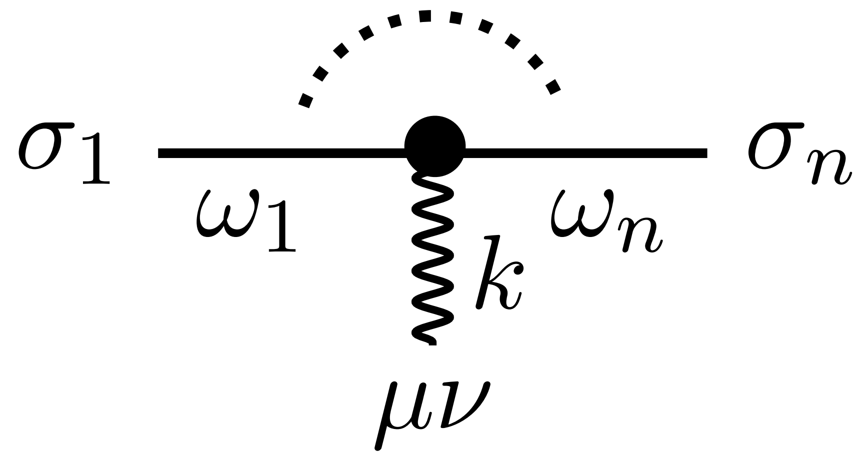
$\int_{-\infty}^{\tau} \mathrm{d}\tau' \int_{-\infty}^{\tau'} \mathrm{d}\tau''$

Causal (retarded) boundary conditions!

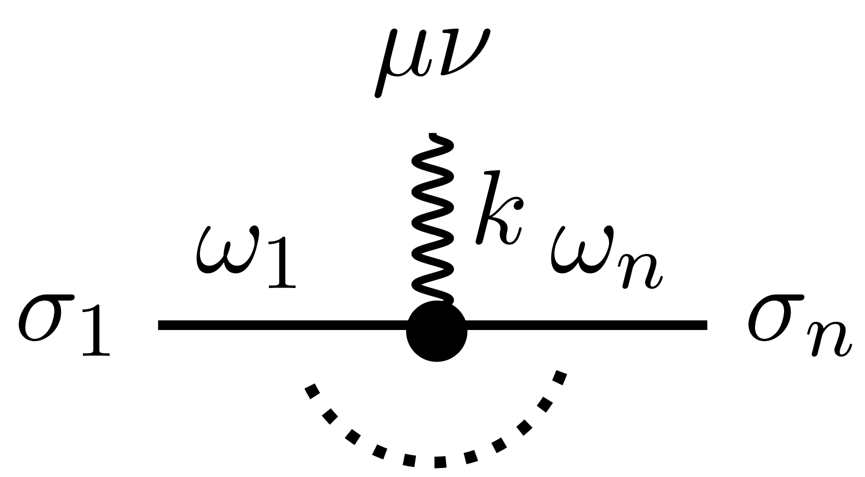
$\partial_\mu\partial^\mu h \ldots \sum_i \ldots \Big|_0 \ldots \Big|_0 \ldots \Big|_0 \ldots .\Big]$

$$+\frac{\sqrt{G}}{2}\frac{\delta^2\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)}{\delta h^2}\bigg|_0 h^2 + \frac{G}{6}\frac{\delta^3\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)}{\delta h^3}\bigg|_0 h^3 + \ldots$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^2 z_i = \sqrt{G}\left(\frac{\delta\Gamma\dot{x}_i\dot{x}_i}{\delta h}\bigg|_0 h + \frac{\delta^2\Gamma\dot{x}_i\dot{x}_i}{\delta h\delta z_i}\bigg|_0 hz + \frac{1}{2}\frac{\delta^3\Gamma\dot{x}_i\dot{x}_i}{\delta h\delta z_i^2}\bigg|_0 hz^2 + \ldots\right)$$



$$\sim \sqrt{G} m_1 \delta\left(k \cdot v_1 + \sum \omega_i\right) e^{-ik\cdot b_1} v_1^\mu v_1^\nu k_1^{\sigma_1} \dots k_n^{\sigma_n}$$



$$\sim \sqrt{G} m_2 \delta\left(k \cdot v_2 + \sum \omega_i\right) e^{-ik\cdot b_2} v_2^\mu v_2^\nu k_1^{\sigma_1} \dots k_n^{\sigma_n}$$

$$\partial_\mu \partial^\mu h = \sqrt{G} \sum_i \left[T_i\Big|_0 + \frac{\delta T_i}{\delta z_i}\Big|_0 z_i + \frac{1}{2}\frac{\delta^2 T_i}{\delta z_i^2}\Big|_0 z_i^2 + \dots \right]$$

$$+ \frac{\sqrt{G}}{2} \frac{\delta^2\left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}\right)}{\delta h^2}\Big|_0 h^2 + \frac{G}{6} \frac{\delta^3\left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}\right)}{\delta h^3}\Big|_0 h^3 + \dots$$

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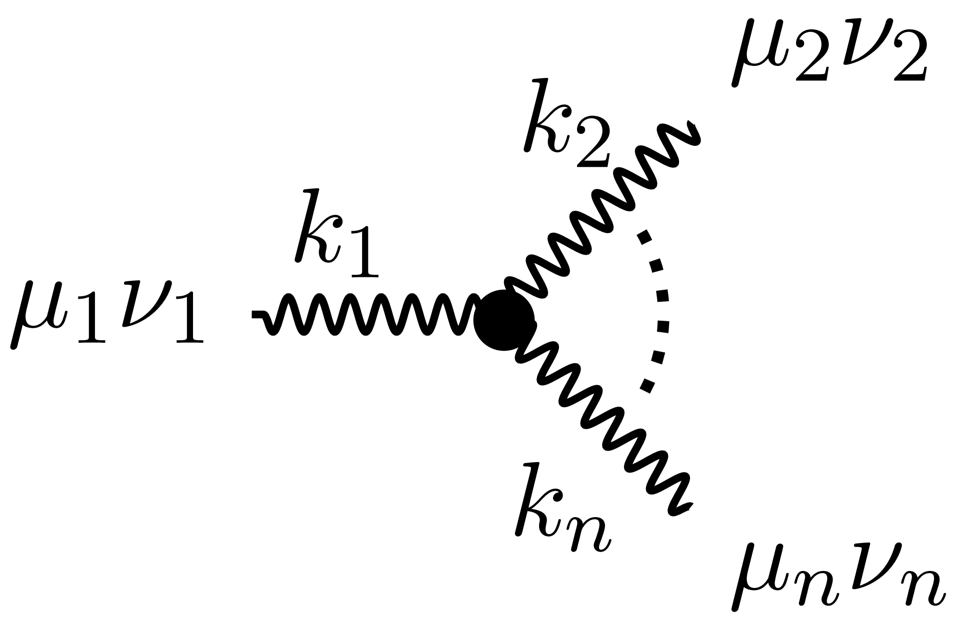

$$\sim \sqrt{G}^{n-2}\delta^4\left(\sum k_i^\sigma\right)\left(k_1^{\mu_1}k_2^{\mu_2}\eta^{\nu_1\nu_2}\eta^{\mu_n\nu_n} + \ldots\right)$$

$$\partial_\mu\partial^\mu h = \sqrt{G}\sum_i \left[T_i\Big|_0 + \frac{\delta T_i}{\delta z_i}\Big|_0 z_i + \frac{1}{2}\frac{\delta^2 T_i}{\delta z_i^2}\Big|_0 z_i^2 + \ldots\right]$$

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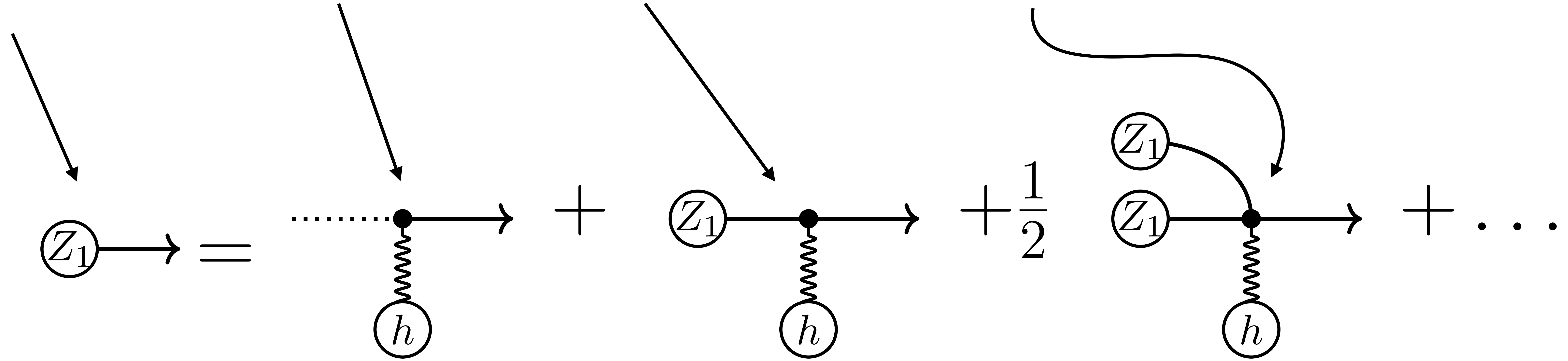
$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^2 z_i = \sqrt{G} \left(\frac{\delta \Gamma \dot{x}_i \dot{x}_i}{\delta h}\Big|_0 h + \frac{\delta^2 \Gamma \dot{x}_i \dot{x}_i}{\delta h \delta z_i}\Big|_0 hz + \frac{1}{2}\frac{\delta^3 \Gamma \dot{x}_i \dot{x}_i}{\delta h \delta z_i^2}\Big|_0 hz^2 + \dots \right)$$



$$\partial_\mu \partial^\mu h = \sqrt{G} \sum_i \left[T_i \Big|_0 + \frac{\delta T_i}{\delta z_i}\Big|_0 z_i + \frac{1}{2}\frac{\delta^2 T_i}{\delta z_i^2}\Big|_0 z_i^2 + \dots \right]$$

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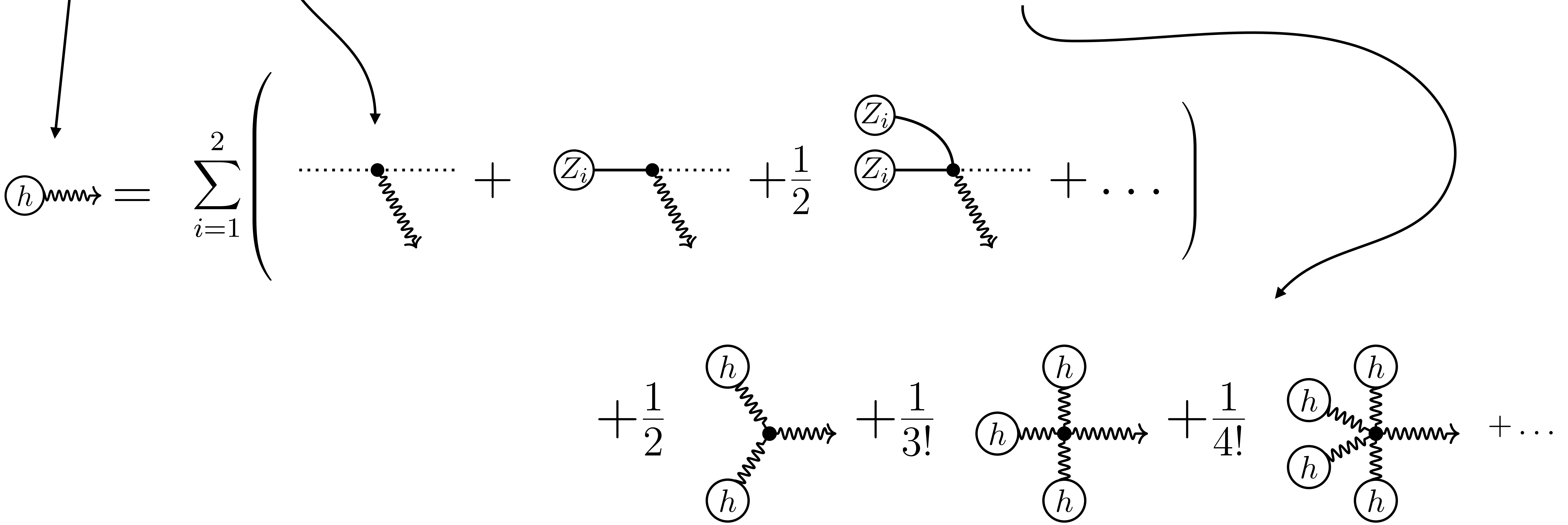
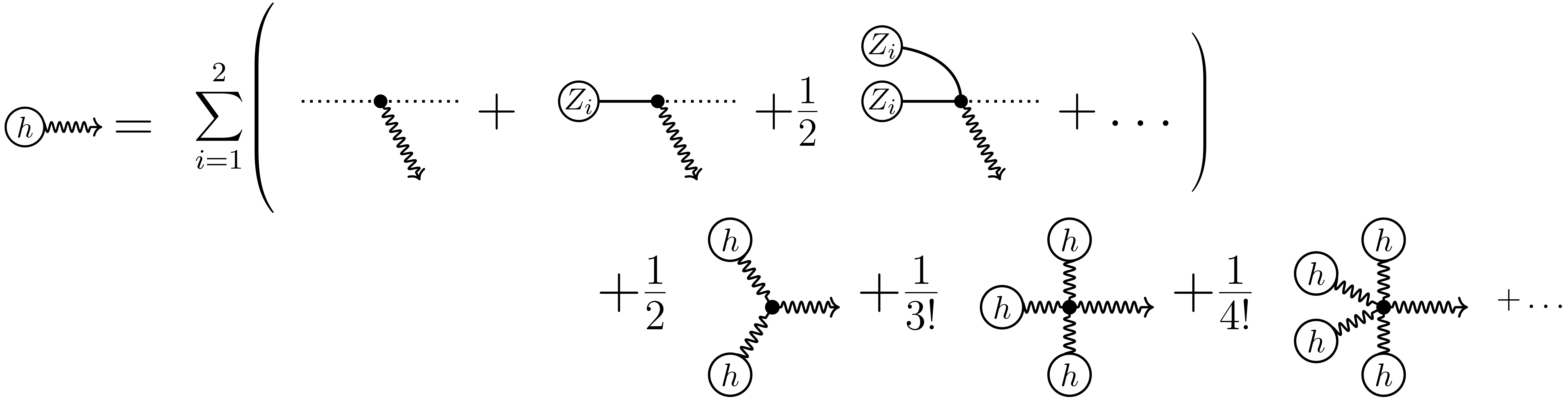
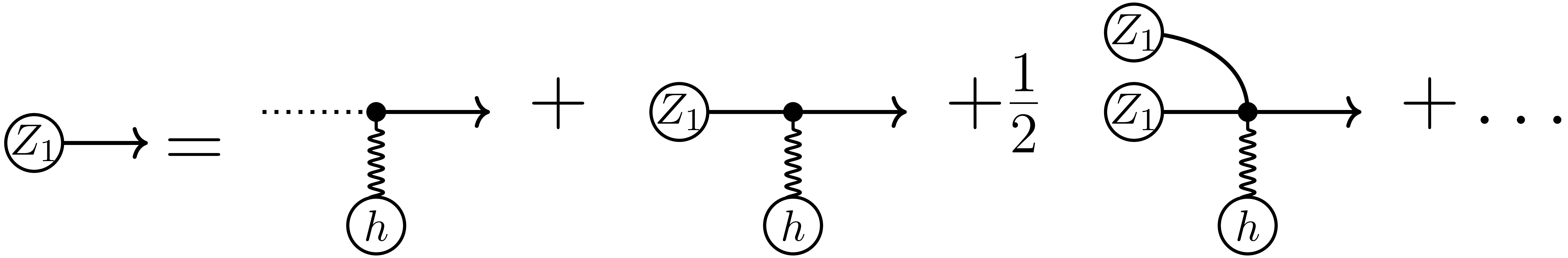
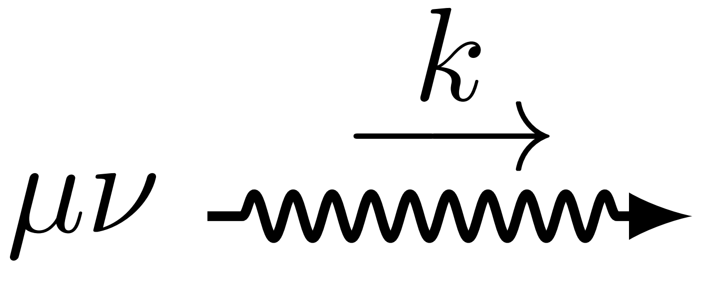


Equations of motion in the diagrammatic (worldline) QFT language

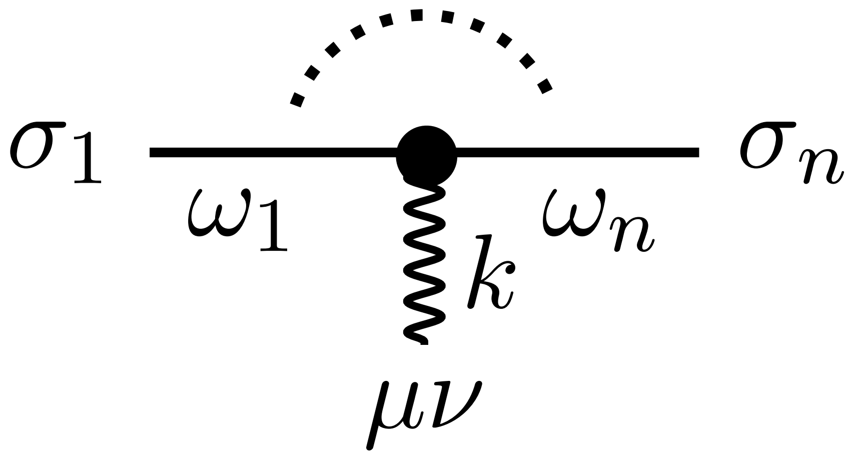




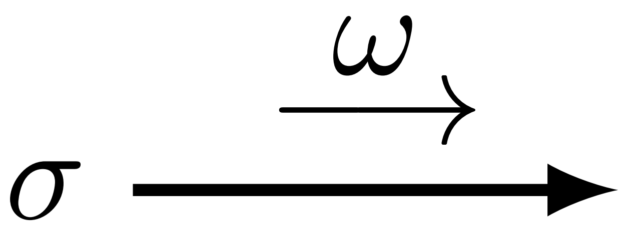
Everything follows from the action!



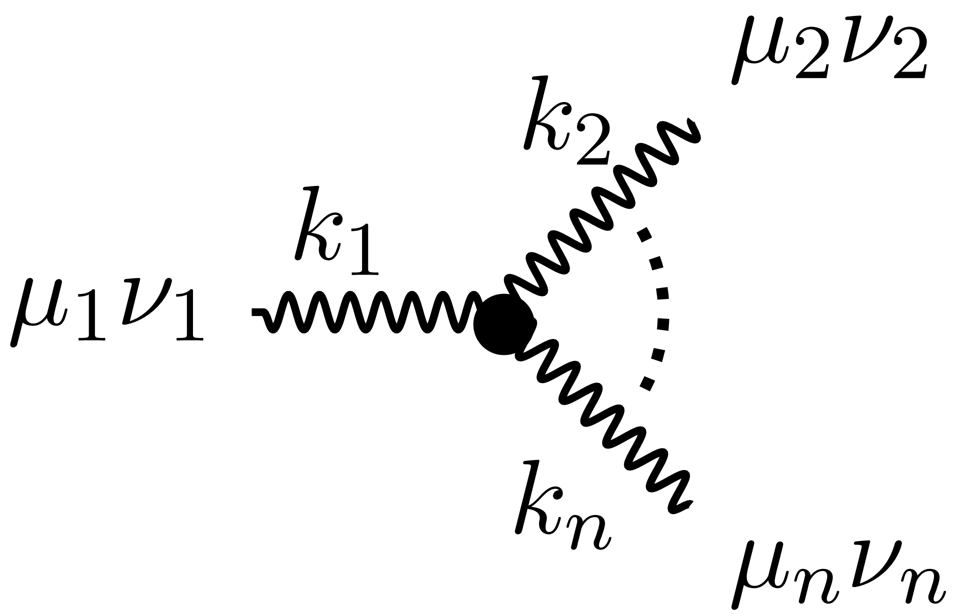
$$= \left(\frac{\delta^2 S}{\delta h^2}\right)^{-1}$$



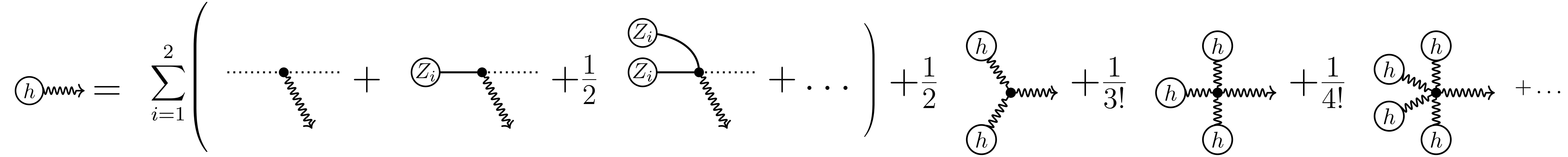
$$= \frac{\delta^{n+1} S}{\delta h \delta z^n}$$

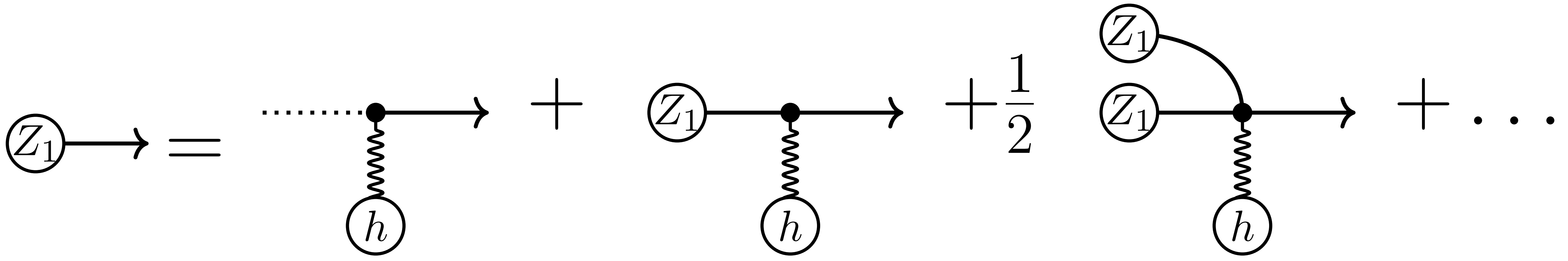


$$= \left(\frac{\delta^2 S}{\delta z^2}\right)^{-1}$$

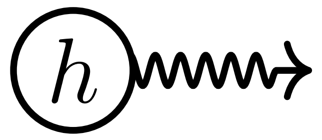


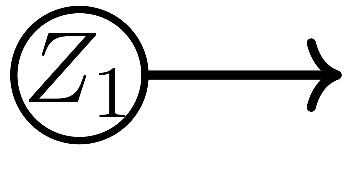
$$= \frac{\delta^n S}{\delta h^n}$$



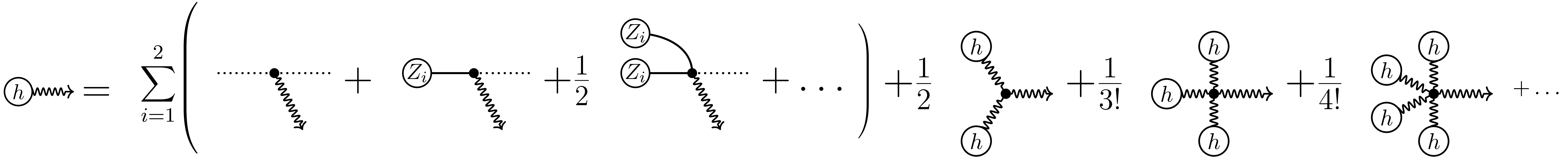


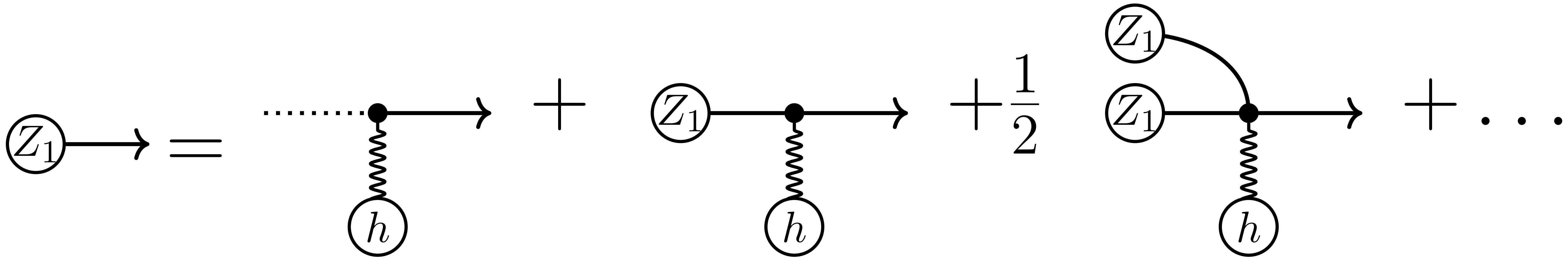
Post-Minkowskian expansion of perturbative fields:

$$\sqrt{G}$$  $$= \sqrt{G} h_{\mu\nu}(x) = \sum_{n=1}^{\infty} G^n h^{(n)}_{\mu\nu}(x)$$



$$= z_i^{\mu}(\tau) = \sum_{n=1}^{\infty} G^n z_i^{(n)\mu}(\tau)$$





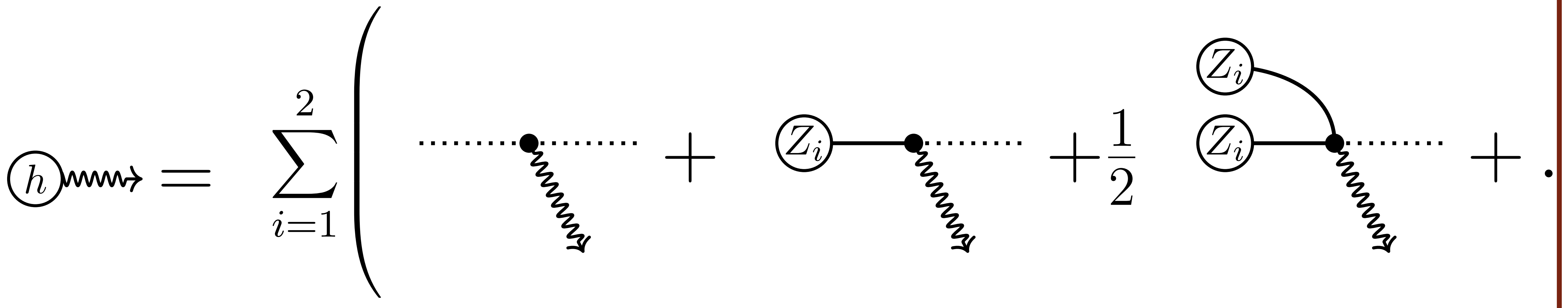
Post-Minkowskian expansion of perturbative fields:

$$\sqrt{G}\,\text{(h)}\rightsquigarrow = \sqrt{G}h_{\mu\nu}(x) = \sum_{n=1}^{\infty} G^n h_{\mu\nu}^{(n)}(x)$$

$$\text{(}Z_1\text{)}\longrightarrow = z_i^{\mu}(\tau) = \sum_{n=1}^{\infty} G^n z_i^{(n)\mu}(\tau)$$

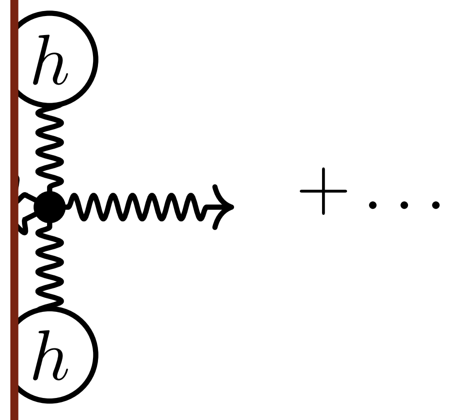
For m-PM accuracy solve for

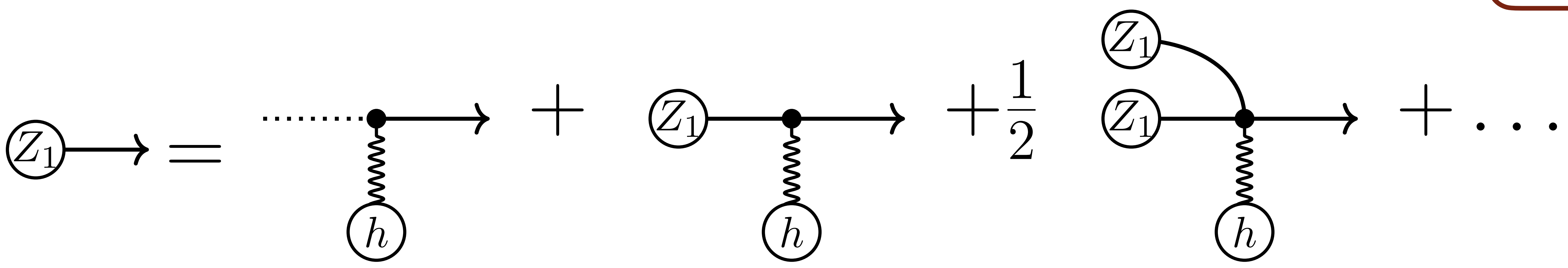
$$z_i^{(m)\mu}(\tau) \qquad h_{\mu\nu}^{(m)}(x)$$



After break, solve for m=3

$$z_i^{(3)\mu}(\tau)$$





Part II: 3PM impulse and dissipation of energy

$$\Delta p_i^\mu = m_i \int_{-\infty}^{\infty} \mathrm{d}\tau \ddot{x}_i^\mu(\tau) = -m_i \lim_{\omega \to 0} \omega^2 z_i^\mu(\omega)$$

Impulse of i'th black hole

$$= p_i^\mu(\infty) - p_i^\mu(-\infty)$$

$$-P^\mu_{\mathrm{rad}} = \Delta p_1^\mu + \Delta p_2^\mu$$

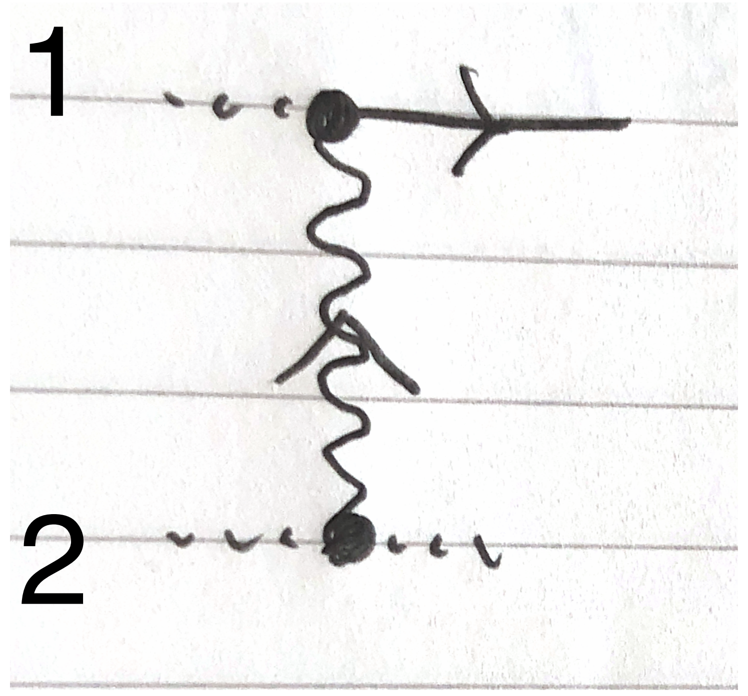
Total radiated four-momentum
by gravitational waves

Appears at 3PM order (and 1SF order)

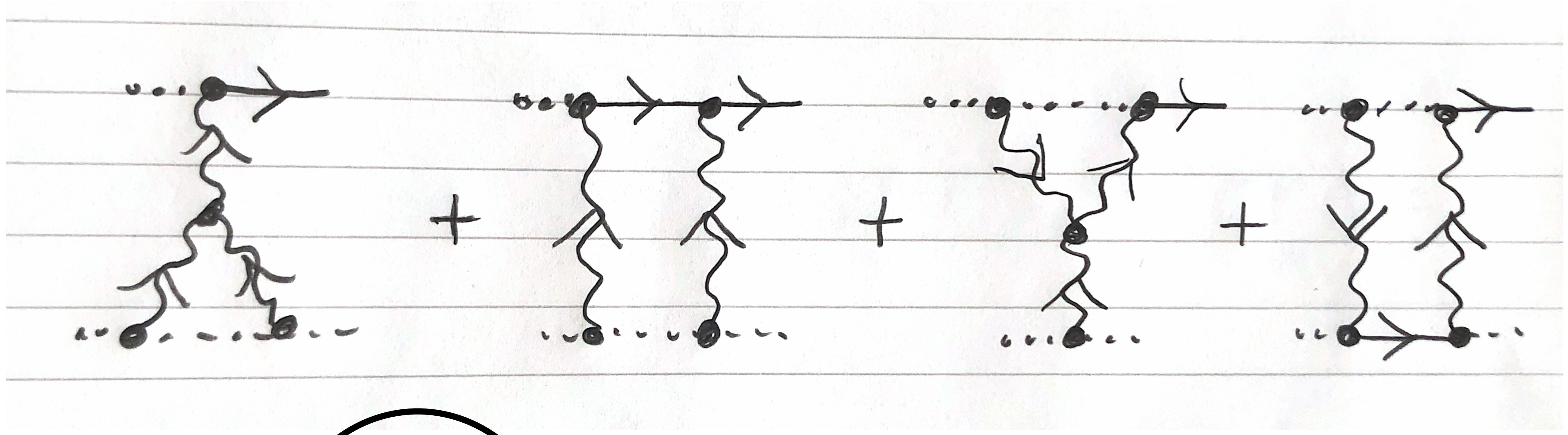
Black hole 1

Gravity

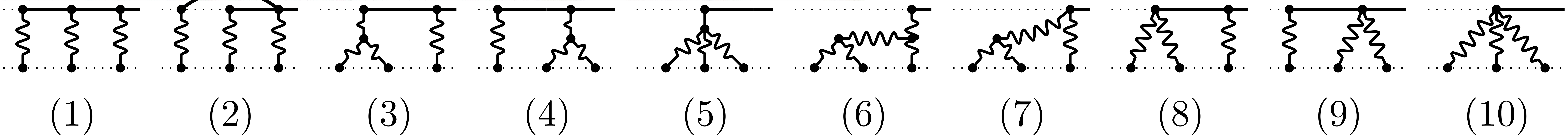
Black hole 2



1PM $z_i^{(1)\mu}(\omega)$

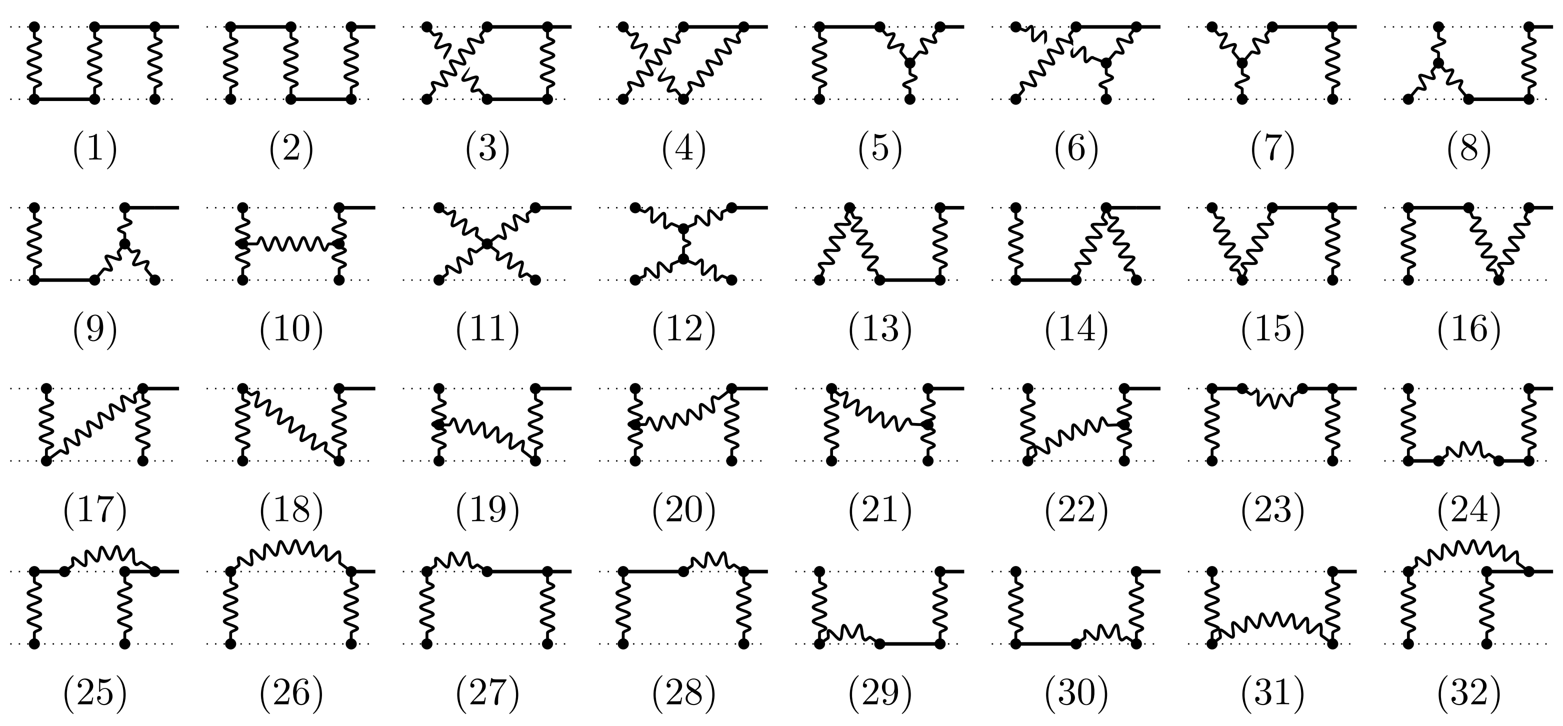


2PM $z_i^{(2)\mu}(\omega)$



3PM, 0SF

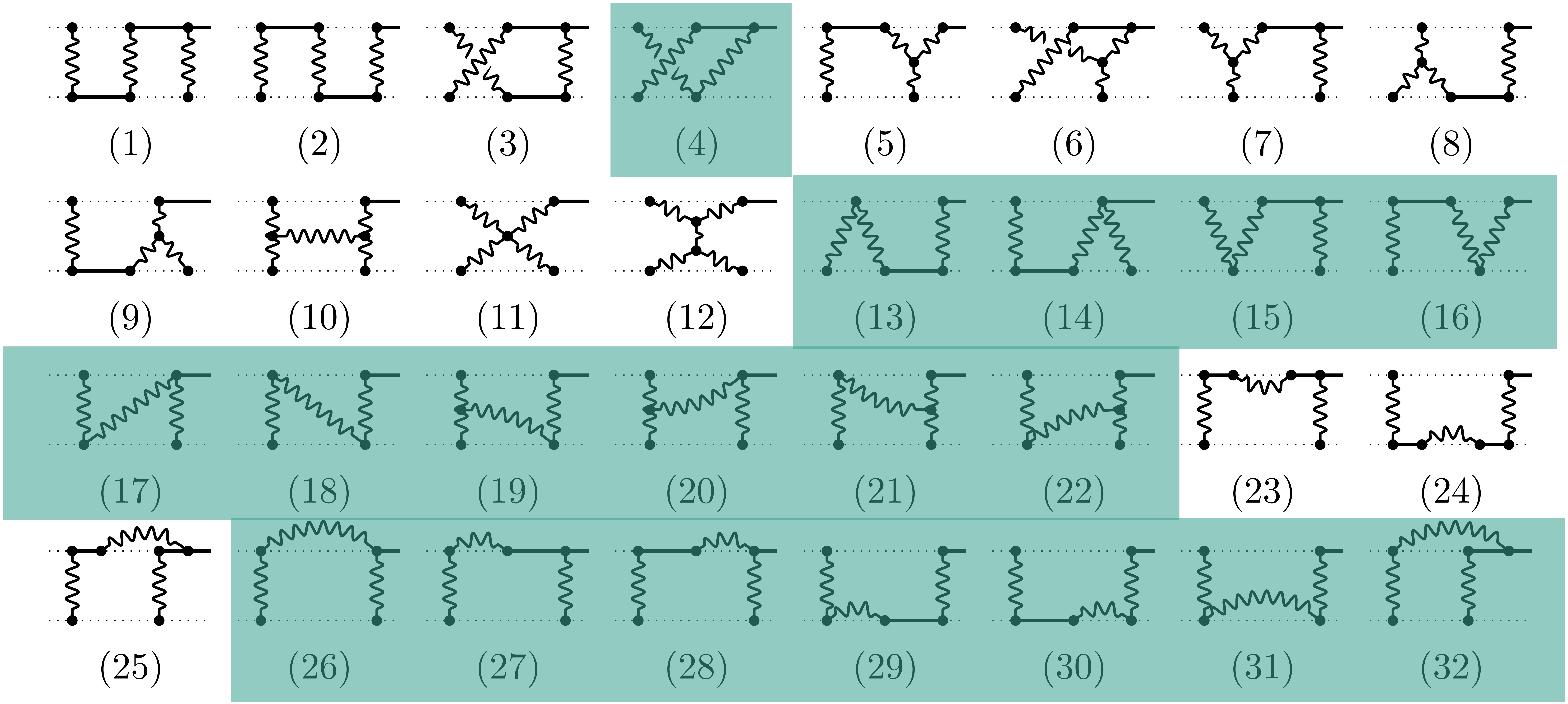
$z_i^{(3)\mu}(\omega)$



3PM, 1SF

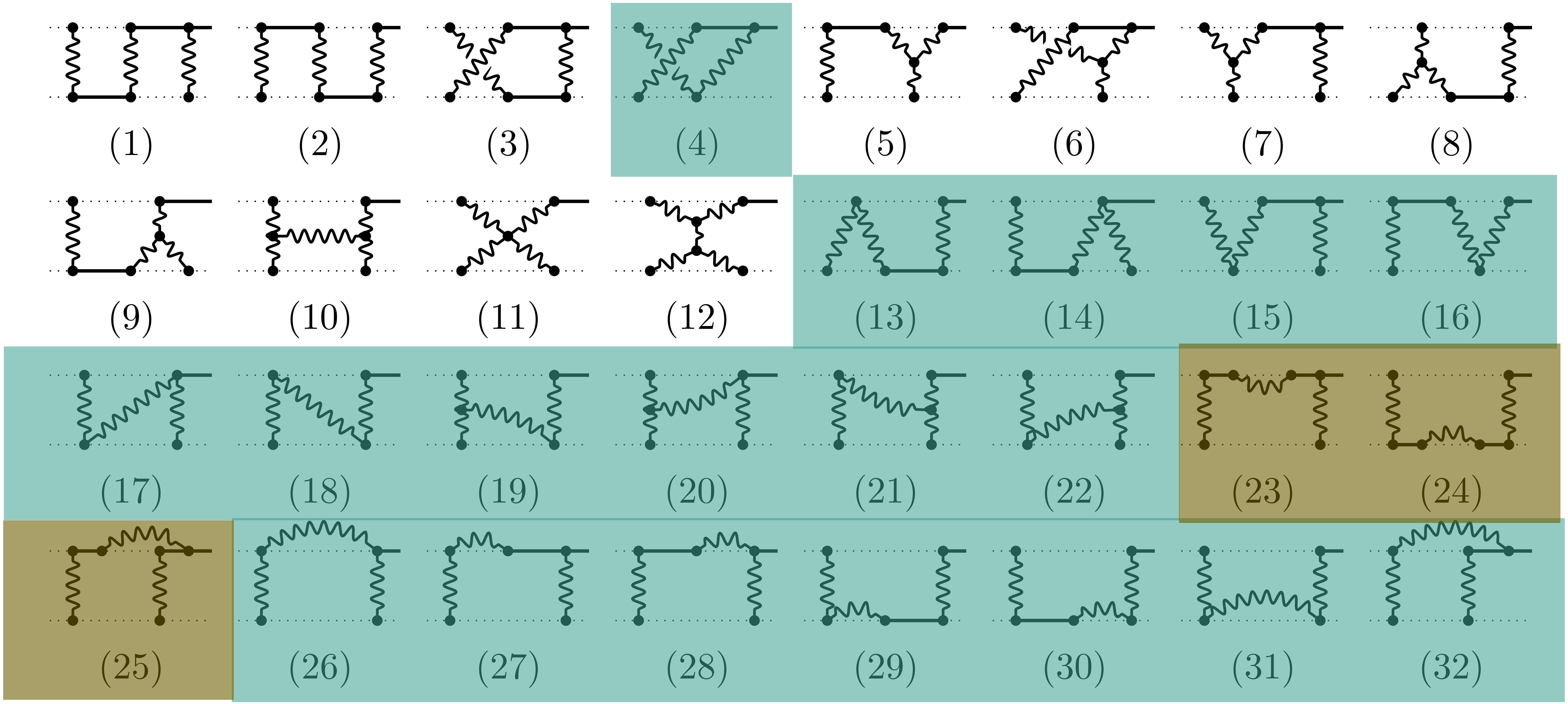
14 spinless graphs at 3PM, 1SF

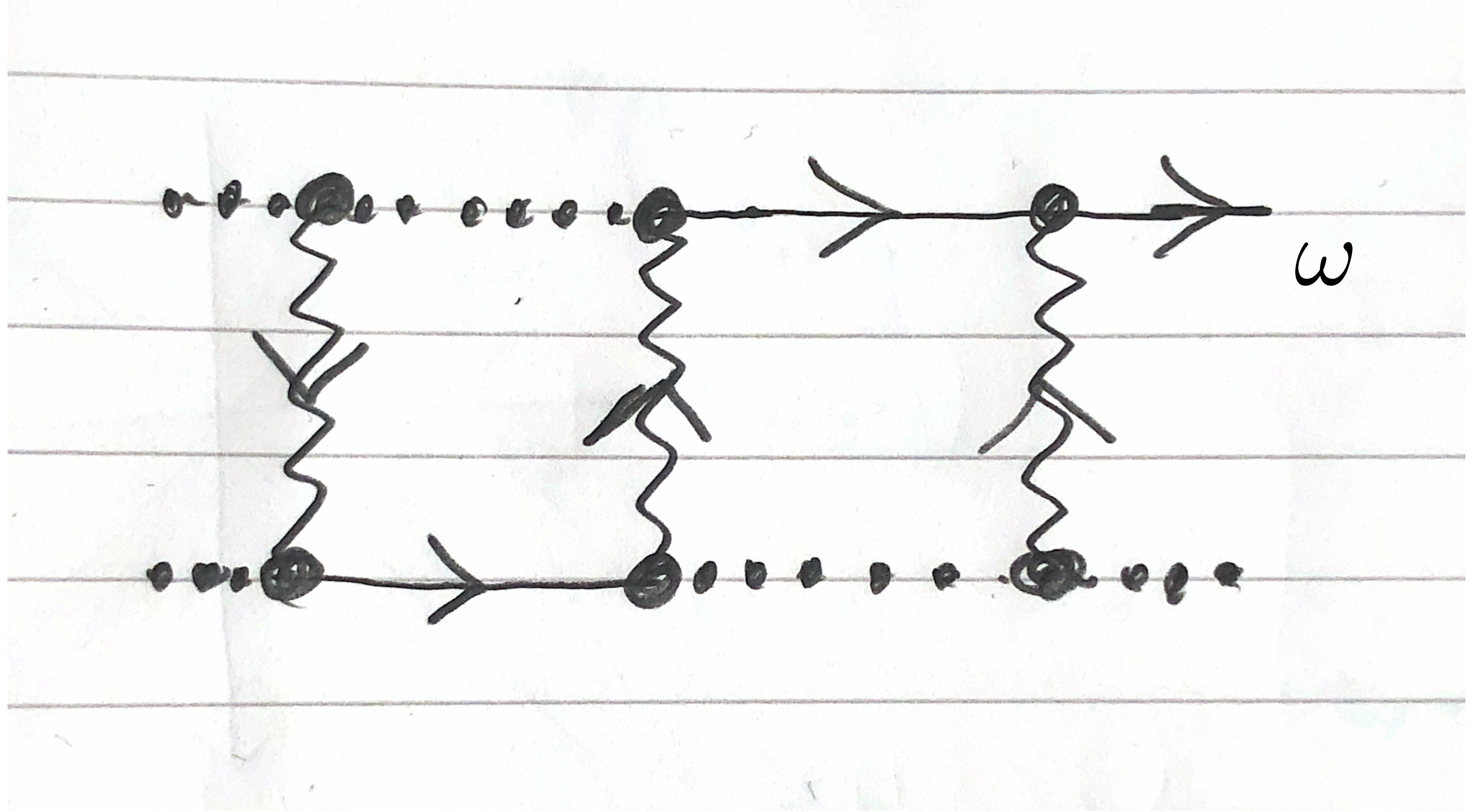
= spin and finite size effects



14 spinless graphs at 3PM, 1SF

= pure dissipation



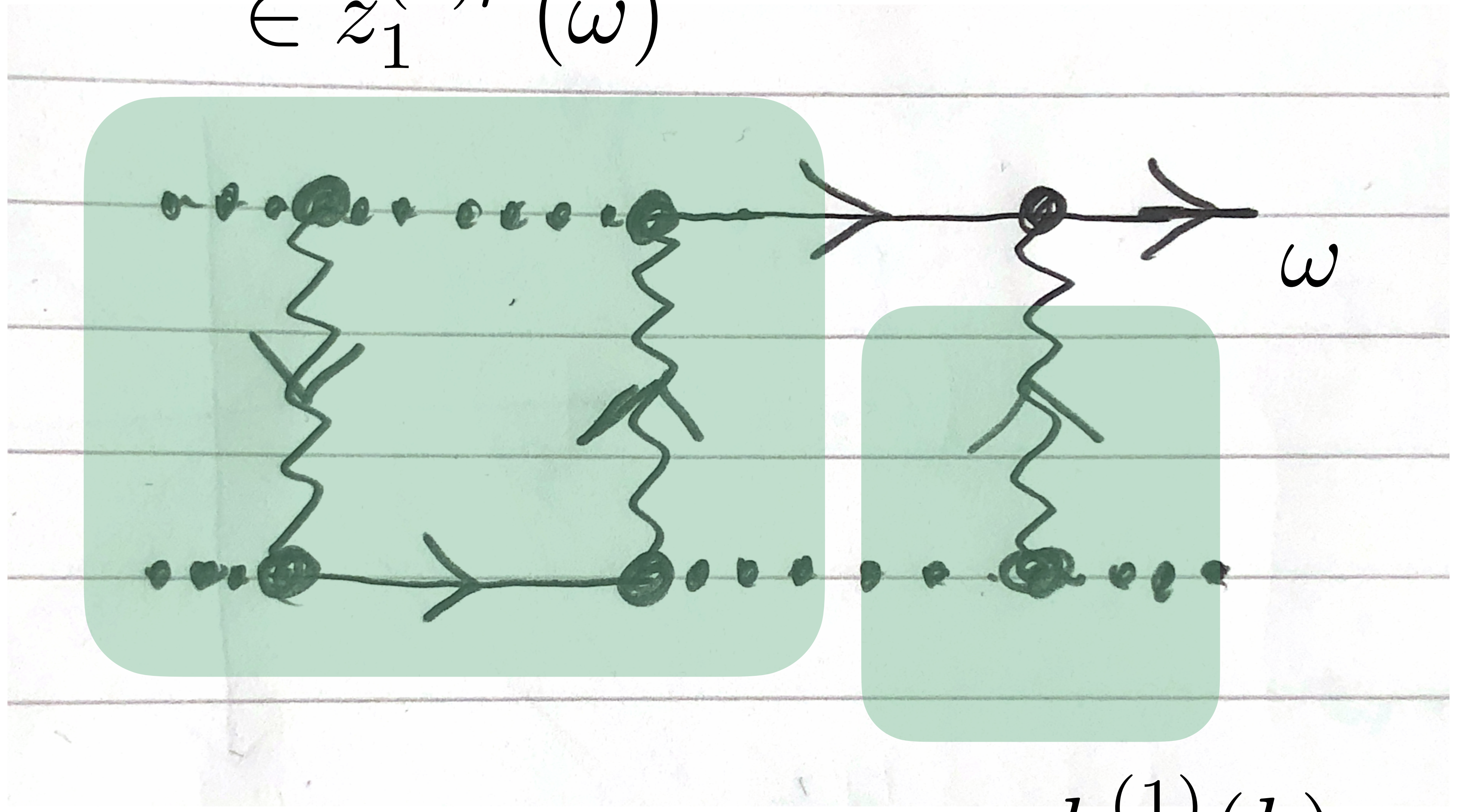


$$\in z_1^{(3)\mu}(\omega)$$

$$\sqrt{G}\ \textcircled{h}\!\rightsquigarrow \;=\; \sqrt{G} h_{\mu\nu}(x) = \sum_{n=1}^{\infty} G^n h_{\mu\nu}^{(n)}(x)$$

$$\textcircled{Z_1}\!\longrightarrow \;=\; z_i^{\mu}(\tau) = \sum_{n=0}^{\infty} G^n z_i^{(n)\mu}(\tau)$$

$\in z_1^{(2)\mu}(\omega)$



ω

$\in z_1^{(3)\mu}(\omega)$

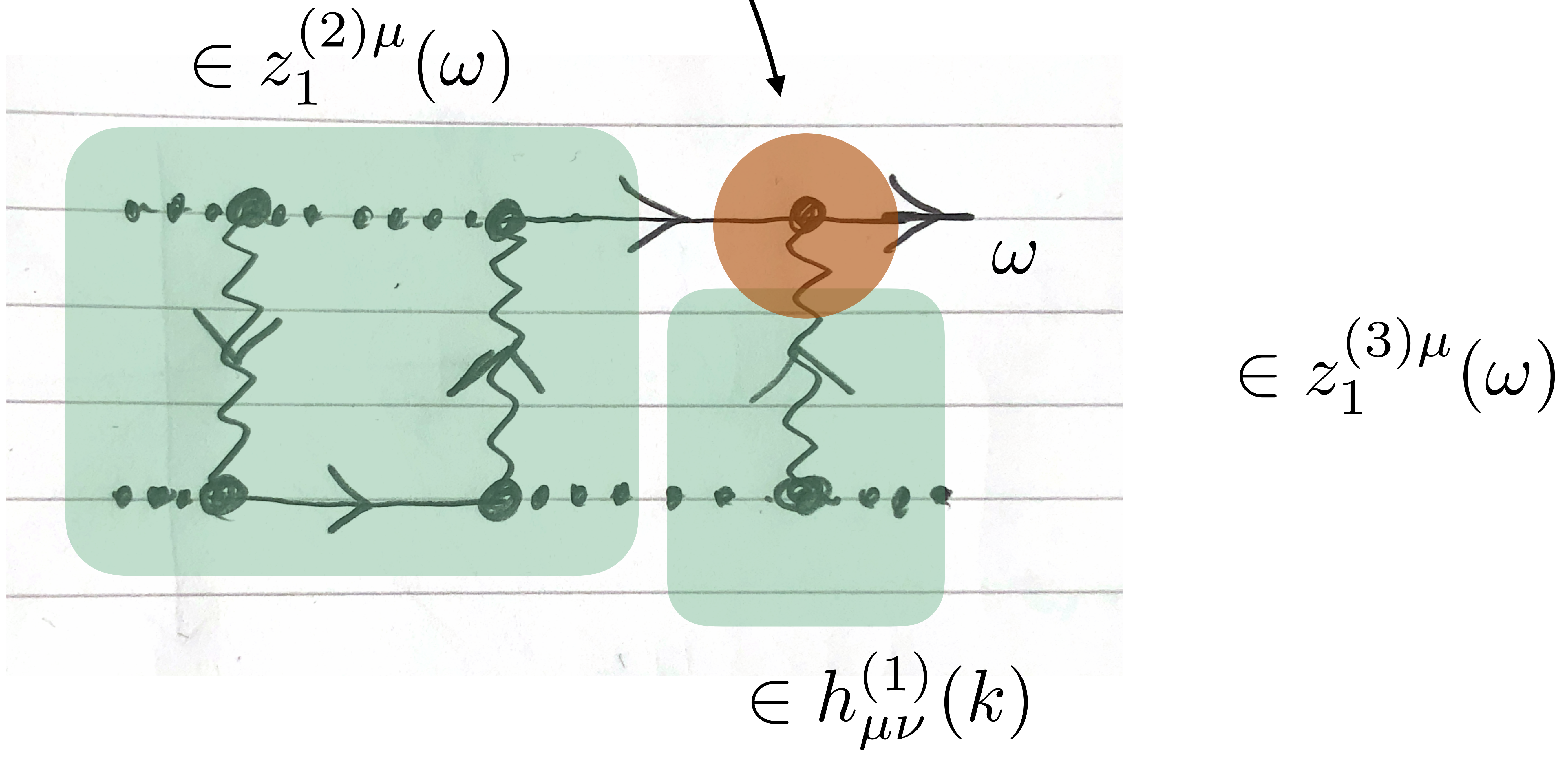
$\in h_{\mu\nu}^{(1)}(k)$

$$\sqrt{G}\ (h)\!\rightsquigarrow = \sqrt{G} h_{\mu\nu}(x) = \sum_{n=1}^{\infty} G^n h_{\mu\nu}^{(n)}(x)$$

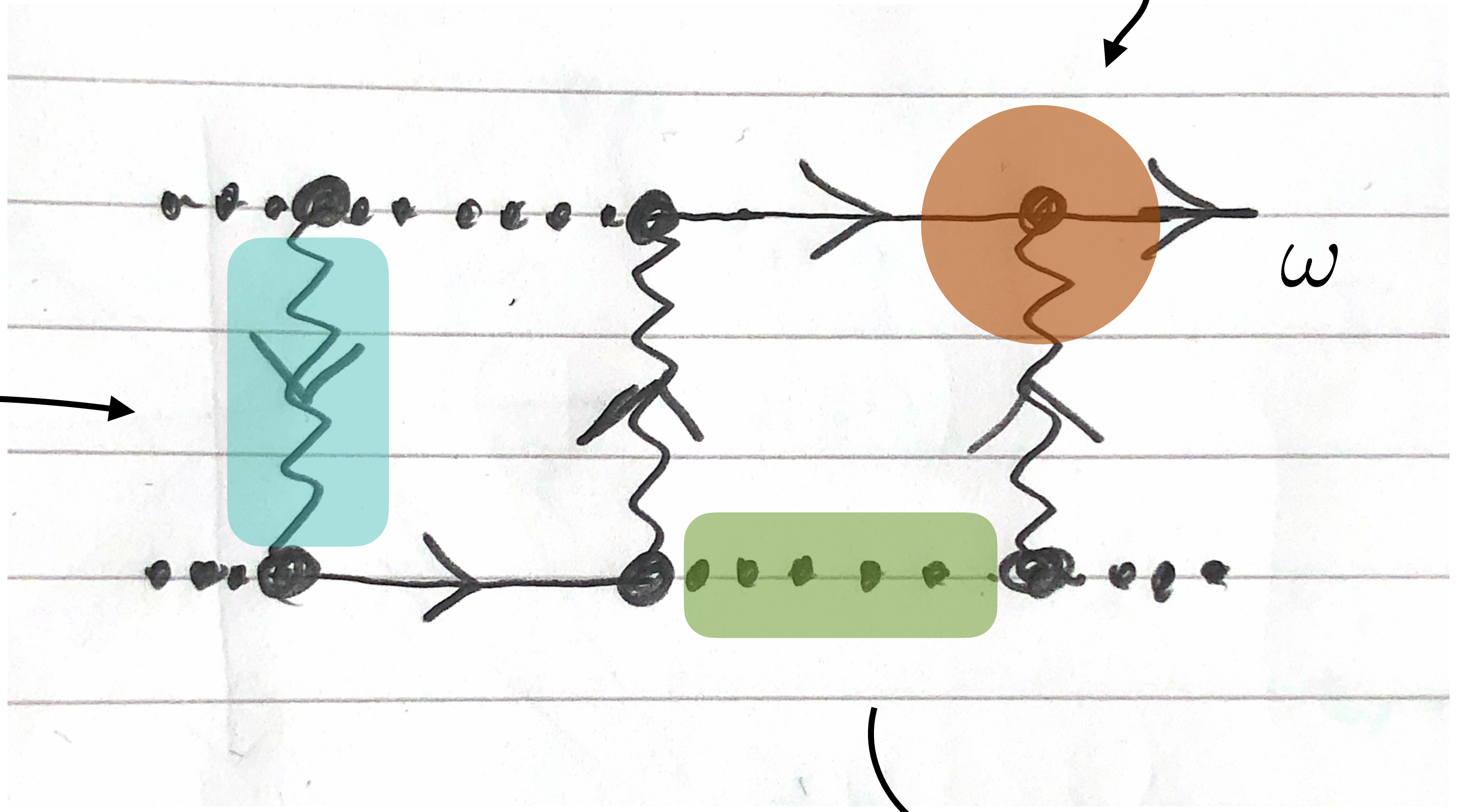
$$(Z_1)\!\longrightarrow = z_i^{\mu}(\tau) = \sum_{n=0}^{\infty} G^n z_i^{(n)\mu}(\tau)$$

Interaction vertex

$$\left.\frac{\delta^3 S}{\delta z^2 \delta h}\right|_0 \sim \left.\frac{\delta^2 \Gamma^{\mu}_{\alpha\beta}}{\delta z \delta h}\right|_0$$



Interaction vertices
Differentiation, i.e. momenta in numerator



$\in z_1^{(3)\mu}(\omega)$

Propagators
Integration, i.e. momenta in denominator

Static straight-line motion
Non-dynamical

Constructing and simplifying $-m \lim_{\omega \to 0} \omega^2 z_i^{(3)\mu}(\omega) = \Delta p_i$

Draw all diagrams and insert Feynman rules

Use tensor manipulation software, e.g. FORM, FeynCalc, xAct

Simplify numerators (interaction vertices)

Tensor reduction: For example any IBP program
IBP reduction: KIRA, LiteRed, FIRE

Reduce to a minimal basis of required integrals (master integrals)

IBP reduction: KIRA, LiteRed, FIRE

Integration-by-parts (IBP) reduction

- Technique from collider physics
- Linear identities between loop integrals (stemming from integration by parts identities)
- Reduces an (in principle) infinite family of integrals to a finite basis of master integrals
- Leads to a significantly simpler integrand
- Implemented in public packages: KIRA, LiteRed, FIRE

Dimensional regularization

$$d = 4 - 2\epsilon$$

Master integrals

$$\Delta p_i^{(3)\mu} = G^3 m_1^2 m_2^2 \lim_{\epsilon \to 0} \sum_{i=1}^{11} c_i^\mu(v_i^\mu, b^\mu, \epsilon)\mathcal{I}_i(\gamma, b, \epsilon) + [0\mathrm{SF}]$$

Rational coefficients

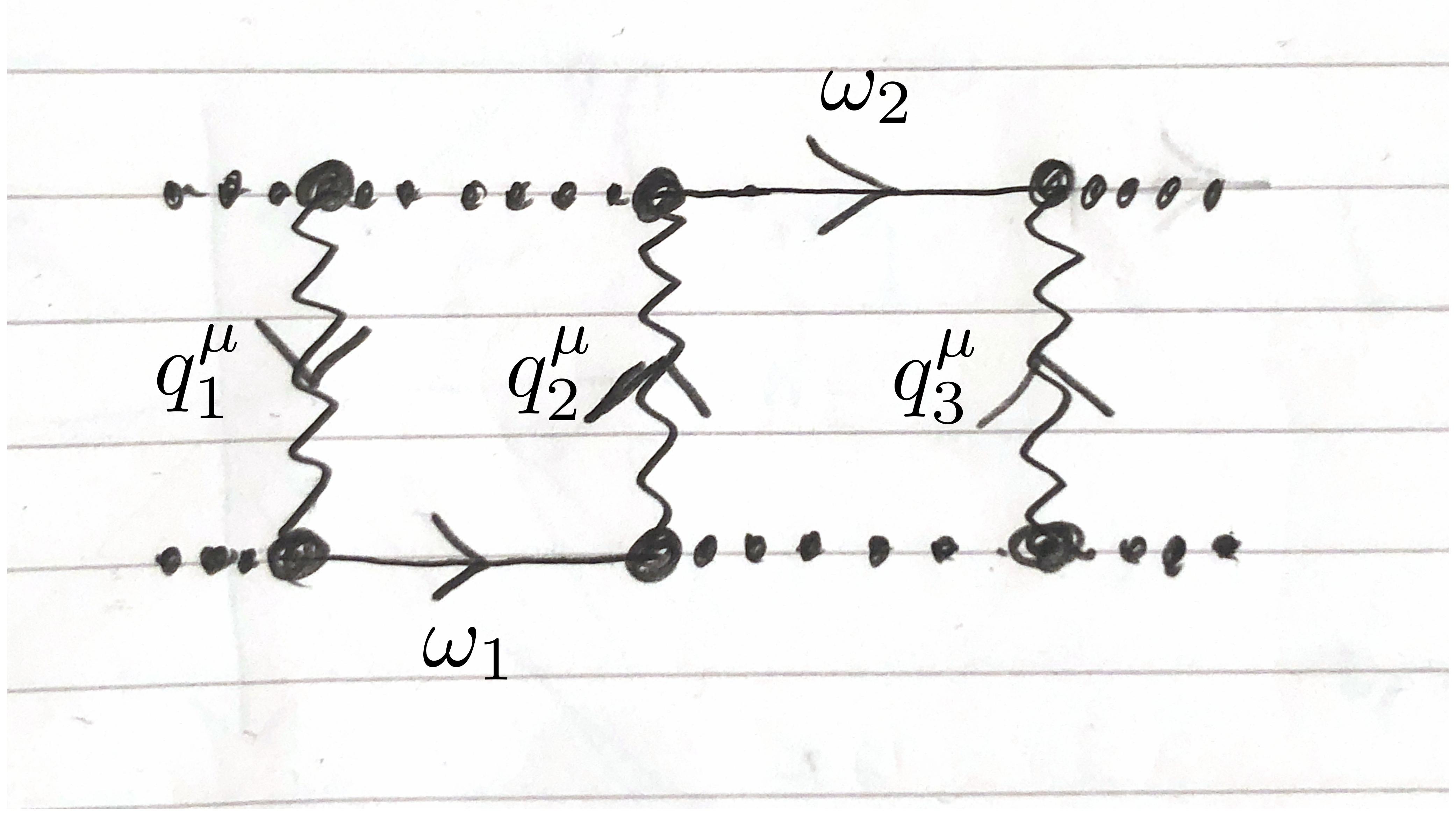
Relative Lorentz factor: $\gamma = v_1 \cdot v_2$

Scale (impact parameter): b

Regulator: ϵ

$$\Delta p_i^{(3)\mu} = G^3 m_1^2 m_2^2 \lim_{\epsilon\to 0} \sum_{i=1}^{11} c_i^\mu(v_i^\mu, b^\mu, \epsilon)\mathcal{I}_i(\gamma, b, \epsilon) + [0\mathrm{SF}]$$

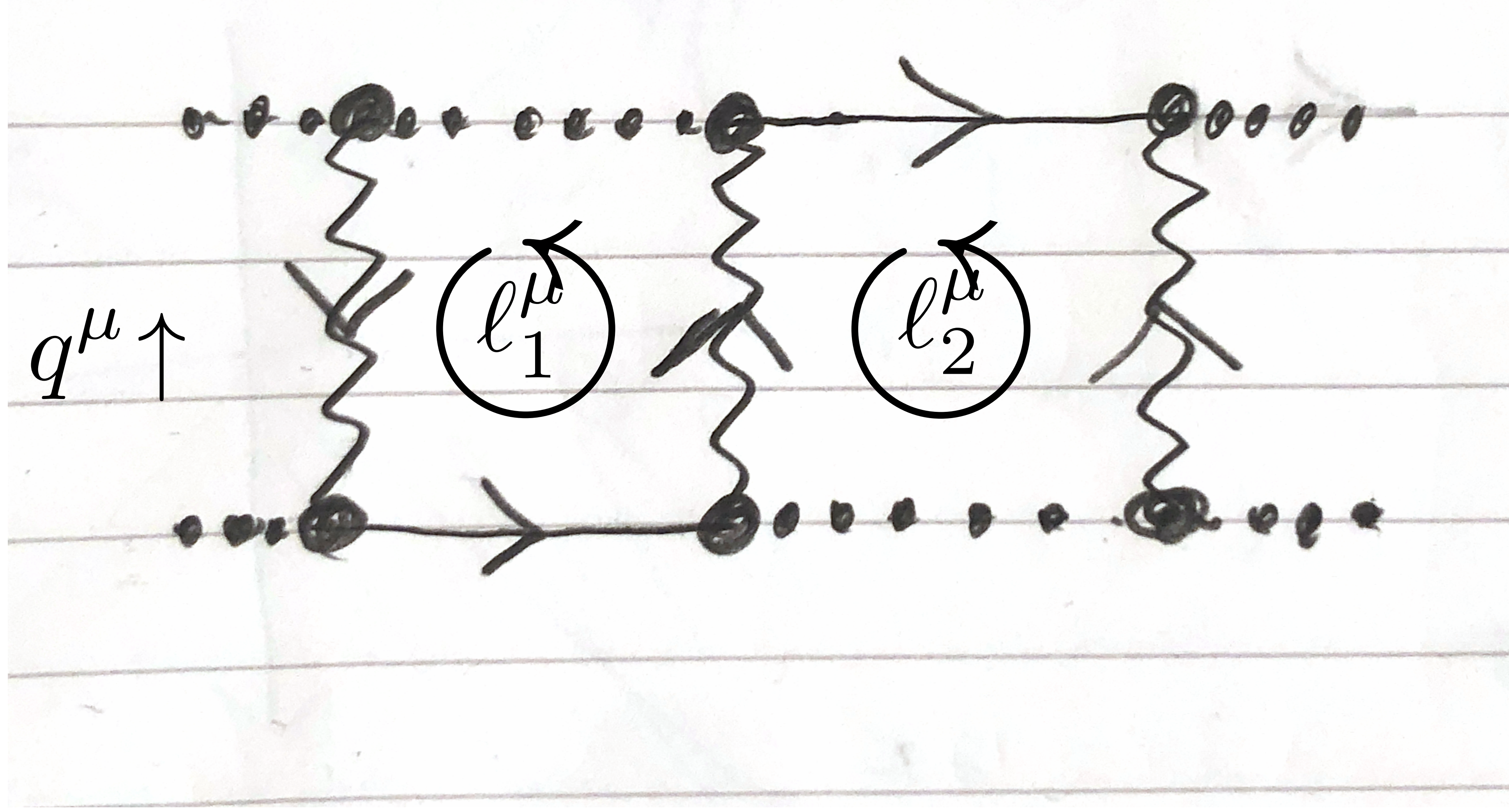
$$\mathcal{I}_1(\gamma, b, \epsilon) =$$



$$\mathcal{I}_1(\gamma, b, \epsilon) = \int \mathrm{d}^d q_1 \mathrm{d}^d q_2 \mathrm{d}^d q_3 \mathrm{d}\omega_1 \mathrm{d}\omega_2 e^{-ib^\mu(-q_1+q_2+q_3)_\mu}$$

$$\times \frac{\delta(q_1\cdot v_1)\delta(q_1\cdot v_2-\omega_1)\delta(\omega_1-q_2\cdot v_2)\delta(q_2\cdot v_1-\omega_2)\delta(\omega_2+q_3\cdot v_1)\delta(q_3\cdot v_2)}{q_1^2 q_2^2 q_3^2(\omega_1+i0^+)^2(\omega_2+i0^+)^2}$$

$$q_i^2 = (q_i^0 + i0^+)^2 - \boldsymbol{q}_i^2$$



$$\mathcal{I}_1(\gamma, q, \epsilon) = \int \mathrm{d}^{d-2} q e^{i q^\mu b_\mu} \mathcal{I}_1(\gamma, b, \epsilon)$$

$$= \int \mathrm{d}^d \ell_1 \mathrm{d}^d \ell_2 \frac{\delta(\ell_1 \cdot v_1)\delta(\ell_2 \cdot v_2)\delta(q \cdot v_1)\delta(q \cdot v_2)}{(\ell_1 - q)^2 (\ell_1 - \ell_2)^2 \ell_2^2 (\ell_1 \cdot v_2 + i0^+)(-\ell_2 \cdot v_1 + i0^+)}$$

$$\mathcal{I}_1(\gamma, b, \epsilon) = \int \mathrm{d}^d q_1 \mathrm{d}^d q_2 \mathrm{d}^d q_3 \mathrm{d}\omega_1 \mathrm{d}\omega_2 e^{-i b^\mu (-q_1 + q_2 + q_3)_\mu}$$

$$\times \frac{\delta(q_1 \cdot v_1)\delta(q_1 \cdot v_2 - \omega_1)\delta(\omega_1 - q_2 \cdot v_2)\delta(q_2 \cdot v_1 - \omega_2)\delta(\omega_2 + q_3 \cdot v_1)\delta(q_3 \cdot v_2)}{q_1^2 q_2^2 q_3^2 (\omega_1 + i0^+)^2 (\omega_2 + i0^+)^2}$$

Method of differential equations

Master integrals: $\vec{I} = \begin{pmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \\ \mathcal{I}_3 \\ \mathcal{I}_4 \\ \dots \\ \mathcal{I}_{11} \end{pmatrix}$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\vec{I}(\gamma, b, \epsilon) = M(\gamma, \epsilon) \cdot \vec{I}(\gamma, b, \epsilon)$$

Change of integral basis
("Canonicalization")

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\vec{I}'(\gamma, b, \epsilon) = \epsilon M'(\gamma) \cdot \vec{I}'(\gamma, b, \epsilon)$$

Software:
Canonica, Initial, Fuchsia, more

Ideal basis for epsilon expansions

$$\vec{I}'(\gamma, b, \epsilon) = \mathcal{P} e^{\epsilon \int_1^\gamma \mathrm{d}\gamma' M(\gamma')} \vec{I}'_0(\epsilon, b)$$

$$\vec{I}'(\gamma, b, \epsilon) = \mathcal{P}e^{\epsilon \int_1^\gamma \mathrm{d}\gamma' M(\gamma')} \vec{I}'_0(\epsilon, b)$$
$$= \vec{I}'_0(\epsilon, b) + \epsilon \int_1^\gamma \mathrm{d}\gamma' M(\gamma') \cdot \vec{I}'_0(\epsilon, b) + \dots$$

Integration (boundary) constants

Boundary chosen in the post-Newtonian limit:

$$\vec{I}'_0(\epsilon, b) = \lim_{\gamma \to 1} \vec{I}'(\gamma, b, \epsilon) \; = b^{[\text{dimension}]} \epsilon^{[\text{integer}]} \Big(c_0 + \epsilon c_1 + \epsilon c_2 + \dots \Big)$$

Expansion is (usually) asymptotic and method of regions must be used

Usually, the first time “genuine” integration is required…

Results

$$\Delta p_1^\mu = p_\infty \sin\theta \frac{b^\mu}{|b|} + (\cos\theta - 1)\frac{m_1 m_2}{E^2}[(\gamma m_1 + m_2)v_1^\mu - (\gamma m_2 + m_1)v_2^\mu] - v_2 \cdot P_{\rm rad}\, w_2^\mu$$

$$\frac{\theta_{\rm cons}}{\Gamma} = \frac{GM}{|b|}\frac{2(2\gamma^2-1)}{\gamma^2-1} + \left(\frac{GM}{|b|}\right)^2 \frac{3\pi(5\gamma^2-1)}{4(\gamma^2-1)}$$
$$+ \left(\frac{GM}{|b|}\right)^3 \left(2\frac{64\gamma^6-120\gamma^4+60\gamma^2-5}{3(\gamma^2-1)^3}\Gamma^2\right.$$
$$\left. - \frac{8\nu\gamma(14\gamma^2+25)}{3(\gamma^2-1)} - 8\nu\frac{(4\gamma^4-12\gamma^2-3)}{(\gamma^2-1)}\frac{\operatorname{arccosh}\gamma}{\sqrt{\gamma^2-1}}\right) + \mathcal{O}(G^4)\,,$$

$$\frac{\theta_{\rm rad}}{\Gamma} = \left(\frac{GM}{|b|}\right)^3 \frac{4\nu(2\gamma^2-1)^2}{(\gamma^2-1)^{3/2}}\left(-\frac{8}{3} + \frac{1}{v^2} + \frac{(3v^2-1)}{v^3}\operatorname{arccosh}\gamma\right) + \mathcal{O}(G^4)\,,$$

Results: Radiation

$$P^{\mu}_{\text{rad}} = \frac{G^3 m_1^2 m_2^2 \pi}{|b|^3} \frac{v_1^{\mu} + v_2^{\mu}}{\gamma + 1} \mathcal{E}(\gamma) + \mathcal{O}(G^4) \,,$$

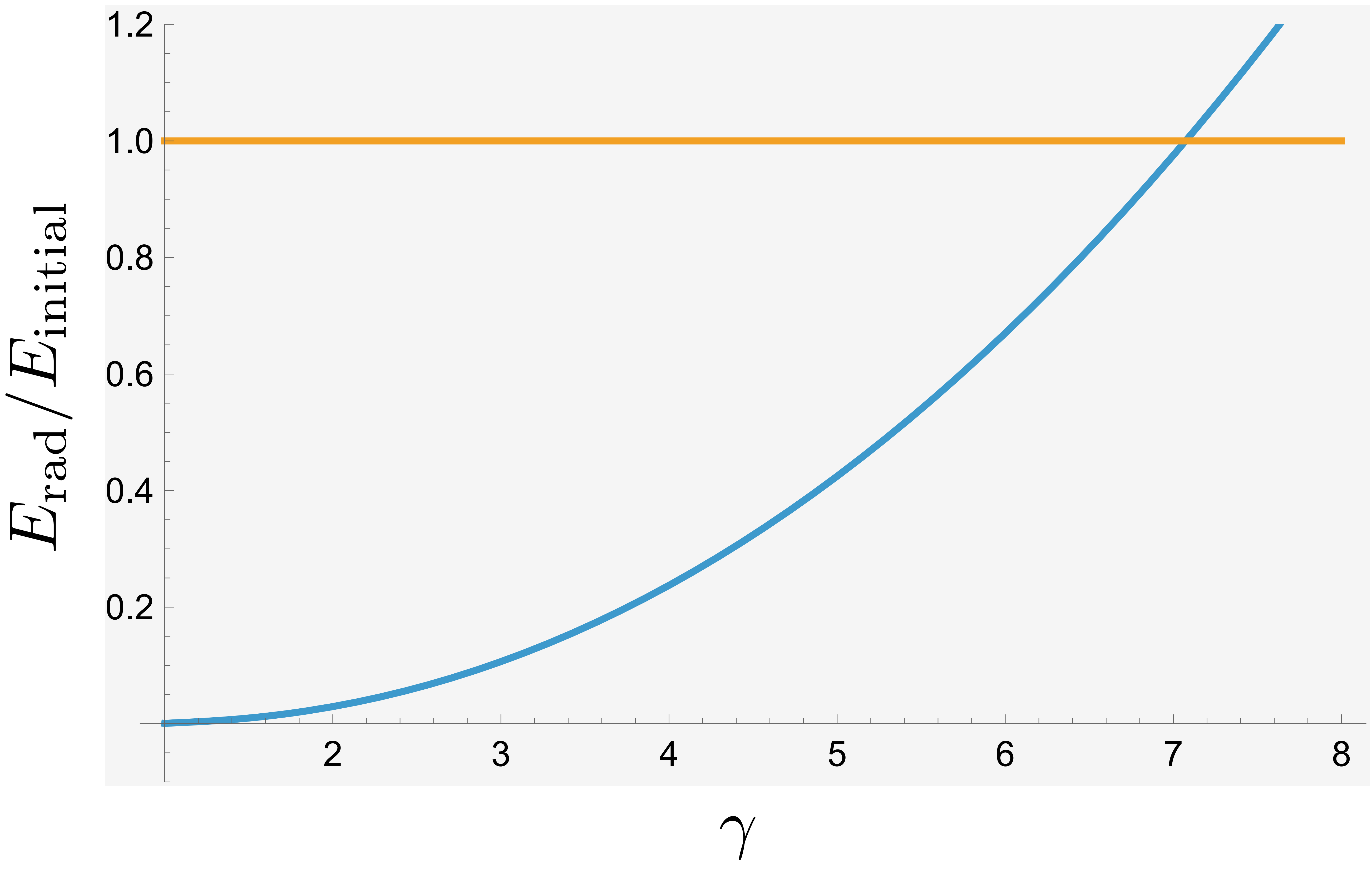
$$\mathcal{E}(\gamma) = e_1 + e_2 \log\left(\frac{\gamma+1}{2}\right) + e_3 \frac{\text{arccosh}\gamma}{\sqrt{\gamma^2 - 1}} \,.$$

$$\begin{aligned} e_1 &= \frac{210\gamma^6 - 552\gamma^5 + 339\gamma^4 - 912\gamma^3 + 3148\gamma^2 - 3336\gamma + 1151}{48(\gamma^2-1)^{3/2}} \,, \\ e_2 &= -\frac{35\gamma^4 + 60\gamma^3 - 150\gamma^2 + 76\gamma - 5}{8\sqrt{\gamma^2-1}} \,, \\ e_3 &= \frac{\gamma(2\gamma^2 - 3)\,(35\gamma^4 - 30\gamma^2 + 11)}{16(\gamma^2-1)^{3/2}} \,. \end{aligned}$$

$$E_{\rm rad} = P^0_{\rm rad}$$

$$m_1 = m_2$$

$$\frac{Gm_1}{b} = \frac{Gm_2}{b} = \frac{1}{10}$$



$$P^\mu_{\rm rad} = \frac{G^3 m_1^2 m_2^2 \pi}{|b|^3} \frac{v_1^\mu + v_2^\mu}{\gamma+1} \mathcal{E}(\gamma) + \mathcal{O}(G^4)\,,$$

$$\mathcal{E}(\gamma) = e_1 + e_2 \log\left(\frac{\gamma+1}{2}\right) + e_3 \frac{\text{arccosh}\gamma}{\sqrt{\gamma^2-1}}\,.$$

$$e_1 = \frac{210\gamma^6 - 552\gamma^5 + 339\gamma^4 - 912\gamma^3 + 3148\gamma^2 - 3336\gamma + 1151}{48(\gamma^2-1)^{3/2}}\,,$$

$$e_2 = -\frac{35\gamma^4 + 60\gamma^3 - 150\gamma^2 + 76\gamma - 5}{8\sqrt{\gamma^2-1}}\,,$$

$$e_3 = \frac{\gamma(2\gamma^2-3)\,(35\gamma^4 - 30\gamma^2 + 11)}{16(\gamma^2-1)^{3/2}}\,.$$

Conclusion and perspectives

- Efficient PM expansion of impulse with worldlines for black holes and QFT methods

- Spin, finite size and absorption effects may be incorporated (but are subleading in PM)

- Other observables include the scattering waveform, the spin kick and the radiated angular momentum

- Intricate analytic functions such as complete elliptics and Calabi-Yau periods appear at high orders in the impulse

- Computational bottleneck of impulse is IBP reductions

- PM results have been used in both unbound and bound EOB models