Calabi-Yau Periods in QED

based on work in progress with Andreas von Manteuffel, Christoph Nega, Lorenzo Tancredi

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Motivation

- Task: compute scattering amplitudes as a Laurent expansion in the dimensional regulator ϵ, through differential equations
- Strategy: Use properties of scattering amplitudes (e.g. logarithmic singularities, iterated integrals) to organize the computation in a clever way!

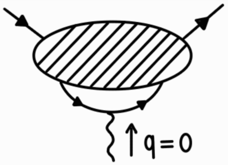
$\rightarrow$ *canonical basis*, that admits *canonical differential equations*:

- ϵ-factorized
- only simple poles
- independent differential forms, no hidden relations

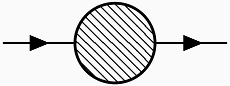
Goal: study this for QED self-energies

The $(g-2)$ and the Self-Energies in QED

- Calabi-Yau constants known to appear in the (g-2) of the electron: [Laporta, Remiddi, 1996] [Laporta, 2017]

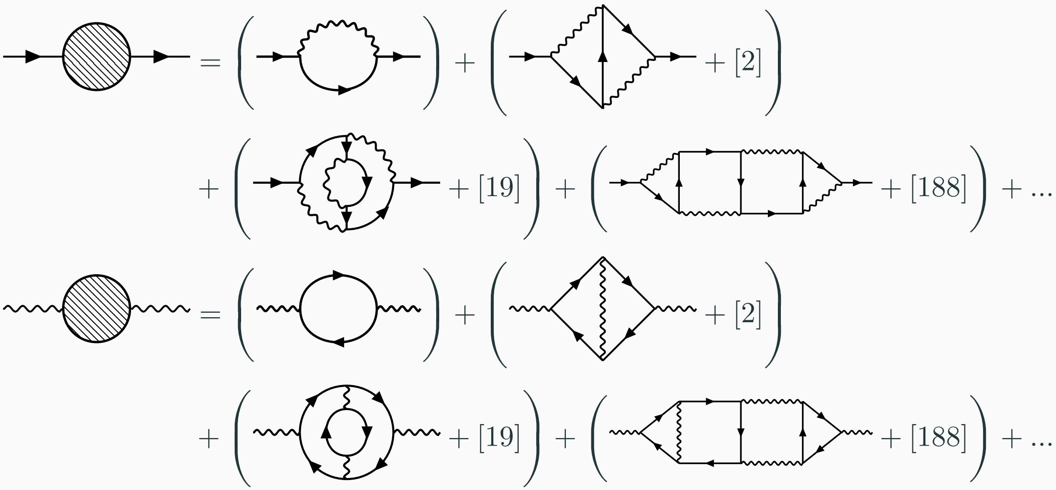


- related to the electron self-energy (1-particle-irreducible 2-point function)



$\rightarrow$ Calabi-Yau geometries
$\rightarrow$ two-point function $\Rightarrow$ only one variable $x \equiv p^2/m^2$ $\Rightarrow$ higher loop orders
$\rightarrow$ compute full correlation functions in a "real" theory (simplifications?)

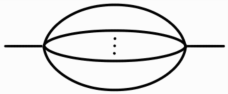
Loop Expansion



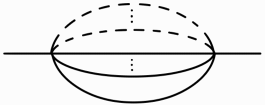
Note: can only cut odd (even) numbers of massive lines for electron (photon)

Banana Integrals

- Calabi-Yau geometries will appear (at least) through banana integrals
- equal-mass l-loop banana integral $\leftrightarrow$ $(l-1)$-dimensional Calabi-Yau



- adding massless lines does not change the geometry



- banana with n lines of equal mass and arbitrarily many massless lines
  $\leftrightarrow$ $(n-2)$-dimensional Calabi-Yau

  $\rightarrow$ odd-dimensional CYs for the electron

  $\rightarrow$ even-dimensional CYs for the photon

QED Self-Energies – An Overview

	electron self-energy	photon self-energy
1 loop	trivial	trivial
2 loop	elliptic [Sabry, 1962] [Hönemann, Tempest, Weinzierl, 2018]	trivial
3 loop	elliptic [Duhr et al., 2024]	K3 [Forner, Nega, Tancredi, 2025]
4 loop	CY3 and elliptic (+...?)	K3 (+...?)

(Canonical) Differential Equations

Scalar Feynman Integrals

Scattering amplitudes are in general tensorial objects, but can be written as

$$\mathcal{M} = \sum (\text{tensorial coefficients}) \times (\text{scalar Feynman integrals})$$

$\rightarrow$ Ultimately, need to compute only scalar Feynman integrals (typically $d = 4 - 2\epsilon$)

$$\mathcal{I}_{\nu_1,\ldots,\nu_m} = \int \left(\prod_{j=1}^{l} \frac{\mathrm{d}^d k_j}{i\pi^{d/2}} \right) \frac{1}{D_1^{\nu_1} \cdots D_m^{\nu_m}}$$

with *inverse propagators* of the form $D_j = q_j^2 - m_j^2$

$\rightarrow$ for a given set of propagators, the set of integrals $\mathcal{I}_{\nu_1,\ldots,\nu_m}$ forms an *integral family*

Master Integrals

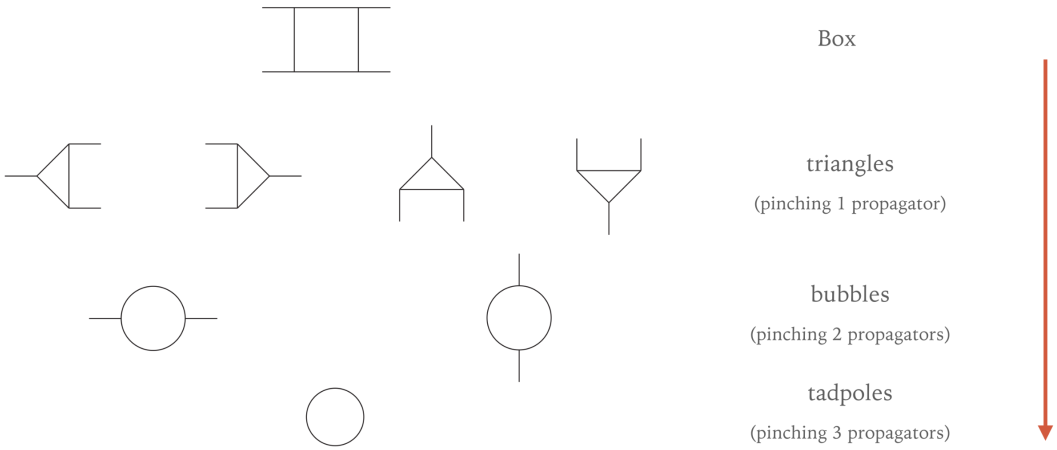
- integration by parts identities (IBPs):

$$\int \frac{\mathrm{d}^d k}{i\pi^{d/2}} \frac{\partial}{\partial k_\mu} \left(\frac{v_\mu}{D_1^{\nu_1} \cdots D_m^{\nu_m}} \right) = 0, \qquad v_\mu = a k_\mu + c_\mu.$$

$\rightarrow$ generate large number of equations among members of an integral family
$\rightarrow$ reduce the set of scalar integrals to a minimal set of *master integrals* $\vec{\mathcal{I}}$

- integrals containing the same denominators form a *sector*
- e.g., $\mathcal{I}_{1,1,0,0}$, $\mathcal{I}_{2,1,0,0}$, $\mathcal{I}_{1,1,-1,0}$ belong to the same sector, while $\mathcal{I}_{1,1,1,1}$ does not
- the sector containing $\mathcal{I}_{1,1,0,0}$, $\mathcal{I}_{2,1,0,0}$, $\mathcal{I}_{1,1,-1,0}$ is a *subsector* of $\mathcal{I}_{1,1,1,1}$

The "Tree" of Master Integrals



Differential Equations for Feynman Integrals

- then, use IBPs to derive differential equations for master integrals:
  [Kotikov, 1993] [Remiddi, 1997] [Gehrmann, Remiddi, 2000]

$$\frac{\mathrm{d}\vec{\mathcal{I}}}{\mathrm{d}s_i} = A(\epsilon; \{s_j\})\,\vec{\mathcal{I}}$$

  - matrix with rational entries
  - *dimensional regulator*; $d = d_0 - 2\epsilon$
  - kinematic invariants; here: $p^2 =: s$ and m^2
- in our case, $d_0 = 4$ or 2 ($\to$ dimensional shift relations)

$$\mathbf{A} = \begin{pmatrix} * & & & & & & \\ & * & & & 0 & & \\ & & * & & & & \\ & & & * & & & \\ & & & & * & & \\ & * & & & & \ddots & \\ & & & & & & * \end{pmatrix}$$

maximal cuts solve the homogeneous differential equation [Primo, Tancredi, 2017]

Canonical Bases and ϵ-Factorization

- goal: solve for $\vec{\mathcal{I}}$ as expansion in ϵ
- change of basis: $\vec{\mathcal{J}} = T\vec{\mathcal{I}}$ satisfies

$$\frac{\mathrm{d}\vec{\mathcal{J}}}{\mathrm{d}x} = B(\epsilon; x)\,\vec{\mathcal{J}}, \qquad B = TAT^{-1} + \frac{\mathrm{d}T}{\mathrm{d}x}T^{-1}$$

- strategy: find ϵ-*factorized* (*canonical*) basis: [Henn, 2013]

$$\frac{\mathrm{d}\vec{\mathcal{I}}_C}{\mathrm{d}x} = \epsilon A_C(x)\,\vec{\mathcal{I}}_C,$$

- yields formal solution at every order ϵ^n in terms of iterated integrals,

$$\frac{\mathrm{d}\vec{\mathcal{I}}_C^{(n)}}{\mathrm{d}x} = A_C\,\vec{\mathcal{I}}_C^{(n-1)},$$

  up to boundary constants

Canonical Bases in the Polylogarithmic Case

Canonical integrals: iterated integrals over logarithmic differential forms ("dlog"), with a constant prefactor ("leading singularity")

($\rightarrow$ they have uniform transcendental weight and are pure functions)
[Arkani-Hamed, Bourjaily, Cachazo, Trnka, 2012] [Kotikov, 2013]

E.g. consider the bubble for $\epsilon = 0$ (in Baikov representation):

$$\begin{aligned} \mathcal{I}_{1,1} &\sim \int \mathrm{d}z_1 \wedge \mathrm{d}z_2 \frac{1}{z_1 z_2 \sqrt{x^2 - 2x(z_1 + z_2 + 2) + (z_1 - z_2)^2}} \\ &= \int \mathrm{d}\log(f(z_1, z_2)) \wedge \mathrm{d}z_2 \frac{1}{2z_2 \sqrt{x^2 - 2x(z_2 + 2) + z_2^2}} \end{aligned}$$

QED Self-Energies at Higher Loop Orders

Feynman Diagrams and Master Integrals

	Feynman diagrams		Master integrals		Sectors	
	electron	photon	electron	photon	electron	photon
1 loop	1	1	2	2	2	2
2 loop	3	3	8	5	6	4
3 loop	20	20	51	36	31	21
4 loop	189	189	$\sim$1000	$\sim$500		

Electron Self-Energy at Two Loops

Can map all scalar integrals to one family,

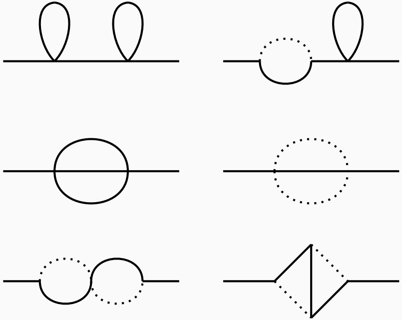
$$\mathcal{I}_{\nu_1,\ldots,\nu_5} = \int \frac{\mathrm{d}^d k_1}{i\pi^{d/2}} \int \frac{\mathrm{d}^d k_2}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}$$

$\mathcal{I}_{0,1,1,0,0}$, $\mathcal{I}_{1,1,0,1,0}$

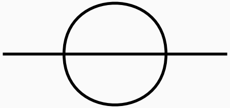
$\mathcal{I}_{0,1,1,1,0}$, $\mathcal{I}_{1,0,1,0,1}$

$\mathcal{I}_{0,2,1,1,0}$, $\mathcal{I}_{1,0,1,-1,1}$

$\mathcal{I}_{1,1,0,1,1}$, $\mathcal{I}_{1,1,1,1,1}$



Electron Self-Energy at Two Loops – The Sunrise Graph



The maximal cut (homogeneous solution) of the sunrise graph is *elliptic*:

$$\text{MC}\left(\mathcal{I}^{d=2}_{0,1,1,1,0}\right) \sim \int \mathrm{d}z_1 \wedge \mathrm{d}z_2 \frac{1}{2\left(x^2 - x(z_1 z_2 + 2) + z_1^2 z_2 + z_1(z_2 - 3)z_2 + 1\right)}$$

$$\sim \int \mathrm{d}z_2 \frac{1}{\sqrt{z_2(z_2-4)(z_2-(\sqrt{x}-1)^2)(z_2-(\sqrt{x}+1)^2)}} \log(f(z_1, z_2))$$

square root of 4th-order polynomial $\rightarrow$ elliptic curve $y^2 = P^{(4)}_{\text{Sun}}(z_2; x)$

new differential forms

$$\frac{\mathrm{d}z}{\sqrt{P^{(4)}_{\text{Sun}}(z;x)}}, \quad \frac{z^2\mathrm{d}z}{\sqrt{P^{(4)}_{\text{Sun}}(z;x)}} \quad \text{(derivative)}$$

Wronskian Splitting

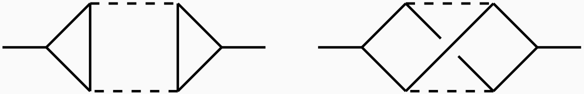
- even if we have the differential forms under control, we cannot get rid of the double poles
- starting from the derivative basis, we need a further basis rotation to reach ϵ-factorisation
- crucial step: split the Wronskian matrix, [Görges, Nega, Tancredi, Wagner, 2023] [Duhr et al., 2025]

$$\underbrace{\begin{pmatrix} \omega_0 & \omega_1 \\ \partial_x \omega_0 & \partial_x \omega_1 \end{pmatrix}}_{W} = \underbrace{\begin{pmatrix} \omega_0 & 0 \\ \partial_x \omega_0 & \frac{\Delta}{\omega_0} \end{pmatrix}}_{W_{\text{ss}}} \cdot \underbrace{\begin{pmatrix} 1 & \frac{\omega_1}{\omega_0} \\ 0 & 1 \end{pmatrix}}_{W_{\text{u}}},$$

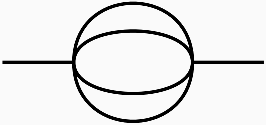
  and rotate with W_{ss}^{-1}
- In a limit where the geometry degenerates to a sphere, this construction gives a canonical basis in the polylogarithmic sense!

Photon Self-Energy at Three Loops

- 2 integral families, 36 master integras, 21 sectors
- top sectors

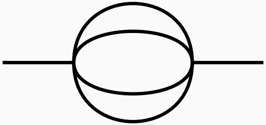


- again, there is only one sector which is not polylogarithmic on the maximal cut: the equal-mass banana sector



$\rightarrow$ first, consider only its maximal cut

The Equal-Mass Three-Loop Banana Graph



On the maximal cut we find the form:

$$\frac{\mathrm{d}z_1 \mathrm{d}z_2}{\sqrt{P^{(4)}(z_1, z_2; x)}}$$

$\to$ K3 surface $\to$ differential forms

$$\frac{\mathrm{d}z_1 \mathrm{d}z_2}{\sqrt{P^{(4)}(z_1, z_2; x)}} \quad \text{and derivatives}$$

Canonical Differential Equations for the Three-Loop Banana

Follow same procedure as before:

- transform to the derivative basis
- split up the Wronskian matrix,

$$W = W_{ss} \cdot \begin{pmatrix} 1 & \frac{\varpi_1}{\varpi_0} & \frac{\varpi_2}{\varpi_0} \\ 0 & 1 & \frac{\varpi_1}{\varpi_0} \\ 0 & 0 & 1 \end{pmatrix},$$

  and rotate with W_{ss}^{-1}
- use Griffiths transversality to obtain the quadratic relation

$$\varpi_0''(x) = \frac{1}{2}\left(-\frac{(x-8)}{(x-16)(x-4)x}\varpi_0(x) - \frac{4\left(x^2-15x+32\right)}{(x-16)(x-4)x}\varpi_0'(x) + \frac{\varpi_0'(x)^2}{\varpi_0(x)}\right),$$

  to eliminate $\varpi_0''(x)$

Additional Functions

To reach ϵ-factorization, we need a further rotation with two new functions

$$G_1(x) = -\int_0^x \mathrm{d}u \frac{(u-8)(u+8)^3 \varpi_0(u)^2}{32(u-16)^2(u-4)^2} = \frac{x}{32} + \frac{x^2}{64} + \frac{93x^3}{16384} + O(x^4)\,,$$

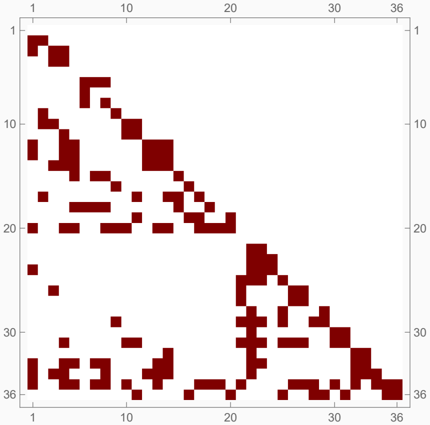
$$G_2(x) = \int_0^x \mathrm{d}u \frac{8G_1(u)}{\sqrt{(4-u)(16-u)}u\varpi_0(u)} = \frac{x}{32} + \frac{19x^2}{2048} + \frac{167x^3}{65536} + O(x^4)$$

A Look at The Full Differential Equation

Couplings of higher sectors require also

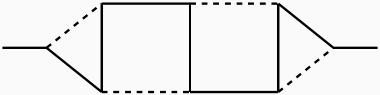
$$G_3(x) = \int_0^x \mathrm{d}u \frac{(u+2)\varpi_0(u)}{(4-u)^{3/2}\sqrt{u}}$$

$$= \frac{\sqrt{x}}{2} + \frac{5x^{3/2}}{32} + \frac{75x^{5/2}}{2048} + O(x^{7/2})$$

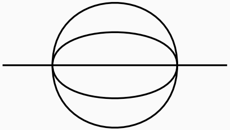


Electron Self-Energy at Four Loops: ”Ladder” Family

Consider the integral family with top sector



- 160 master integrals, 72 sectors
- contains the equal-mass banana as a subsector $\rightarrow$ periods of a CY threefold



holomorphic form

$$\frac{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3}{\sqrt{P_4(z_1,z_2,z_3;x)}}$$

+ tower of derivatives

$\rightarrow$ Wronskian splitting

$\rightarrow$ but we also encounter many elliptic sectors (on the maximal cut!)

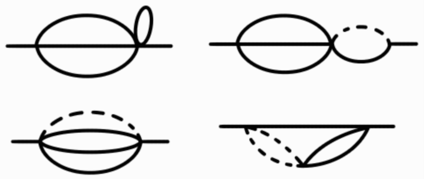
Sectors With Non-Trivial Geometries

Photon

- K3 at 3 loops (equal-mass banana)

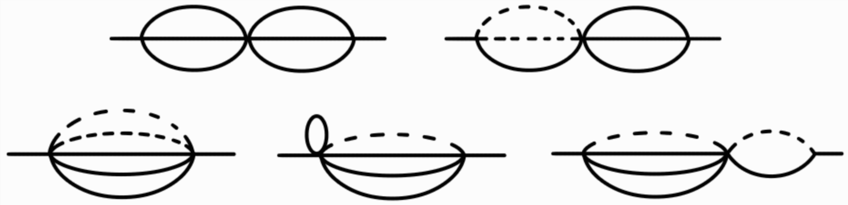
Electron

- 2 loop: sunrise (electron)
- 3 loop: four elliptic sectors [Duhr et al., 2024]:



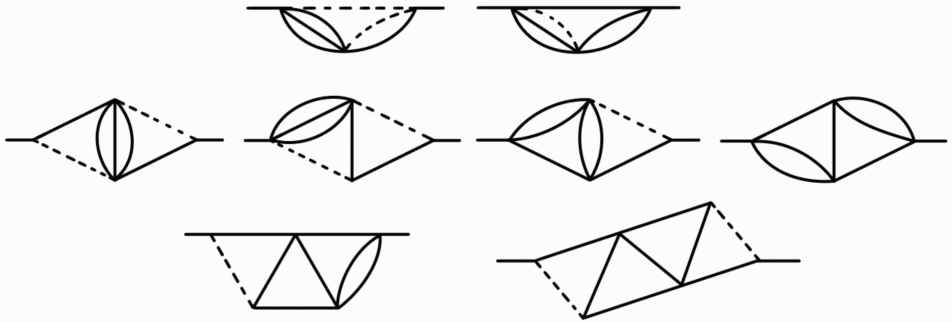
  Elliptic curves all given by $y^2 = P^{(4)}_{\text{Sun}}(z_2; x)$
- 4 loop ladder family of the electron self-energy: CY3 (equal-mass banana), but there are also 13 elliptic maximal cuts

Elliptic Sectors of the 4-Loop Ladder Family



These are obviously related to the sunrise!

Elliptic Sectors of the 4-Loop Ladder Family



But also these all relate back to the elliptic curve of the sunrise! (on the max cut)

Many of them couple to the CY3!

Summary and Outlook

- Calabi-Yau varieties play a fundamental role in perturbative QFT
- today we saw this explicitly in QED
- organize differential equations in a convenient way with canonical basis
- Many open challenges (which become especially relevant at 4 loops):
    - integrand analysis: how to find the right forms and the right parametrizations?
    - couplings between non-trivial geometries
    - new ”G”-functions: independence and geometric interpretation?
    - Why do all elliptic curves map to the sunrise?

      ⋮

Thank you! ☺