Parametric annihilators and differential equations for twisted integrals

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Single-Valued Periods in Scattering Amplitudes — 16.01.2026

Menu of the day

- Introduction & notation
- Parametric annihilators
  - How do we compute them?
- A similar technique for differential operators
- Applications
- Conclusions & outlook

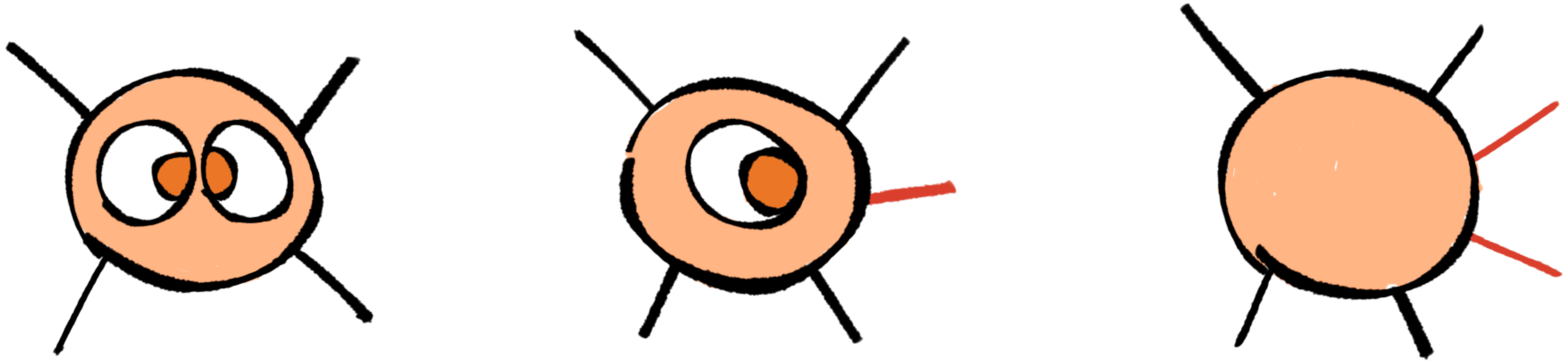
Introduction

Precision era @ colliders

- Precision physics as
  - test of the Standard model
  - gate to **new physics**

- High-Lumi upgrade of LHC :
- theory and experiments must have comparable uncertainties
  - needed: %-level accuracy:
    - perturbation theory @ NNLO and often N3LO
    - diagrams with increasing no. of **loops, legs & mass scales**

$@\mathcal{O}(\alpha_s^2)$:



Loop Integrals

- LEGO® blocks of perturbative QFT beyond tree level
- Key ingredient of phenomenological predictions
- Rich and interesting mathematical structures

Thousands of loop integrals appear when studying perturbative predictions!

- **Crucial:** Finding relations between them
- Loop integrals admit various integral representations with different tradeoffs and mathematical properties

This work

- Study & elaborate on the method of **parametric annihilators** for finding integral identities,
  - Focus on **parametric representations** of loop integrals
  - Extend applications to **different** representations
  - Similar technique for finding **differential equations**
- Provide an **implementation** of annihilators & differential operators based on modern linear solvers relying on cutting edge **finite-fields** techniques
- Implementation in the public Mathematica package: **`CALICO`**

Computing **A**nnihilators from **L**inear **I**dentities
Constraining (differential) **O**perators

Some useful definitions

$$\left.\begin{array}{l}\text{A list of variables: } \mathbf{z}=(z_1,\ldots,z_n)\\ \text{A list of exponents: } \alpha=(\alpha_1,\ldots,\alpha_n)\end{array}\right\}\longrightarrow \quad \mathbf{z}^{\alpha}=\prod_j z_j^{\alpha_j} \qquad |\alpha|=\sum_j \alpha_j$$

Monomials

total degree

We are interested in **integral families** of the form

These include **many parametrizations** of loop integrals

$$I_\alpha=\int \mathrm{d}^n\mathbf{z}\,\varphi_\alpha(\mathbf{z})\,u(\mathbf{z})$$

Rational factors raised to integer powers

Assumption:

$\varphi_\alpha(\mathbf{z})$ closed under monomial multiplication & differentiation

Twist
Multivalued

$$\begin{cases} u(\mathbf{z})=\prod_j B_j(\mathbf{z})^{\gamma_j} \\ u(\mathbf{z})=\exp F(\mathbf{z})\,\prod_j B_j(\mathbf{z})^{\gamma_j} \end{cases}$$

What do we want to do?

- Finding and solving **linear relations** satisfied by integrals having the form of I_α
- Express integrals within a family as a linear combination of a set of independent **master integrals (MIs)**

$$I_\alpha = \sum_{\beta \in \mathrm{MIs}} c_{\alpha\beta} I_\beta$$

Crucial ingredient are **Integration-By-Parts identities (IBP):**

$$\int \mathrm{d}^n \mathbf{z}\, \partial_j \Big(\varphi_\alpha(\mathbf{z})\, u(\mathbf{z}) \Big) = 0$$

(Regulated integrals vanish at the integration boundary)

$$\int \mathrm{d}^n\mathbf{z}\,\partial_j\Big(\varphi_\alpha(\mathbf{z})\,u(\mathbf{z})\Big) = 0$$

By expanding the LHS of the IBP relation we get :

$$\int \mathrm{d}^n\mathbf{z}\,\partial_j\Big(\varphi_\alpha(\mathbf{z})\,u(\mathbf{z})\Big) = \boxed{\int \mathrm{d}^n\mathbf{z}\,\left(\partial_j\varphi_\alpha(\mathbf{z})\right)\,u(\mathbf{z}) + \int \mathrm{d}^n\mathbf{z}\,\varphi_\alpha(\mathbf{z})\,\left(\partial_j\log u(\mathbf{z})\right)\,u(\mathbf{z})}$$

We get:

- Non-trivial identities among the integrals
- $\partial_j \log u(\mathbf{z})$ usually creates terms that cannot be absorbed in $\varphi_\alpha(\mathbf{z})$

not a relation within the original integral family!

Parametric annihilators

Integral identities via parametric annihilators

Parametric annihilator of order o of $u(\mathbf{z})$

[Baikov (1996); Lee (2014); Bitoun, Bogner, Klausen, Panzer (2017)]

$$\begin{aligned}\hat{A} = & c_0(\mathbf{z}) + \sum_j c_j(\mathbf{z})\,\partial_j + \sum_{j_1 \leq j_2} c_{j_1 j_2}(\mathbf{z})\,\partial_{j_1}\partial_{j_2} \\ & + \cdots + \sum_{j_1 \leq \cdots \leq j_o} c_{j_1 \cdots j_o}(\mathbf{z})\,\partial_{j_1} \cdots \partial_{j_o}\end{aligned}$$

$c_{j_1 j_2 \ldots}(\mathbf{z})$ polynomials in $\mathbf{z}$

Such that $\boxed{\hat{A}\,u(\mathbf{z}) = 0}$

$$\hat{A}\, u(\mathbf{z}) = 0$$

For any annihilator $\hat{A}$, we have infinitely many integral identities

$$\int \mathrm{d}^n\mathbf{z}\, \varphi_\alpha(\mathbf{z})\, \hat{A}\, u(\mathbf{z}) = 0, \ \ \forall \alpha$$

symbolic α

Using IBPs on derivatives, we get a **template identity** for symbolic α

$$\int u\, (\varphi_\alpha c_0) - \sum_j \int u\, (\partial_j c_j \varphi_\alpha) + \cdots + (-1)^o \sum_{j_1 \leq \cdots \leq j_o} \int u\, \partial_{j_1} \cdots \partial_{j_o} (c_{j_1 \cdots j_o} \varphi_\alpha) = 0$$

`All the integrals belong to the family` I_α

Laporta algorithm

[Chetyrkin, Tkachov (1981); Laporta (2000)]

$$\int u\,(\varphi_\alpha c_0) - \sum_j \int u\,(\partial_j c_j \varphi_\alpha) + \cdots + (-1)^o \sum_{j_1\leq\cdots\leq j_o} \int u\,\partial_{j_1}\cdots\partial_{j_o}(c_{j_1\cdots j_o}\varphi_\alpha) = 0$$

Seeding the template eq.s : replacing symbolic α with integer numbers

- Applying each template identity to a large number of seed integrals: obtain a **linear system** of equations
- Choice of an **ordering**: express **complex** integrals as a function of **simple** ones
- solving it: reduction to **master integrals**★

$$I_\alpha = \sum_{\beta\in\text{MIs}} c_{\alpha\beta} I_\beta$$

Computational bottleneck in state-of-the-art calculations

★ Additional relations may exist, such as symmetry relations

Properties of parametric annihilators

$\hat{A}$ is an annihilator $\Rightarrow$ $\mathbf{z}^{\beta}\hat{A}$ and $\partial_j\hat{A}$ are annihilators

Set of annihilators of $u(\mathbf{z})$ is a D-module

$$\int \mathrm{d}^n\mathbf{z}\,\varphi_\alpha(\mathbf{z})\left(\mathbf{z}^{\beta}\hat{A}\,u(\mathbf{z})\right) = \int \mathrm{d}^n\mathbf{z}\left(\mathbf{z}^{\beta}\varphi_\alpha(\mathbf{z})\right)\hat{A}\,u(\mathbf{z})$$

$$\int \mathrm{d}^n\mathbf{z}\,\varphi_\alpha(\mathbf{z})\left(\partial_j\hat{A}\,u(\mathbf{z})\right) = -\int \mathrm{d}^n\mathbf{z}\left(\partial_j\varphi_\alpha(\mathbf{z})\right)\hat{A}\,u(\mathbf{z})$$

$\varphi_\alpha(\mathbf{z})$ closed under monomial multiplication & differentiation

interested in a **minimal** set of **generators**:

set of annihilators $\hat{A}$ independent modulo linear combinations of $\mathbf{z}^{\beta}\hat{A}$ and $\partial_j\hat{A}$

Differential operators

Differential equations

[Barucchi, Ponzano (1973); Kotikov (1991); Bern, Dixon, Kosower (1994); Gehrmann, Remiddi (2000)]

- Integrals in the form of I_α also depend on additional **free parameters** x (e.g. kinematic invariants)
- Studying of **analytic structure** & their numerical or analytical **evaluation**
- Reducing the derivative of MIs with respect to x to MIs, write a **system of differential equations** satisfied by the MIs themselves

$$\partial_x I_\alpha = \sum_{\beta \in \mathrm{MIs}} M_{\alpha\beta} I_\beta\,, \qquad \text{for } \alpha \in \mathrm{MIs}.$$

x **free parameter of the integrals**

Differential equations via differential operators

- Derive an operator $\hat{O}_x$ that realizes differentiation with respect to x

$$\hat{O}_x = c_0^{(x)}(\mathbf{z}) + \sum_j c_j^{(x)}(\mathbf{z})\,\partial_j + \sum_{j_1 \le j_2} c_{j_1 j_2}^{(x)}(\mathbf{z})\,\partial_{j_1}\partial_{j_2}$$

$$+ \cdots + \sum_{j_1 \le \cdots \le j_o} c_{j_1 \cdots j_o}^{(x)}(\mathbf{z})\,\partial_{j_1} \cdots \partial_{j_o}$$

$c_{j_1 j_2 \cdots}^{(x)}$ polynomials

Such that $\boxed{\hat{O}_x\, u(\mathbf{z}) = \partial_x\, u(\mathbf{z})}$

Differential equations via differential operators

- Explicitly, integrating by parts all derivatives in $\hat{O}_x$

$$\begin{aligned}\partial_x I_\alpha &= \int (\hat{O}_x u)\,\varphi_\alpha + \int u\,(\partial_x\varphi_\alpha)\\ &= \int u\,(\varphi_\alpha c_0^{(x)}) - \sum_j \int u\,(\partial_j c_j^{(x)}\varphi_\alpha) + \cdots\\ &\quad + (-1)^o \sum_{j_1\le\cdots\le j_o} \int u\,\partial_{j_1}\cdots\partial_{j_o}(c^{(x)}_{j_1\cdots j_o}\varphi_\alpha) + \int u\,(\partial_x\varphi_\alpha)\end{aligned}$$

- Our preferred way to compute derivatives
  - Alternatively: recurrence relations

How to compute parametric annihilators

Computing annihilators via linear constraints

- Computing parametric annihilators up to a certain order and polynomial degree
- implemented in the `CALICO` package

Step 1 : From annihilators to syzygies

$$\mathbf{f}(\mathbf{z}) \cdot \mathbf{g}(\mathbf{z}) = 0$$ Syzygy equation

known polynomials
$$\mathbf{f}(\mathbf{z}) = \{f_1(\mathbf{z}), \ldots, f_n(\mathbf{z})\}$$

unknown polynomials
$$\mathbf{g}(\mathbf{z}) = \{g_1(\mathbf{z}), \ldots, g_n(\mathbf{z})\}$$

Syzygy sol.s form a module:

$\mathbf{g}^{(k)}(\mathbf{z})$ set of solutions $\Rightarrow \mathbf{g}(\mathbf{z}) = \sum_k p_j(\mathbf{z})\mathbf{g}^{(k)}(\mathbf{z})$ is also a solution

$p_j(\mathbf{z})$ arbitrary polynomials

Start from $\hat{A}\, u(\mathbf{z}) = 0 \;\Rightarrow$ Identify $c_{j_1,\ldots,j_k}(\mathbf{z})$ as the unknown polynomials $\mathbf{g}(\mathbf{z})$ of a syzygy $\mathbf{f}(\mathbf{z}) \cdot \mathbf{g}(\mathbf{z}) = 0$

$$\frac{1}{u(\mathbf{z})}\,\hat{A}\, u(\mathbf{z}) = 0$$

- Insert the expression for $\hat{A} = c_0(\mathbf{z}) + \sum c_j(\mathbf{z})\partial_j + \ldots$
- Collect LHS under a common denominator
- Imposing vanishing of the numerator ⇒ obtain a syzygy equation for $c_{j_1,\ldots,j_k}(\mathbf{z})$

- Example : **first-order annihilator**

$$\hat{A} = c_0(\mathbf{z}) + \sum_j c_j(\mathbf{z})\,\partial_j \qquad\qquad u(\mathbf{z}) = \prod_j B_j(\mathbf{z})^{\gamma_j}$$

$$c_0(\mathbf{z}) \prod_k B_k(\mathbf{z}) + \sum_{j=1}^{n} c_j(\mathbf{z}) \sum_k \gamma_k \left(\partial_j B_k(\mathbf{z})\right) \prod_{l \neq k} B_l(\mathbf{z}) = 0$$

Form of a syzygy equation

$$\mathbf{f}(\mathbf{z}) \cdot \mathbf{g}(\mathbf{z}) = 0$$

Step 2: Solving syzygy equations via linear constraints

[Schabinger (2015)]

- Ansatz for the solution

$$\mathbf{g}(\mathbf{z}) = \sum_j \sum_\alpha c_{j\alpha}\, \mathbf{z}^\alpha\, \hat{e}_j$$

$|\alpha| \leq d$ for some max degree d

$\hat{e}_j$ unit vector in the j-th direction

- Plug it into $\mathbf{f}(\mathbf{z}) \cdot \mathbf{g}(\mathbf{z}) = 0$
- Impose coeff.s of each monomial $\mathbf{z}$ vanish

$\Rightarrow \Big\{$ **Linear system for $c_{j\alpha}$**

`CALICO uses the efficient linear solver of FiniteFlow, based on finite-field methods`

[Peraro (2019)]

Applications

Hypergeometric ${}_2F_1$

$$I_\alpha = \int_0^1 \mathrm{d}z\, \varphi_\alpha(z)\, u(z)$$

$$\varphi_\alpha(z) = z^\alpha, \qquad u(z) = z^{b_2-1}\,(1-z)^{b_3-b_2-1}\,(1-x\,z)^{-b_1}$$

Related to the hypergeometric ${}_2F_1$

$$I_\alpha = \frac{\Gamma(b_2+\alpha)\Gamma(b_3-b_2)}{\Gamma(b_3+\alpha)}\,{}_2F_1(b_1,\, b_2+\alpha,\, b_3+\alpha;\, x)$$

1. `1 first-order annihilator → reduction to 2 MIs` $\{I_0, I_1\}$
2. `First-order differential operator`

$$\partial_x \begin{pmatrix} I_1 \\ I_0 \end{pmatrix} = \begin{pmatrix} \frac{b_1\,x-b_3}{(1-x)\,x} & \frac{b_2}{(1-x)\,x} \\ \frac{b_1-b_3}{1-x} & \frac{b_2}{1-x} \end{pmatrix} \cdot \begin{pmatrix} I_1 \\ I_0 \end{pmatrix}$$

Generalised to Hypergeometric ${}_{n+1}F_n$

Loop integrals

Momentum-space representation

$$J_\alpha = J_{\alpha_1 \cdots \alpha_n} = \int \prod_{i=1}^{\ell} \frac{\mathrm{d}^d k_i}{i\pi^{d/2}} \frac{1}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}}$$

$$D_{F,j} = l_j \cdot v_j - m_j^2$$

$$D_{F,j} = l_j^2 - m_j^2$$

l_j linear combination of k_j,
v_j linear combination of p_j

$D_j s$ are generalised denominators

- **Proper denominators:** D_j such that $\alpha_j > 0$
- Irreducible scalar products (ISPs): D_j such that $\alpha_j \leq 0$

IBPs in momentum space

[Tkachov (1981), Chetyrkin, Tkachov (1981)]

$$\int \prod_{i=1}^{\ell} \frac{\mathrm{d}^d k_i}{i\pi^{d/2}} \frac{\partial}{\partial k_j^\mu} \frac{v^\mu}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}} = 0, \quad \text{with } v^\mu = k_i^\mu, p_i^\mu$$

Parametric representations of loop integrals

Baikov

$$I_\alpha = \int \mathrm{d}^n\mathbf{z}\, \frac{1}{\mathbf{z}^\alpha}\, B(\mathbf{z})^\gamma$$

Also **Loop-by-Loop Baikov**
& **Duals of loop integrals**

Lee-Pomeransky

$$I_\alpha = \int \mathrm{d}^n\mathbf{z} \left(\prod_{j=1}^{n} \frac{1}{\Gamma(\alpha_j)} \right) \mathbf{z}^{\alpha-1} G(\mathbf{z})^{-d/2}$$

$$G(\mathbf{z}) = \mathcal{U}(\mathbf{z}) + \mathcal{F}(\mathbf{z})$$

Schwinger

$$I_\alpha = \int \mathrm{d}^n\mathbf{z} \left(\prod_{j=1}^{n} \frac{1}{\Gamma(\alpha_j)} \right) \mathbf{z}^{\alpha-1} \exp\left[-\mathcal{F}(\mathbf{z})/\mathcal{U}(\mathbf{z}) \right] \mathcal{U}(\mathbf{z})^{-d/2}$$

Baikov representation

$$I_\alpha = \int \mathrm{d}^n\mathbf{z}\, \varphi_\alpha(\mathbf{z})\, u(\mathbf{z})$$

Reminder

Baikov

$$I_\alpha = \int \mathrm{d}^n\mathbf{z}\, \frac{1}{\mathbf{z}^\alpha}\, B(\mathbf{z})^\gamma$$

Loop-By-Loop Baikov [Frellesvig, Papadopoulos (2017)]

$$I_\alpha = \int \mathrm{d}^n\mathbf{z}\, \frac{1}{\mathbf{z}^\alpha} \prod_{j=1}^{2\ell-1} B_j(\mathbf{z})^{\gamma_j}$$

Change of integration vars from loop momenta to generalised denominators

- Require a full set of ISPs
- $\deg B = \min(\ell + e, 2\ell)$
- Less variables in LBL (Baikov change of var.s is done one loop at a time)

Intersection theory in a nutshell

[Mastrolia, Mizera (2019)]

[Brunello, Cacciatori, Caron-Huot, Chestnov, Crisanti, Duhr, Frellesvig, GF, Gasparotto, Giroux, Laporta, Maggio, Mandal, Mastrolia, Matsubara-Heo, Mattiazzi, Mizera, Munch, Peraro, Pokraka, Porkert, Semper, Smith, Sohnle, Stavinski, Takayama (2019 – Ongoing)]

Introduction of a scalar product, **intersection number**, between

- **Integrals I_α**

$$I_\alpha = \int \mathrm{d}^n\mathbf{z} \, \frac{1}{\mathbf{z}^\alpha} \, B(\mathbf{z})^\gamma$$

- **Dual integrals $I^\star_\alpha$**

$$I^\star_\alpha = \int \mathrm{d}^n\mathbf{z} \, \frac{1}{\mathbf{z}^\alpha} \, B(\mathbf{z})^{-\gamma}$$

- Calculation of intersection numbers $\longrightarrow$ standard procedure: **recursive** algorithm in the integration var.s

- Focus on **dual integrals**
  - Linear relations
  - Differential equations (necessary step in the intersection numbers calculation)

Dual integrals & regulators

$$I^\star_\alpha = \int \mathrm{d}^n\mathbf{z}\,\frac{1}{\mathbf{z}^\alpha}\,B(\mathbf{z})^{-\gamma}$$

- $(z_1, \ldots, z_m)$ proper denominators
- $(z_{m+1}, \ldots, z_n)$ ISPs

$\Big\}$ $\alpha_j \leq 0$ for $j > m$

Problem: need of additional regulators $\dfrac{1}{z_j^{\alpha_j}} \to \dfrac{1}{z_j^{\alpha_j-\rho_j}}$ for $j \leq m$

[GF, Peraro (2023)]

$$I^\star_\alpha = \underbrace{\prod_{j=1}^{m} \rho_j^{\Theta(\alpha_j-1/2)} \int \mathrm{d}^n\mathbf{z}\,\frac{1}{\mathbf{z}^{\alpha-\rho}}}_{\varphi_\alpha(\mathbf{z})}\;\underbrace{B(\mathbf{z})^{-\gamma}}_{u(\mathbf{z})}$$

work on the leading coefficients $\rho_j \to 0$

Reduction of dual integrals

- Method of annihilator can be applied straightforwardly (Baikov, LBL Baikov)
- Tmp ids require the knowledge of $\varphi_\alpha(\mathbf{z})$ under
    - Multiplication by monomials

$$z_j^{\beta_j}\,\varphi_\alpha(\mathbf{z}) = \begin{cases} \varphi_{\alpha-\beta_j\hat{e}_j}(\mathbf{z}) & \text{if } \alpha_j > \beta_j \text{ or } \alpha_j \leq 0 \\ 0 & \text{if } 0 < \alpha_j \leq \beta_j\,. \end{cases}$$

    - Differentiation

$$\partial_j\,\varphi_\alpha(\mathbf{z}) = \begin{cases} -(\alpha_j - \delta_{\alpha_j 0})\,\varphi_{\alpha+\hat{e}_j}(\mathbf{z}) & \text{if } j \leq m \\ -\alpha_j\,\varphi_{\alpha+\hat{e}_j}(\mathbf{z}) & \text{if } j > m. \end{cases}$$

- Approach tested for both
    - Reduction of dual integrals
    - Computation of DEQ satisfied by dual integrals in fewer variables
- Successful comparison on all examples of [GF, Peraro (2023)]

Lee-Pomeransky and Schwinger rep.s

Lee-Pomeransky

$$I_\alpha = \int \mathrm{d}^n \mathbf{z} \left(\prod_{j=1}^{n} \frac{1}{\Gamma(\alpha_j)} \right) \mathbf{z}^{\alpha-1} G(\mathbf{z})^{-d/2}$$

$$G(\mathbf{z}) = \mathcal{U}(\mathbf{z}) + \mathcal{F}(\mathbf{z})$$

Schwinger

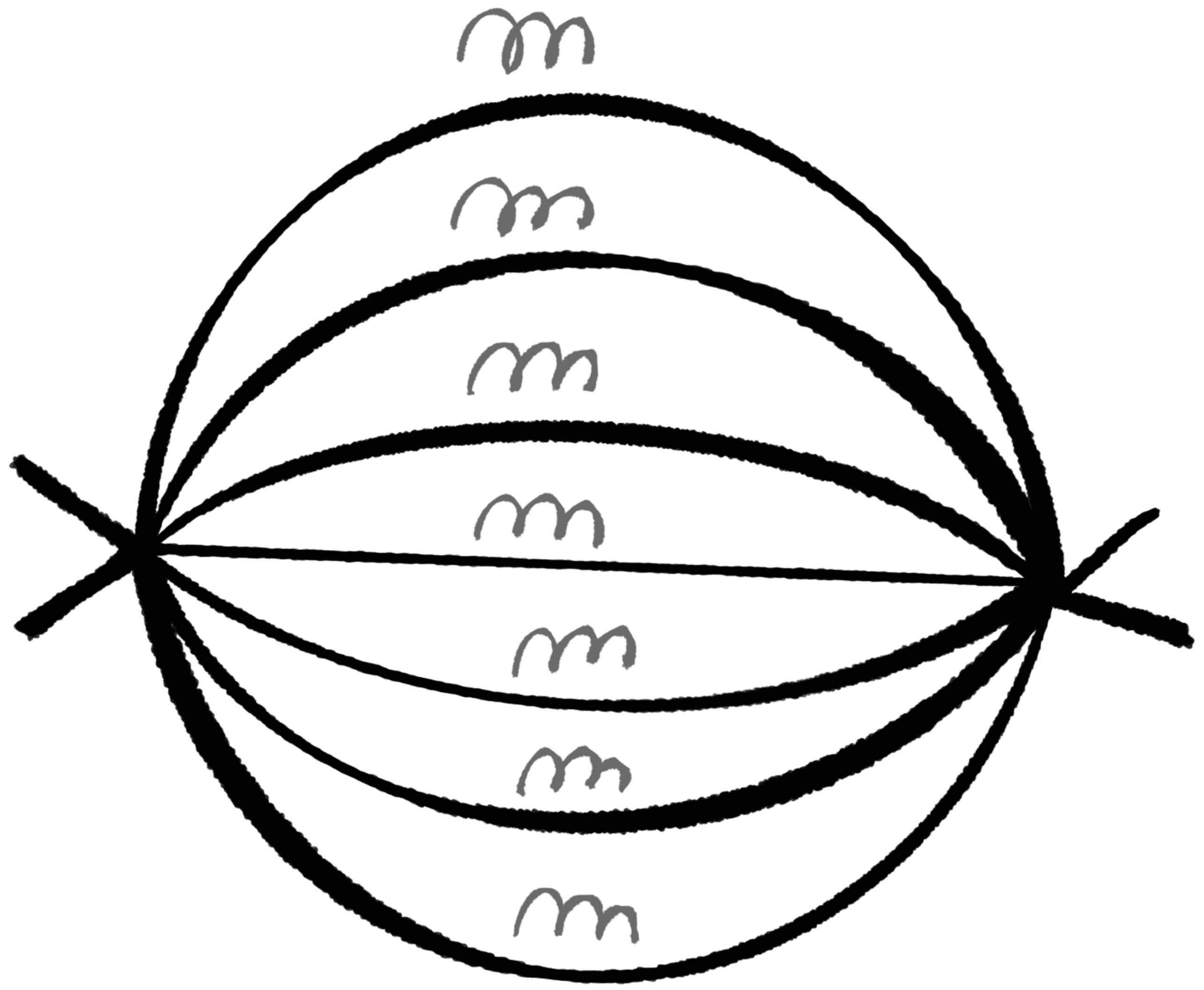
$$I_\alpha = \int \mathrm{d}^n \mathbf{z} \left(\prod_{j=1}^{n} \frac{1}{\Gamma(\alpha_j)} \right) \mathbf{z}^{\alpha-1} \exp\left[- \mathcal{F}(\mathbf{z})/\mathcal{U}(\mathbf{z}) \right] \mathcal{U}(\mathbf{z})^{-d/2}$$

- n integration variables (# number of proper denominators)
- no need of introducing ISPs (can be added if required)
- $\deg G = \ell + 1$ at ℓ loops

- twist includes exp. factor
- empirically:
  - annihilators have lower degree
  - More frequent use of 2nd order ann.
  - tmp id.s often fewer and simpler

L loop bananas

ℓ-loop & one internal mass m, defined by the set of $\ell + 1$ proper denominators:



$$D_j = k_j^2 - m^2 \quad \text{for } j = 1,\ldots,\ell$$

$$D_{\ell+1} = (k_1 + \cdots + k_\ell - p)^2 - m^2$$

Momentum space has $(\ell + 2)(\ell - 1)/2$ ISPs → for $\ell = 6$, it has 20 ISPs

Use Lee-Pomeranski or Schwinger representation to

- Reduce to MIs
- Derive DEQs

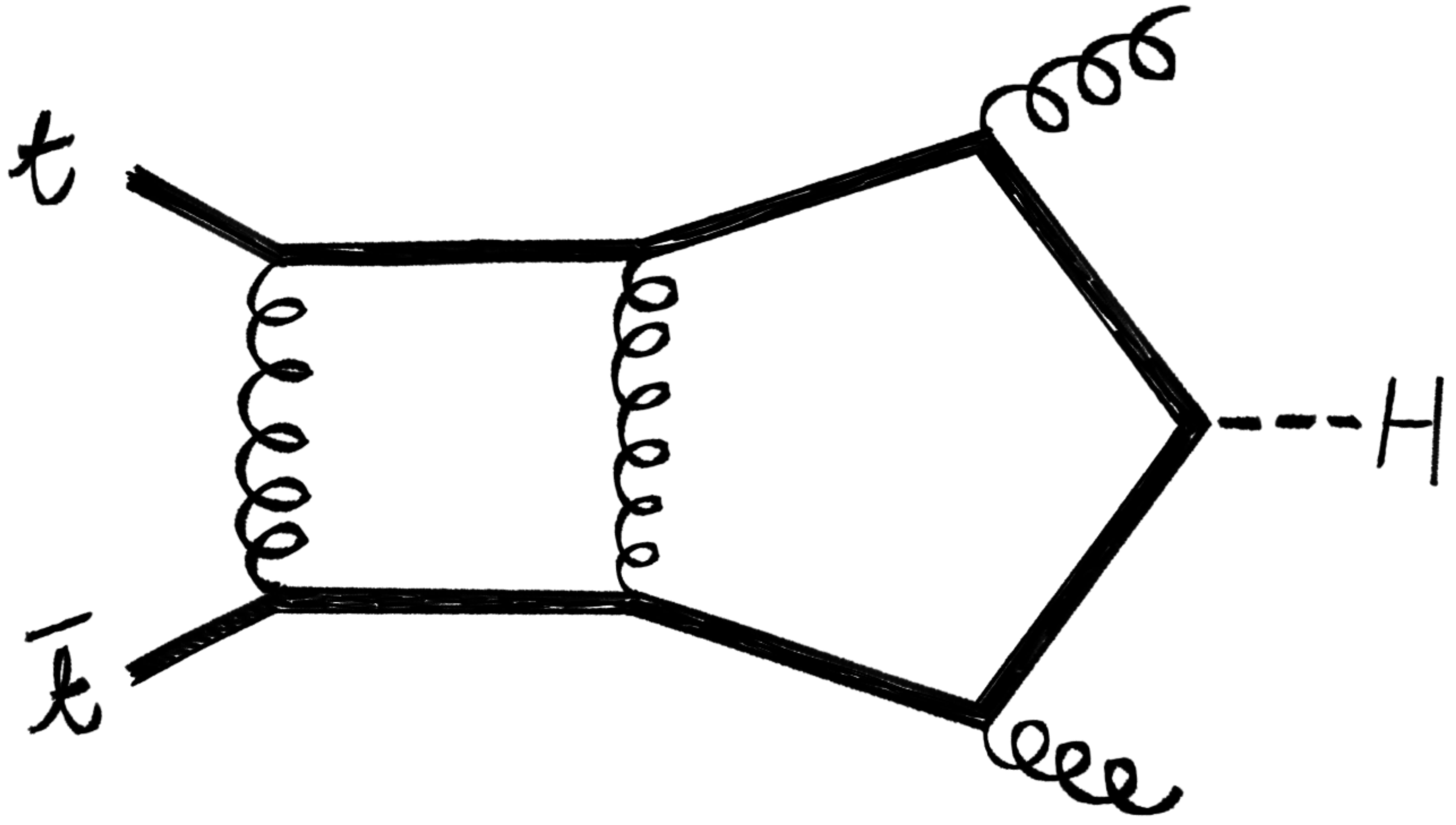
Without the need of additional ISPs!

Done in a couple of minutes on a laptop up to 6 loops (mostly spent computing annihilators and template eqs)

Family for $t\bar{t}H$ production

Cutting-edge example

- Many different scales
- Many external legs



Using the integral representations:

- Schwinger (More efficient!)
- Lee-Pomeranski

Tested: numerical reduction on a laptop with up to 3 extra powers of denominators

Finding relations between integrals with constant numerator and higher powers of proper denominators

Useful for finding integrals with good properties

- Quasi-finite [von Manteuffel, Panzer, Schabinger (2014)]
- Pure functional form [Henn (2013)]

Conclusions & Outlook

- Parametric annihilators are a useful tool for finding linear relations between integrals

- Allow to use integral parametrizations tailored to specific problems

    - New applications to LBL Baikov, duals of loop integrals, Schwinger parametrisation

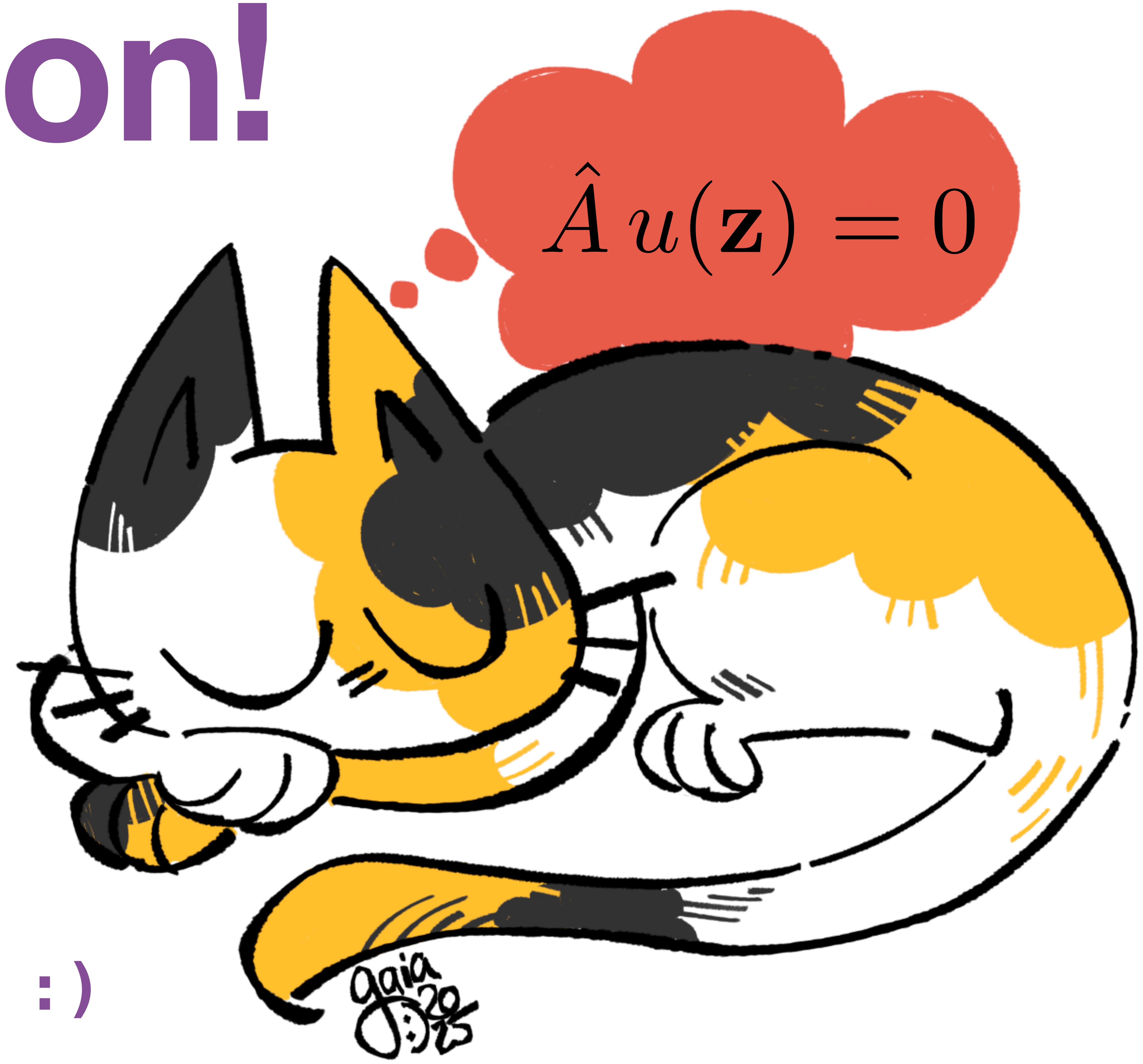
- Introduced similar technique for deriving differential equations

- Implementation released in the public package `CALICO`

★ **Bonus:** can also solve syzygy equations and polynomial decomposition problems

https://github.com/fontana-g/calico

Thank you for your attention!



… `more drawings @qftoons :)`