Single-Valued Periods in Feynman Integrals

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MITP-Youngstars Program on "Single-Valued Periods in Scattering Amplitudes"

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* **Periods:**

[Kontsevich, Zagier]

$$\int_{\gamma} \omega$$

ω: Rational/algebraic differential form

γ: cycle defined by polynomial inequalities

Example: $\pi = \int_{x^2+y^2\leq 1} dx\,dy$, $\log z = \int_1^z \frac{dt}{t}$, $K(\lambda) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\lambda t^2)}}$

Multiple polylogarithms (MPLs):

$$G(a_1,\dots,a_n;z) = \int_0^z \frac{dt}{t-a_1}\, G(a_2,\dots,a_n;t) \quad ; \quad G(a_1;z) = \log\left(1-\frac{z}{a_1}\right)$$

$$G(0;z) = \log z\,, \qquad G(\underbrace{0,\dots,0}_{n-1},1;z) = -\mathrm{Li}_n(z)$$

* Feynman Integrals

$$\int \frac{d^D k_1 \ldots d^D k_L}{D_1 \ldots D_P} \qquad D_j = q_j^2 - m_j^2 + i\varepsilon$$

→ DimReg: $D = d_0 - 2\varepsilon$ → Laurent expansion

* Facts:

* Feynman Integrals are periods
[Belkale, Brosnan; Bogner, Weinzierl; Brown, ...]

* UNITARITY ⇒ Feynman Integrals are multivalued

SINGLE-VALUED PERIODS???

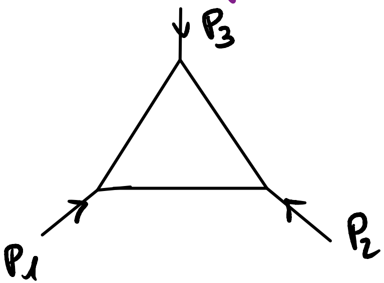
Feynman Integrals &

Single-Valued

Multiple Polylogarithms

* Many Feynman integrals evaluate to MPLs.

* **Example**: One-loop triangle with massless propagators in $D=4$

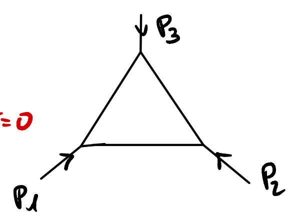


$$\sim 2\,\mathrm{Li}_2(z) - 2\,\mathrm{Li}_2(\bar{z}) + \log(z\bar{z})\log\frac{1-z}{1-\bar{z}}$$

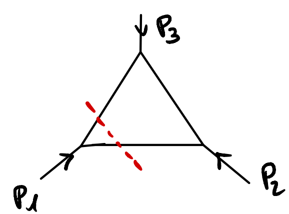
$$\frac{p_1^2}{p_3^2} = z\bar{z} \qquad\qquad \frac{p_2^2}{p_3^2} = (1-z)(1-\bar{z})$$

UNITARITY: There are branch cuts starting at $p_i^2 = 0$

$\mathrm{Disc}_{p_1^2=0}$



$\sim$



A useful tool to study MPLs are symbols

[Goncharov, Spradlin, Vergu, Volovich]

$$\mathcal{S}(\log x) = x$$

$$\mathcal{S}(\mathrm{Li}_n(x)) = -(1-x) \otimes x \otimes \dots \otimes x$$

$$\mathcal{S}(A \cdot B) = \mathcal{S}(A) \sqcup\!\sqcup\, \mathcal{S}(B)$$

↳ shuffle product

Ex: $a \sqcup\!\sqcup\, b = a \otimes b + b \otimes a$

The symbol turns functional identities among MPLs into algebraic relations!

* Symbols & discontinuities : First entries encode branch cuts :

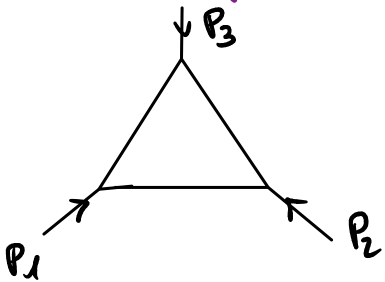
$$\mathcal{S}(\mathrm{Li}_n(x)) = -(1-x)\otimes x \otimes \ldots \otimes x \rightarrow$$ Branch cut starts at $x=1$

$$\mathcal{S}(\log x \log(1-x)) = x \otimes (1-x) + (1-x)\otimes x \rightarrow$$ 2 Branch cuts, starting at $x=1$ and $x=0$

* Symbols of Feynman Integrals encode unitarity in their first entries

First entry condition : The first entries of Feynman integrals with massless propagators must be (ratios of) Mandelstam invariants

*** Example :**



$$\frac{p_1^2}{p_3^2} = z\bar{z} \qquad \frac{p_2^2}{p_3^2} = (1-z)(1-\bar{z})$$

$$\sim 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log(z\bar{z})\log\frac{1-z}{1-\bar{z}}$$

$$\text{Symbol} = (1-z)(1-\bar{z}) \otimes \frac{\bar{z}}{z} + (z\bar{z}) \otimes \frac{1-\bar{z}}{1-\bar{z}}$$

$$= \frac{p_2^2}{p_3^2} \otimes \frac{\bar{z}}{z} + \frac{p_1^2}{p_3^2} \otimes \frac{1-\bar{z}}{1-z}$$

First entry condition

$\leftrightarrow$ Correct branch cut structure

CRUCIAL OBSERVATION:

$$\text{Symbol} = (1-z)(1-\bar{z}) \otimes \frac{\bar{z}}{z} + (z\bar{z}) \otimes \frac{1-\bar{z}}{1-\bar{z}}$$

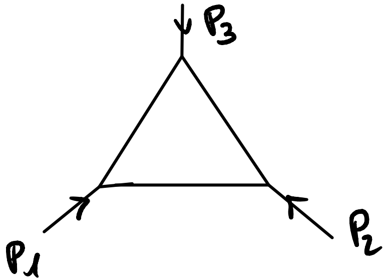
$$= \frac{p_2^2}{p_3^2} \otimes \frac{\bar{z}}{z} + \frac{p_1^2}{p_3^2} \otimes \frac{1-\bar{z}}{1-z}$$

→ In some region, z and $\bar{z}$ will be complex conjugates of each other

→ $z\bar{z} = |z|^2$, $(1-z)(1-\bar{z}) = |1-z|^2$ → SV !

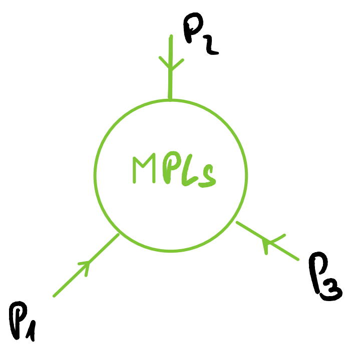
CONCLUSION:

$$\boxed{\begin{array}{c}\text{CORRCT BRANCH CUTS in } p_i^2 \\ \Longleftrightarrow \\ \text{SV as a function of complex } z\end{array}}$$



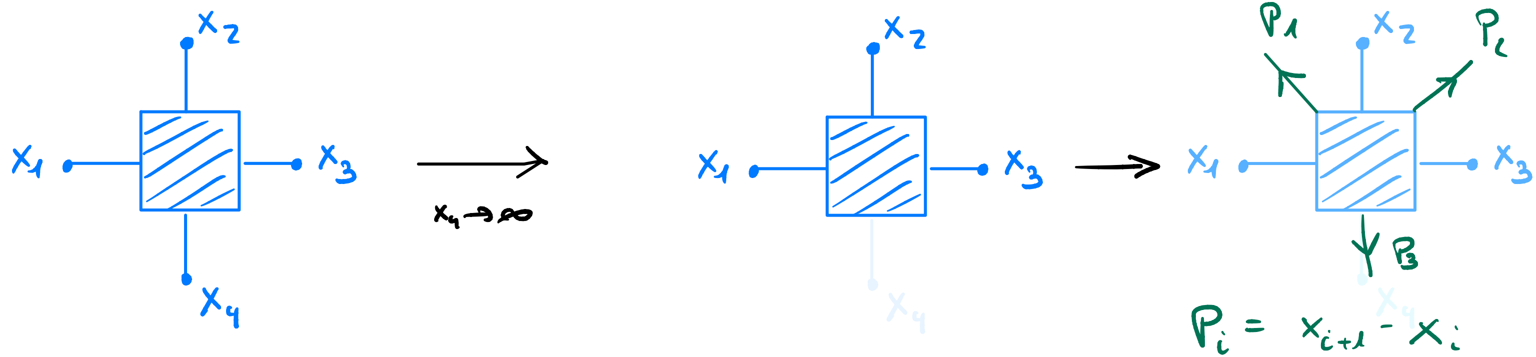
$$\sim \underbrace{2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log(z\bar{z}) \log\frac{1-z}{1-\bar{z}}}_{\text{SVMPL}\;,\;\sim\ \text{Bloch-Wigner dilog}}$$

N.B: This reasoning holds for all 3-pt. functions with massless props (provided it can be expressed in terms of MPLs), independently of loop order!



$\longleftrightarrow$ SVMPLs in $(z, \bar{z})$

This argument immediately extends to conformal 4-point functions :



Can only depend on conformal cross ratios :

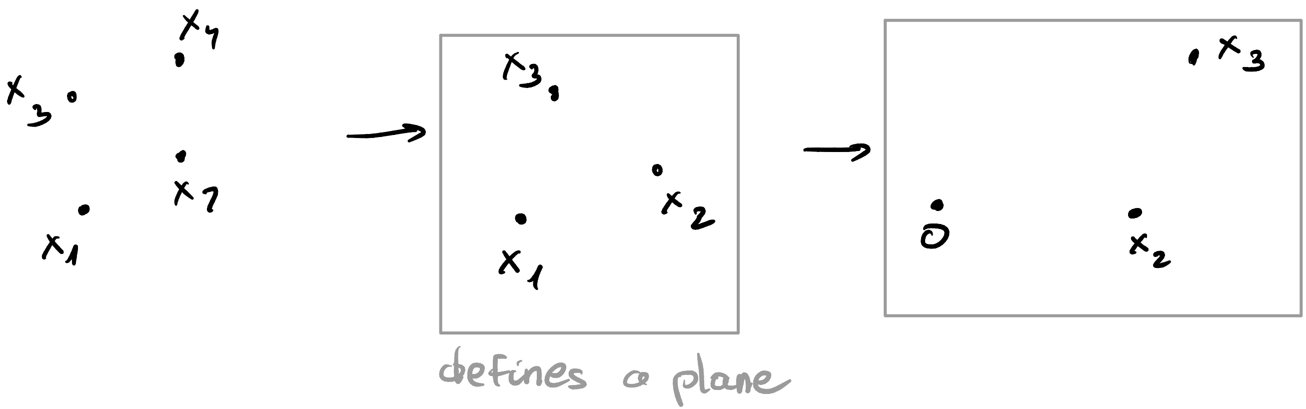
$$u = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2} = z\bar{z} \qquad v = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2} = (1-z)(1-\bar{z})$$

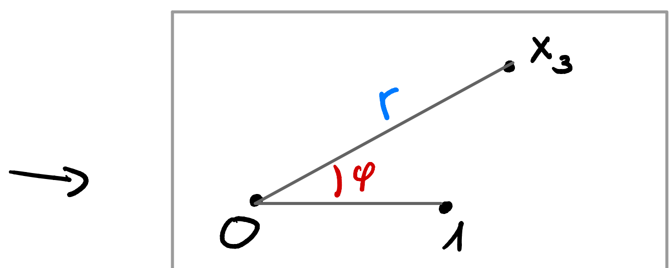
Prime example: 4D ladder graphs [Davydychev, Ussyukina]

$\Longleftrightarrow$ SV classical MPLs

For conformal 4-pt functions, there is another nice interpretation of z: [Schnetz]

$$(x_1,x_2,x_3,x_4) \longrightarrow (x_1,x_2,x_3,\infty) \longrightarrow (0,x_2,x_3,\infty)$$





$$z = r\, e^{i\varphi}$$

SVMPLs : Combinations of holomorphic & antiholomorphic MPLs, s.t. all discontinuities cancel:

→ Can be constructed explicitly! [Brown]

"Single-value map" SV : Assign to an MPL its "SV-analogue"

* linear
* Respects products
* Same holomorphic derivative
* single-valued
* non-holomorphic

symbol letters are all either purely holomorphic or anti-holomorphic [Ex: $z, \bar{z}, 1-z, 1-\bar{z}$]

Example: $SV(\log z) = \log|z|^2 = \log z + \log \bar{z}$

Are these SVMPLs sufficient for 3 & 4-pt function?

NO! There are examples which involve symbol letters that are neither holomorphic nor anti-holomorphic. [Chavez, CD; Schetz; CD, Eden, Heslop, Smirnov; ...]

Examples: 1-Loop triangle contains $z-\bar{z}$ to higher orders in ε

N.B: There is a general theory of generalised SVMPLs that may contain symbol letters

$$a + bz + c\bar{z} + dz\bar{z}$$

[Schetz]

Are there other Feynman Integrals that can be expressed in terms of SVMPLs ?

YES ! ALL one-loop n-point integrals in $D=n$ or $D=n+1$ dimensions [Ren, Rudenko, Spradlin, Volovich]

Main ingredient :

One-loop integrals $\longleftrightarrow$ volumes in hyperbolic space $\longleftrightarrow$ SVMPLs

Davydychev

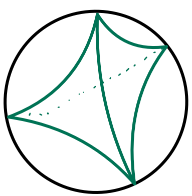
Delbourgo;

Schetz

Kellerhals;

Goncharov;

Rudenko



[Can be used as a stepping stone to $D\neq n$; but SV property is lost [CD, Mork; Mork (in preparation)]]

So far we discussed individual Feynman integrals.

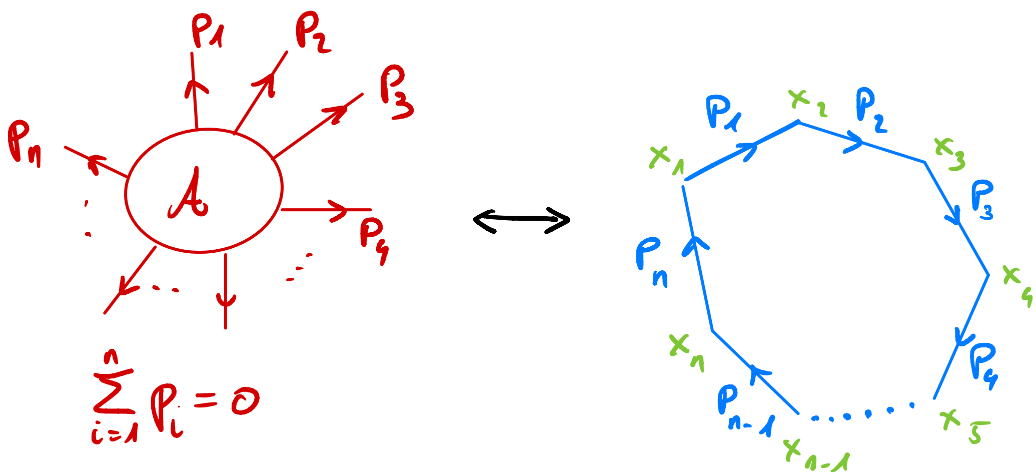
What about full scattering amplitudes?

Multi-Regge Kinematics (MRK)

in planar $\mathcal{N}=4$ SYM.

Planar $\mathcal{N}=4$ SYM

- Supersymmetric "cousin" of QCD
- Dual via AdS/CFT to string theory on $AdS_5 \times S^5$.
- Expected to be integrable
- Dual conformal invariance:



$$p_i = x_{i+1} - x_i$$

Consequence

- Non-trivial part of amplitudes only depend on cross ratios:

$$\frac{x_{ij}^2\, x_{kl}^2}{x_{ik}^2\, x_{jl}^2}$$

- Non-trivial cross-ratios appear for the first time for $n=6$!
- Many results known for amplitudes with many loops & legs!

Multi-Regge Kinematics (MRK)

light-cone coordinates: $p_j^{\pm} = p_j^0 \pm p_j^z$, $\quad p_{j\perp} = p_j^x + i p_{jy}^y$

$$p_j^+ p_j^- = |p_{j\perp}|^2$$

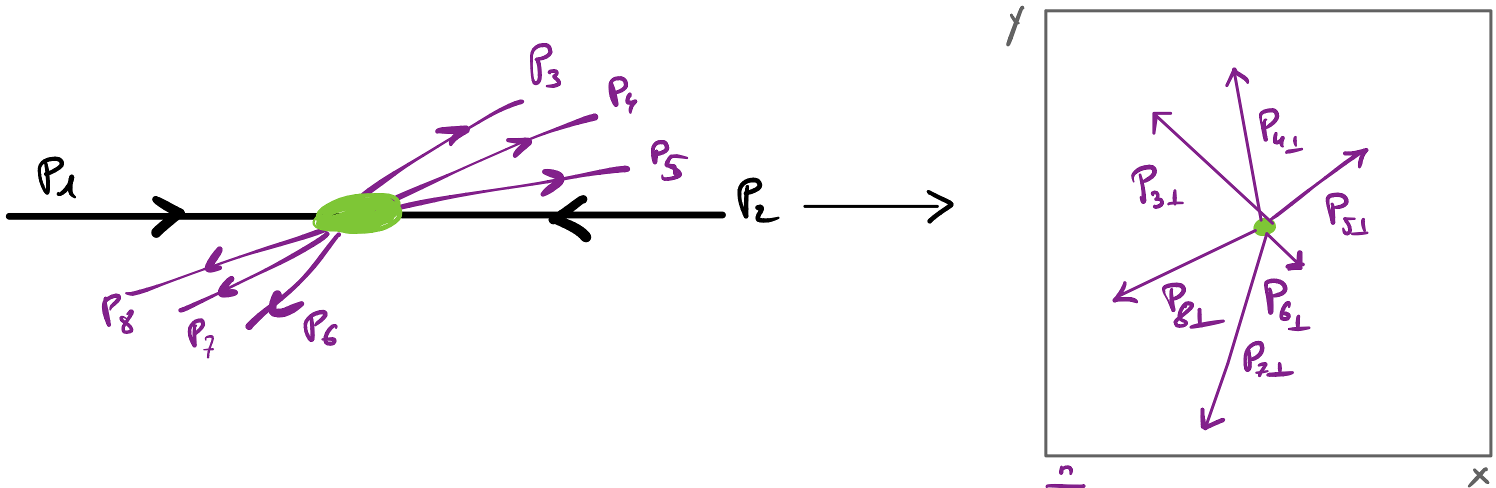
MRK: Consider p_1, p_2 "in-going". MRK

$$p_2^+ \simeq p_3^+ \gg p_4^+ \gg \ldots \gg p_n^+ \gg p_1^+ ,$$

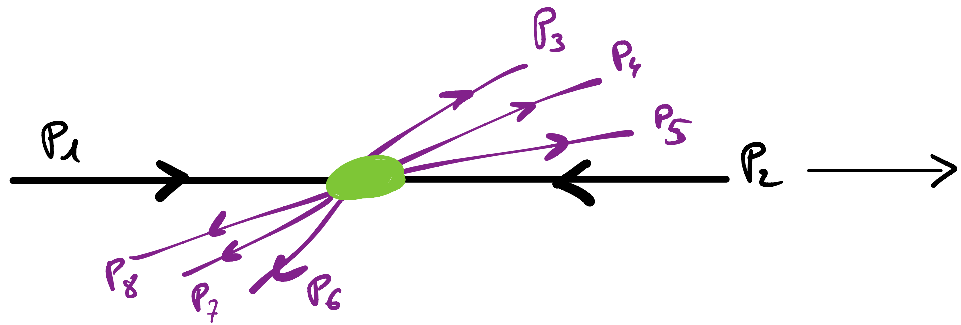
$$|p_{3\perp}| \simeq |p_{4\perp}| \simeq \ldots \simeq |p_{n\perp}|$$

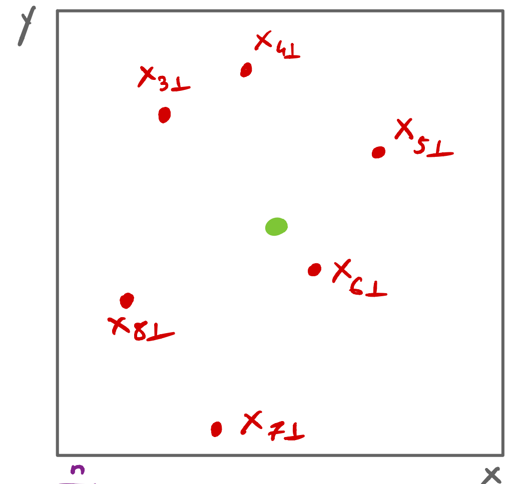
$\Rightarrow \quad p_1^- p_2^+ \simeq s$ is largest invariant

$\rightsquigarrow$ high-energy limit / forward scattering



$$\sum_{i=3}^{n} P_{i\perp} = 0$$





$$\sum_{i=3}^{n} p_{i\perp} = 0$$

- Kinematics in transverse space encoded in $x_{j\perp} \in \mathbb{C}$

- Dual conformal invariance in transverse space $\longleftrightarrow$ $SL(2,\mathbb{C})$
- Soft limits : $p_{i\perp} \to 0 \iff x_{i+1\perp} = x_{i\perp}$
- Configurations of points in $\mathbb{P}^1$ modulo $SL(2,\mathbb{C}) \leftrightarrow \mathcal{M}_{0,n-2}$

$\longrightarrow$ Natural function space : MPLs

If you add first entry condition → SVMPLs!

- First systematically observed and used for $n=6$
  [Dixon, CD, Pennington]

  → SVHPLs [symbol letters $z, \bar{z}, 1-z, 1-\bar{z}$]

- Can be extended to all n (and all helicity configurations)
  [Del Duca, Drummond, CD, Marzucca, Papathanasiou, Verbeek; Broedel, Sprenger, Orjuela]

  → Requires full machinery of SVMPLs!

[Outside $\mathcal{N}=4$: also generalised SVMPLs with letter $1+z\bar{z}$]

We even get more:

* Integrability predicts a general expression for all MRK amplitudes in planar $\mathcal{N}=4$ SYM as "Fourier-Mellin integral, to all orders!

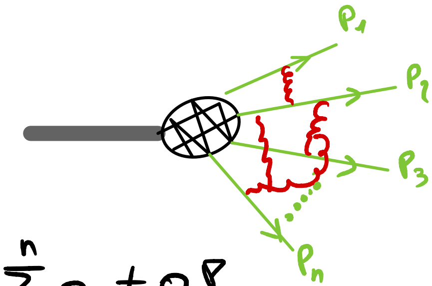
* SVMPLs allow one to solve this integral order by order in an algorithmic fashion!

→ boundary data for amplitudes bootstrap

→ Proofs of general properties in this limit (e.g. uniform weight, etc)

SVMPLs & Eikonal limits

* Consider hard massless gluons, which interact via soft gluon exchange



$\sum_{i=1}^{n} p_i \neq 0$!

hard partons → Wilson lines

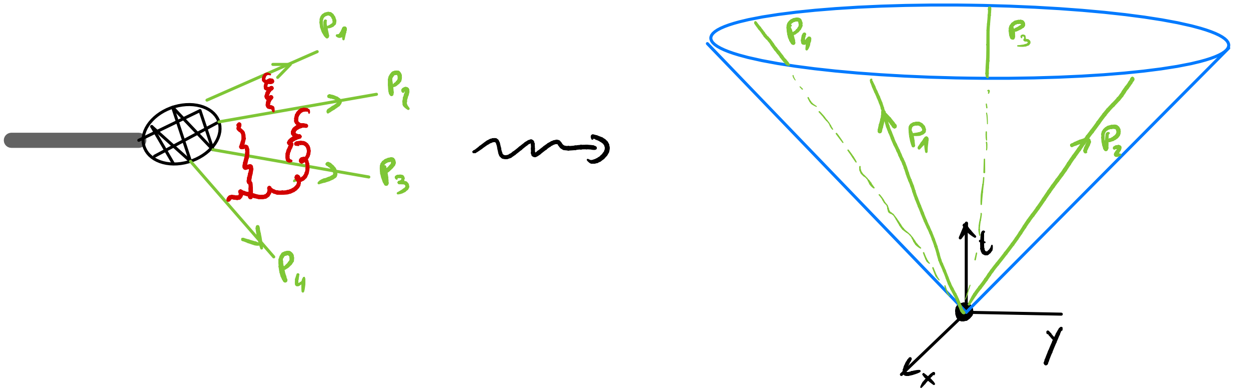
Wilson line correlator

~ IR singularities of amplitudes

* Rescaling Symmetry of Wilson lines : $p_i \to \lambda p_i$

⇝ "Non-trivial" colour connections can only depend on 'cross ratios' $\dfrac{(p_i p_j)(p_k p_l)}{(p_i p_k)(p_j p_l)}$ [Becher, Neubert; Gardi, Magnea]

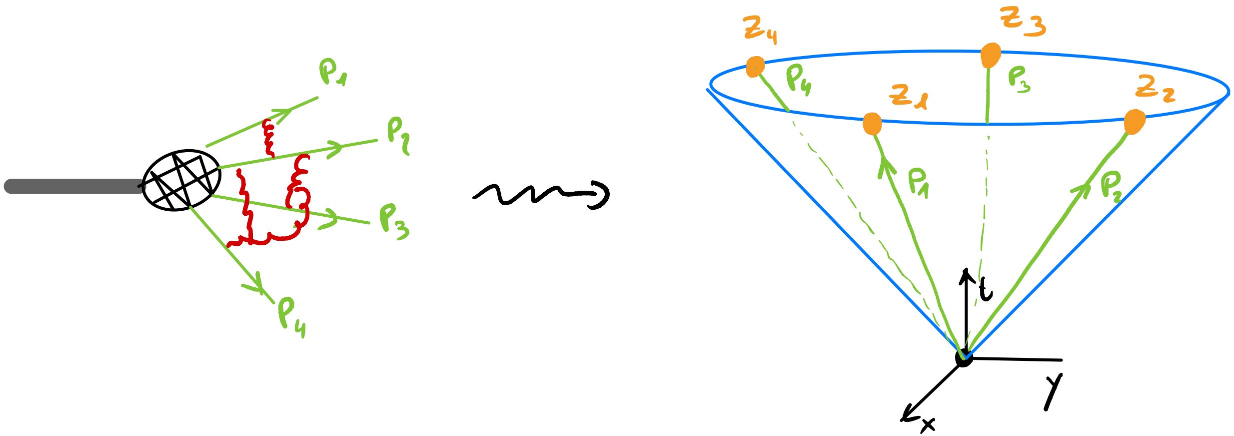
⇝ Needs 3-loops



* Kinematic encoded in: 4 Wilson lines on lightcone

* Boundary of lightcone in 4D Minkowski $\simeq$ 2D sphere $\simeq \mathbb{P}^1$

  ↳ celestial sphere



* Kinematic encoded in: 4 Wilson lines on lightcone

* Boundary of lightcone in 4D Minkowski $\simeq$ 2D sphere $\simeq \mathbb{P}^1$
  ↳ celestial sphere
* Kinematics is encoded in a configuration of points on celestial sphere

  + 1st entry condition

  $\rightsquigarrow \mathcal{M}_{0,n} \rightsquigarrow$ MPLs $\rightsquigarrow$ SVMPLs

* This is indeed what the explicit 3-loop computation revealed!

[Almelid, CD, Gardi]

⇝ only SUHPLs are involved!

$n=4$ points $\Rightarrow$ $(z_1, z_2, z_3, z_4) = (0, 1, \infty, z)$

* Was used (a posteriori) to bootstrap the 3-loop result.

[Almelid, CD, Gardi, Mcleod]

Active Research: Can we bootstrap 4-loop result?

⇝ Can show that SVHPLs are not enough for $n=4$!

⇝ generalised SVMPLs?

[individual diagrams for 3-loop need $z-\bar{z}$, but they cancel...]

Similar Argument can be used to explain appearance of SVHPLs in

* Soft-current for 3 Wilson lines + 1 soft gluon
  [Dixon, Herrmann, Yan, Zhu]

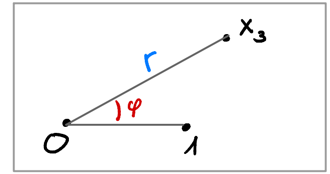
  because gluon soft, so only its direction matters

  [Fails for 2 soft gluons, because now ratio of 2 soft gluon energy matters, see [Zhu]]

* Triple-energy correlator (EEEC) in triple collinear limit
  [Yan, Zhang]

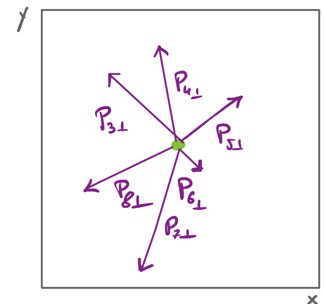
Where SVMPLs are known to appear:

* 3pt functions / conformal 4pt

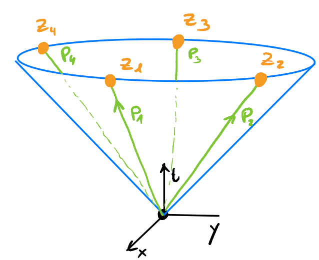


* hyperbolic volumes
  1-loop integrals

* Amplitudes in MRK



* Soft amplitudes
  (only directions / points on celestial sphere matter)



2D kinematic seems to play a role ???

Feynman Integrals

in 2D

* Consider integrals in 2D (euclidean)

$$I \sim \int \frac{d^2k_1 \ldots d^2k_L}{D_1^{\alpha_1} \ldots D_p^{\alpha_p}} \qquad D_j = \left(\sum_{j=1}^{L} \varepsilon_j k_j + \sum_r p_r \right)^2$$

$\in \mathbb{R} \setminus \mathbb{Z}$

[If in $\mathbb{Z}$, there are divergences, cf. Schnetz]

* We may introduce complex coordinates

external momenta

$$x_j = k_j^1 + i k_j^2 \qquad z_j = p_j^1 + i p_j^2$$

$$|z|^2 = x^2 + y^2 = z\bar{z}$$

$$\rightsquigarrow D_j = \underbrace{\left(\sum_j \varepsilon_j x_j + \sum_r z_r \right)}_{d_j} \underbrace{\left(\sum_j \varepsilon_j \bar{x}_j + \sum_r \bar{z}_r \right)}_{\bar{d}_j}$$

* Then

$$I \sim \int_{\mathbb{C}^L} \left(\prod_{j=1}^{L} dx_j \wedge d\bar{x}_j\right) \frac{1}{|d_1|^{2\alpha_1} \cdots |d_p|^{2\alpha_p}}$$

$$\sim \int_{\mathbb{C}^L} \Omega \wedge \overline{\Omega} \quad , \quad \underbrace{\Omega = \frac{dx_1 \wedge \ldots \wedge dx_L}{d_1^{\alpha_1} \cdots d_p^{\alpha_p}}}_{\text{purely holomorphic}}$$

* "Double-copy formula" [Brown, Dupont]

[Very handwavy argument] Insert an identity

$$I \sim \langle \Omega | \overline{\Omega} \rangle = \langle \Omega | \underbrace{\gamma_i] (H^{-1})_{ij} [\gamma_j}_{= \mathbb{1},\ \{\gamma_i\} = \text{basis of cycles}} | \overline{\Omega} \rangle$$

* Then

$$I \sim \int_{\mathbb{C}^L} \left(\prod_{j=1}^{L} dx_j \wedge d\bar{x}_j\right) \frac{1}{|d_1|^{2\alpha_1} \cdots |d_p|^{2\alpha_p}}$$

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* "Double-copy formula" [Brown, Dupont]

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$H_{ij} \sim [\gamma_i | \gamma_j]$: intersection pairing

$$I \sim I_i \, (H^{-1})_{ij} \, \overline{I}_j$$

I_i : "1D Feynman integral"; can be expressed as Aomoto Gel'fand - hypergeometric function

Appearance of H makes it SV

Conclusion: Feynman integrals with massless propagators and generic α_j evaluate to SV analogues of Aomoto-Gel'fand hypergeometric functions

[CD, Porkert; Mimachi, Yoshida ($_2F_1$)]

Special case: 2D conformal Fishets $\Leftrightarrow$ CY volumes

[CD, Klemm, Loebbert, Nega, Porkert]

Comments:

* The double-copy formula between 1D & 2D Feynman integrals was studied in more detail by [Ferrando, Loebbert, Pitters, Stawinski]

* Aomoto – Gel'fand hypergeometric functions describe open string Amplitudes in AdS [Alday, Nocchi, Strömholm]

* SV - Aomoto – Gel'fand hypergeometric functions seem to govern AdS double-copy (open/closed strings)
  [Kakkad, Ochirov, Zhang]

Conclusions

* SVMPLs govern many integrals & amplitudes where we can identify an underlying 2D kinematic picture
* Deep connections to Aomoto-Gel'fand functions

* Open Question: What about other classes of "single-valued periods", like equivariant Eisenstein integrals? [see Axel's talk]

→ So far not observed in Feynman integrals

⇝ where would the antiholomorphic part come from?