

# Generating non-holomorphic modular forms

**Single-Valued Periods in Scattering Amplitudes, MITP**

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With Claude Duhr [[arXiv:2502.04752](https://arxiv.org/abs/2502.04752)]

WIP with Claude Duhr

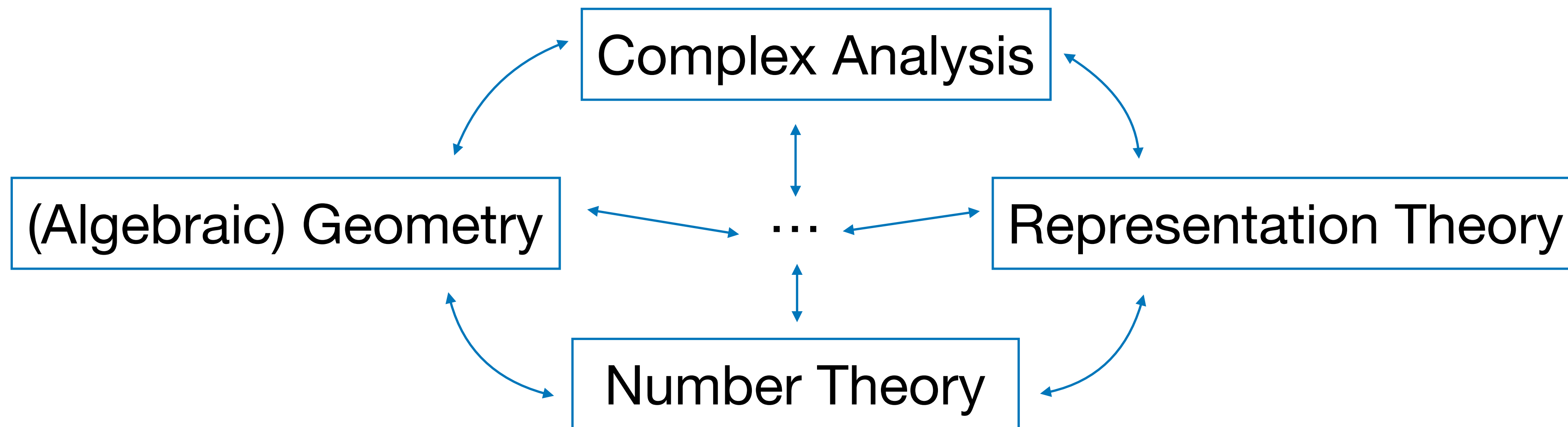
**Franca Lippert**



Bethe Center for  
Theoretical Physics

# Motivation

- Open to closed string amplitudes
- Modular graph forms as equivariant Eisenstein integrals
- Connects different areas in mathematics



# Outline

- single-valued periods in genus 0
- single-valued and equivariant integrals in genus 1
- generating non-holomorphic modular forms from single-valued  ${}_2F_1$

# Recap: single-valued periods in genus 0

Multiple polylogarithms (MPLs):

$$G(a_1, \dots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t)$$

multivalued functions  $G(0; z) = \log(z)$

$$G(\underbrace{0, \dots, 0}_{n-1}, 1; z) = \text{Li}_n(z)$$

single-valued versions of MPLs [Brown '04; Wojtkowiak '89; Gangl, Zagier '00]

$$\log(z) \xrightarrow{\text{sv}} \log |z|^2$$

$$\text{Li}_2(z) \xrightarrow{\text{sv}} 2i \text{Im}(\text{Li}_2(z) + \log |z| \log(1 - z)) - 2 \log |z| \log |1 - z|$$

# Recap: single-valued periods in genus 0

MPLs in terms of generating series:

$$\begin{aligned} \mathbb{G}_{\mathbf{e}_0, \mathbf{e}_1}(z) &= \text{Pexp} \left( \int_0^z dt \left( \frac{\mathbf{e}_0}{t} + \frac{\mathbf{e}_1}{t-1} \right) \right) \\ &= 1 + \underbrace{G(0; z)}_{= \log(z)} \mathbf{e}_0 + \underbrace{G(1; z)}_{= \log(1-z)} \mathbf{e}_1 + \sum_{a_1, a_2 \in \{0, 1\}} G(a_1, a_2; z) \mathbf{e}_{a_1} \mathbf{e}_{a_2} + \dots \end{aligned}$$

single-valued MPLs [Brown '04; Frost et al. '23]

$$\begin{aligned} \mathbb{G}^{sv}(z) &= (\mathbb{M}^{sv})^{-1} \overline{\mathbb{G}(z)} \mathbb{M}^{sv} \mathbb{G}(z) \\ &= 1 + \log |z|^2 \mathbf{e}_0 + \log |1-z|^2 \mathbf{e}_1 + \dots \end{aligned}$$

$\mathbb{M}^{sv}$ : generating series in single-valued zetas

# Genus 1: modular forms

$\mathrm{SL}_2(\mathbb{Z})$ -action on upper half plane  $\tau \mapsto \frac{a\tau + b}{c\tau + d}$   
 $\mathrm{Im}\tau > 0$

## congruence subgroups:

Principal congruence subgroup

$$\Gamma(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

↑  
level

**modular forms** of weight  $k$ :  $f: \mathfrak{H} \rightarrow \mathbb{C}$

- holomorphic
- modularity  $f(\gamma\tau) = (c\tau + d)^k f(\tau)$
- growth condition at  $\infty$  and the real line

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2 \times 2} \mid ad - bc = 1 \right\}$$

## Examples:

$$\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$$

$\Gamma(2)$  generated by

$$T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

## Example:

Eisenstein series of weight 4

$$G_4(\tau) = \sum'_{(m,n) \in \mathbb{Z}^2} \frac{1}{(m\tau + n)^4}$$

# Genus 1: Eisenstein series

Let  $k \geq 3$  and  $N \in \mathbb{N}$

cusp: element of order  $N$  in  $(\mathbb{Z}/N\mathbb{Z})^2$

$$G_k^v(\tau) = \sum'_{(a,b) \equiv_N v} \frac{1}{(a\tau + b)^k} \xrightarrow{N=1} G_k(\tau) = \sum'_{(a,b)} \frac{1}{(a\tau + b)^k}$$

Extended modularity:  $G_k^v(\gamma\tau) = (c\tau + d)^k G_k^{v\gamma}(\tau)$  for  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$   
 modularity if  $\gamma \in \Gamma(N)$

## Non-holomorphic Eisenstein series

$$G_{r,s}^v(\tau) = \sum'_{(a,b) \equiv_N v} \frac{\mathrm{Im}(\tau)}{(a\tau + b)^{r+1} (a\bar{\tau} + b)^{s+1}}$$

Extended modularity:  $G_{r,s}^v(\gamma\tau) = (c\tau + d)^r (c\bar{\tau} + d)^s G_{r,s}^{v\gamma}(\tau)$   
 „modular of weights  $(r, s)$ “

# Genus 1: equivariant integrals

Differential form

$$\underline{G}_k^v(z) = 2\pi i G_k^v(z) (X - zY)^{k-2} dz$$

Modular equivariant: „extended modular equivariance“

$$\underline{G}_k^{v\gamma^{-1}}(\gamma z) |_\gamma = \underline{G}_k^v(z)$$

action on polynomial variables:  
 $(X, Y) |_\gamma = (aX + bY, cX + dY)$

Integrate  $\mathcal{G}_k^v(\tau) := \int_{\tau}^{i\infty} \underline{G}_k^v(z)$

↙ tangential base point regularisation

modular equivariant?

$$\mathcal{G}_k^{v\gamma^{-1}}(\gamma\tau) |_\gamma = \int_{\gamma\tau}^{i\infty} \underline{G}_k^{v\gamma^{-1}}(z) |_\gamma \stackrel{?}{=} \int_{\tau}^{i\infty} \underline{G}_k^v(z) + \int_{i\infty}^{\gamma^{-1}i\infty} \underline{G}_k^v(z)$$

Cocycle  $C_k^v(\gamma)$

# Genus 1: equivariant integrals

**Level 1:** [Brown '17; Dorigoni et al. '24]

$$\operatorname{Re} \int_{i\infty}^{\gamma^{-1}i\infty} \underline{G}_k(z) = P_k|_{\gamma} - P_k$$

Example:  $k = 4$       $P_4 = \frac{(2\pi i)^2}{3} \zeta(3) Y^2$

**arbitrary level:** [Duhr, FL '25]

$$\operatorname{Re} \int_{i\infty}^{\gamma^{-1}i\infty} \underline{G}_k^v(z) = P_k^{v\gamma^{-1}}|_{\gamma} - P_k^v$$

$$P_4^v = \frac{(2\pi i)^2}{3N^2} \sum_{m=0}^{N-1} \cos\left(\frac{2\pi m v_2}{N}\right) \underset{\uparrow}{Cl_3}\left(\frac{2\pi m}{N}\right) Y^{2 \stackrel{N \equiv 1}{=} 1} P_4$$

Clausen function:

$$Cl_{2n}(x) = \operatorname{Im}(\operatorname{Li}_{2n}(e^{ix}))$$

$$Cl_{2n+1}(x) = \operatorname{Re}(\operatorname{Li}_{2n+1}(e^{ix}))$$

**Theorem**

$$\mathcal{E}_k^v(\tau) = \operatorname{Re} \int_{\tau}^{i\infty} \underline{G}_k^v(z) - P_k^v \text{ is modular equivariant and}$$

$$\mathcal{E}_k^v(\tau) = -\frac{2\pi}{k-1} \sum_{r+s=k-2} G_{r,s}^v(\tau) (X - \tau Y)^r (X - \bar{\tau} Y)^s$$

$\uparrow$   
 non-holomorphic modular forms

# Single-valued and equivariant [Dorigoni et al. '24]

single-valuedness:

$\mathbb{I}^{sv}$  is invariant under monodromy

equivariance:

$\mathbb{I}^{eqv}$  is equivariant under  $SL_2(\mathbb{Z})$ -action

$$\mathbb{I}^{eqv}(\gamma\tau) = M(\gamma)^{-1} \mathbb{I}^{eqv}(\tau) M(\gamma)$$

In level 1 we can construct these objects as iterated integrals as follows:

$$\mathbb{I}^{sv}(\epsilon_k; \tau) = \mathbb{M}^{sv}(\sigma_i)^{-1} \overline{\mathbb{I}(\epsilon_k; \tau)} \mathbb{M}^{sv}(\sigma_i) \mathbb{I}(\epsilon_k; \tau)$$

$$\mathbb{I}(\epsilon_k; \tau) = \text{Pexp} \left( \int_{\tau}^{i\infty} \text{generating series in } G_k \epsilon_k \right)$$

$$\mathbb{I}^{eqv}(\epsilon_k; \tau) = \mathbb{M}^{sv}(z_i)^{-1} \overline{\mathbb{I}(\epsilon_k; \tau)} \mathbb{M}^{sv}(\sigma_i) \mathbb{I}(\epsilon_k; \tau)$$

$\mathbb{M}^{sv}(\sigma) = \text{generating series in } \zeta_n^{sv}$

$$\sigma_i = z_i - \frac{1}{(i-1)!} \epsilon_{i+1}^{(i-1)} + \dots$$

Lie algebra generators

reversion of integration order

$$\mathbb{I}^{eqv}(\epsilon_k; \tau) = \mathbb{M}^{sv}(z_i)^{-1} \mathbb{M}^{sv}(\sigma_i) \mathbb{I}^{sv}(\epsilon_k; \tau)$$

What happens for congruence subgroups?

# Single-valued hypergeometric function [Brown, Dupont '19]

period matrix of  $x(x-1)(x-\lambda)$

$$P(z) = \begin{pmatrix} \mathcal{F}(a, b, c; z) & \mathcal{F}(1+a, 1+b, 2+c; z) \\ \mathcal{G}(a, b, c; z) & \mathcal{G}(1+a, 1+b, 2+c; z) \end{pmatrix}$$

**single-valued version:**

$$P_{\mathbf{a}}^{sv}(z) = P_{\mathbf{a}}(z) \overline{P(z)}_{-\mathbf{a}}^{-1}$$

$\mathbf{a} = (a, b, c)$

Set  $\mathbf{a} = (\frac{1}{2} - \alpha\epsilon, \frac{1}{2} + \gamma\epsilon, c = 1 - \alpha\epsilon - \beta\epsilon)$

and perform a change of variables from  $z$  to  $\tau \in \mathfrak{S}$

$$P^{sv}(\tau, \epsilon) = P(\tau, \epsilon) \overline{P(\tau, -\epsilon)}^{-1} \xrightarrow{\text{expansion in } \epsilon} \text{iterated integrals of Eisenstein series}$$

Normalized Gauss hypergeometric function

$$\mathcal{F}(a, b, c; z) = \int_0^1 x^b (1-x)^{c-b} (1-zx)^{-a} \frac{dx}{x(1-x)}$$

companion function

$$\mathcal{G}(a, b, c; z) = \int_{\infty}^{z^{-1}} x^b (1-x)^{c-b} (1-zx)^{-a} \frac{dx}{x(1-x)}$$

# Canonical form

There is a base change matrix  $R$ , such that  $K(\tau, \epsilon) = R(\tau, \epsilon)P(\tau, \epsilon)$  satisfies

[Broedel et al. '18]

$$dK(\tau, \epsilon) = \epsilon A(\tau)K(\tau, \epsilon)$$

$$A(\tau) = \begin{pmatrix} \frac{d\tau}{2\pi i} g_2(\tau) & \frac{d\tau}{2\pi i} g_0(\tau) \\ \frac{d\tau}{2\pi i} g_4(\tau) & \frac{d\tau}{2\pi i} g_2(\tau) \end{pmatrix}$$

modular forms for  $\Gamma(2)$

Notation:  $\overline{X(\tau, \epsilon)} := \overline{X(\bar{\tau}, -\epsilon)}$

$$L(\tau, \epsilon) := R(\tau, \epsilon)P^{sv}(\tau, \epsilon)\overline{R(\tau, \epsilon)}^{-1}$$

$SL_2(\mathbb{Z})$ - action on the base change:  $R(\gamma\tau, \epsilon) = M(\gamma, \tau, \epsilon)R(\tau, \epsilon)$

$$L(\gamma\tau, \epsilon) = M(\gamma, \tau, \epsilon)L(\tau, \epsilon)\overline{M(\gamma, \tau, \epsilon)}^{-1}$$

almost equivariant!

$$M(\gamma_1\gamma_2, \tau) = M(\gamma_1, \gamma_2\tau)M(\gamma_2, \tau)$$

Can we get rid of the  $\tau$ -dependence in  $M$ ?

# From single-valued to equivariant

$$L(\gamma\tau, \epsilon) = M(\gamma, \tau, \epsilon)L(\tau, \epsilon)\overline{M(\gamma, \tau, \epsilon)}^{-1} \quad \text{almost equivariant}$$

Gauge transform the connection  $A(\tau)$ :  $X_0(\tau, \epsilon) = \begin{pmatrix} 1 & -2\pi i \epsilon \tau \\ 0 & 1 \end{pmatrix}$

$$B(\tau, \epsilon) = (X_0 \epsilon A(\tau) + dX_0) X_0^{-1} = \frac{\epsilon d\tau}{2\pi i} \begin{pmatrix} g_2 - 2\pi i \epsilon \tau g_4 & 4\pi^2 \epsilon^2 \tau^2 g_4 \\ g_4 & g_2 + 2\pi i \epsilon \tau g_4 \end{pmatrix}$$

$$J(\tau, \epsilon) := X_0 L \overline{X_0}^{-1} \Rightarrow dJ = BJ$$

$$X_0(\gamma\tau)M(\gamma, \tau, \epsilon)X_0(\tau)^{-1} = \begin{pmatrix} a & -2\pi i \epsilon b \\ -\frac{c}{2\pi i \epsilon} & d \end{pmatrix} =: N(\gamma)$$

$$\text{SL}_2(\mathbb{Z})\text{-action: } J(\gamma\tau) = \underset{\substack{\uparrow \\ \tau\text{-independent}}}{N(\gamma)} J(\tau) \overline{N(\gamma)}^{-1} = N(\gamma) J(\tau) N(\gamma)^{-1} \quad \text{equivariant!}$$

# Generating modular Eisenstein integrals

$$N \text{ can be decomposed } \quad N = E \gamma^{-1,t} E^{-1} \quad E = \begin{pmatrix} 0 & 2\pi i \epsilon \\ 1 & 0 \end{pmatrix}$$

$$J^{eqv}(\tau) := EJ(\tau)E^{-1} \quad \Rightarrow J^{eqv}(\gamma\tau) = \gamma^{-1,t} J^{eqv}(\tau) \gamma^t$$

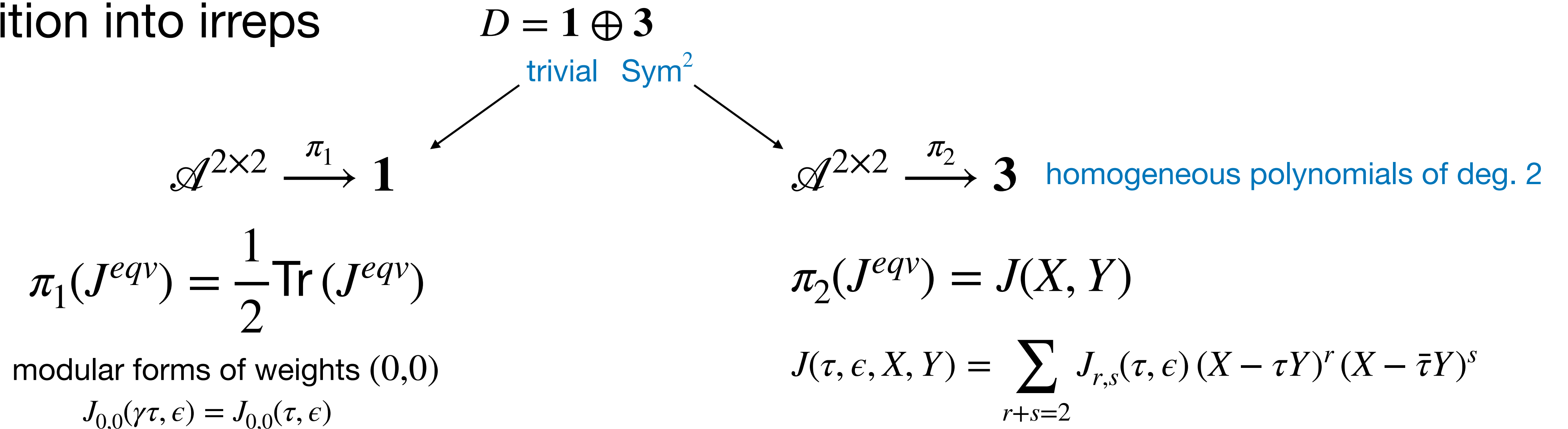
$$\begin{aligned} J_{1,1}^{eqv}(\tau) = & \frac{2\pi i}{\epsilon} + 8(\alpha + \beta) \left( \operatorname{Re} \int_{\tau}^{i\infty} g_2^{(0,1)}(z) dz + \pi i \log(2) \right) - 8(\alpha - \gamma) \operatorname{Re} \int_{\tau}^{i\infty} g_2^{(1,1)}(z) dz \\ & + \epsilon \left( \frac{i}{4\pi} \operatorname{coeff}_0^2 - \frac{48i}{\pi} \left( (\alpha + \beta)^2 \operatorname{Re} \int_{\tau}^{i\infty} G_4^{(0,1)}(z) z dz + (\beta + \gamma)^2 \operatorname{Re} \int_{\tau}^{i\infty} G_4^{(1,0)}(z) z dz \right. \right. \\ & \left. \left. + (\alpha + \gamma)^2 \operatorname{Re} \int_{\tau}^{i\infty} G_4^{(1,1)}(z) z dz \right) \right) + \dots \end{aligned}$$

How can we extract non-holomorphic modular forms?

# Extracting non-holomorphic modular forms

4-dimensional  $\mathfrak{sl}_2$ -representation  $\mathcal{A}^{2 \times 2} \xrightarrow{D(\gamma)} \mathcal{A}^{2 \times 2}$  ← space of 2x2 matrices of real analytic functions  
 $J^{eqv} \mapsto \gamma^{-1,t} J^{eqv} \gamma^t$

Decomposition into irreps



$J_{r,s}$  modular of weights  $(r, s)$

$$J_{r,s}(\gamma\tau, \epsilon) = (c\tau + d)^r (c\bar{\tau} + d)^s J_{r,s}(\tau, \epsilon)$$

# Recap

- $K = RP$  solving the differential equation in canonical form
- $L := K\bar{K}^{-1}$  transforms like  $L(\gamma\tau) = M^{-1}LM$ , not a group action yet
- gauge transform connection  $A$  to  $B$  by  $X_0$
- $J := X_0L\bar{X}_0^{-1}$  solves differential equation for  $B$
- $N = X_0(\gamma\tau)MX_0^{-1}$  is  $\tau$ -independent and can be decomposed  $N = E\gamma^{-1,t}E^{-1}$
- define new equivariant function  $J^{eqv} = EJE^{-1} \Rightarrow J^{eqv}(\gamma\tau) = \gamma^{-1,t}J^{ev}(\tau)\gamma^t$
- decompose obtained representation into irreducible representations
- projection onto irreps gives non-holomorphic modular forms

# Conclusion

- equivariant integrals for subgroups:
  - depth 1
  - examples for arbitrary depth
- WIP: find formulas in terms of generating series

Thank you!