Single-valued integration, a cohomological viewpoint

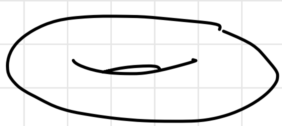
(joint work with Francis Brown)

$$\begin{tikzcd}
 & \text{Cohomology} \arrow[dl] \arrow[dr] & \\
\text{Single-valued periods} & \subseteq & \text{Periods}
\end{tikzcd}$$

(1) Cohomology & periods

X = an algebraic variety defined over some field $k \subset \mathbb{C}$.

(example: $X = \{(x,y) \in \mathbb{C}^2 \mid y^2 = x^3 + ux + v\}$ $\quad u, v \in k$)



- Betti (co)homology : $H_n^B(X) = \dfrac{\{n\text{-cycles on } X\}}{\{n\text{-boundaries on } X\}}$ a $\mathbb{Q}$-vs of finite dimension
singular

$$H_n^B(X) = \frac{\ker(\partial : C_n(X) \to C_{n-1}(X))}{\operatorname{im}(\partial : C_{n+1}(X) \to C_n(X))} \quad ; \quad H^n_B(X) := H_n^B(X)^\vee$$

- algebraic de Rham (co)homology (X affine & smooth) algebraic n-forms on X

$$H^n_{dR}(X) = \frac{\ker\left(d : \Omega^n(X) \to \Omega^{n+1}(X)\right)}{\operatorname{im}\left(d : \Omega^{n-1}(X) \to \Omega^n(X)\right)}$$

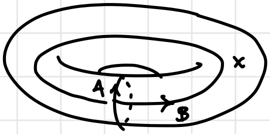
k-v.s. of finite dimension

- Comparison isomorphism (de Rham, Grothendieck)

$$\begin{array}{ccc} H^n_{dR}(X) \otimes_k \mathbb{C} & \xrightarrow[\simeq]{\int} & H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{C} \\ \omega & \longmapsto & \left(\sigma \mapsto \int_\sigma \omega\right) \end{array}$$

If $k \subset \overline{\mathbb{Q}}$, the coefficients are called <u>periods</u>.

<u>Ex</u>: $X = \{ y^2 = x^3 + ux + v \}$



$H^1(X) = 2$-dim

Period matrix:

$$\begin{array}{c|cc} & \frac{dx}{y} & x\frac{dx}{y} \\ \hline A & \int_A \frac{dx}{y} & \int_A x\frac{dx}{y} \\ B & \int_B \frac{dx}{y} & \int_B x\frac{dx}{y} \end{array}$$

Important variants:

- Relative (co)homology: $H^n(X, Y)$ $\quad Y \subseteq X$ closed subvariety.

e.g. $H^1(\mathbb{C}\setminus\{0\}, \{1,a\})$ $\quad a \in \mathbb{C}\setminus\{0,1\}$ $\quad \dim = 2$.

$$\begin{array}{cc} ? & \frac{dx}{x} \\ \end{array}$$
$$\begin{pmatrix} 1 & \log(a) \\ 0 & 2\pi i \end{pmatrix}$$

- Parameters:

$$\begin{array}{ccc} X & & X_s \\ \downarrow & & \downarrow \\ S = \text{space of parameters} & \ni & s \end{array}$$

$X = \{(u,v,x,y) \in \mathbb{C}^2 \times \mathbb{C}^2 \mid y^2 = x^3 + ux + v\}$
$\downarrow$
$S = \{(u,v) \in \mathbb{C}^2 \mid 4u^3 + 27v^2 \neq 0\}$

- Betti side: $\{H^n_B(X_s), s \in S\}$ forms a local system

  Monodromy: period matrix $P \rightsquigarrow \underbrace{M}_{\text{monodromy matrix}} P$

- de Rham side: $\{H^n_{dR}(X_s), s \in S\}$ forms a vector bundle with integrable connection

  Differential equation: period matrix $dP = P\Omega$

<u>Example</u>: $S = \mathbb{C}^* \ni a$

$$\begin{pmatrix} 1 & \log(a) \\ 0 & 2\pi i \end{pmatrix} \underset{\text{a winds around 0}}{\rightsquigarrow} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \log(a) \\ 0 & 2\pi i \end{pmatrix}$$

$$d\begin{pmatrix} 1 & \log(a) \\ 0 & 2\pi i \end{pmatrix} = \begin{pmatrix} 1 & \log(a) \\ 0 & 2\pi i \end{pmatrix}\begin{pmatrix} 0 & \frac{da}{a} \\ 0 & 0 \end{pmatrix}.$$

(2) Single-valued periods

Start with $k \subset \mathbb{R} \rightsquigarrow$ complex conjugation $\begin{array}{l} X \longrightarrow X \\ x \longmapsto \bar{x} \end{array}$

Induces a linear involution $F_\infty : H^n_B(X) \longrightarrow H^n_B(X)$

<u>Def</u>: Single-valued period isomorphism

$$\begin{tikzcd} H^n_{dR}(X)\otimes_k \mathbb{C} \arrow[r, "\sim"] \arrow[rrr, bend right, "sv"] & H^n_B(X)\otimes_{\mathbb{Q}} \mathbb{C} \arrow[r, "F_\infty \otimes id"] & H^n_B(X)\otimes_{\mathbb{Q}} \mathbb{C} \arrow[r, "\sim"] & H^n_{dR}(X)\otimes_k \mathbb{C} \end{tikzcd}$$

In bases, single-valued period matrix $S = P^{-1} F_\infty P = P^{-1}\overline{P} = \overline{P}^{-1} P$.

Makes sense for general k, makes sense with parameters.

Entries of S are single-valued functions of the parameters:

$$S = P^{-1}\overline{P} \rightsquigarrow (MP)^{-1}\overline{MP} = P^{-1}\underbrace{M^{-1}\overline{M}}_{id}\overline{P} = P^{-1}\overline{P} = S$$

Example: $P = \begin{pmatrix} 1 & \log(a) \\ 0 & 2\pi i \end{pmatrix}$

$$S = P^{-1}\overline{P} = \begin{pmatrix} 1 & 2\log|a| \\ 0 & -1 \end{pmatrix}$$

$P = (2\pi i)$ $\quad H^1(\mathbb{C}^\times)$

$$P^{-1}\overline{P} = (-1)$$

(3) How to compute single-valued periods?

$$\mathrm{sv} : \underbrace{H^n_{dR}(X)^\vee}_{?} \times H^n_{dR}(X) \longrightarrow \mathbb{C}$$

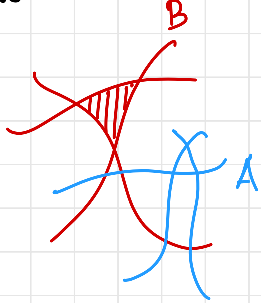
Poincaré: X smooth + compact of dim n, $H^n_{dR}(X)^\vee \simeq H^n_{dR}(X)$

in this case $\mathrm{sv} : H^n_{dR}(X) \times H^n_{dR}(X) \longrightarrow \mathbb{C}$

$$(\nu \,,\, \omega) \longmapsto \frac{1}{(2\pi i)^n}\int_X \nu \wedge \overline{\omega}$$

Generalization (w/ F. Brown): $H^n(X \setminus A, B \setminus A\cap B)$

X smooth + compact of dim n, A, B normal crossing in X



Poincaré-Verdier (Grothendieck):

$$H^n(X\setminus A, B \setminus A\cap B)^\vee \simeq H^n(X \setminus B, A \setminus A \cap B)$$

$$\mathrm{sv}(\nu \otimes \omega) = \frac{1}{(2\pi i)^n}\int_X \nu \wedge \overline{\omega}$$

ν: n-form on X, $\log B$

ω: n-form on X, $\log A$

[⚠ $\nu \wedge \overline{\omega}$ not a smooth form $\frac{dz}{z} \wedge d\overline{z}$]

<u>Example</u>: $\underbrace{H^1(\mathbb{P}^1_{\mathbb{C}}}_{X} \setminus \underbrace{\{0,\infty\}}_{A}, \underbrace{\{1,a\}}_{B})$ $\qquad \omega = \dfrac{dz}{z}$

$\nu = d\log\left(\dfrac{z-a}{z-1}\right)$

$$\frac{1}{2\pi i} \iint_{\mathbb{P}^1_{\mathbb{C}}} \underbrace{d\log\left(\frac{z-a}{z-1}\right)}_{\text{"canonical form" of } \underset{1}{\times}\longrightarrow\underset{a}{\times}} \wedge \frac{d\bar{z}}{\bar{z}} = 2\log|a|.$$

(Arkani-Hamed, Bai-Lam ...)

<u>Double copy formula</u>:

$$\iint_X \nu \wedge \bar{\omega} = \sum_{\gamma,\delta} \langle \gamma^\vee, \delta^\vee \rangle \int_\gamma \nu \; \overline{\int_\delta \omega}$$

γ: basis of $H^B_n(X\setminus B, A\setminus A\cap B)$

δ: basis of $H^B_n(X\setminus A, B\setminus A\cap B)$

$Li_2(a)$

$$P = \begin{pmatrix} 1 & -\log(1-a) & Li_2(a) \\ 0 & 2\pi i & 2\pi i \log(a) \\ 0 & 0 & (2\pi i)^2 \end{pmatrix}$$

Bloch-Wigner dilog

$$\log(D\, P^{-1} \overline{P}) = \begin{pmatrix} 0 & -2\log|1-a| & \\ 0 & 0 & -2\log|a| \\ 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$