Cocycles for Graph Complexes via Feynman Integrals

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OUTLINE / PROGRAM :

- INTRO to $\mathcal{GC}_3$ "KONTSEVICH'S ODD commutative graph complex" → combinatorial chain complex

- ORIENTATION FORMS associated to graphs
  → closed differential forms defined via matrices

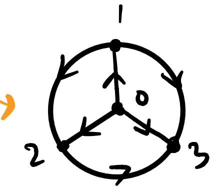
- "FEYNMAN-ESQUE" INTEGRALS → an integration pairing to detect non-trivial homology classes

oriented graphs (G, or) $\mapsto$ $\displaystyle\int_\sigma \underbrace{\phi \wedge \omega}_{\text{"orientation forms"}}$ $\mapsto$??

① WHAT IS GC_3? "ODD GRAPH COMPLEX"

$$GC_3 := \bigoplus_{(G,or)} \mathbb{Q}(G,or) \Big/ \sim$$

oriented graphs



(G,or)

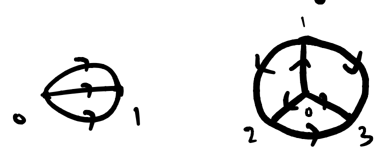
$\ell = 3$

$k = deg_3 = -3$

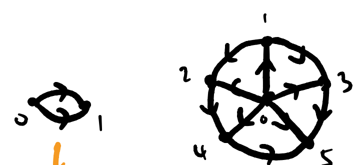
- G : vertices of valency ≥ 3 $\quad$ or : ordering vertices + edge directions
- bigraded by $\ell = E - V + 1$, $\quad k := deg_3 = E - 3\ell$
- $\sim$: graphs up to isomorphism + even automorphisms i.e. $(G,or) \sim -(G,-or)$

• 

do not need to consider self-loops

• 

✓ but

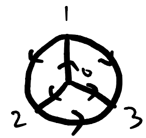
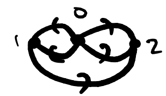


= 0 ✗

all odd-loop dipoles are 0 

sum over all edge contractions

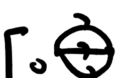
$(\mathcal{G}C_3, \partial)$ defines a chain complex

- ∂  $= 3$  $= 0$
- ∂  $= 6$ 

and we want to consider it's homology:

$$H_\bullet(\mathcal{G}C_3) := \frac{\ker \partial}{\operatorname{img} \partial} = \bigoplus_{\ell, k} gr_\ell \, H_k(\mathcal{G}C_3)$$

bigraded (ℓ, k)

$[$  $] \in gr_2 \, H_{-3}(\mathcal{G}C_3)$

KONTSEVICH '90s ~ applications to topological physics

perturbative Chern-Simons theory

"graph" homology

knot invariants

odd commutative

odd-dimesional manifold invariants

$\mathfrak{grt}_1$, Grothendieck-Teichmüller Lie algebra

even commutative

top-weight cohomology of $\mathcal{M}_g$

(via tropical curves)

cohomology of $\mathcal{M}_{g,n}$ & mapping class groups

ribbon graphs (associative)

Outer space

& cohomology of $Out(F_n)$

Lie graphs

KNOWN DIMENSIONS of $H^\bullet(GC_3)$:

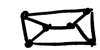
ℓ	H^{-12}	H^{-11}	H^{-10}	H^{-9}	H^{-8}	H^{-7}	H^{-6}	H^{-5}	H^{-4}	H^{-3}
2									0	1
3							0	0	0	1
4					0	0	0	0	0	1
5			0	0	0	0	0	0	0	2
6	0	0	0	0	0	0	1	0	0	2
7	0	0	0	0	0	0	1	0	0	3
8	0	0	0	0	0	0	2	0	0	4
9	0	0	0	0	0	0	3	0	0	5
10	0	0	0	0	0	0	5	0	0	6
11	0	?	?	?	?	?	?	?	?	8
12	?	?	?	?	?	?	?	?	?	9

Table 2: Dimensions of the cohomology $\mathrm{gr}_\ell H^k(\mathsf{GC}_3)$, bigraded by degree k (columns) and loop number ℓ (rows), known from computer calculations [BNM01; KWŽ17; BW24]. Empty cells indicate trivial zeroes (no graphs), whereas 0 indicates vanishing in the range $k < -\ell$ known from (1.8). The column $k = -3$ highlights the algebra of 3-graphs; e.g. $\mathrm{gr}_2 H^{-3}(\mathsf{GC}_3) \cong \mathbb{Q}$ is spanned by the theta graph.

algebra of 3-graphs
"Jacobi diagrams"

$\ell=2$: 

$\ell=3$: 

$\ell=4$: 

$\longrightarrow$ It's hard to compute!

② Q : HOW TO DETECT CLASSES ?

BROWN : construct a cocycle $I(\omega) : \mathcal{GC}_3 \to \mathbb{R}$

via associating convergent integrals to graphs

$$\| \quad G \mapsto \int_{\sigma_G} \phi_G \wedge \omega_G \quad \leftarrow \text{ closed differential forms}$$

↪ I is a linear functional s.t. $I(\omega) \in \mathcal{GC}_3 \otimes \mathbb{R} = \mathrm{Hom}(\mathcal{GC}_3, \mathbb{R})$ with the cocycle condition $\delta I(\omega) = 0$

Then a non-zero pairing of $I(\omega)$ with a cycle $[G] \in H_\cdot(\mathcal{GC}_3)$

$\uparrow$ $\partial G = 0$

$\Rightarrow \quad [I(\omega)] \neq 0 \in H^\cdot(\mathcal{GC}_3) \otimes \mathbb{R}$

$[G] \neq 0 \in H_\cdot(\mathcal{GC}_3)$

INGREDIENTS :

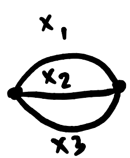
- WORK WITH METRIC GRAPHS :

  - associate to each edge $e \in E(G)$ a variable x_e ↙ schwinger parameters

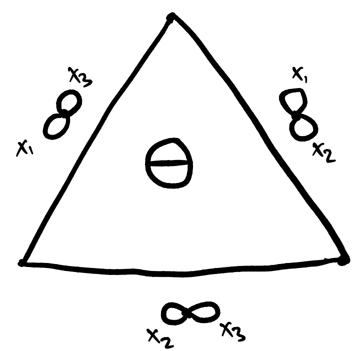
    for $x_e \in \mathbb{R}_{>0}$ and s.t. $\sum x_e = 1$

  - σ_G open simplex of "edge lengths" ↙ dim $E-1$

    $\{[x_1 : \cdots : x_n] : x_e > 0\} \subset \mathbb{P}(\mathbb{R}^E)$



$\longmapsto \quad \overline{\sigma}_G =$



$\subset \mathbb{P}(\mathbb{R}^3)$

- dual Laplacian matrix of a graph G with n edges

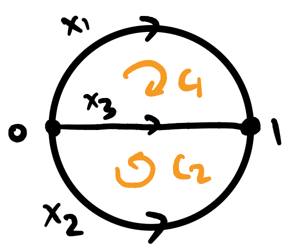
$$\| \quad \Lambda_c = C^T \cdot \mathbb{D} \cdot C$$

C: edge-cycle incidence matrix with respect to cycle basis

$\mathbb{D}$: $\operatorname{diag}(x_1, \dots, x_n)$

$\rightarrow$ positive definite symmetric $\ell \times \ell$ matrix

with $\Psi := \det \Lambda_c = \sum_{T \text{ spanning tree}} \prod_{e \notin T} x_e$ (the (first) Symanzik polynomial)



$$\mapsto \quad \Lambda_c = \begin{pmatrix} x_1 + x_3 & x_3 \\ x_3 & x_1 + x_2 \end{pmatrix}$$

$$\Psi = x_1 x_2 + x_1 x_3 + x_2 x_3$$

- orientation forms $\phi_X \wedge \omega_X$ defined for positive definite symmetric $2m \times 2m$ matrices via

Pfaffian of even-dim skew-sym matrix M: $\mathrm{Pf}^2(M) = \det(M)$

$$\phi_X^{2m} := \frac{\mathrm{Pf}(dX \cdot X^{-1} \cdot dX)}{\sqrt{\det X}}$$

$$\beta_X^{4k+1} := \mathrm{tr}\left((X^{-1} dX)^{4k+1} \right), \quad \omega_X \in \Lambda^{\bullet}\left(\bigoplus_{k \geq 1} \mathbb{Q}\, \beta_X^{4k+1} \right)$$

↪ "orientation" as $\phi^{2n}_{P^T X P} = \mathrm{sgn}(\det P) \cdot \phi_X^{2n}$

$\omega_{P^T X P} = \omega_X$ ← invariant form

↪ nice properties: closed, smooth, scale invariant

$d\phi_X = 0 = d\omega_X$ $\quad \det X \neq 0$ $\quad$ under $\mathbb{R}^{\times}_{>0}$ action

$\phi_{\lambda X} = \phi_X$, $\omega_{\lambda X} = \omega_X$

APPLICATIONS of ORIENTATION FORMS / INTEGRALS

- GIVES REPRESENTATIVES of CLASSES in the UNSTABLE COHOMOLOGY of $GL_{2n}(\mathbb{Z})$

  (Brown, SH, Panzer) " compact-type forms on the locally symmetric space $P_{2n} / GL_{2n}(\mathbb{Z})$ w/ coeffs in orientation bundle "

- DETECTING NON-TRIVIAL GRAPH HOMOLOGY CLASSES in GC_3

  " forms on the moduli space of metric graphs "

- ϕ_G is the " parametric integrand of Feynman integrals computing violations of BRST-closedness of a 1D TQFT "

  (Balduf, SH)

  $\phi_G \wedge \phi_G = 0$

ORIENTATION FORMS ASSOCIATED TO GRAPHS :

For a graph G and a cycle basis C,

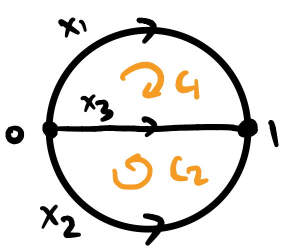
Λ_C : (dual) Laplacian matrix

$$\phi_G := \frac{\mathrm{Pf}\left(d\Lambda_C \cdot \Lambda_C^{-1} \cdot d\Lambda_C\right)}{\sqrt{4}} \quad \& \quad \omega_G^{4k+1} := \mathrm{tr}\left(\left(\Lambda_C^{-1}\, d\Lambda_C\right)^{4k+1}\right)$$

$\in \Omega^{\ell}(\sigma_G)$ $\qquad$ $\in \Omega^{4k+1}(\sigma_G)$

→ we will be interested in top-dimensional forms $\phi_G \wedge \omega_G^{4k+1}$

↳ $\dim(\sigma_G) = E-1 = \ell + 4k+1$



$$\mapsto \quad \phi_{\Lambda_C} = \frac{\mathrm{Pf}\left(d\Lambda_C \cdot \Lambda_C^{-1} \cdot d\Lambda_C\right)}{\sqrt{4}}$$

$$-\frac{\Omega_3}{\psi^{3/2}}$$

$$= \frac{-x_1\, dx_2 \wedge dx_3 + x_2\, dx_1 \wedge dx_3 - x_3\, dx_1 \wedge dx_2}{(x_1x_2 + x_2x_3 + x_1x_3)^{3/2}}$$

THEOREM (FB, EP, SH):

For any oriented graph $G \in \mathcal{G}C_3$ w/ $E = \ell + 4k + 2$ ← want even loop number

$$I_G(\omega^{4k+1}) := \frac{1}{(-2\pi)^{\ell/2}} \int_{\sigma_G} \phi_G \wedge \omega_G^{4k+1}$$

is absolutely convergent and defines a linear functional $I : \mathcal{G}C_3 \to \mathbb{R}$.

Furthermore, when restricted to $\ell = 2k+4$, they define cocycles

$$[\underbrace{gr_{2k+4}\, I(\omega^{4k+1})}_{[\Sigma_k]}] \in H^{-6}(GC_3) \otimes \mathbb{R}$$

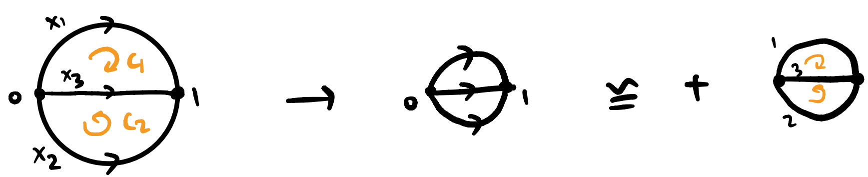
Back to $\mathcal{GC}_3$:

what we wanted was to construct a linear functional:

$$I(\omega) : \mathcal{GC}_3 \longrightarrow \mathbb{R}$$
$$G \longmapsto \int_{\sigma_G} \phi_G \wedge \omega_G^{4k+1}$$

only depends on edge orderings + (ordered) cycle basis

why does this work? ~ equivalent notion of orientation data + "orientation" forms

- 

$$\Lambda_c = \begin{pmatrix} x_1+x_3 & x_3 \\ x_3 & x_1+x_2 \end{pmatrix}$$

- changes of cycle basis amounts to $\mapsto$ $\det P$ $\phi_{\Lambda_c} \wedge \omega_{\Lambda_c}$

  edge orderings $\mapsto$ $\operatorname{sgn}(\alpha)\, \sigma_G$

↳ exact sign needed $(G, or) \mapsto (G, or')$ for or' induced by action of P, α on or

COCYCLE CONDITION :

one key property of $\phi_G \wedge \omega_G$ is that they are closed

$\Rightarrow$ we can use STOKES' THEOREM

$\omega \in \Lambda^{\cdot}\left(\bigoplus_{k\geq 1} \Omega\beta^{4k+1}\right)$

$$0 = \int_{\tilde{\sigma}_G} d\left(\phi_G \wedge \omega_G\right) = \int_{\partial\tilde{\sigma}_G} \phi_G \wedge \omega_G$$

a compactification of σ_G

"Feynman polytope"

associated to γ and G/γ

$$= \sum_{e \in E} I_{G/e}(\omega) \quad + \quad \text{other boundary terms}$$

$I_{\partial G}(\omega)$

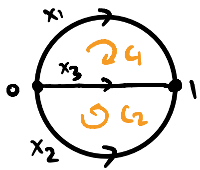
when we restrict to ω_G^{4k+1} (invariant trace)

and to graphs with loop number $\ell = 2k+4$

these terms vanish !

i.e. $\deg_3 = E - 3\ell$
$= -6$

③ SIMPLEST EXAMPLE:



$E = 3, \ \ell = 2$

$\deg_3 = -3$

$$\mapsto \quad I_G(1) = \frac{1}{(-2\pi)} \int_{\sigma_G} \phi_{\Lambda_c}$$

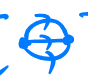
Pfaffian form:

$$\frac{\mathrm{Pf}\left(d\Lambda_c \cdot \Lambda_c^{-1} \cdot d\Lambda_c\right)}{\sqrt{\psi}}$$

$$-\frac{\Omega_3}{\psi^{3/2}}$$

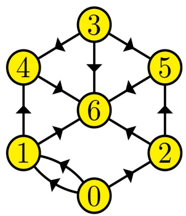
$$= \frac{1}{(-2\pi)} \int_{\sigma_G} \frac{-x_1\, dx_2 \wedge dx_3 + x_2\, dx_1 \wedge dx_3 - x_3\, dx_1 \wedge dx_2}{(x_1x_2 + x_2x_3 + x_1x_3)^{3/2}}$$

$$= 1$$

$\Leftrightarrow$ [] $\neq 0 \in H_{-3}(GC_3)$

FEYNMAN-ESQUE INTEGRALS!

For non-trivial ω^{4k+1} the integrals are much more complicated:

$G =$ 

$$\mapsto \quad I_G(\beta^5) = -80 + \frac{140}{9}\pi^2 - \frac{4}{3}\pi^4 + 210\zeta(3) - 20\pi^2 \ln 2 + \frac{20}{3}\left[(\ln 2)^4 + 24\,\mathrm{Li}_4\left(\tfrac{1}{2}\right)\right] - \frac{40}{9}\left[\pi^2 \mathcal{C} + 24\,\mathrm{Im}\,\mathrm{Li}_4(\mathrm{i})\right]$$

$E = 12$, $\ell = 6$

$k = \deg_3 = -6$

polylogarithm

Catalan's constant

and generally of the form:

$$I_G(\omega) = \frac{1}{(-2\pi)^{\ell/2}} \int_{\sigma_G} \frac{Q(x)}{\Psi^{\frac{\ell+1}{2}}} \Omega_G$$

some homogeneous polynomial in the x_e variables

projective volume form

symanzik polynomial

We were able to show :

$* \int \phi_G^6 \wedge \omega_G^5 \qquad \left(\begin{smallmatrix} \ell = 6 \\ E = 12 \end{smallmatrix} \right)$

THEOREM (FB, EP, SH) :

> The class $[\, gr_6 I(\omega^5) \,] \in H^{-6}(GC_3) \otimes \mathbb{R}$ is non-trivial.
>
> For a $X \in GC_3$ s.t. $\partial X = 0$, its orientation integral is
>
> $$I_X(\omega^5) = 40 \cdot \left(13\,\zeta(3) - 2\pi^2 \ln 2 \right)$$

proof

H^{-8}	H^{-7}	H^{-6}	H^{-5}	H^{-4}	H^{-3}
				0	1
		0	0	0	1
0	0	0	0	0	1
0	0	0	0	0	2
0	0	1	0	0	2
0	0	1	0	0	3
0	0	2	0	0	4

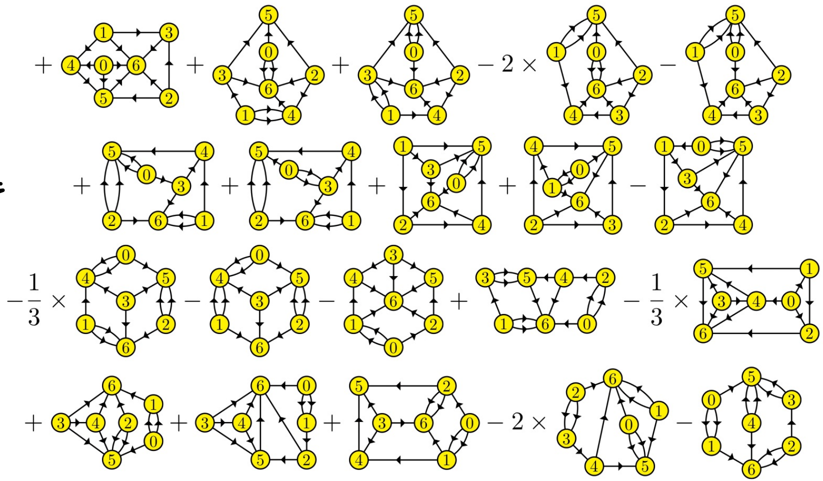
$gr_2 I(1)$

$\Leftrightarrow \quad \mapsto \frac{1}{(-2\pi)} \int_{\sigma_G} \phi_G = 1$

$[\Gamma_1] \Leftrightarrow X$

$[\Gamma_6] \in H^{-6}(GC_3)$

$X =$

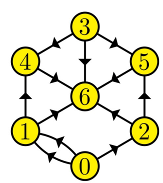


$I_X(w^5)$

$\mapsto \;= 40\cdot\left(13\,\zeta(3) - 2\pi^2 \ln 2\right)$

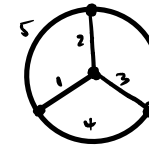
$\phi_G \wedge \beta_G^5 = \frac{Q_G}{\Psi^{5/2}}\,\Omega_{12}$

vs.

$G =$  $\mapsto$

$$I_G(\beta^5) = -80 + \tfrac{140}{9}\pi^2 - \tfrac{4}{3}\pi^4 + 210\zeta(3) - 20\pi^2\ln 2 + \tfrac{20}{3}\left[(\ln 2)^4 + 24\,\mathrm{Li}_4\left(\tfrac{1}{2}\right)\right] - \tfrac{40}{9}\left[\pi^2\mathcal{C} + 24\,\mathrm{Im}\,\mathrm{Li}_4(\mathrm{i})\right]$$

ANALOGOUS STORY for GC_2: " EVEN GRAPH COMPLEX "



- use only invariant differential forms $\omega \in \Lambda^{\bullet}\left(\bigoplus_{k \geq 1} \mathbb{Q}\, \beta^{4k+1} \right)$

- in degree 0, more known structure $\mathbb{L}(\sigma_3, \sigma_5, \dots) \hookrightarrow \mathfrak{grt}_1 \cong H^0(GC_2)$

(BROWN '12) (WILLWACHER '15)

with infinite family of known cycles/classes:

$$0 \neq [W_n] \in H_0(GC_2)$$

(ROSSI-WILLWACHER '14)

$W_n =$ 

$\mapsto \displaystyle\int_{\sigma_{W_n}} \omega^{2n-1} = n \binom{2n}{n} \zeta(n)$ (BROWN-SCHNETZ '24)

n odd

$\hookrightarrow$  $\mapsto \displaystyle\int_{\sigma_{W_3}} \omega^{5} = 60\, \zeta(3)$

PERIODS from INVARIANT DIFFERENTIAL FORMS:

$$\phi_X^{2m} := \frac{\mathrm{pf}\left(dX \cdot X^{-1} \cdot dX\right)}{\sqrt{\det X}} \qquad \beta_X^{4k+1} := \mathrm{tr}\left((X^{-1} dX)^{4k+1} \right)$$

- $\phi_G^{2m} \mapsto (2\pi)^m$
- $\beta_G^{4k+1} \mapsto$ includes all odd zetas $\zeta(3), \zeta(5), \ldots \zeta(2k+1)$

in upcoming work by J-L. PORTNER:

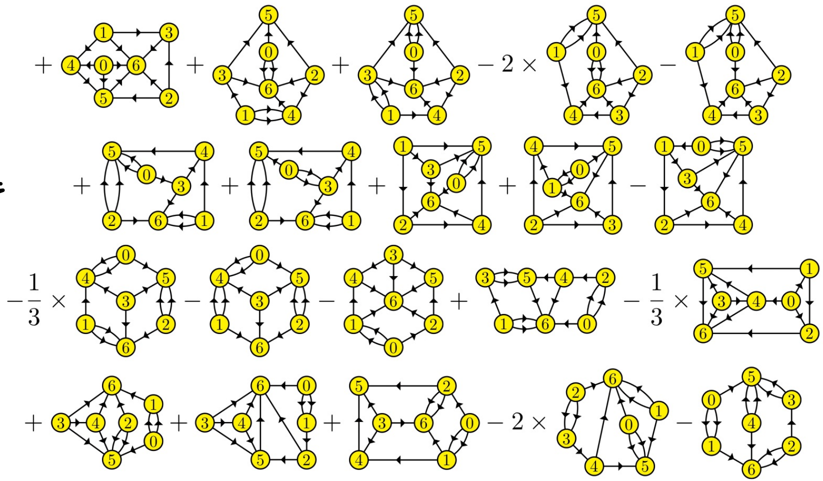
$$\int_{\sigma_G} \beta_G^{4k+1}$$ are all single-valued MZVs!

- $\phi_G^6 \wedge \beta_G^5 \mapsto$ lin. comb of polylogarithms (w/ 2π normalization)

$$10 \cdot \left(\frac{\lambda_1}{3} + \frac{\lambda_2}{9}\pi^2 + \lambda_3\zeta(3) + \frac{\lambda_4}{6}\left(2\pi^2 \ln 2 - 21\zeta(3)\right) + \frac{\lambda_5}{180}\pi^4 + \frac{\lambda_6}{3}\left((\ln 2)^4 + 24\,\mathrm{Li}_4(1/2)\right) + \frac{\lambda_7}{9}\left(\pi^2 \mathcal{C} + 24\,\mathrm{Im}\,\mathrm{Li}_4(\mathrm{i})\right)\right)$$

- $\beta_G^5 \wedge \beta_G^9 \mapsto$ includes $\zeta(3)\cdot\zeta(5)$ lin comb of π^8, $\zeta(3)\cdot\zeta(5)$, $\zeta(3,5)$

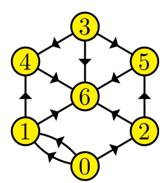
THANKS!

$X =$ 

$$I_X(\beta^5) \mapsto = 40 \cdot \left(13\zeta(3) - 2\pi^2 \ln 2\right)$$

$$\phi_G \wedge \beta_G^5 = \frac{Q_G}{\psi^{5/2}} \Omega_{12}$$

vs.

$G =$  $\mapsto$

$$I_G(\beta^5) = -80 + \tfrac{140}{9}\pi^2 - \tfrac{4}{3}\pi^4 + 210\zeta(3) - 20\pi^2 \ln 2 + \tfrac{20}{3}\left[(\ln 2)^4 + 24\,\mathrm{Li}_4\left(\tfrac{1}{2}\right)\right] - \tfrac{40}{9}\left[\pi^2 \mathcal{C} + 24\,\mathrm{Im}\,\mathrm{Li}_4(\mathrm{i})\right]$$

(c) TOPOLOGICAL QUANTUM FIELD THEORY (TQFT)

THEOREM (P-H B, SH):

$$\phi_G := * \frac{\mathrm{Pf}(d\Lambda \cdot \Lambda^{-1} d\Lambda)}{\sqrt{\psi}} = * \int \cdots \int_{\mathbb{R}^{|V|-1}} \bigwedge_{e \in E} e^{-s_e^2} \, ds_e =: \alpha_G$$

$$s_e = \frac{Y_e^+ - Y_e^-}{\sqrt{x_e}}$$

- α_G topological form introduced by (Gaiotto, Kulp, Wu)

in the context of Holomorphic-Topological QFTs ← flat spacetime w/ structure $\mathbb{C}^H \times \mathbb{R}^T$

where $\int_{\sigma_G} \alpha_G$ is computing violations to BRST-closedness

- $\alpha_G \wedge \alpha_G = 0$ ⇒ "no quantum loop corrections in theories w/ ≥ 2 topological directions" i.e. 2D TQFTs