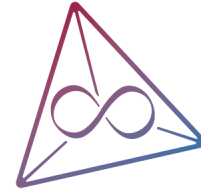




MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES



HYPERGEOMETRIC INTEGRALS ON A PUNCTURED RIEMANN SURFACE

Based on [2509.01598]
Andrzej Pokraka, Lecheng Ren, Carlos Rodriguez



MAIN QUESTION: ARE THERE g -LOOP KLT RELATIONS?

- Or more specifically, does the configuration space integrals over $\text{Conf}(n, \Sigma_g)$ in n -point, g -loop closed-string amplitudes exhibit a double-copy formula? These integrals are over the position of \underline{n} punctures.



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- If someone gives us such a formula, how do we check it? How do we evaluate [talk by Marco] such integrals? Can we α' -expand these into interesting periods?
- In this talk: Integrals over one puncture position exhibit a double-copy relation. It comes from twisted (co)homology.



OUR FAVORITE EXAMPLE: THE BETA FUNCTION

For $s, t \in \mathbb{R}$, $s, t > 0$, the Euler beta function $\beta(s, t)$ is given by the integral over $X = \mathbb{C}P^1 - \{0, 1, \infty\}$, a Riemann sphere with 3 punctures:

$$\beta(s, t) = \int_0^1 z^s (1-z)^t \frac{dz}{z(1-z)} = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$$

Physicists know $\beta(s, t)$ as the Veneziano amplitude. An amplitude of 4 open strings.

A closely related integral is the complex beta function $\beta_{\mathbb{C}}(s, t)$, given by the integral:

$$\beta_{\mathbb{C}}(s, t) = \int_{\mathbb{C}P^1 - \{0, 1, \infty\}} |z|^{2s} |1-z|^{2t} \frac{dz \wedge d\bar{z}}{|z|^2 |1-z|^2}$$

whenever $s, t > 0$, $s + t < 0$.

Physicist know $\beta_{\mathbb{C}}(s, t)$ as the Virasoro-Shapiro amplitude. An amplitude of 4 closed strings.

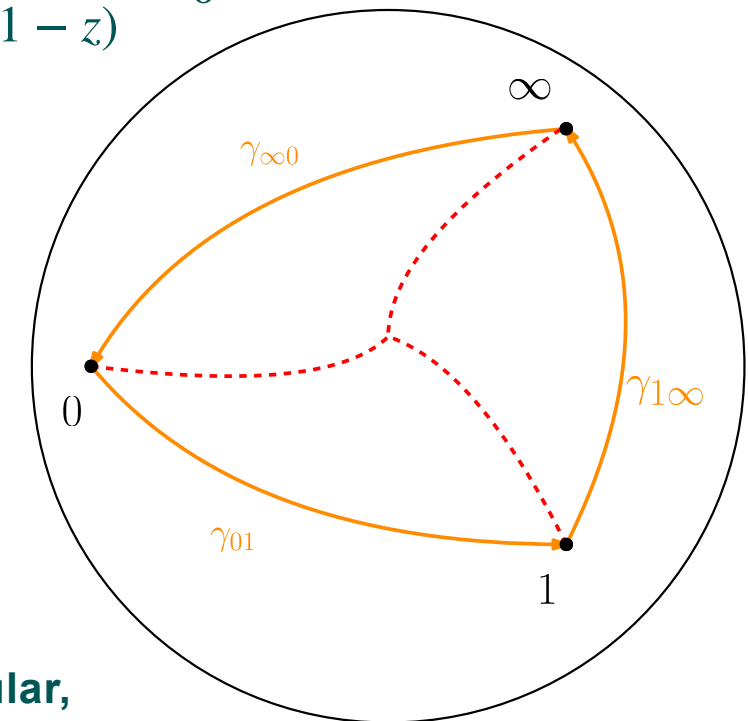
LINEAR AND QUADRATIC RELATIONS FOR THE BETA FUNCTION

The beta function satisfies linear relations:

$$\int_0^1 z^s(1-z)^t \frac{dz}{z(1-z)} + e^{\pi it} \int_1^\infty z^s(z-1)^t \frac{dz}{z(1-z)} + e^{-\pi is} \int_{-\infty}^0 (-z)^s(1-z)^t \frac{dz}{z(1-z)} = 0$$

And a quadratic - or double copy - relation:

$$\beta_{\mathbb{C}}(s, t) = \left(\frac{2 \sin(\pi s) \sin(\pi t)}{i \sin(\pi(s+t))} \right) [\beta(s, t)]^2$$



Both of these can be explained with tools of twisted (co)homology! In particular, the double copy relation is obtained via linear algebra.



SO WHERE IS THE PHYSICS?

- $\beta(s, t)$ corresponds to a scattering amplitude of 4 open strings in flat spacetime.
- $\beta_{\mathbb{C}}(s, t)$ corresponds to a scattering amplitude of 4 closed strings in flat spacetime.
- The linear relations among the different contours relates different amplitudes of open strings. These are known as monodromy relations [Plahte; Bjerrum-Bohr, Damgaard, Vanhove; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove; Stieberger].
- The double copy relations relates an amplitude of closed strings to a bilinear combination of amplitudes of open strings [Kawai, Lewellen, Tye, '85; Dotsenko, Fateev '85, Aomoto '87]. These are known to physicists as KLT relations.
- What about loop-level string amplitudes - i.e. g -loop KLT relations? There are some encouraging results at genus $g = 1$ [Stieberger; Mazloumi, Stieberger].

HYPERGEOMETRIC INTEGRALS, FOLLOWING AOMOTO

The beta function is a hypergeometric integral, i.e. an integral over $X = \mathbb{C} - \{0,1\}$ given by:

$$\beta(s, t) = \int_{\gamma} T(z)\varphi(z)$$

Where $T(z_1)$ is a multivalued function on X , $\varphi(z)$ a 1-form holomorphic on X .

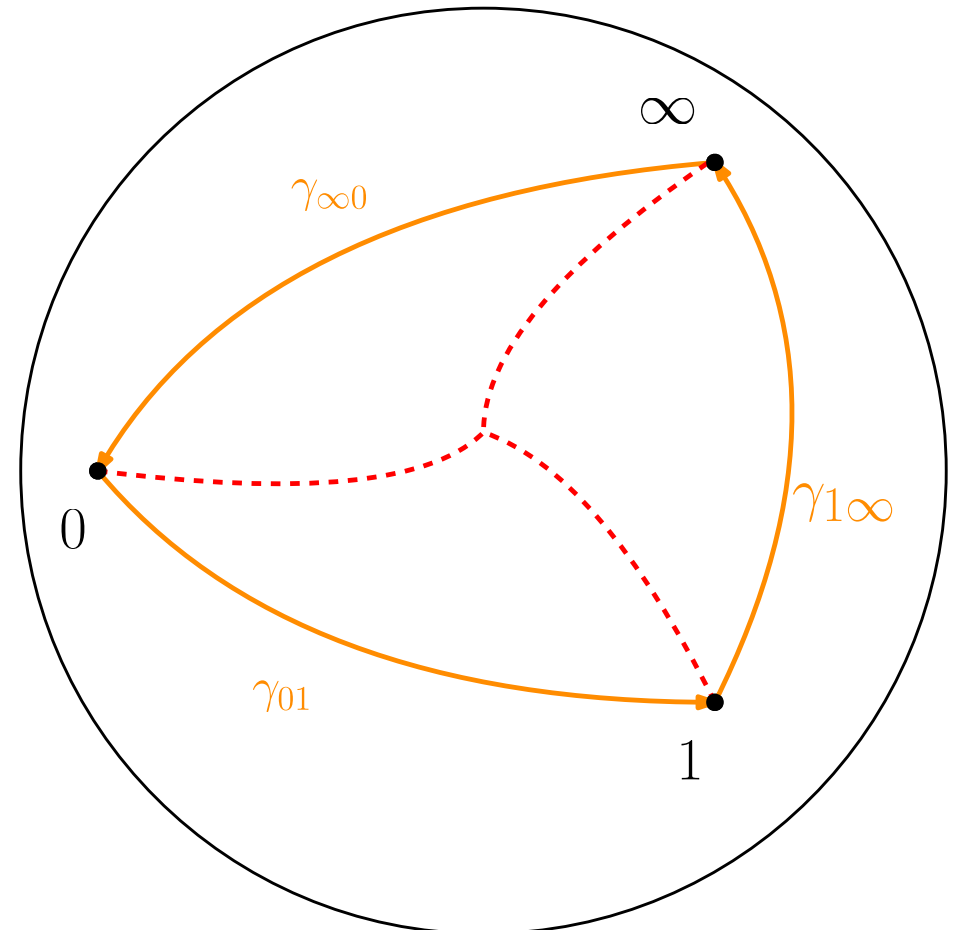
We moreover have:

$[\varphi] \in H^1(X, \nabla_{\omega})$, a twisted cohomology group

$[\gamma] \in H_1(X, \check{\mathcal{L}}_{\omega})$, a twisted cohomology group

Both of these (co)homology groups are determined by $T(z)$.

Our goal will be to find similar integral, and (co)homology groups, for a punctured Riemann surface of genus- g .





REMINDER: FUNCTIONS ON A COMPACT RIEMANN SURFACE Σ_g

- Any such Riemann surface has a g linearly independent holomorphic differentials $\omega_I, I = 1, 2, \dots, g$.

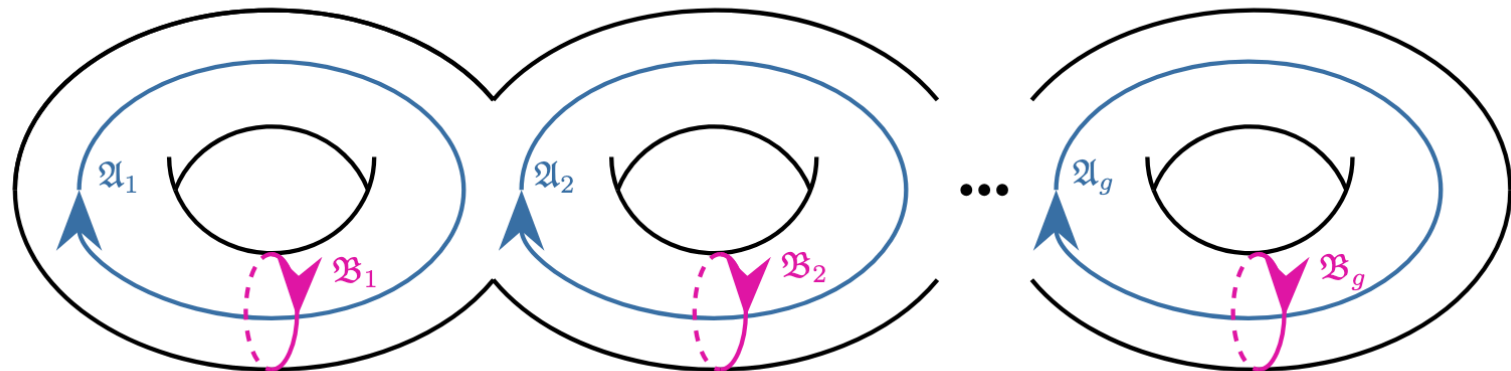
- The first homology group $H_1(\Sigma_g, \mathbb{Z})$ is of rank $2g$. We can choose a symplectic basis of cycles $\{\mathfrak{A}_J, \mathfrak{B}_J\}, J = 1, 2, \dots, g$.

- The holomorphic differentials we choose are normalized such that:

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ},$$

where the $\Omega_{IJ} = \Omega_{JI}$ is the period matrix of Σ_g .

- One more function: the prime form $E(x, y)$. This is the component of a $(-1/2, -1/2)$ -form on Σ_g , locally behaves like $(x - y)$.



WHAT'S A MULTIVALUED FUNCTION?

Let $\Sigma_g^* = \Sigma_g - \{z_2, z_3, \dots, z_n\}$ be a Riemann surface of genus g with $(n - 1)$ points removed. Let

$$\rho : \pi_1(\Sigma_g^*, p) \rightarrow \mathbb{C}^*$$

be a representation of its fundamental group.

A ρ -multivalued function on $\tilde{\Sigma}_g^*$ is a function

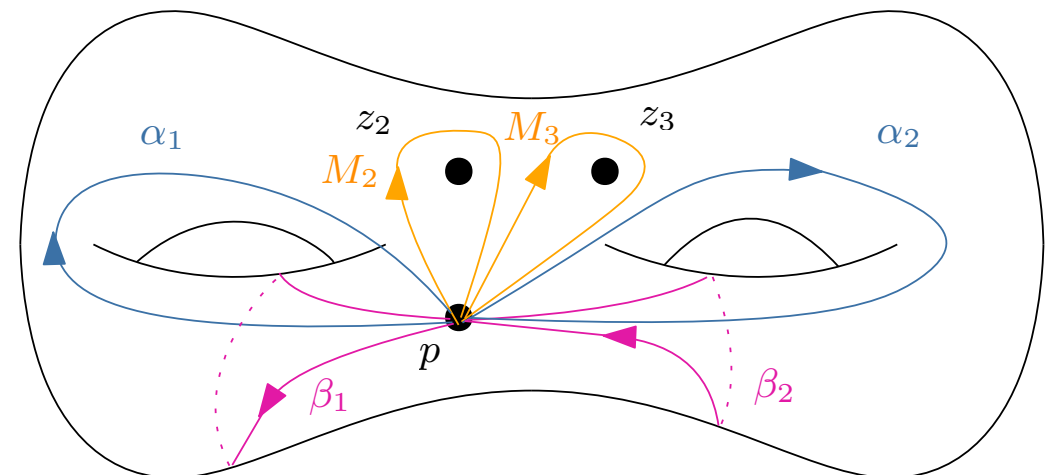
$$\begin{aligned} T : \tilde{\Sigma}_g^* &\rightarrow \mathbb{C} \\ z_1 &\mapsto T(z_1) \end{aligned}$$

such that

$$T(\alpha \circ z_1) = \rho(\alpha)T(z_1), \quad \forall \alpha \in \pi_1(\Sigma_g^*, p)$$

Note that $\pi_1(\Sigma_g^*) = \langle M_2, \dots, M_n, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g \rangle$, where:

- M_j is the loop that goes around the puncture z_j .
- α_I is the loop that goes around an \mathfrak{A}_I -cycle.
- β_I is the loop that goes around an \mathfrak{B}_I -cycle.





WHAT IS $T(z_1)$? THE MULTIVALUED FUNCTION ON Σ_g^*

Let $s_{1j}, s_{1A_J} \in \mathbb{R}, j = 2, \dots, n, J = 1, \dots, g$ subject to $\sum_{j=2}^n s_{1j} = 0$. Then we define

Koba-Nielsen factor in
chiral splitting
[D'Hoker, Phong]

$$T(z_1) = \prod_{j=2}^n [E(z_1, z_j)]^{s_{1j}} \exp \left[2\pi i \sum_{J=1}^g s_{1A_J} \int_{t=P}^{t=z_1} \omega_J(t) \right],$$

$s_{1A_J} = k_1 \cdot \ell_J$

where $E(x, y)$ is the prime form on Σ_g , and the ω_J are holomorphic differentials on Σ_g .

We can read off a representation ρ as before:

$$\begin{aligned} T(M_j \circ z_1) &= \rho_j T(z_1) = e^{2\pi i s_{1j}} T(z_1) \\ T(\alpha_J \circ z_1) &= \rho_{A_J} T(z_1) = e^{2\pi i s_{1A_J}} T(z_1) \\ T(\beta_J \circ z_1) &= \rho_{B_J} T(z_1) = e^{2\pi i s_{1B_J}} T(z_1), \end{aligned}$$

where $s_{1B_J} \in \mathbb{C}$ is given by

$$s_{1B_J} = \sum_{I=1}^g \Omega_{IJ} s_{1A_I} + \sum_{j=2}^n s_{1j} \int_{t=P}^{t=z_j} \omega_J(t)$$

Remark: [Humihiko Watanabe, 2016] have studied the twisted cohomology groups $H^1(\Sigma_g^*, \nabla_\omega) \ni [\varphi]$.



DEFINING TWISTED HOMOLOGY AND COHOMOLOGY GROUPS

Let $T(z_1)$ be a ρ -multivalued function, and denote its rank-1 local system by $\check{\mathcal{L}}_\omega$.

Note that the 1-form $\omega = d \log T(z)$ is well-defined on Σ_g^* .

We form the twisted de Rham operator

$$\nabla_\omega = d + \omega \wedge$$

which is closed, i.e. $\nabla_\omega \circ \nabla_\omega = 0$. Then, we have a chain complex:

$$0 \xrightarrow{\nabla_\omega} \Omega^1(\Sigma_g^*) \xrightarrow{\nabla_\omega} \Omega^2(\Sigma_g^*) \xrightarrow{\nabla_\omega} \dots$$

- The k -th cohomology group of the above chain complex is the twisted cohomology group, $H^k(\Sigma_g^*, \nabla_\omega)$.
- The k -th singular homology group with coefficients in $\check{\mathcal{L}}_\omega$ form the twisted homology group $H_k(\Sigma_g^*, \check{\mathcal{L}}_\omega)$.

Theorem (Watanabe, 2016)

For generic ρ (i.e. certain products $\rho_i \neq 1, i = 2, \dots, n$), the groups $H^k(\Sigma_g^*, \nabla_\omega)$ all vanish except for $k = 1$. We have $\dim H^1(\Sigma_g^*, \check{\mathcal{L}}_\omega) = |\chi(\Sigma_g^*)| = (2g + n - 3)$. Moreover, $[\omega_J] \in H^1(\Sigma_g^*, \check{\mathcal{L}}_\omega), J = 1, \dots, g$.

HYPERGEOMETRIC INTEGRALS ON Σ_g^* , FOLLOWING AOMOTO

We are interested in integrals on a complex 1-dimensional manifold Σ_g^* :

$$I_\gamma^\varphi = \int_\gamma T(z_1)\varphi(z_1)$$

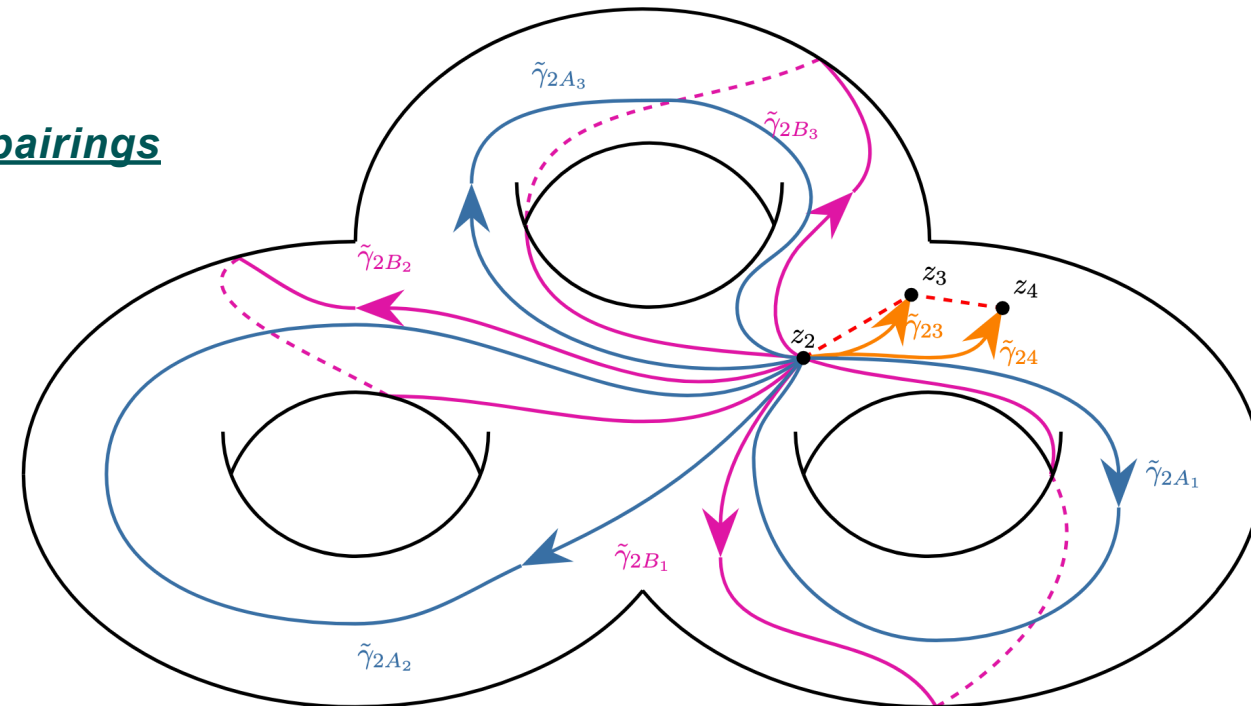
The integrals I_γ^φ are a natural, non-degenerate bilinear pairings between twisted (co)homology groups, for

$$[\gamma] \in H_1(\Sigma_g^*, \check{\mathcal{L}}_\omega),$$

$$[\varphi] \in H^1(\Sigma_g^*, \nabla_\omega).$$

Key idea: We can construct other non-degenerate pairings among twisted (co)homology groups.

Then use linear algebra to find the double-copy relations.





A MULTITUDE OF BILINEAR PAIRINGS AND TWISTED (CO)HOMOLOGY GROUPS

Theorem [Cho, Matsumoto, Ito, Yoshida, Hanamura...]

When $\rho : \pi_1(\Sigma_g^*) \rightarrow U(1)$, the following bilinear pairings are non-degenerate (for $X = \Sigma_g^*$):

$$[\varphi_j] \in H^1(X, \nabla_\omega) \cong H^1(X, \nabla_{\bar{\omega}}) \ni [\bar{\varphi}_k]$$

\Downarrow

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$$[\gamma_a] \in H_1(X, \check{\mathcal{L}}_\omega) \cong H_1(X, \check{\mathcal{L}}_{\bar{\omega}}) \ni [\bar{\gamma}_b]$$



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$$\int_{\gamma_a} T \varphi_j \quad \cong \quad \int_{\gamma_b} \overline{T \varphi_k}$$

$$[\gamma_a] \in H_1(X, \check{\mathcal{L}}_\omega) \cong H_1(X, \check{\mathcal{L}}_{\overline{\omega}}) \ni [\overline{\gamma_b}]$$

$$[\check{\gamma}_b | \gamma_a]$$



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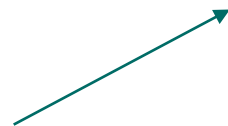
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Fun bilinear: homology intersection number

$$[\check{\gamma}_b | \gamma_a] \in \mathbb{Q}(\rho) = \mathbb{Q}(e^{2\pi i s_{1j}}, e^{2\pi i s_{1Aj}}, e^{2\pi i s_{1Bj}})$$



ENSURING SINGLE-VALUEDNESS OF $|T(z_1)|^2$

For the double-copy to work, we need that $s_{1B_j} \in \mathbb{R}$:

$$s_{1B_j} = \sum_{I=1}^g \Omega_{IJ} s_{1AI} + \sum_{j=2}^n s_{1j} \int_{t=p}^{t=z_j} \omega_J(t)$$

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We can ensure this via

$$s_{1A_I} = - \sum_{J=1}^g Y^{IJ} \sum_{j=2}^n s_{1j} \mathfrak{I}[\nu_J(z_j)], \quad \forall I \in \{1, \dots, g\},$$

where $Y_{IJ} = \mathfrak{I}(\Omega_{IJ})$, and $Y^{IJ} Y_{JK} = \delta_K^I$.



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where $Y_{IJ} = \mathfrak{I}(\Omega_{IJ})$, and $Y^{IJ} Y_{JK} = \delta_K^I$.

But remembering that $s_{1A_j} = k_1 \cdot \ell_j$, we have:

$$p^I \rightarrow - \sum_{J=1}^g Y^{IJ} \sum_{j=2}^n k_j \mathfrak{I}[\nu_J(z_j)], \quad \forall I \in \{1, \dots, g\}$$

This resembles the result of integrating out the loop momentum in chiral splitting!

NEW RESULT ON TWISTED (CO)HOMOLOGY

Theorem [Pokraka, Ren, R, 2025]

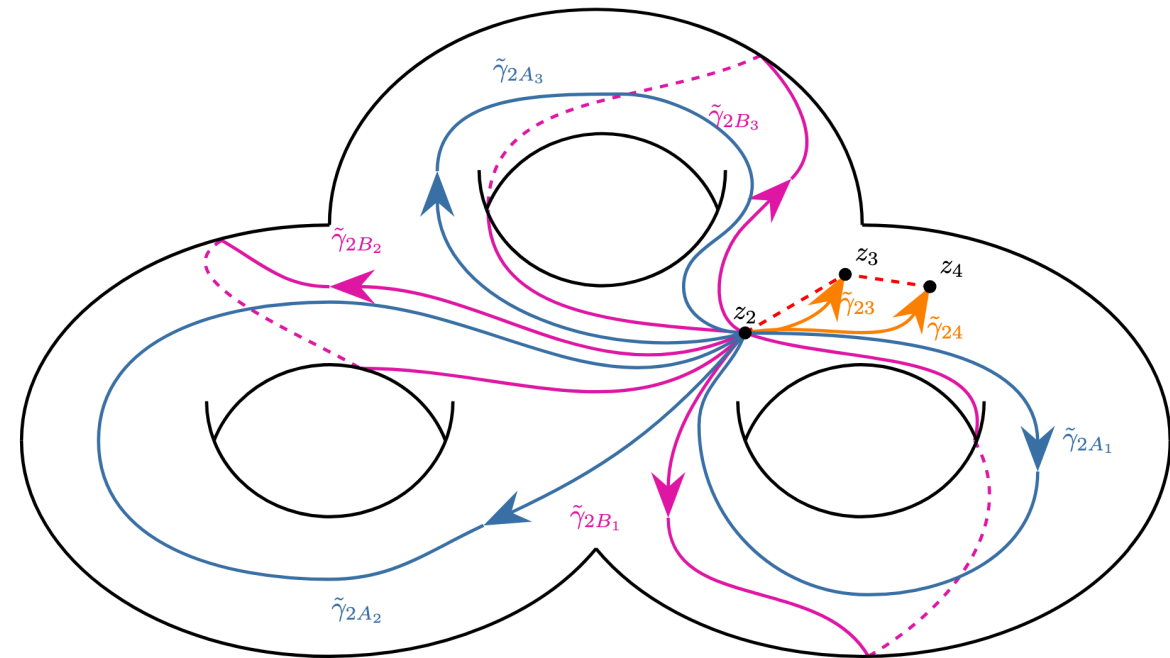
$H_1(\Sigma_g^*, \check{\mathcal{L}}_\omega)$ is of dimension $(2g + n - 3)$. It is generated by $(2g + n - 2)$ twisted cycles $[\gamma_a]$ subject to one linear relation. To prove this, we use homology intersection numbers $[\bullet | \bullet]$.

Corollary

The hypergeometric integrals on Σ_g^* satisfy double copy relations!

$$\int_{\Sigma_g^*} |T(z_1)|^2 \overline{\omega_I} \wedge \omega_J = \sum_{a,b \in \mathcal{K}} H_{(2,2)}^{a,b} \left[\int_{\gamma_a} T(z_1) \omega_I \right] \left[\int_{\gamma_b} T(z_1) \omega_J \right],$$

where $H_{(2,2)}^{a,b}$ is the inverse of a matrix of homology intersection numbers, and \mathcal{K} labels your favorite basis of twisted cycles.

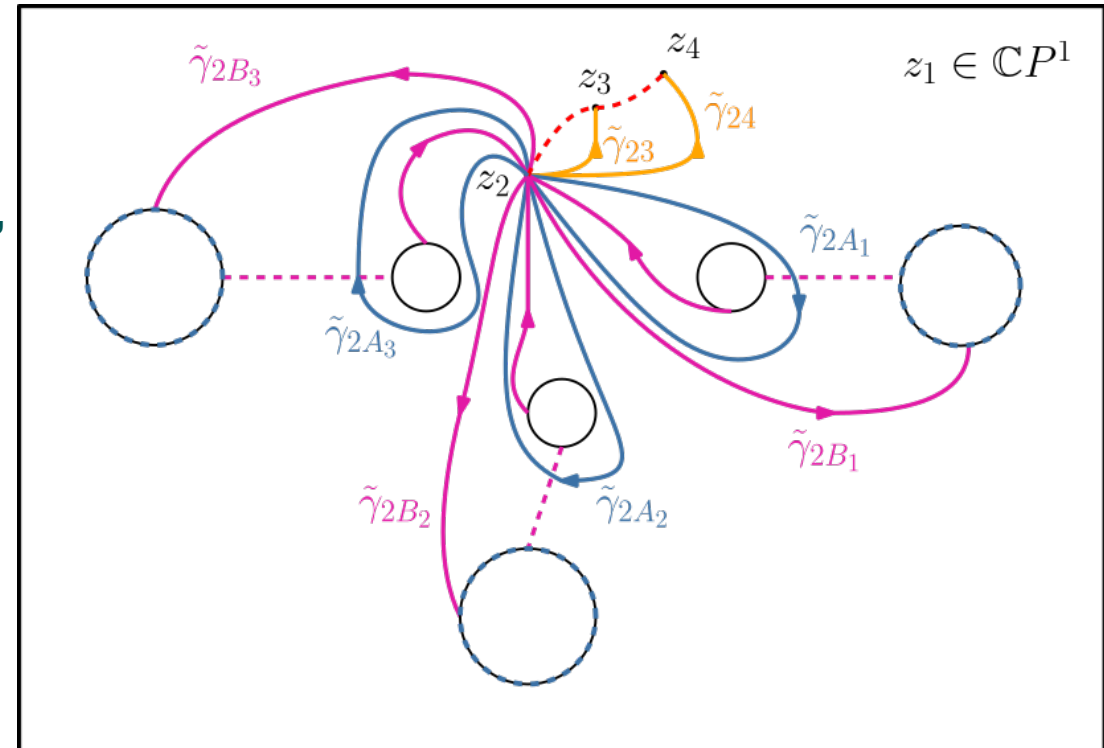


CONCLUSIONS AND REMARKS

- There are hypergeometric functions on Σ_g^* for which all the goodness of twisted (co)homology works well: intersection numbers, double copy, etc...
- These integrals I_γ^φ naturally extend the case $g = 1$ [Mano,Watanabe;Ghazouani-Pirio; Goto] to $g > 1$.
- We predict concrete linear and quadratic relations. We can check these numerically.
- It's also possible to use twisted cohomologies with Abelian Kronecker forms [Baune, Broedel, Im, Lisitsy, Zerbini;D'Hoker, Enriquez, Schlotterer, Zerbini; D'Hoker, Hidding, Schlotterer], generating function of higher-genus polylog kernels:

$$\omega_J \rightarrow F_J(z_1, z_j, \alpha_I)$$

$$F_J(z_1, z_j, \alpha_I) \alpha_J = \omega_J + g_J^{I_1}(z_1, z_j) \alpha_{I_1} + g_J^{I_1 I_2}(z_1, z_j) \alpha_{I_1} \alpha_{I_2} + \dots$$



FUTURE DIRECTIONS AND CHALLENGES

- Twisted cohomology works for these integrals of 1 puncture... what about the full configuration space integrals (i.e. n punctures)? Can we obtain g -loop KLT relations? The story at genus-one [Stieberger, Mazloumi-Stieberger] suggests this is possible.

- How do we obtain the α' -expansion of the integrals

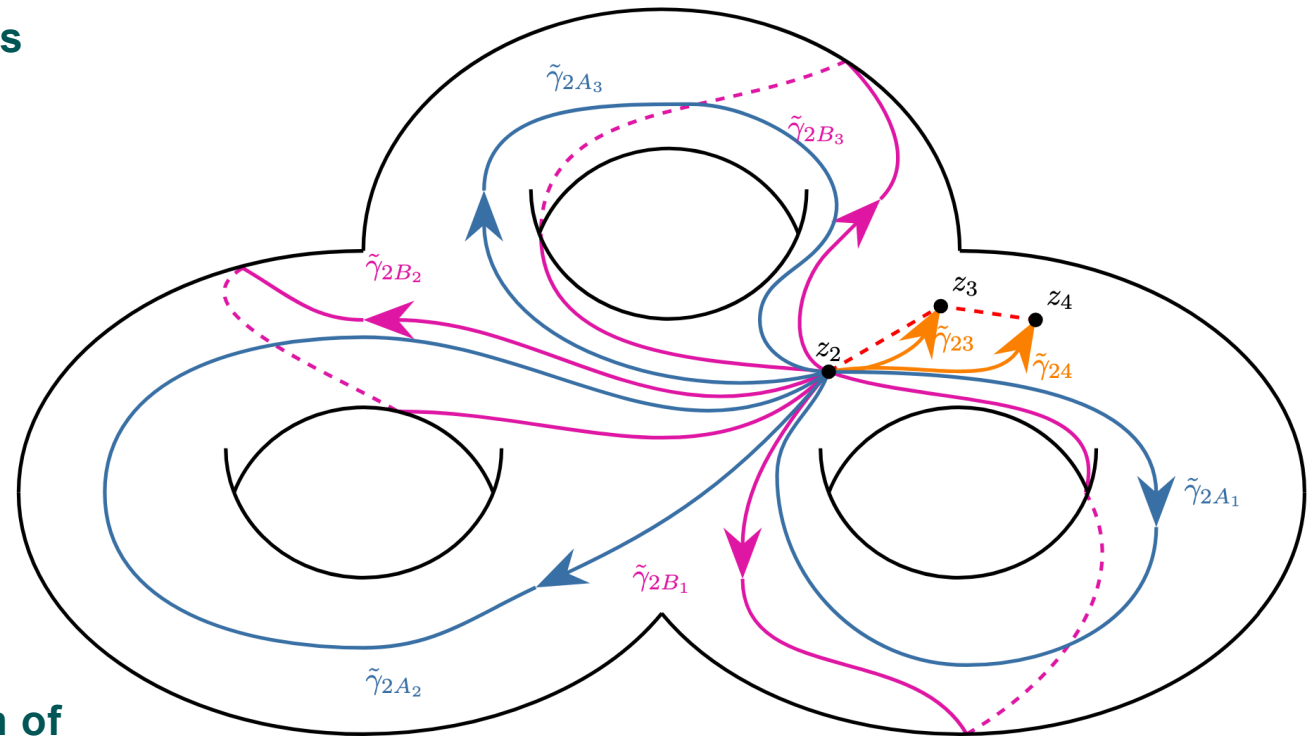
$$I_\gamma^\varphi = \int_\gamma T(z_1)\varphi(z_1)$$

and

$$J^{\bar{\varphi}_i, \varphi_k} = \int_{\Sigma^*} |T(z_1)|^2 \bar{\varphi}_i \wedge \varphi_k ?$$

The double-copy relations would then give interesting relations among periods. This is still open at genus-one!

- Expand the numerical toolset we to the evaluation of higher-genus MZVs! [Baune, Broedel, Im, Ji, Moeckli]





EXTRA SLIDES

THANK YOU FOR YOUR ATTENTION!

