# Rademacher evaluation of modular integrals

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Based on work in collaboration with

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[2501.13827, 2507.22105, WIP]

# Motivation

**Very little is known about higher-loop amplitudes exactly in $\alpha'$**

Main technical difficulty: define contour+hard integrals (numerically unstable!)

# Literature status so far:

Small $s$ $\rightarrow$ modular graph forms (one-loop)

Large $s$ $\rightarrow$ saddle-point approximation

**Our work: exact in $\alpha'$ at 1-loop!**

# Plan of this talk:

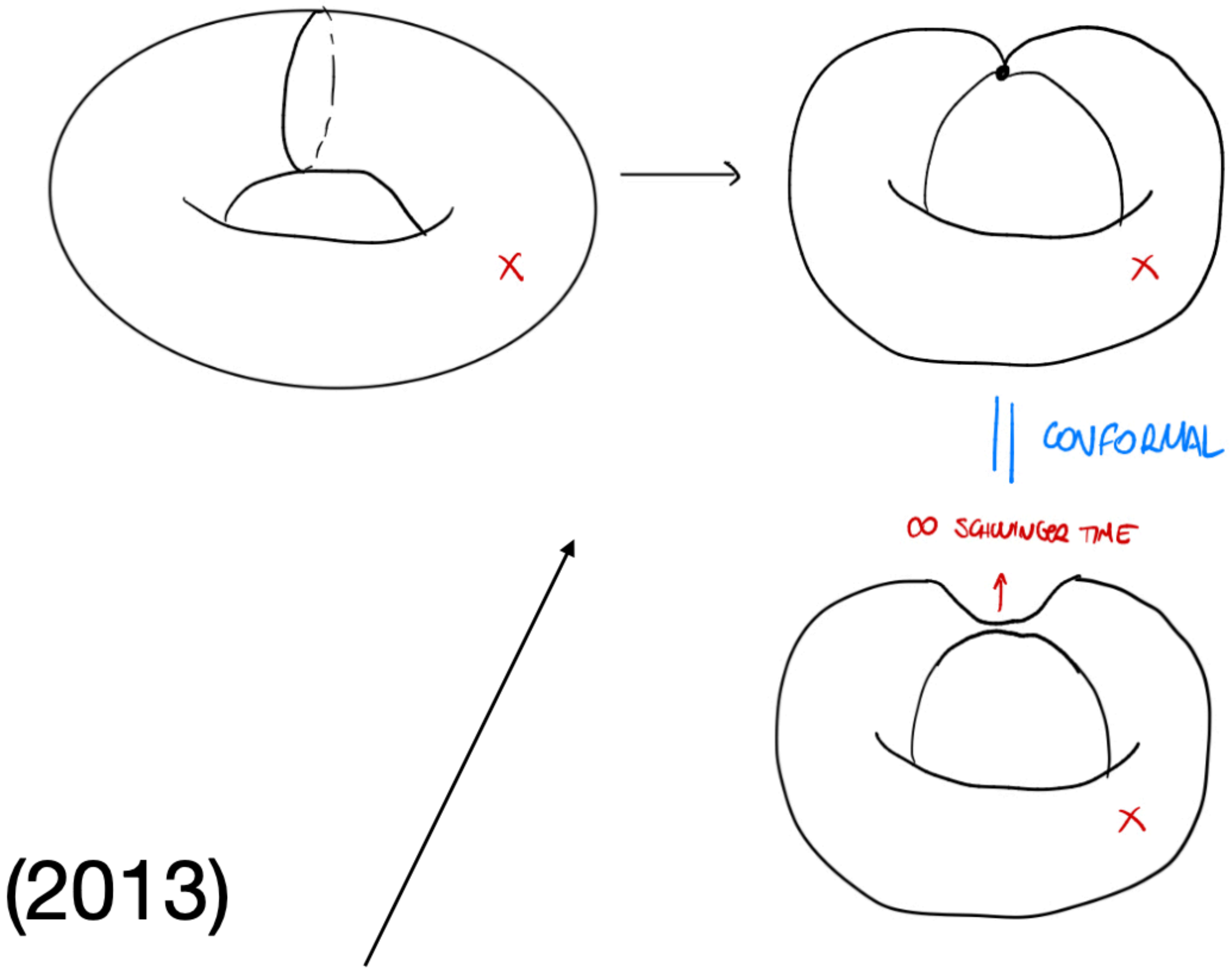
- Part 1: Understanding the contour problem in string theory ($i\varepsilon$-prescription)
- Part 2: 1-loop: define **Lorentzian** contours and evaluate integrals in one-go: Rademacher evaluation of modular integrals [2501.13827]
- Part 3: Interesting physics [2507.22105, WIP]

# Part 1. The contour problem

# String scattering amplitudes

$$A = \int_{\Gamma \subset \mathfrak{M}^{\mathbb{C}}_{g,n}} (\text{CFT correlator})$$

Contour consistent with Lorentzian signature




How? E. Witten **The Feynman $i\varepsilon$ in string theory** (2013)

Should modify contour when worldsheet degenerates = when intermediate particles go on-shell

# Intuition from QFT

$$\frac{-i}{p^2+m^2-i\varepsilon} = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-it(p^2+m^2)-\varepsilon t}$$

- $t =$ Schwinger time = length of the propagator $\Rightarrow$ $\mathbb{R}_{\geq 0} \sim$ toy $\mathfrak{M}_{g,n}$
- Choice of $i\varepsilon$ - prescription = contour choice in $\mathbb{R}^{\mathbb{C}}_{\geq 0}$
- No contour choice $\Rightarrow$ integral diverges when $p^2 = -m^2$ ($t = \infty$ worldline degenerates)

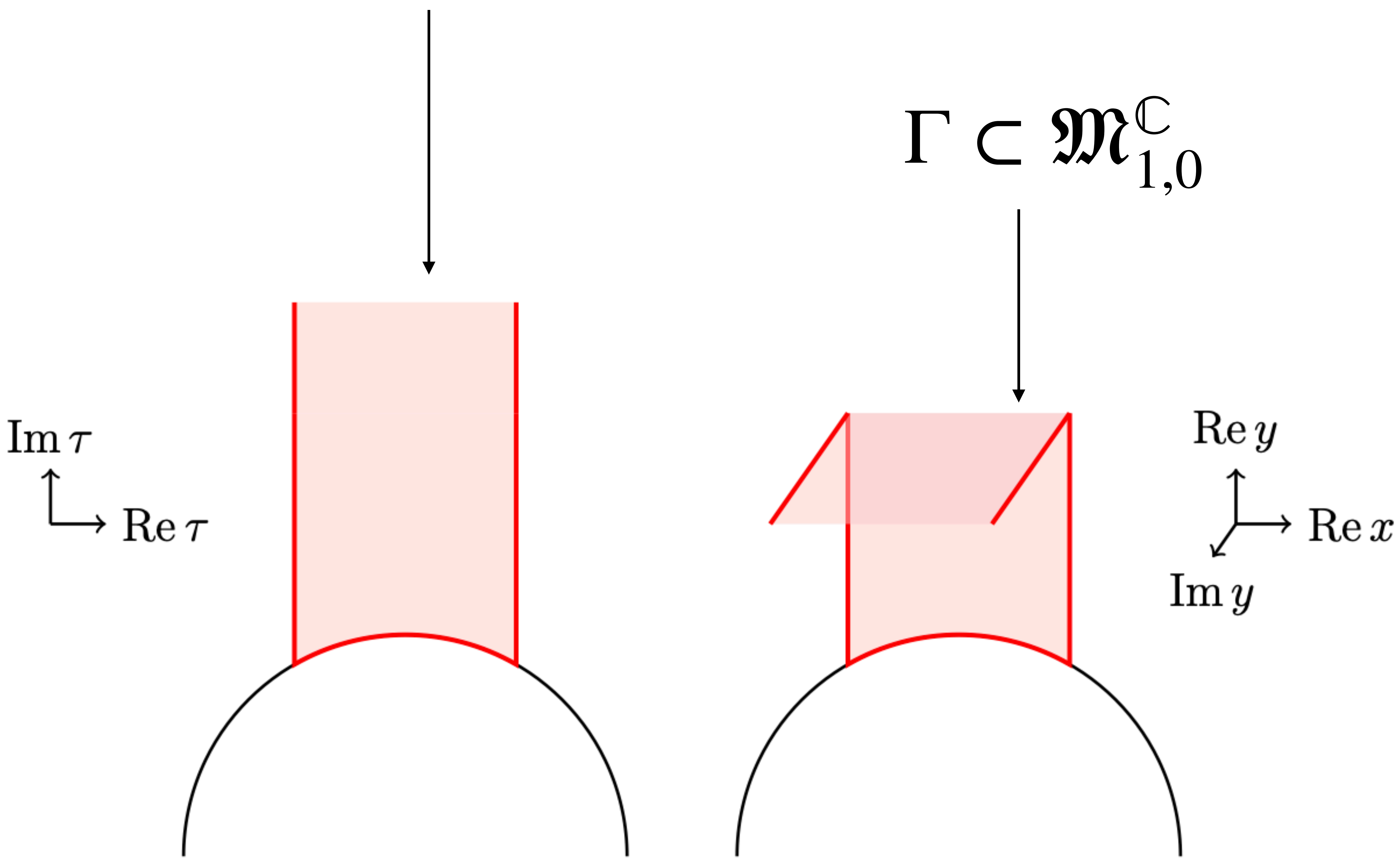
# Part 2. Rademacher evaluation of modular integrals

# Our setting: Torus (1-loop)

- Schwinger time is $\mathrm{Im}(\tau)$
- After some contourology, we get a ready-to-use formula!
- This formula computes integrals of forms over the fundamental domain in general !!

$$\mathfrak{M}_{1,0} = \frac{\mathbb{H}}{\mathrm{SL}(2,\mathbb{Z})} \Rightarrow \mathfrak{M}^{\mathbb{C}}_{1,0} = \frac{\mathbb{H}\times\mathbb{H}}{\mathrm{SL}(2,\mathbb{Z})}$$

$\Gamma \subset \mathfrak{M}^{\mathbb{C}}_{1,0}$



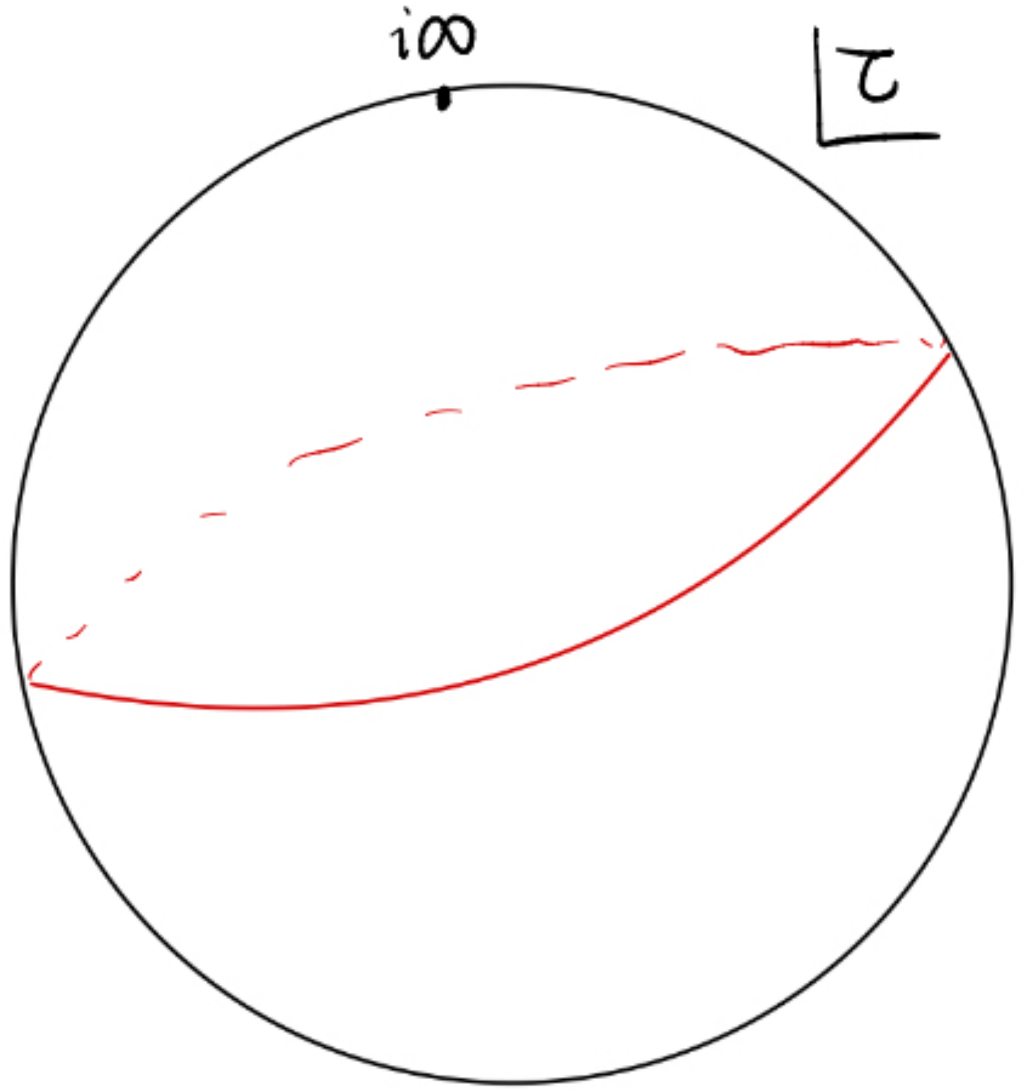
# The Rademacher formula

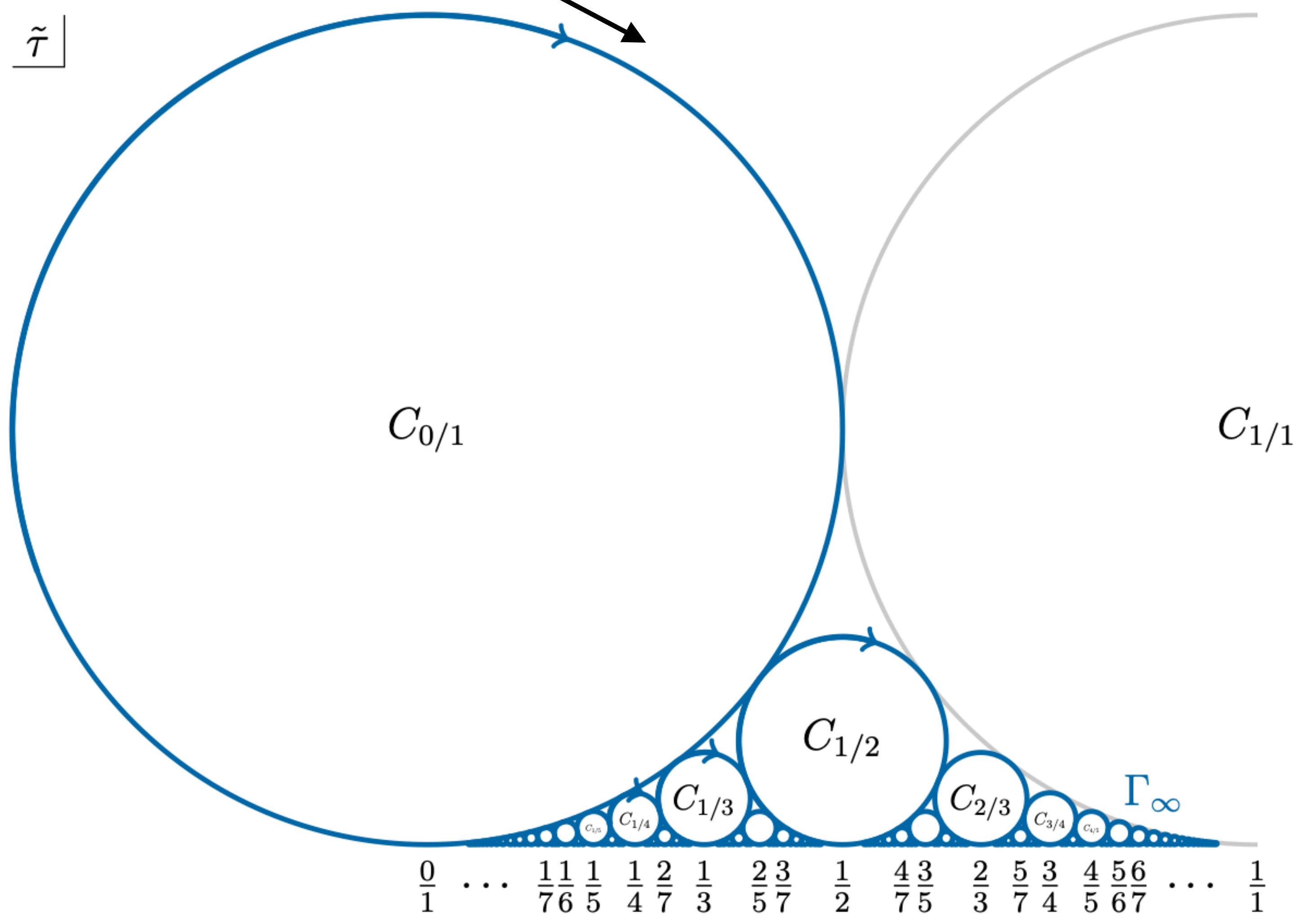
MMB, Chandra, Eberhardt, Hartman, Mizera ‘25

$s(a,c)$ Dedekind sum

$$\int_\Gamma \mathrm{d}^2\tau\, f(\tau,\tilde\tau) = \sum_{c=1}^{\infty}\sum_{\substack{a=0\\(a,c)=1}}^{c-1}\int_{\longrightarrow} \mathrm{d}\tau \int_{C_{a/c}} \mathrm{d}\tilde\tau \left[\frac{1}{12i}\left(\tau-\tilde\tau+\frac{2a}{c}\right)+is(a,c)-\frac{1}{2}\right] f(\tau,\tilde\tau)$$

Imaginary part




A modular transformation maps $C_{a/c}$ to $\longrightarrow$

$\Rightarrow$ pick polar terms and extract two residues (easy)

# Fix ideas: bosonic string partition function

$$Z = \int_\Gamma \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^{14}\,|\eta(\tau)^{24}|^2}$$

$$\tau = \frac{a\tau' + b}{c\tau' + d}$$ Maps $C_{a/c}$ to $\longrightarrow$

All you need to know about $\eta(\tau)$:

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)^{24} = (c\tau+d)^{12}\eta(\tau)^{24},$$

$$\eta(\tau)^{-24} = \mathrm{e}^{-2\pi i\tau} + \dots \quad \text{as} \quad \mathrm{Im}\,\tau \to \infty$$

$$\int_{\longrightarrow} \mathrm{d}\tau \int_{C_{a/c}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^{14}\,|\eta(\tau)^{24}|^2} = 2^{13}\int_{\longrightarrow}\frac{\mathrm{d}\tilde{\tau}}{\eta(\tilde{\tau})^{24}}\int_{\longrightarrow}\frac{\mathrm{d}\tau}{(-(a\tilde{\tau}+b+\tau(c\tilde{\tau}+d))^{14}\,\eta(\tau)^{24}}$$

$$= 2^{13}\int_{\longrightarrow}\mathrm{d}\tilde{\tau}\;\mathrm{e}^{2\pi i\tilde{\tau}}\int_{\longrightarrow}\frac{\mathrm{d}\tau}{(-(a\tilde{\tau}+b+\tau(c\tilde{\tau}+d))^{14}}\mathrm{e}^{2\pi i\tau}$$

# Main takeaways

- Rademacher formula gives the one-loop Lorentzian contour in $\mathfrak{M}^{\mathbb{C}}_{1,0}$
- It is holomorphically factorized ⇒ easy to do contour deformations (saddle point analysis later!)
- The formula is ready-to-use!

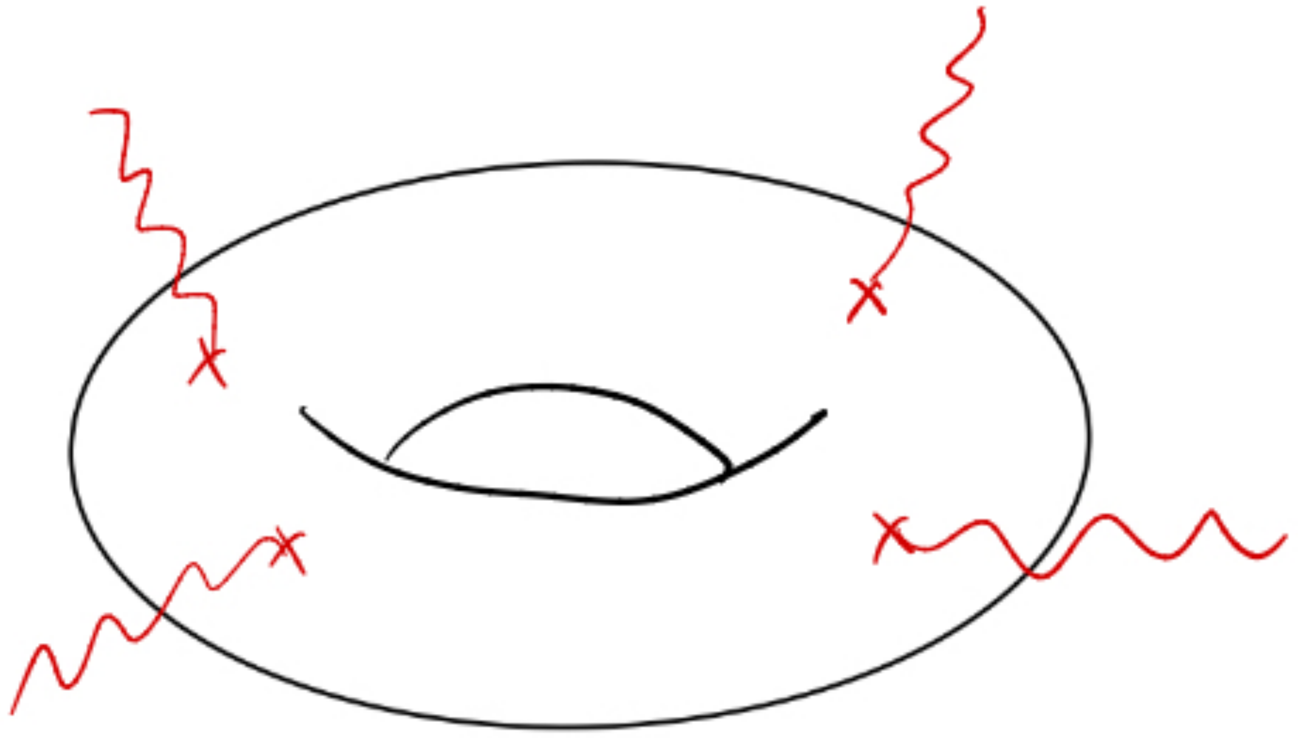
$$\int_\Gamma \mathrm{d}^2\tau\, f(\tau,\tilde{\tau}) = \sum_{c=1}^{\infty} \sum_{\substack{a=0\\(a,c)=1}}^{c-1} \int_{\longrightarrow} \mathrm{d}\tau \int_{C_{a/c}} \mathrm{d}\tilde{\tau} \left[\frac{1}{12i}\left(\tau-\tilde{\tau}+\frac{2a}{c}\right)+is(a,c)-\frac{1}{2}\right] f(\tau,\tilde{\tau})$$

# Part 3. Interesting physics

# Focus on 2-2 gravitons scattering in type II

$$A=\int_{\Gamma}\frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^5}\int_{\mathbb{T}^2}\prod_{j=1}^{3}\mathrm{d}^2 z_j\prod_{1\leqslant i<j\leqslant 4}\left|\vartheta_1\left(z_{ij}\mid\tau\right)\right|^{-2s_{ij}}\mathrm{e}^{\frac{2\pi s_{ij}(\mathrm{Im}\,z_{ij})^2}{\mathrm{Im}\,\tau}}$$

Jacobi theta function



Highly oscillatory near $\mathrm{Im}(\tau)=\infty \Rightarrow$ awuful numerics!

# Rademacher × 2-2 gravitons at one-loop

$$A \sim \sum_{c=1}^{\infty} \sum_{\substack{a=0 \\ \gcd(a,c)=1}}^{c-1} \sum_{n_{\mathrm{L}},n_{\mathrm{D}},n_{\mathrm{R}}=0}^{c-1} \mathrm{e}^{i(\text{phase})} A_{a/c}^{n_{\mathrm{L}},n_{\mathrm{D}},n_{\mathrm{R}}}$$

⟶ Complicated expression

$$1 \lesssim s \lesssim \mathcal{O}(10-70)$$

Convergence gets bad for small $s$

Sum over polarizations becomes hard

Try for yourself: `StringQMC.cpp` ⟶ Instructions on [2507.22105]

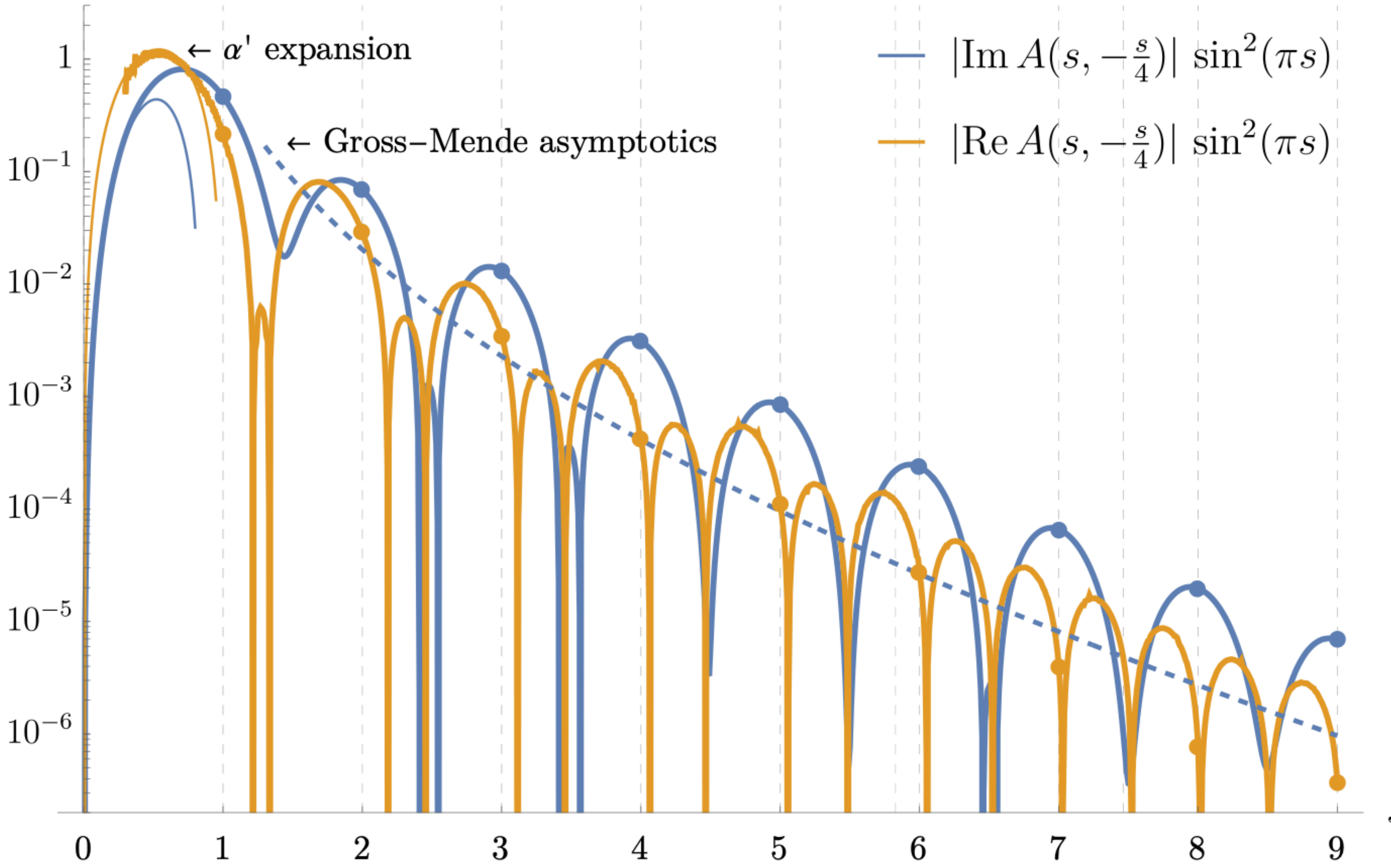
# Example: Fixed angle @ 60 degrees

This plot was not accessible before!

Decay predicted but never shown Gross-(Mañes)Mende ('89)

We see oscillations on top of it

Rademacher allows to include Lorentzian saddles! (WIP)

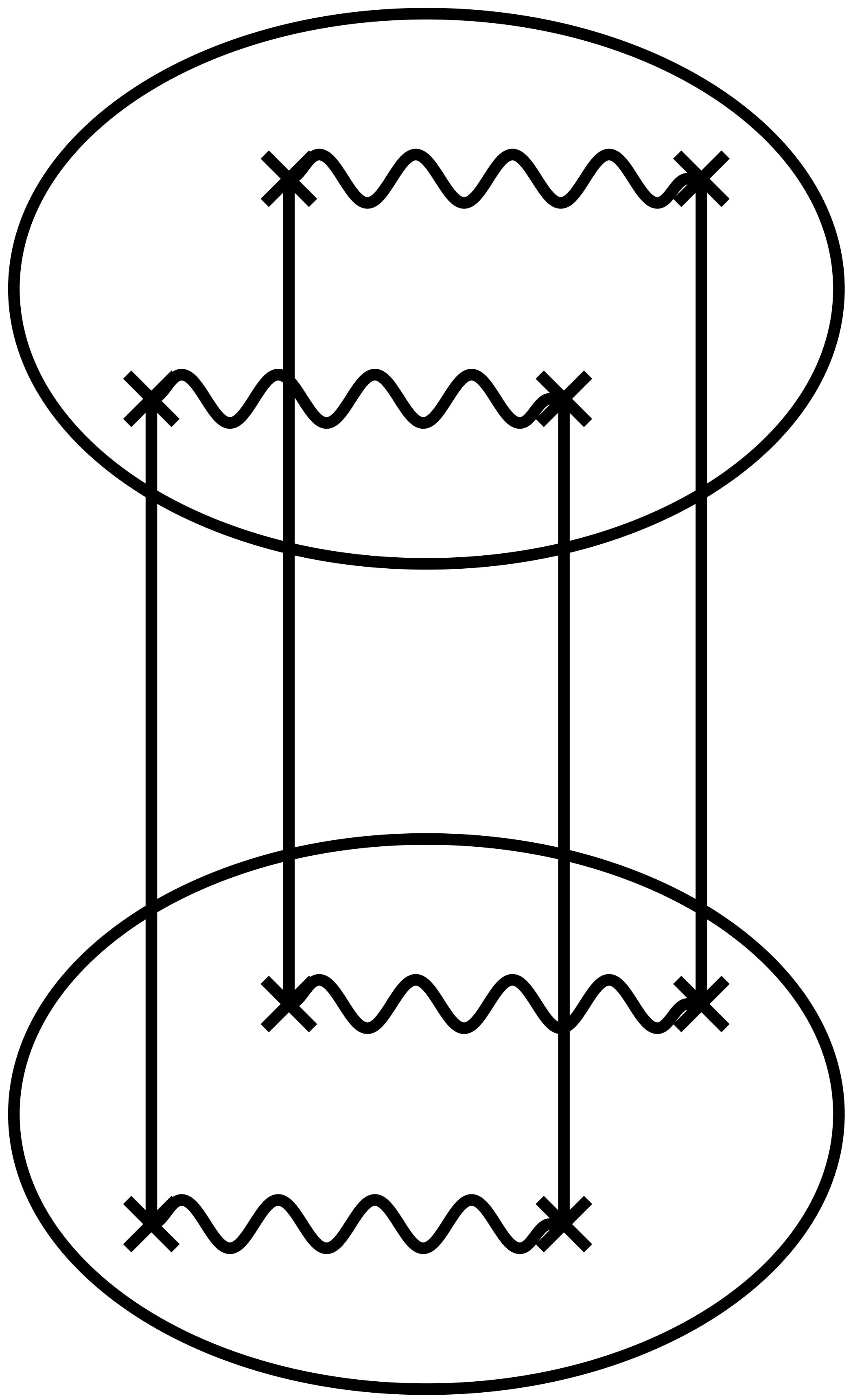



MMB, Eberhardt, Mizera '25

# High energy: Gross-Mende argument

Gross-(Mañes)Mende ('89) saddle point of string amplitudes

They find one saddle: covering of tree level



$$S_* = \frac{1}{g+1} S_{\text{tree}} = \frac{1}{g+1}\big(s\log(s) + t\log(-t) + u\log(-u)\big)$$

$$A_g \sim s^{1-g} e^{-\frac{s}{g+1} f(\theta)}$$

But you can braid punctures to get an infinite set of saddles!

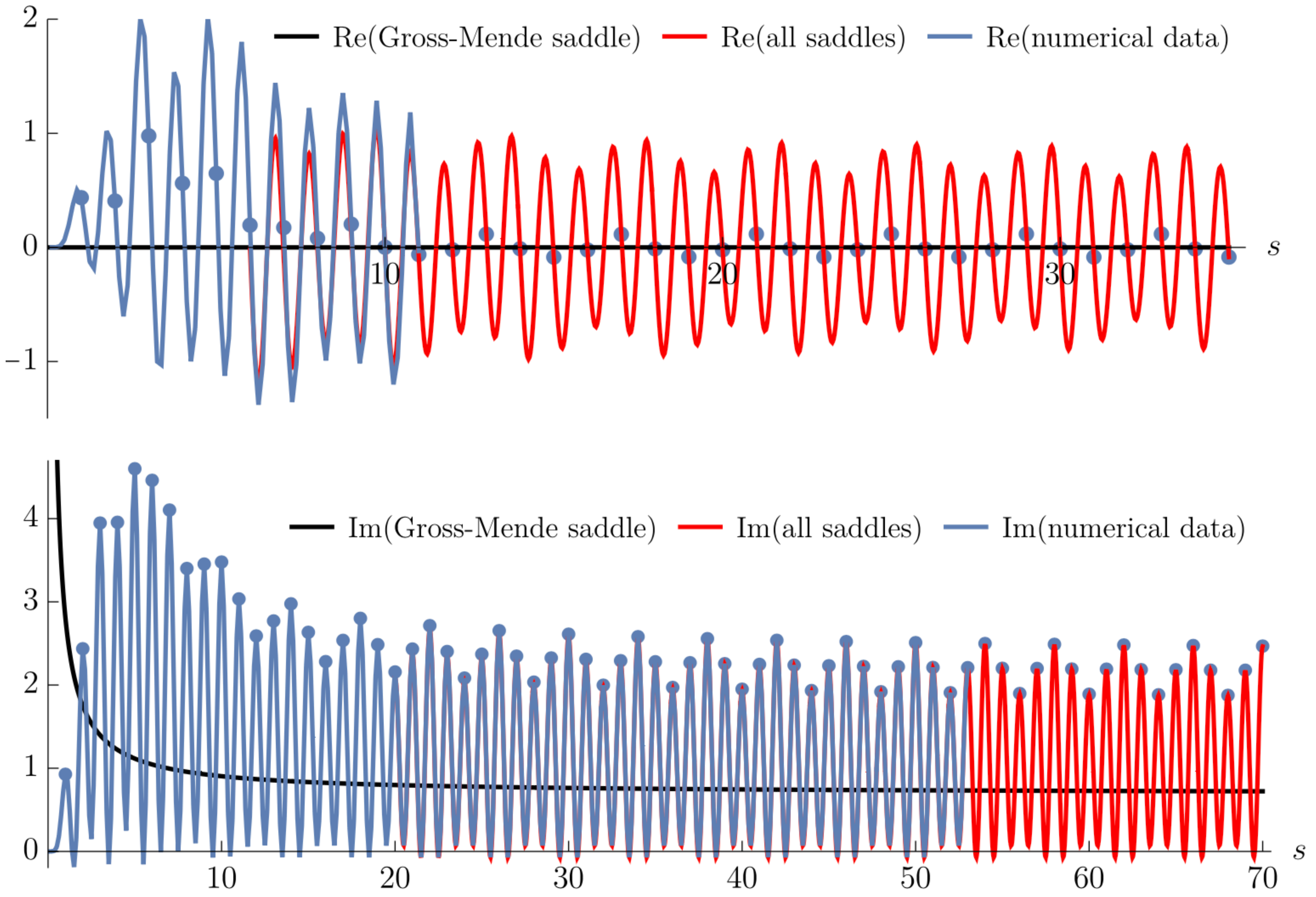
They give oscillatory contributions.

# Large $s$: saddle-point approximation (WIP)

Scattering angle: 90 degrees

Rademacher contour is holomorphically factorized

⇒ Steepest descend contour easier to study



# Conclusions

- Rademacher formula makes available new computations in string theory at one-loop!
- Rademacher is Lorentzian
- Holomorphically factored nature allows to explore the UV of string theory
- Future directions: Low-energy matching? High-energy open strings? Higher-loops?

# Backup: 2-2 scattering expression

$$A_{a/c}^{n_{\mathrm{L}},n_{\mathrm{D}},n_{\mathrm{R}}}$$

$$= \sum_{\substack{\sqrt{m_{\mathrm{D}}}+\sqrt{m_{\mathrm{U}}}\leqslant\sqrt{s} \\ \sqrt{\tilde{m}_{\mathrm{D}}}+\sqrt{\tilde{m}_{\mathrm{U}}}\leqslant\sqrt{s}}} \frac{(\Delta\tilde{\Delta})^{\frac{9}{4}}}{720\sqrt{2}c^{\frac{7}{2}}s^{\frac{11}{2}}}\, \mathrm{e}^{\frac{2\pi i}{c}(n_{\mathrm{D}}(m_{\mathrm{D}}+d\tilde{m}_{\mathrm{D}})+n_{\mathrm{U}}(m_{\mathrm{U}}+d\tilde{m}_{\mathrm{U}}))-\frac{\pi i}{cs}\Delta^{(2)}}$$

$$\times \int_{\mathbb{D}} \mathrm{d}x\,\mathrm{d}y \int_{\mathbb{D}} \mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}\, Q\tilde{Q}\, \mathrm{e}^{\frac{\pi i\sqrt{\Delta\tilde{\Delta}}}{sc}(x\tilde{x}+y\tilde{y})}((1-x^2-y^2)(1-\tilde{x}^2-\tilde{y}^2))^{\frac{5}{4}}$$

$$\times \left[\sqrt{\frac{\tilde{P}}{P}}\left(J_{\frac{3}{2}}\left(\frac{4\pi\sqrt{P\tilde{P}}}{c}\right) - J_{\frac{7}{2}}\left(\frac{4\pi\sqrt{P\tilde{P}}}{c}\right)\right) + 12ic\, s(a,c) J_{\frac{5}{2}}\left(\frac{4\pi\sqrt{P\tilde{P}}}{c}\right)\right]\Bigg|_{\substack{P\to\frac{\Delta}{4s}(1-x^2-y^2) \\ \tilde{P}\to\frac{\tilde{\Delta}}{4s}(1-\tilde{x}^2-\tilde{y}^2)}}$$

$$\times \left(\frac{\Gamma(-t_{\mathrm{L}})\Gamma(s+t_{\mathrm{L}}-m_{\mathrm{D}}-m_{\mathrm{U}})\Gamma(-\tilde{t}_{\mathrm{L}})\Gamma(s+\tilde{t}_{\mathrm{L}}-\tilde{m}_{\mathrm{D}}-\tilde{m}_{\mathrm{U}})}{\Gamma(s)^2}\right.$$

$$\left.\times \begin{cases} \mathrm{e}^{2\pi i t_{\mathrm{L}}\left(\left(\frac{n_{\mathrm{L}}}{c}\right)\right)+2\pi i\tilde{t}_{\mathrm{L}}\left(\left(\frac{dn_{\mathrm{L}}}{c}\right)\right)}, & n_{\mathrm{L}} \neq 0 \\ \frac{\mathrm{e}^{\pi i(t_{\mathrm{L}}-\tilde{t}_{\mathrm{L}})}+\mathrm{e}^{\pi i(\tilde{t}_{\mathrm{L}}-t_{\mathrm{L}})}-\mathrm{e}^{\pi i(t_{\mathrm{L}}+\tilde{t}_{\mathrm{L}})}-\mathrm{e}^{-\pi i(2s+t_{\mathrm{L}}+\tilde{t}_{\mathrm{L}})}}{1-\mathrm{e}^{-2\pi i s}}, & n_{\mathrm{L}} = 0 \end{cases}\right)$$

$$\times (\mathrm{L} \leftrightarrow \mathrm{R})\ .$$

# Backup: Sketch of proof

Key conceptual insight:

$$\mathfrak{M}_{1,0} \cong \mathfrak{M}_{0,4}$$

$$I = \int_{\mathcal{F}(\mathbb{H}/\mathrm{SL}(2,\mathbb{Z}))} \mathrm{d}^2\tau \, f(\tau, \tilde{\tau}) = \frac{1}{6} \int_{\mathcal{F}(\mathbb{H}/\Gamma(2))} \mathrm{d}^2\tau \, f(\tau, \tilde{\tau})$$

- We work with $\mathbb{H}/\Gamma(2)$, which has genus 0!
- ⇒Bijection with Riemann sphere via $z = \lambda(\tau)$ (multivalued)
- Some contourology lands us on a **holomorphically-factorized contour**

$$I = \frac{1}{6} \int_{\tau \in [0, i\infty)} \int_{\tilde{\tau} \in [-1,1]} \mathrm{d}^2\tau \, f(\tau, \tilde{\tau})$$