Elliptic modular graph forms, equivariant iterated integrals and single-valued elliptic polylogarithms

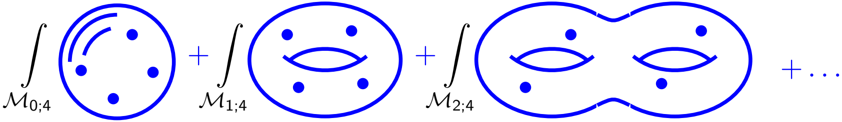
Yi-Xiao Tao (Tsinghua University)

arXiv:2511.15883 with Oliver Schlotterer and Yoann Sohnle

Table of Contents

String perturbation theory

The perturbative expansion of string amplitudes leads to a decomposition in integrals over the moduli space of punctured Riemann surfaces:



Low energy expansion of string amplitudes

At each order, string amplitudes can be expanded as a series of α'. E.g., at the genus-0 level, the Veneziano amplitude (4pt massless open string amplitude) can be expanded as

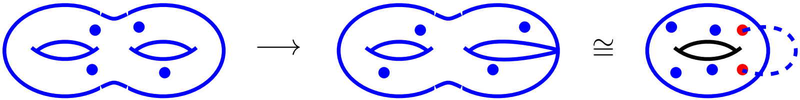
$$\begin{aligned}\frac{\Gamma(s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} &= \frac{1}{s_{12}}\exp\left(\sum_{n=2}^{\infty}\frac{\zeta_n}{n}(-1)^n[s_{12}^n+s_{23}^n-(s_{12}+s_{23})^n]\right)\\ &= \frac{1}{s_{12}}-\zeta_2 s_{23}+\zeta_3 s_{23}(s_{12}+s_{23})+\dots\end{aligned} \tag{1}$$

where $s_{ij}=2\alpha' k_i\cdot k_j$.

Expansion in α' in genus-0 case $\longrightarrow$ Riemann zeta values ζ_n

String integrals with extra marked points

We are also interested in the string integrals with some extra marked points which are unintegrated. E.g., when considering the non-separating degeneration of the genus-2 surface, we will introduce two new marked points:



Genus-0 open string amplitudes: MZVs and MPLs

For higher-point genus-0 open string amplitudes, we will get the multiple zeta values (MZVs) $\zeta_{n_1,n_2,\ldots,n_r} = \sum_{0<k_1<k_2<\cdots<k_r} k_1^{n_1} k_2^{n_2} \ldots k_r^{n_r}$ [Schlotterer-Stieberger '12; Broedel-Schlotterer-Stieberger-Terasoma '13]. These MZVs can be generated by MPLs:

$$G(a_1, a_2, \ldots, a_w; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_w; z) \tag{2}$$

with $a_i = \{0, 1\}$ and $G(\emptyset; z) = 1$. And

$$\zeta_{n_1,n_2,\ldots,n_r} = (-1)^r G(\vec{0}_{n_r-1}, 1, \ldots, \vec{0}_{n_2-1}, 1, \vec{0}_{n_1-1}; 1) \tag{3}$$

Genus-0 open string amplitudes: more marked points

MPLs will also appear in the expansion of open string integrals with more marked points. Generally, we can define the generating series of n-variable MPLs:

$$\begin{aligned}\mathbb{G}_n &= \mathbb{G}_{A=\{0,1,z_1,\ldots,z_{n-1}\}}(z_n; e_i^{(n)}) \\ &= \sum_{r=0}^{\infty} \sum_{\substack{a_1,\ldots,a_r \\ \in\{0,1,z_1,\ldots,z_{n-1}\}}} e_{a_1}^{(n)} \ldots e_{a_r}^{(n)} G(a_r,\ldots,a_1; z_n)\end{aligned} \tag{4}$$

$e_i^{(n)}$: generators of the braid algebra.

Genus-0 open string integrals with n extra marked points relate to $\mathbb{G}_i$ with all $1 \leq i \leq n$ and MZVs. In different cases, $e_i^{(n)}$ have different representations [Britto-Mizera-Rodriguez-Schlotterer '21].

Single-valued map for MPLs: From open to close strings

After the low-energy expansion, close string amplitudes can be related to the open string amplitudes via a single-value map. In the genus-0 case [Schnetz '13; Brown '13; Frost-Hidding-Kamlesh-Rodriguez-Schlotterer-Verbeek '23]:

$$\text{MPL} \xrightarrow{\text{single-value map}} \text{sv MPL} \xrightarrow{\text{generate}} \text{sv MZVs}$$

$$\mathbb{G}_{e_0,e_1}(z) \xrightarrow{\text{single-value map}} \mathbb{G}^{\mathrm{sv}}_{e_0,e_1}(z) = (\mathbb{M}^{\mathrm{sv}}_0)^{-1}\, \overline{\mathbb{G}_{e_0,e_1}(z)^T}\, \mathbb{M}^{\mathrm{sv}}_0\, \mathbb{G}_{e_0,e_1}(z) \tag{5}$$

Single-valued map for MPLs: From open to close strings

with

$$\mathbb{M}_0^{\rm sv} = \sum_{r=0}^{\infty} \sum_{\substack{i_1,\ldots,i_r \\ \in 2\mathbb{N}+1}} \rho^{-1}\big(\mathrm{sv}(f_{i_1}\ldots f_{i_r})\big)\, M_{i_1}\ldots M_{i_r} \tag{6}$$

$$= 1 + 2\sum_{i_1\in 2\mathbb{N}+1} \zeta_{i_1} M_{i_1} + 2\sum_{i_1,i_2\in 2\mathbb{N}+1} \zeta_{i_1}\zeta_{i_2} M_{i_1} M_{i_2} + \ldots$$

Here M_i are the zeta generators and satisfy

$$\big[e_0, M_w\big] = 0\,, \qquad \big[e_1, M_w\big] = \big[P_w(e_0,e_1), e_1\big] \tag{7}$$

with

$$P_w(e_0,e_1) = \mathbb{G}_{e_0,e_1}(1)\,\Big|_{\zeta_w} = -P_w(e_1,e_0)\,, \qquad w \geq 3 \text{ odd} \tag{8}$$

Single-valued map for MPLs: more variables

For multi-variable MPLs, which correspond to genus-0 string integrals with more marked points, the single-value map is defined in [Frost-Hidding-Kamlesh-Rodriguez-Schlotterer-Verbeek '23].
E.g., 2-variable MPLs:

$$\mathbb{G}^{\mathrm{sv}}_{e_0',e_1',e_y'}(y,z) = \\ \left(\mathbb{G}^{\mathrm{sv}}_{e_0,e_1}(y)\right)^{-1} (\mathbb{M}_0^{\mathrm{sv}})^{-1}\, \overline{\mathbb{G}_{e_0',e_1',e_y'}(y,z)^T}\, \mathbb{M}_0^{\mathrm{sv}}\, \mathbb{G}^{\mathrm{sv}}_{e_0,e_1}(y) \mathbb{G}_{e_0',e_1',e_y'}(y,z) \tag{9}$$

Genus-1 open string amplitudes: eMZVs and eMPLs

Following the normalization conventions ($q = e^{2\pi i\tau}$)

$$\theta_1(z,\tau) = q^{1/8}(e^{i\pi z} - e^{-i\pi z}) \prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi iz}q^n)(1-e^{-2\pi iz}q^n) \tag{10}$$

The meromorphic Kronecker-Eisenstein series $F(z,\alpha,\tau)$ and the associated integration kernels $g^{(n)}(z,\tau)$:

$$F(z,\alpha,\tau) = \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)} = \sum_{n=0}^{\infty}\alpha^{n-1}g^{(n)}(z,\tau) \tag{11}$$

A doubly-periodic but non-meromorphic completion:

$$\Omega(z,\alpha,\tau) = \exp\left(2\pi i\alpha\,\frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau}\right)F(z,\alpha,\tau) = \sum_{n=0}^{\infty}\alpha^{n-1}f^{(n)}(z,\tau) \tag{12}$$

Genus-1 open string amplitudes: eMZVs and eMPLs

In the expansion of the moduli integrand of genus-1 open string amplitudes, there will be elliptic MZVs (eMZVs) and their generating function: elliptic MPLs (eMPLs), which will also appear in the expansion of the open string integrals with 1 extra marked point [Kaderli-Rodriguez '22]. The meromorphic 1-variable eMPL is defined by

$$\begin{aligned} &\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix} ; z, \tau \right) = \int_0^z \mathrm{d}t \, g^{(n_1)}(t-a_1, \tau) \tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_r \\ a_2 & \dots & a_r \end{smallmatrix} ; t, \tau \right) , \\ &\tilde{\Gamma}\left(\begin{smallmatrix} \emptyset \\ \emptyset \end{smallmatrix} ; z, \tau \right) = 1 \end{aligned} \tag{13}$$

and the A-eMZVs are

$$\omega(n_1, \dots, n_r; \tau) = \tilde{\Gamma}\left(\begin{smallmatrix} n_r & \dots & n_1 \\ 0 & \dots & 0 \end{smallmatrix} ; 1, \tau \right) \tag{14}$$

Genus-1 close string amplitudes: MGFs

For genus-1 close string amplitudes, the low-energy expansion of the moduli integrand will give us modular graph forms (MGFs), which comes from the integrals

$$\int \frac{\mathrm{d}^2 z}{\mathrm{Im}\tau} \exp(\sum s_{ij} \mathcal{G}(z_{ij}|\tau)) \tag{15}$$

with the Green function on the torus:

$$\begin{aligned} \mathcal{G}(z|\tau) &= -\log\left|\frac{\theta_1(z,\tau)}{\eta(\tau)}\right|^2 - \frac{\pi}{2\mathrm{Im}\,\tau}(z-\bar{z})^2 \\ \eta(\tau) &= e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty}(1-e^{2n\pi i\tau}) \end{aligned} \tag{16}$$

E.g.

$$\int \frac{\mathrm{d}^2 z}{\mathrm{Im}\tau} \mathcal{G}(z|\tau)^2 = \left(\frac{\mathrm{Im}\tau}{\pi}\right)^2 \sum_{p\in\Lambda'=\mathbb{Z}+\tau\mathbb{Z}/\{0\}} \frac{1}{|p|^4} \tag{17}$$

Genus-1 close string amplitudes: MGFs

The generating series of MGFs can be written as a product of iterated Eisenstein integral [Dorigoni-Doroudiani-Drewitt-Hidding-Kleinschmidt-Schlotterer-Schneps-Verbeek '24]:

$$\mathbb{I}^{\mathrm{eqv}}_{\epsilon^{\mathrm{TS}}}(\tau) = (\mathbb{M}^{\mathrm{sv}}_{z})^{-1}\,\overline{\mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau)^{T}}\,\mathbb{M}^{\mathrm{sv}}_{\sigma}\,\mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau) \tag{18}$$

where

$$\mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau) = \mathrm{Pexp}\left(\int_{\tau}^{i\infty} \mathbb{D}_{\epsilon^{\mathrm{TS}}}(\tau_1)\right) \tag{19}$$

and

$$\mathbb{D}_{\epsilon^{\mathrm{TS}}}(\tau) = 2\pi i d\tau \sum_{k=4}^{\infty} \frac{(k-1)}{(2\pi i)^k} \sum_{j=0}^{k-2} \frac{(-1)^j}{j!} (2\pi i \tau)^j \mathrm{G}_k(\tau) \epsilon_k^{(j)\mathrm{TS}} \tag{20}$$

Eisenstein series:

$$G_k(\tau) = \sum_{p \in \Lambda'} \frac{1}{p^k} \tag{21}$$

Genus-1 close string with 1 extra marked point: eMGFs

Similarly, the genus-1 close string integrals (τ integrand) with an extra marked point will produce the elliptic MGFs (eMGFs). E.g., Zagier's single-valued elliptic polylogarithms [Zagier '90]

$$\mathcal{D}^{+}\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\tau) = \frac{(\mathrm{Im}\tau)^a}{\pi^b} \sum_{(m,n)\in\Lambda'_\tau} \frac{e^{2\pi i(nu-mv)}}{(m\tau+n)^a(m\bar{\tau}+n)^b} \tag{22}$$

which are special cases of the dihedral eMGFs (with $z_i = z$ or 0 in our case) [D'Hoker-Kleinschmidt-Schlotterer '20]

$$\begin{aligned} \mathcal{C}^{+}\left[\begin{smallmatrix} a_1 & \dots & a_r \\ b_1 & \dots & b_r \\ z_1 & \dots & z_r \end{smallmatrix}\right](\tau) &= \frac{(\mathrm{Im}\tau)^{a_1+\dots+a_r}}{\pi^{b_1+\dots+b_r}} \times \\ &\sum_{(m_1,n_1),\dots,(m_r,n_r)\in\Lambda'_\tau} \delta\Big(\sum_{j=1}^{r}(m_j,n_j)\Big) \prod_{i=1}^{r} \frac{e^{2\pi i(n_i u_i - m_i v_i)}}{(m_i\tau+n_i)^{a_i}(m_i\bar{\tau}+n_i)^{b_i}} \end{aligned} \tag{23}$$

Main results

We will answer the following questions:

1. What is the generating series of eMGFs?
2. What is the single-value map of the generating series of eMPLs?

A new basis of eMGFs: equivariant series in modular frame

Define the integration kernel:

$$\omega_+\left[\begin{smallmatrix} j \\ k \\ z \end{smallmatrix};\tau,\tau_1\right] = -\frac{\mathrm{d}\tau_1}{2\pi i}(-1)^k\left(\frac{\tau-\tau_1}{4\pi\mathrm{Im}\,\tau}\right)^{k-2-j}(\bar\tau-\tau_1)^j f^{(k)}(u\tau_1+v,\tau_1)$$

$$\omega_-\left[\begin{smallmatrix} j \\ k \\ z \end{smallmatrix};\tau,\tau_1\right] = \frac{\mathrm{d}\bar\tau_1}{2\pi i}\left(\frac{\tau-\bar\tau_1}{4\pi\mathrm{Im}\,\tau}\right)^{k-2-j}(\bar\tau-\bar\tau_1)^j \overline{f^{(k)}(u\tau_1+v,\tau_1)} \tag{24}$$

and

$$\beta_+\left[\begin{smallmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \\ z & z & \cdots & z \end{smallmatrix};\tau\right] = \tag{25}$$

$$\int_\tau^{i\infty}\omega_+\left[\begin{smallmatrix} j_\ell \\ k_\ell \\ z \end{smallmatrix};\tau,\tau_\ell\right]\cdots\int_{\tau_3}^{i\infty}\omega_+\left[\begin{smallmatrix} j_2 \\ k_2 \\ z \end{smallmatrix};\tau,\tau_2\right]\int_{\tau_2}^{i\infty}\omega_+\left[\begin{smallmatrix} j_1 \\ k_1 \\ z \end{smallmatrix};\tau,\tau_1\right] \tag{26}$$

$$\beta_-\left[\begin{smallmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \\ z & z & \cdots & z \end{smallmatrix};\tau\right] = \tag{27}$$

$$\int_{\bar\tau}^{-i\infty}\omega_-\left[\begin{smallmatrix} j_\ell \\ k_\ell \\ z \end{smallmatrix};\tau,\tau_\ell\right]\cdots\int_{\bar\tau_3}^{-i\infty}\omega_-\left[\begin{smallmatrix} j_2 \\ k_2 \\ z \end{smallmatrix};\tau,\tau_2\right]\int_{\bar\tau_2}^{-i\infty}\omega_-\left[\begin{smallmatrix} j_1 \\ k_1 \\ z \end{smallmatrix};\tau,\tau_1\right]$$

A new basis of eMGFs: equivariant series in modular frame

The modular equivariance object β^{eqv} can be regarded as the modular completion of $\beta_{\pm}$. It has the following property:

$$\beta^{\mathrm{eqv}}\left[\begin{smallmatrix} j_1 & \cdots & j_r \\ k_1 & \cdots & k_r \\ \frac{z}{c\tau+d} & \cdots & \frac{z}{c\tau+d} \end{smallmatrix};\frac{a\tau+b}{c\tau+d}\right]=\left(\prod_{i=1}^{r}(c\bar{\tau}+d)^{k_i-2j_i-2}\right)\beta^{\mathrm{eqv}}\left[\begin{smallmatrix} j_1 & \cdots & j_r \\ k_1 & \dots & k_r \\ z & \dots & z \end{smallmatrix};\tau\right] \tag{28}$$

A new basis of eMGFs: equivariant series in modular frame

Similar to the MGF case [Dorigoni-Doroudiani-Drewitt-Hidding-Kleinschmidt-Matthes-Schlotterer-Verbeek '22], eMGFs can be written as combinations of β^{eqv}'s. Examples:

$$\mathcal{C}^+\left[\begin{smallmatrix}1&1&1\\1&1&1\\z&0&0\end{smallmatrix}\right](\tau) = 8\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}1&0\\3&3\\z&z\end{smallmatrix};\tau\right] - 10\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}2\\6\end{smallmatrix};\tau\right] - 20\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}2\\6\\z\end{smallmatrix};\tau\right] + \zeta_3$$

$$\nabla_z\mathcal{C}^+\left[\begin{smallmatrix}2&1&1\\2&1&1\\z&0&0\end{smallmatrix}\right](\tau) = 12\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}2&0\\4&3\\z&z\end{smallmatrix};\tau\right] - 60\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}3\\7\\z\end{smallmatrix};\tau\right] \tag{29}$$

$$- 12\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}2\\4\end{smallmatrix};\tau\right]\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}0\\3\\z\end{smallmatrix};\tau\right] \tag{30}$$

A new basis of eMGFs: equivariant series in modular frame

with (just give 1 example)

$$\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}1 & 0\\ 3 & 3\\ z & z\end{smallmatrix};\tau\right]=\beta_{+}\left[\begin{smallmatrix}1 & 0\\ 3 & 3\\ z & z\end{smallmatrix};\tau\right]+\beta_{-}\left[\begin{smallmatrix}0 & 1\\ 3 & 3\\ z & z\end{smallmatrix};\tau\right]+\beta_{-}\left[\begin{smallmatrix}1\\ 3\\ z\end{smallmatrix};\tau\right]\beta_{+}\left[\begin{smallmatrix}0\\ 3\\ z\end{smallmatrix};\tau\right]+\frac{1}{4}\zeta_3 B_2(u)-\frac{1}{8}\zeta_3 \tag{31}$$

The empty slot means the corresponding $f^{(k)} \to -G_k$.

The generating series of eMGFs $\leftrightarrow$ The generating series of β^{eqv}'s.

But how to figure out what combination of $\beta_{\pm}$'s and ζ_w will appear in the corresponding β^{eqv}? We need some algebras to help us select!

Genus-1 Lie algebra structure

Recall the dz part of the CEE connection [Calaque-Enriquez-Etingof '09]:

$$\tilde{\mathbb{K}}_{x,y}(z,\tau) = \mathrm{d}z\,\mathrm{ad}_x F(z, \mathrm{ad}_{\frac{x}{2\pi i}}, \tau) y = \mathrm{d}z \sum_{n=0}^{\infty} (2\pi i)^{1-n} g^{(n)}(z,\tau)\,\mathrm{ad}_x^{\,n}(y) \tag{32}$$

There are two generators x and y. Define the following generators, which will appear in the differential equations of the A-elliptic KZB associator:

$$[\epsilon_0, x] = y\,, \qquad [\epsilon_0, y] = 0 \tag{33}$$

as well as ($\ell \in \mathbb{N}$)

$$[\epsilon_{2\ell}^{\mathrm{TS}}, x] = \mathrm{ad}_x^{\,2\ell}(y) \tag{34}$$

$$[\epsilon_{2\ell}^{\mathrm{TS}}, y] = [y, \mathrm{ad}_x^{\,2\ell-1}(y)] + \sum_{j=1}^{\ell-1} (-1)^j [\mathrm{ad}_x^{\,j}(y), \mathrm{ad}_x^{\,2\ell-1-j}(y)]$$

These operators accompany the Eisenstein series $G_k(\tau)$, which are known as Tsunogai's derivations [Tsunogai '95].

Genus-1 Lie algebra structure

Moreover, certain combinations of Tsunogai derivations starting from degree 14 obey *Pollack relations* [Pollack '09] related to holomorphic cusp forms, for instance

$$0 = [\epsilon_4^{\mathsf{TS}}, \epsilon_{10}^{\mathsf{TS}}] - 3[\epsilon_6^{\mathsf{TS}}, \epsilon_8^{\mathsf{TS}}] \tag{35}$$

$$\begin{aligned} 0 &= 80[\epsilon_4^{(1)\mathsf{TS}}, \epsilon_{12}^{\mathsf{TS}}] + 16[\epsilon_{12}^{(1)\mathsf{TS}}, \epsilon_4^{\mathsf{TS}}] - 250[\epsilon_6^{(1)\mathsf{TS}}, \epsilon_{10}^{\mathsf{TS}}] - 125[\epsilon_{10}^{(1)\mathsf{TS}}, \epsilon_6^{\mathsf{TS}}] \\ &\quad + 280[\epsilon_8^{(1)\mathsf{TS}}, \epsilon_8^{\mathsf{TS}}] - 462[\epsilon_4^{\mathsf{TS}}, [\epsilon_4^{\mathsf{TS}}, \epsilon_8^{\mathsf{TS}}]] - 1725[\epsilon_6^{\mathsf{TS}}, [\epsilon_6^{\mathsf{TS}}, \epsilon_4^{\mathsf{TS}}]] \end{aligned}$$

where

$$\epsilon_k^{(j)\mathsf{TS}} = \mathrm{ad}_{\epsilon_0}^{j}(\epsilon_k^{\mathsf{TS}}) \tag{36}$$

Genus-1 Lie algebra structure

The non-vanishing $\epsilon_k^{(j)\mathsf{TS}}$ at fixed k and $0 \leq j \leq k-2$ form $(k-1)$-dimensional multiplets with respect to the $\mathfrak{sl}_2$ algebra generated by ϵ_0 and the additional derivation $\epsilon_0^\vee$ of degree zero

$$[\epsilon_0^\vee, x] = 0\,, \qquad [\epsilon_0^\vee, y] = x \tag{37}$$

More specifically, ϵ_0 and $\epsilon_0^\vee$ act as ladder operators of the $\mathfrak{sl}_2$ with

$$[\epsilon_0, \epsilon_k^{(j)\mathsf{TS}}] = \epsilon_k^{(j+1)\mathsf{TS}}\,, \qquad [\epsilon_0^\vee, \epsilon_k^{(j)\mathsf{TS}}] = j(k-j-1)\epsilon_k^{(j-1)\mathsf{TS}} \tag{38}$$

and their commutator yields the Cartan generator h of the $\mathfrak{sl}_2$

$$\mathrm{h} = [\epsilon_0, \epsilon_0^\vee]\,, \quad [\mathrm{h}, x] = -x\,, \quad [\mathrm{h}, y] = y\,, \quad \big[\mathrm{h}, \epsilon_k^{(j)\mathsf{TS}}\big] = (2j-k+2)\epsilon_k^{(j)\mathsf{TS}} \tag{39}$$

In view of these Cartan eigenvalues, we refer to $\epsilon_k^{(k-2)\mathsf{TS}}$ and $\epsilon_k^{(0)\mathsf{TS}} = \epsilon_k^{\mathsf{TS}}$ as highest-weight vectors and lowest-weight vectors, respectively,

$$[\epsilon_0, \epsilon_k^{(k-2)\mathsf{TS}}] = 0\,, \qquad [\epsilon_0^\vee, \epsilon_k^{\mathsf{TS}}] = 0\,, \qquad k \geq 2 \tag{40}$$

Genus-1 Lie algebra structure

Now let us consider the whole CEE connection [Calaque-Enriquez-Etingof '09]:

$$\mathbb{K}_{x,y,\epsilon}(z,\tau) = 2\pi i \mathrm{d}z \sum_{k=0}^{\infty} \frac{g^{(k)}(z,\tau)}{(2\pi i)^k} \,\mathrm{ad}_x^k(y) \tag{41}$$

$$+ 2\pi i \mathrm{d}\tau \left(\epsilon_0 + \sum_{k=2}^{\infty} \frac{(k-1)}{(2\pi i)^k} \left[\mathrm{G}_k(\tau)\epsilon_k - g^{(k)}(z,\tau) b_k \right] \right)$$

with

$$b_k = -\mathrm{ad}_x^{k-1}(y) \,, \qquad k \geq 2 \tag{42}$$

From the differential equation and its flatness, it is not hard to find

$$\epsilon_k^{\mathrm{TS}} = \epsilon_k + b_k \,, \qquad k \geq 2 \text{ even} \tag{43}$$

Genus-1 Lie algebra structure

We still have

$$b_k^{(j)} = \mathrm{ad}_{\epsilon_0}^j(b_k)\,, \qquad \epsilon_k^{(j)} = \mathrm{ad}_{\epsilon_0}^j(\epsilon_k) \tag{44}$$

and

$$\begin{aligned} [\epsilon_0, b_k^{(j)}] &= b_k^{(j+1)} & [\epsilon_0, \epsilon_k^{(j)}] &= \epsilon_k^{(j+1)} \\ [\epsilon_0^\vee, b_k^{(j)}] &= j(k-j-1)b_k^{(j-1)} & [\epsilon_0^\vee, \epsilon_k^{(j)}] &= j(k-j-1)\epsilon_k^{(j-1)} \\ [\mathrm{h}, b_k^{(j)}] &= (2j-k+2)b_k^{(j)} & [\mathrm{h}, \epsilon_k^{(j)}] &= (2j-k+2)\epsilon_k^{(j)} \end{aligned} \tag{45}$$

The element of the generating series

Formally, we can write down the generating series as follows:

$$\mathbb{H}^{+}_{\epsilon,b} =$$

$$\mathrm{Pexp}\Bigg(\int_{0}^{i\infty} \mathrm{d}\tau_1 \sum_{k=2}^{\infty} (k-1) \sum_{j=0}^{k-2} \frac{(-1)^j}{j!} \Big(\omega_+ \left[\begin{smallmatrix} j \\ k \\ z \end{smallmatrix} ; \tau, \tau_1\right] \, b_k^{(j)} + \omega_+ \left[\begin{smallmatrix} j \\ k \end{smallmatrix} ; \tau, \tau_1\right] \epsilon_k^{(j)}$$

$$= 1 + \sum_{k_1=2}^{\infty} (k_1-1) \sum_{j_1=0}^{k_1-2} \frac{(-1)^{j_1}}{j_1!} \left\{ \beta_+ \left[\begin{smallmatrix} j_1 \\ k_1 \end{smallmatrix} ; \tau\right] \epsilon_{k_1}^{(j_1)} + \beta_+ \left[\begin{smallmatrix} j_1 \\ k_1 \\ z \end{smallmatrix} ; \tau\right] b_{k_1}^{(j_1)} \right\}$$

$$+ \sum_{k_1,k_2=2}^{\infty} (k_1-1)(k_2-1) \sum_{j_1=0}^{k_1-2} \sum_{j_2=0}^{k_2-2} \frac{(-1)^{j_1+j_2}}{j_1! j_2!} \Bigg\{ \beta_+ \left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix} ; \tau\right] \epsilon_{k_1}^{(j_1)} \epsilon_{k_2}^{(j_2)}$$

$$+ \beta_+ \left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \\ z & z \end{smallmatrix} ; \tau\right] b_{k_1}^{(j_1)} b_{k_2}^{(j_2)} + \beta_+ \left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \\ & z \end{smallmatrix} ; \tau\right] \epsilon_{k_1}^{(j_1)} b_{k_2}^{(j_2)} + \beta_+ \left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \\ z & \end{smallmatrix} ; \tau\right] b_{k_1}^{(j_1)} \epsilon_{k_2}^{(j_2)} \Bigg\}$$

$$+ \dots$$

The element of the generating series

And similarly for $\mathbb{H}^-$. Also

$$\mathbb{H}^{\mathrm{eqv}}_{\epsilon,b}(u,v,\tau) = \tag{46}$$

$$\begin{aligned}
&1+\sum_{k_1=2}^{\infty}(k_1-1)\sum_{j_1=0}^{k_1-2}\frac{(-1)^{j_1}}{j_1!}\left\{\beta^{\mathrm{eqv}}\left[\begin{matrix} j_1\\ k_1\end{matrix};\tau\right]\epsilon^{(j_1)}_{k_1}+\beta^{\mathrm{eqv}}\left[\begin{matrix} j_1\\ k_1\\ z\end{matrix};\tau\right]b^{(j_1)}_{k_1}\right\}\\
&\quad+\sum_{k_1,k_2=2}^{\infty}(k_1-1)(k_2-1)\sum_{j_1=0}^{k_1-2}\sum_{j_2=0}^{k_2-2}\frac{(-1)^{j_1+j_2}}{j_1!j_2!}\left\{\beta^{\mathrm{eqv}}\left[\begin{matrix} j_1 & j_2\\ k_1 & k_2\end{matrix};\tau\right]\epsilon^{(j_1)}_{k_1}\epsilon^{(j_2)}_{k_2}\right.\\
&\qquad\left.+\beta^{\mathrm{eqv}}\left[\begin{matrix} j_1\\ k_1\end{matrix}\,\middle|\,\begin{matrix} j_2\\ k_2\\ z\end{matrix}\;;\tau\right]\epsilon^{(j_1)}_{k_1}b^{(j_2)}_{k_2}+\beta^{\mathrm{eqv}}\left[\begin{matrix} j_1 & j_2\\ k_1 & k_2\\ z & z\end{matrix};\tau\right]b^{(j_1)}_{k_1}b^{(j_2)}_{k_2}\right\}+\ldots
\end{aligned}\tag{47}$$

Now the question is: how to express $\mathbb{H}^{\mathrm{eqv}}$ in terms of $\mathbb{H}^{\pm}_{\epsilon,b}$?

From modular frame to holomorphic frame

Things will be easier when we turn to the holomorphic frame. If we do the following transformation

$$U_{\mathrm{mod}}(\tau)^{-1} X U_{\mathrm{mod}}(\tau), \quad U_{\mathrm{mod}}(\tau) = \exp\left(\frac{-\epsilon_0^{\vee}}{4\pi \mathrm{Im}\,\tau}\right) \exp(2\pi i \bar{\tau} \epsilon_0) \tag{48}$$

on any object X in the modular frame, we will get the corresponding holomorphic frame object. E.g., the integration kernel in the holomorphic frame is

$$U_{\mathrm{mod}}(\tau)^{-1} \sum_{k=2}^{\infty} (k-1) \sum_{j=0}^{k-2} \frac{(-1)^j}{j!} \left\{ \omega_+\left[\begin{smallmatrix} j \\ k \\ z \end{smallmatrix}; \tau, \tau_1\right] b_k^{(j)} + \omega_+\left[\begin{smallmatrix} j \\ k \end{smallmatrix}; \tau, \tau_1\right] \epsilon_k^{(j)} \right\} \\ \times U_{\mathrm{mod}}(\tau) = \mathbb{D}_{\epsilon,b}(\tau_1) \tag{49}$$

$$\mathbb{D}_{\epsilon,b}(u,v,\tau) = \\ 2\pi i \mathrm{d}\tau \sum_{k=2}^{\infty} \frac{(k-1)}{(2\pi i)^k} \sum_{j=0}^{k-2} \frac{(-1)^j}{j!} (2\pi i \tau)^j \left[\mathrm{G}_k(\tau) \epsilon_k^{(j)} - f^{(k)}(u\tau + v, \tau) b_k^{(j)} \right] \tag{50}$$

Modular equivariance for generating series in holo. frame

In our $\epsilon^{\mathrm{TS}}, b$ algebraic system, the equivariant generating series can be defined as

$$\mathbb{I}^{\mathrm{eqv}}(\gamma \cdot \tau) = U_\gamma^{-1} \mathbb{I}^{\mathrm{eqv}}(\tau) U_\gamma \;\; \forall \;\; \gamma \in \mathrm{SL}_2(\mathbb{Z}) \tag{51}$$

with

$$U_T = e^{2\pi i \epsilon_0}\,, \qquad U_S = e^{\epsilon_0^\vee} e^{-\epsilon_0} e^{\epsilon_0^\vee} (2\pi i)^{-[\epsilon_0, \epsilon_0^\vee]} = e^{\epsilon_0^\vee/(2\pi i)} e^{-2\pi i \epsilon_0} e^{\epsilon_0^\vee/(2\pi i)} \tag{52}$$

This definition is actually equivalent to the definition for β^{eqv}.

Genus-1 zeta generator

To include the contribution of ζ_w, we need to define the genus-1 zeta generators. After the degeneration $\tau \to i\infty$, the CEE connection becomes

$$\begin{aligned}\tilde{\mathbb{K}}_{x,y}(z,i\infty) &= \mathrm{d}z\left\{2\pi i y + \pi\cot(\pi z)[x,y] - 2\sum_{\ell=1}^{\infty}(2\pi i)^{1-2\ell}\zeta_{2\ell}\,\mathrm{ad}_x^{\,2\ell-1}(y)\right\} \\ &= -\mathrm{d}\sigma\left(\frac{t_{12}}{\sigma-1}+\frac{t_{01}}{\sigma}\right)\end{aligned} \tag{53}$$

where

$$t_{12} = -[x,y]\,, \qquad t_{01} = -\sum_{k=0}^{\infty}\frac{B_k}{k!}\mathrm{ad}_x^k(y) = -\frac{\mathrm{ad}_x}{e^{\mathrm{ad}_x}-1}(y) \tag{54}$$

As the analog of the genus-0 case, we define the genus-1 zeta generator:

$$[t_{12},\sigma_w] = 0\,, \qquad [t_{01},\sigma_w] = \left[P_w(t_{12},t_{01}),t_{01}\right] \tag{55}$$

The generating series of β^{eqv}'s in holo. frame

Define

$$\begin{aligned}\mathbb{I}_{\epsilon,b} &= \mathrm{Pexp}\bigg(\int_0^{i\infty} \mathrm{d}\tau_1 \mathbb{D}_{\epsilon,b}(u,v,\tau_1)\bigg)\\ \mathbb{M}_\sigma^{\mathrm{sv}} &= \sum_{r=0}^{\infty} \sum_{\substack{i_1,\ldots,i_r\\ \in 2\mathbb{N}+1}} \rho^{-1}\big(\mathrm{sv}(f_{i_1}\ldots f_{i_r})\big)\,\sigma_{i_1}\ldots\sigma_{i_r}\end{aligned} \tag{56}$$

The equivariant generating series in the holomorphic frame is

$$\mathbb{I}^{\mathrm{eqv}}_{\epsilon,b}(u,v,\tau) = (\mathbb{M}_z^{\mathrm{sv}})^{-1}\,\overline{\mathbb{I}_{\epsilon,b}(u,v,\tau)^T}\,\mathbb{M}^{\mathrm{sv}}_{\Sigma(u)}\,\mathbb{I}_{\epsilon,b}(u,v,\tau) \tag{57}$$

with

$$\Sigma_w(u) = e^{ux}\big(P_w(t_{12},t_{01}) + \sigma_w\big)e^{-ux}\,, \qquad w \geq 3 \text{ odd} \tag{58}$$

The expansion of $\Sigma(u)$

The expansions of the simplest augmented zeta generators $\Sigma_w(u)$ can be got by the following equation

$$\Sigma_w(u) = \sum_{k=0}^{\infty} \frac{B_k(u)}{k!} \, \mathrm{ad}_x^k(\Sigma_w^{\mathrm{sc}}) \tag{59}$$

with Σ_w^{sc} denoting a u-independent "source term" which is different from $\Sigma_w(u=0)$ but by itself an infinite series in $\epsilon_k^{(j)}$ and $b_k^{(j)}$. E.g.

$$\begin{aligned} \Sigma_3^{\mathrm{sc}} = &-\frac{1}{2}\epsilon_4^{(2)} + z_3 - \frac{1}{4}[b_3, b_3^{(1)}] + \frac{1}{480}[\epsilon_4, \epsilon_4^{(1)}] \\ &+ \frac{1}{1440}[b_2, [b_2, b_4]] - \frac{1}{720}[b_3, [b_2, b_3]] + \dots \end{aligned} \tag{60}$$

The generating series of eMPLs

Now we have found the generating series of eMGFs, but what is the relation between it and the eMPLs?

In this work, we will focus on the following generating series of eMPL:

$$\Gamma_{x,y}(z,\tau) \tag{61}$$

$$= e^{-2\pi i\tau\epsilon_0}\exp\left(\int_{\tau}^{i\infty}\frac{\mathrm{d}\tau_1}{2\pi i}\,\mathrm{G}_2(\tau_1)b_2\right)\mathrm{Pexp}\left(\int_{z}^{0}\tilde{\mathbb{K}}_{x,y}(z_1,\tau)\right)e^{-ux}e^{2\pi i\tau\epsilon_0}$$

$$= \exp\left(\mathcal{E}[\begin{smallmatrix}0\\2\end{smallmatrix};\tau]\,b_2\right)\mathrm{Pexp}\left(\int_{z}^{0}\tilde{\mathbb{K}}_{x-2\pi i\tau y,y}(z_1,\tau)\right)e^{-u(x-2\pi i\tau y)}$$

$$= \exp\left(\mathcal{E}[\begin{smallmatrix}0\\2\end{smallmatrix};\tau]\,b_2\right)\mathrm{Pexp}\left(\int_{z}^{0}\mathbb{J}^{\mathrm{BL}}_{x-2\pi i\tau y,y}(z_1,\tau)\right) \tag{62}$$

The generating series of eMPLs

Let us show the leading terms of the expansion:

$$\Gamma_{x,y}(z,\tau) = 1 - ux - 2\pi i v y + \tfrac{1}{2}u^2x^2 + \tfrac{1}{2}v^2(2\pi i y)^2 + 2\pi i u v y x \\ + b_2\left(i\pi u^2\tau + \tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix};z,\tau) + \mathcal{E}[\begin{smallmatrix}0\\2\end{smallmatrix};\tau] \right) + \dots \qquad (63)$$

where

$$\mathcal{E}[\begin{smallmatrix}0\\2\end{smallmatrix};\tau] = \int_{\tau}^{i\infty} \frac{\mathrm{d}\tau_1}{2\pi i}\, \mathrm{G}_2(\tau_1) \qquad (64)$$

An alternative expression of the generating series

Theorem

The series $\mathbb{I}^{\mathrm{eqv}}_{\epsilon,b}(u,v,\tau)$ *defined by*

$$\mathbb{I}^{\mathrm{eqv}}_{\epsilon,b}(u,v,\tau) = (\mathbb{M}^{\mathrm{sv}}_{z})^{-1}\,\overline{\mathbb{I}_{\epsilon,b}(u,v,\tau)^{T}}\,\mathbb{M}^{\mathrm{sv}}_{\Sigma(u)}\,\mathbb{I}_{\epsilon,b}(u,v,\tau) \qquad (65)$$

can alternatively be rewritten in terms of eMPLs via

$$\mathbb{I}^{\mathrm{eqv}}_{\epsilon,b}(u,v,\tau) = (\mathbb{M}^{\mathrm{sv}}_{z})^{-1}\,\overline{\Gamma_{x,y}(z,\tau)^{T}}\,\mathbb{M}^{\mathrm{sv}}_{z}\,\mathbb{I}^{\mathrm{eqv}}_{\epsilon^{\mathrm{TS}}}(\tau)\,\Gamma_{x,y}(z,\tau) \qquad (66)$$

The total differential of $\mathbb{I}_{\epsilon,b}$

First, introduce the flat connection:

$$\mathbb{J}_{x,y,\epsilon}(u,v,\tau) = x\,\mathrm{d}u + 2\pi i(\tau\,\mathrm{d}u + \mathrm{d}v)\sum_{k=0}^{\infty}\frac{f^{(k)}(u\tau{+}v,\tau)}{(2\pi i)^k}\,\mathrm{ad}_x^k(y)$$

$$+\,2\pi i\mathrm{d}\tau\left(\epsilon_0 + \sum_{k=2}^{\infty}\frac{(k{-}1)}{(2\pi i)^k}\left[\mathrm{G}_k(\tau)\epsilon_k - f^{(k)}(u\tau{+}v,\tau)b_k\right]\right)$$

The total differential of $\mathbb{I}_{\epsilon,b}$

Rewriting $\mathbb{I}_{\epsilon,b}$ as

$$\mathbb{I}_{\epsilon,b}(u,v,\tau) = \mathrm{Pexp}\left(\int_{\tau}^{i\infty} \mathrm{d}\tau_1 \, \mathbb{J}^{(\tau)}_{x,y,\epsilon}(u,v,\tau_1) \right) e^{2\pi i \tau \epsilon_0} \tag{67}$$

and let

$$\mathbb{I}_{\epsilon,b}(u,v,\tau) = e^{ux} \, e^{-2\pi i v t_{01}} \, \mathbb{I}^{\mathrm{red}}_{\epsilon,b}(u,v,\tau) \, e^{2\pi i \tau \epsilon_0} \tag{68}$$

We will get the following differential equation

$$\begin{aligned} \mathrm{d}\mathbb{I}^{\mathrm{red}}_{\epsilon,b}(u,v,\tau) &= -\mathbb{I}^{\mathrm{red}}_{\epsilon,b}(u,v,\tau) \, \mathbb{J}_{x,y,\epsilon}(u,v,\tau) \\ &= -\mathbb{I}^{\mathrm{red}}_{\epsilon,b}(u,v,\tau) \, (\mathbb{J}^{\mathrm{BL}}_{x,y}(z,\tau) + \mathbb{T}_{x,y,\epsilon}(z,\tau)) \end{aligned} \tag{69}$$

with the Brown-Levin connection [Brown-Levin '11]

$$\begin{aligned} \mathbb{J}^{\mathrm{BL}}_{x,y}(z,\tau) = -\frac{x \, \mathrm{d}\bar{z}}{2i \mathrm{Im}\, \tau} + \mathrm{d}z \Bigg\{ & \frac{x}{2i\mathrm{Im}\,\tau} + 2\pi i y \\ & + \sum_{n=1}^{\infty} (2\pi i)^{1-n} f^{(n)}(z,\tau) \mathrm{ad}_x^{\,n}(y) \Bigg\} \end{aligned} \tag{70}$$

Solving differential equations of $\mathbb{I}_{\epsilon,b}$

Since $\mathbb{J}_{x,y,\epsilon}(u,v,\tau)$ is flat, we have

$$\mathbb{I}^{\mathrm{red}}_{\epsilon,b}(u,v,\tau) = \mathbb{I}^{\mathrm{in}}_{x,y,\epsilon}\,\mathrm{Pexp}\left(\int_{\tau}^{i\infty} \lim_{z\to 0} \mathbb{T}_{x,y,\epsilon}(z,\tau_1)\right)\mathrm{Pexp}\left(\int_{z}^{0} \mathbb{J}^{\mathrm{BL}}_{x,y}(z_1,\tau)\right) \tag{71}$$

From the calculation, we know that

$$\mathrm{Pexp}\left(\int_{\tau}^{i\infty} \lim_{z\to 0} \mathbb{T}_{x,y,\epsilon}(z,\tau_1)\right) \tag{72}$$

$$= \mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau)e^{-2\pi i\tau\epsilon_0}\,\mathsf{exp}\left(\int_{\tau}^{i\infty} \frac{\mathrm{d}\tau_1}{2\pi i}\,\mathrm{G}_2(\tau_1)b_2\right) \tag{73}$$

Solving differential equations of $\mathbb{I}_{\epsilon,b}$

Hence the final result is

$$
\begin{aligned}
&\mathbb{I}_{\epsilon,b}(u,v,\tau) \qquad (74)\\
&= e^{ux}\, e^{-2\pi i v t_{01}}\, \mathbb{I}^{\mathrm{in}}_{x,y,\epsilon}\, \mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau)\, \exp\left(\mathcal{E}[\,{}^{0}_{2}\,;\tau]\, b_2\right) \mathrm{Pexp}\Bigg(\int_z^0 \mathbb{J}^{\mathrm{BL}}_{x-2\pi i\tau y,y}(z_1,\tau) \Bigg)\\
&= e^{ux}\, e^{-2\pi i v t_{01}}\, \mathbb{I}^{\mathrm{in}}_{x,y,\epsilon}\, \mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau)\, \Gamma_{x,y}(z,\tau) \qquad (75)
\end{aligned}
$$

Fixing the initial value

We can integrate along the alternative path:

$$\begin{aligned}
&\mathbb{I}^{\mathrm{red}}_{\epsilon,b}(u,v,\tau) &(76)\\
&= \mathbb{I}^{\mathrm{in}}_{x,y,\epsilon}\,\mathrm{Pexp}\left(\int_{(u,v,i\infty)}^{(0,0,i\infty)} \mathbb{J}_{x,y}(u_1,v_1,\tau_1)\right)\mathrm{Pexp}\left(\int_{\tau}^{i\infty} \mathrm{d}\tau_1\,\mathbb{J}^{(\tau)}_{x,y,\epsilon}(u,v,\tau_1)\right)\\
&= \mathbb{I}^{\mathrm{in}}_{x,y,\epsilon}\lim_{\tau\to i\infty}\mathrm{Pexp}\left(\int_{u\tau+v}^{0}\tilde{\mathbb{K}}_{x,y}(z_1,\tau)\right)e^{-ux}\,\mathbb{I}_{\epsilon,b}(u,v,\tau)\,e^{-2\pi i\tau\epsilon_0} &(77)
\end{aligned}$$

Compared with the result before, the consistency condition is

$$\mathbb{I}^{\mathrm{in}}_{x,y,\epsilon}\lim_{\tau\to i\infty}\mathrm{Pexp}\left(\int_{u\tau+v}^{0}\tilde{\mathbb{K}}_{x,y}(z_1,\tau)\right)e^{-2\pi i v t_{01}} = 1 \qquad (78)$$

Evaluating the path exponential, the final result is

$$\mathbb{I}^{\mathrm{in}}_{x,y,\epsilon} = \Phi^{-1}(t_{01},t_{12}) = \Phi(t_{12},t_{01}) \qquad (79)$$

The new expression for eMGF generating series

Then the generating series $\mathbb{I}^{\mathrm{eqv}}_{\epsilon,b}$ can be written as

$$\mathbb{I}^{\mathrm{eqv}}_{\epsilon,b}(u,v,\tau) = (\mathbb{M}^{\mathrm{sv}}_z)^{-1}\,\overline{\Gamma_{x,y}(z,\tau)^T}\,\overline{\mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau)^T}\,\overline{\Phi^{-1}(t_{12},t_{01})^T}e^{-ux}e^{2\pi i v T_{01}(u)} \\ \times \mathbb{M}^{\mathrm{sv}}_{\Sigma(u)}\,e^{-2\pi i v T_{01}(u)}\,e^{ux}\,\Phi^{-1}(t_{12},t_{01})\,\mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau)\,\Gamma_{x,y}(z,\tau) \tag{80}$$

Note that (the proofs are shown in our paper)

$$e^{-ux}\,e^{2\pi i v T_{01}(u)}\,\mathbb{M}^{\mathrm{sv}}_{\Sigma(u)}\,e^{-2\pi i v T_{01}(u)}\,e^{ux} = \mathbb{M}^{\mathrm{sv}}_{\Sigma(0)} \tag{81}$$

and

$$\Phi(-t_{12},-t_{01})\,\mathbb{M}^{\mathrm{sv}}_{\Sigma(0)}\,\Phi^{-1}(t_{12},t_{01}) = \mathbb{M}^{\mathrm{sv}}_{\sigma} \tag{82}$$

Then we have

$$\mathbb{I}^{\mathrm{eqv}}_{\epsilon,b}(u,v,\tau) = (\mathbb{M}^{\mathrm{sv}}_z)^{-1}\overline{\Gamma_{x,y}(z,\tau)^T}\,\overline{\mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau)^T}\,\mathbb{M}^{\mathrm{sv}}_{\sigma}\,\mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau)\,\Gamma_{x,y}(z,\tau) \tag{83}$$

The new expression for eMGF generating series

Using the equation for MGF generating series

$$\overline{\mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau)^{T}}\,\mathbb{M}^{\mathrm{sv}}_{\sigma}\,\mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau) = \mathbb{M}^{\mathrm{sv}}_{z}\,\mathbb{I}^{\mathrm{eqv}}_{\epsilon^{\mathrm{TS}}}(\tau) \tag{84}$$

will lead to the final result

$$\mathbb{I}^{\mathrm{eqv}}_{\epsilon,b}(u,v,\tau) = (\mathbb{M}^{\mathrm{sv}}_{z})^{-1}\overline{\Gamma_{x,y}(z,\tau)^{T}}\,\mathbb{M}^{\mathrm{sv}}_{z}\,\mathbb{I}^{\mathrm{eqv}}_{\epsilon^{\mathrm{TS}}}(\tau)\,\Gamma_{x,y}(z,\tau). \quad \square \tag{85}$$

Equivariance and single-valuedness

The equivariance and single-valuedness of $\mathbb{I}^{\mathrm{eqv}}_{\epsilon,\boldsymbol{b}}$ still need to be proven. We have the following theorem:

Equivariance and single-valuedness

Theorem

The series $\mathbb{I}^{\rm eqv}_{\epsilon,b}(u,v,\tau)$ *in Theorem 1*

(i) *has no monodromy when* z *is moved around the origin,*

(ii) *has no monodromy* $v \rightarrow v{+}1$ *when* z *is transported around the A-cycle of the torus,*

$$\mathbb{I}^{\rm eqv}_{\epsilon,b}(u,v{+}1,\tau) = \mathbb{I}^{\rm eqv}_{\epsilon,b}(u,v,\tau) \tag{86}$$

(iii) *is equivariant in the sense of*

$$\mathbb{I}^{\rm eqv}_{\epsilon,b}\big(\gamma\cdot(u,v,\tau)\big) = U_\gamma^{-1}\,\mathbb{I}^{\rm eqv}_{\epsilon,b}(u,v,\tau)\,U_\gamma \tag{87}$$

(iv) *has no monodromy* $u \rightarrow u{+}1$ *when* z *is transported around the B-cycle of the torus,*

$$\mathbb{I}^{\rm eqv}_{\epsilon,b}(u{+}1,v,\tau) = \mathbb{I}^{\rm eqv}_{\epsilon,b}(u,v,\tau) \tag{88}$$

and thus single-valued in z *on the torus in view of (i) and (ii).*

The sketch of proof

(i) By definition of $\Gamma_{x,y}$, it has the monodromy $\Gamma_{x,y} \to e^{2\pi i[y,x]}\Gamma_{x,y}$, which will be canceled in $\mathbb{I}^{\mathrm{eqv}}_{\epsilon,b}$

(ii) By the definition of the connection $\mathbb{D}_{\epsilon,b}(u,v,\tau)$, we have $\mathbb{I}_{\epsilon,b}(u,v+1) = \mathbb{I}_{\epsilon,b}(u,v)$. Hence $\mathbb{I}^{\mathrm{eqv}}_{\epsilon,b}(u,v,\tau) = (\mathbb{M}^{\mathrm{sv}}_{z})^{-1}\,\overline{\mathbb{I}_{\epsilon,b}(u,v,\tau)^{T}}\,\mathbb{M}^{\mathrm{sv}}_{\Sigma(u)}\,\mathbb{I}_{\epsilon,b}(u,v,\tau)$ keeps this property.

The sketch of proof

(iii) By the modular property of $\Gamma_{x,y}$

$$\Gamma_{x,y}(z,\tau+1) = e^{i\pi b_2/6}\, U_T^{-1}\, \Gamma_{x,y}(z,\tau)\, U_T$$
$$\Gamma_{x,y}\left(\frac{z}{\tau},-\frac{1}{\tau}\right) = e^{-i\pi b_2/2}\, U_S^{-1}\Gamma_{x,y}(z,\tau)\, U_S \tag{89}$$

It is not hard to show $\mathbb{I}^{\mathrm{eqv}}$is modular equivariant.

(iv) Note that $S\cdot(u,v,\tau) = (-v,u,-\frac{1}{\tau})$, then from (ii) and (iii), (iv) is straightforward. $\square$

The single-valued eMPL in the holomorphic frame

New problem: $\mathbb{I}^{\mathrm{eqv}}$ doesn't have a unique expression due to the bracket relations between ϵ and b.

We proposed the following single-valued eMPL

$$\Gamma^{\mathrm{sv}}_{x,y}(z,\tau) = \mathbb{I}^{\mathrm{eqv}}_{\epsilon^{\mathrm{TS}}}(\tau)^{-1}\mathbb{I}^{\mathrm{eqv}}_{\epsilon,b}(u,v,\tau), \tag{90}$$

which is obviously single-valued and equivariant. Moreover, it can be expanded uniquely in $b_k^{(j)}$'s:

$$\Gamma^{\mathrm{sv}}_{x,y}(z,\tau) = 1 + \sum_{k_1=2}^{\infty}(k_1-1)\sum_{j_1=0}^{k_1-2}\frac{(-1)^{j_1}}{j_1!}\Gamma^{\mathrm{sv}}\left[\begin{smallmatrix} j_1 \\ k_1 \end{smallmatrix}; z,\tau\right] b_{k_1}^{(j_1)} \tag{91}$$

$$+ \sum_{k_1,k_2=2}^{\infty}(k_1-1)(k_2-1)\sum_{j_1=0}^{k_1-2}\sum_{j_2=0}^{k_2-2}\frac{(-1)^{j_1+j_2}}{j_1!j_2!}\Gamma^{\mathrm{sv}}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix}; z,\tau\right] b_{k_1}^{(j_1)}b_{k_2}^{(j_2)} + \dots$$

The single-valued eMPL in the holomorphic frame

Expanding $\Gamma^{\rm sv}_{x,y}(z,\tau)$:

$$\Gamma^{\rm sv}_{x,y}(z,\tau) = \mathbb{I}^{\rm eqv}_{\epsilon^{\rm TS}}(\tau)^{-1}\,(\mathbb{M}^{\rm sv}_{z})^{-1}\,\overline{\Gamma_{x,y}(z,\tau)^{T}}\,\mathbb{M}^{\rm sv}_{z}\,\mathbb{I}^{\rm eqv}_{\epsilon^{\rm TS}}(\tau)\,\Gamma_{x,y}(z,\tau) \tag{92}$$

Using the definition$\mathbb{I}^{\rm sv}_{\epsilon^{\rm TS}}(\tau) = \mathbb{M}^{\rm sv}_{\sigma}\overline{\mathbb{I}_{\epsilon^{\rm TS}}(\tau)^{T}}\,\mathbb{M}^{\rm sv}_{\sigma}\,\mathbb{I}_{\epsilon^{\rm TS}}(\tau)$, we have

$$\Gamma^{\rm sv}_{x,y}(z,\tau) = \mathbb{I}^{\rm sv}_{\epsilon^{\rm TS}}(\tau)^{-1}\,(\mathbb{M}^{\rm sv}_{\sigma})^{-1}\,\overline{\Gamma_{x,y}(z,\tau)^{T}}\,\mathbb{M}^{\rm sv}_{\sigma}\,\mathbb{I}^{\rm sv}_{\epsilon^{\rm TS}}(\tau)\,\Gamma_{x,y}(z,\tau) \tag{93}$$

A 1-to-1 map between genus-0 and genus-1

Note that there is a 1-to-1 map between the genus-0 and the genus-1 case:

genus zero	genus one
$\mathbb{G}^{\mathrm{sv}}_{e_i}(z) = (\mathbb{M}^{\mathrm{sv}}_0)^{-1}\,\overline{\mathbb{G}_{e_i}(z)^T}\,\mathbb{M}^{\mathrm{sv}}_0\,\mathbb{G}_{e_i}(z)$	$\mathbb{I}^{\mathrm{sv}}_{\epsilon^{\mathrm{TS}}}(\tau) = (\mathbb{M}^{\mathrm{sv}}_\sigma)^{-1}\,\overline{\mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau)^T}\,\mathbb{M}^{\mathrm{sv}}_\sigma\,\mathbb{I}_{\epsilon^{\mathrm{TS}}}(\tau)$
$\mathbb{G}^{\mathrm{sv}}_{e'_i}(y,z) = \mathbb{G}^{\mathrm{sv}}_{e_i}(y)^{-1}\,(\mathbb{M}^{\mathrm{sv}}_0)^{-1}$ $\times\,\overline{\mathbb{G}_{e'_i}(y,z)^T}\,\mathbb{M}^{\mathrm{sv}}_0\,\mathbb{G}^{\mathrm{sv}}_{e_i}(y)\mathbb{G}_{e'_i}(y,z)$	$\Gamma^{\mathrm{sv}}_{x,y}(z,\tau) = \mathbb{I}^{\mathrm{sv}}_{\epsilon^{\mathrm{TS}}}(\tau)^{-1}\,(\mathbb{M}^{\mathrm{sv}}_\sigma)^{-1}$ $\times\,\overline{\Gamma_{x,y}(z,\tau)^T}\,\mathbb{M}^{\mathrm{sv}}_\sigma\,\mathbb{I}^{\mathrm{sv}}_{\epsilon^{\mathrm{TS}}}(\tau)\,\Gamma_{x,y}(z,\tau)$

Table: We are using shorthands $\mathbb{G}_{e_i}(z) = \mathbb{G}_{e_0,e_1}(z)$ and $\mathbb{G}_{e'_i}(y,z) = \mathbb{G}_{e'_0,e'_1,e'_y}(y,z)$ and similarly in their single-valued versions.

The single-valued eMPL in the modular frame

$$\Lambda^{\rm sv}_{x,y}(z,\tau) = U_{\rm mod}(\tau)\,\Gamma^{\rm sv}_{x,y}(z,\tau)\,U_{\rm mod}(\tau)^{-1} \tag{94}$$

is uniquely expressible as a series

$$\begin{aligned}\Lambda^{\rm sv}_{x,y}(z,\tau) &= 1 + \sum_{k_1=2}^{\infty}(k_1-1)\sum_{j_1=0}^{k_1-2}\frac{(-1)^{j_1}}{j_1!}\,\Lambda^{\rm sv}\left[\begin{smallmatrix} j_1 \\ k_1 \end{smallmatrix};z,\tau\right] b^{(j_1)}_{k_1} \qquad (95)\\ &+ \sum_{k_1,k_2=2}^{\infty}(k_1-1)(k_2-1)\sum_{j_1=0}^{k_1-2}\sum_{j_2=0}^{k_2-2}\frac{(-1)^{j_1+j_2}}{j_1!j_2!}\,\Lambda^{\rm sv}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix};z,\tau\right] b^{(j_1)}_{k_1} b^{(j_2)}_{k_2} + \ldots\end{aligned}$$

in non-holomorphic modular forms $\Lambda[\ldots;z,\tau]$ of purely antiholomorphic modular weights

$$\Lambda^{\rm sv}\left[\begin{smallmatrix} j_1 & \ldots & j_r \\ k_1 & \ldots & k_r \end{smallmatrix};\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right] = \left(\prod_{i=1}^{r}(c\bar\tau+d)^{k_i-2j_i-2}\right)\Lambda^{\rm sv}\left[\begin{smallmatrix} j_1 & \ldots & j_r \\ k_1 & \ldots & k_r \end{smallmatrix};z,\tau\right] \tag{96}$$

The single-valued eMPL in the modular frame

$$\Lambda^{\rm sv}\left[\begin{smallmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{smallmatrix};z,\tau\right] = \tag{97}$$

$$\sum_{p_1=0}^{k_1-j_1-2}\sum_{p_2=0}^{k_2-j_2-2}\cdots\sum_{p_\ell=0}^{k_\ell-j_\ell-2}\binom{k_1-j_1-2}{p_1}\binom{k_2-j_2-2}{p_2}\cdots\binom{k_\ell-j_\ell-2}{p_\ell}$$

$$\times\left(\frac{1}{4\pi{\rm Im}\,\tau}\right)^{p_1+\cdots+p_\ell}\sum_{r_1=0}^{j_1+p_1}\sum_{r_2=0}^{j_2+p_2}\cdots\sum_{r_\ell=0}^{j_\ell+p_\ell}\binom{j_1+p_1}{r_1}\binom{j_2+p_2}{r_2}\cdots\binom{j_\ell+p_\ell}{r_\ell}$$

$$\times(-2\pi i\bar\tau)^{r_1+\cdots+r_\ell}\,\Gamma^{\rm sv}\left[\begin{smallmatrix} j_1+p_1-r_1 & j_2+p_2-r_2 & \cdots & j_\ell+p_\ell-r_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{smallmatrix};z,\tau\right] \tag{98}$$

eMGFs as Λ^{sv}

eMGFs can also be expressed as a combination of Λ^{sv}. Here we only give examples of the relations between β^{eqv} and Λ^{sv}.

$$\begin{aligned}
\Lambda^{\mathrm{sv}}\left[\begin{smallmatrix}1 & 0\\ 3 & 3\end{smallmatrix}; z,\tau\right] &= \beta^{\mathrm{eqv}}\left[\begin{smallmatrix}1 & 0\\ 3 & 3\\ z & z\end{smallmatrix};\tau\right] \qquad (99)\\
\Lambda^{\mathrm{sv}}\left[\begin{smallmatrix}2 & 1\\ 4 & 4\end{smallmatrix}; z,\tau\right] &= \beta^{\mathrm{eqv}}\left[\begin{smallmatrix}2 & 1\\ 4 & 4\\ z & z\end{smallmatrix};\tau\right] + \beta^{\mathrm{eqv}}\left[\begin{smallmatrix}2 & 1\\ 4 & 4\end{smallmatrix};\tau\right] - \beta^{\mathrm{eqv}}\left[\begin{smallmatrix}2\\ 4\end{smallmatrix};\tau\right]\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}1\\ 4\\ z\end{smallmatrix};\tau\right]\\
\Lambda^{\mathrm{sv}}\left[\begin{smallmatrix}2 & 1\\ 5 & 3\end{smallmatrix}; z,\tau\right] &= \beta^{\mathrm{eqv}}\left[\begin{smallmatrix}2 & 1\\ 5 & 3\\ z & z\end{smallmatrix};\tau\right] - \frac{3}{2}\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}2 & 1\\ 4 & 4\end{smallmatrix};\tau\right] - \frac{3}{4}\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}1 & 2\\ 4 & 4\end{smallmatrix};\tau\right]
\end{aligned}$$

Expansion around the cusp $\tau \to i\infty$

Theorem

The $\tau \to i\infty$ asymptotics of the series $\mathbb{H}^{\mathrm{eqv}}_{\epsilon,b}(u,v,\tau)$ at fixed values of $0 \le u, v < 1$ is given by

$$\begin{aligned} \mathbb{H}^{\mathrm{eqv}}_{\epsilon,b}(u,v,\tau) &= U_{\mathrm{mod}}(\tau)\, \mathbb{I}^{\mathrm{eqv}}_{\epsilon,b}(u,v,\tau)\, U_{\mathrm{mod}}(\tau)^{-1} \\ &= \exp\left(-\frac{\epsilon_0^\vee}{Y}\right) (\mathbb{M}^{\mathrm{sv}}_z)^{-1}\, \mathbb{M}^{\mathrm{sv}}_{\Sigma(u)}\, \mathbb{G}^{\mathrm{sv}}_{E_0(u),E_1(u)}(e^{2\pi i z})\, e^{-Y\epsilon_0} \exp\left(\frac{\epsilon_0^\vee}{Y}\right) \\ &\quad + \mathcal{O}(q^{1-u}, \bar{q}^{1-u}) \end{aligned} \tag{100}$$

where $Y = 4\pi \mathrm{Im}\,\tau$, and

$$\begin{aligned} E_0(u) &= \frac{1}{u}\left(-\epsilon_0 + \sum_{k=4}^{\infty}(k-1)\frac{B_k}{k!}\epsilon_k + \sum_{k=2}^{\infty}(k-1)\frac{B_k(u)}{k!} b_k\right), \\ E_1(u) &= \sum_{k=2}^{\infty} \frac{u^{k-2}}{(k-2)!}\, b_k \end{aligned} \tag{101}$$

The sketch of proof

Only need to prove

$$\mathbb{I}_{\epsilon,b}(u,v,\tau) = e^{-2\pi i v E_0(u)}\,\mathbb{G}_{E_0(u),E_1(u)}(e^{2\pi i z})\,e^{2\pi i \epsilon_0 \tau} + \mathcal{O}(q^{1-u}) \tag{102}$$

and

$$[E_0(u),\Sigma_w(u)] = 0,\ \left[E_1(u),\Sigma_w(u)\right] = \left[P_w\left(E_0(u),E_1(u)\right),E_1(u)\right] \tag{103}$$

which means $\mathbb{G}^{\mathrm{sv}}_{E_0(u),E_1(u)}(e^{2\pi i z}) =$ $(\mathbb{M}^{\mathrm{sv}}_{\Sigma(u)})^{-1}\overline{\mathbb{G}_{E_0(u),E_1(u)}(e^{2\pi i z})}^{T}\mathbb{M}^{\mathrm{sv}}_{\Sigma(u)}\mathbb{G}_{E_0(u),E_1(u)}(e^{2\pi i z})$.

Thank you!