Single-valued periods in string theory

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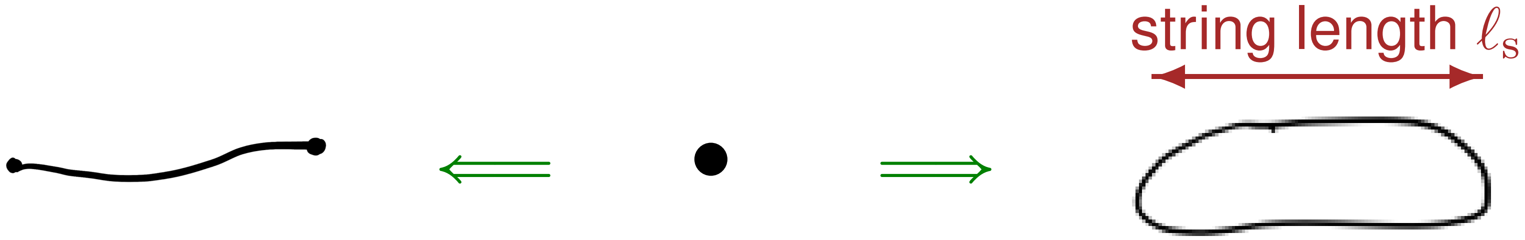
Single-valued periods in scattering amplitudes
MITP YOUNGST@RS (online), 14 Jan 2026

Overview

String theory: Elementary objects are one-dim’l strings



string length ℓ_s

open $\Longleftarrow$ QFT $\Longrightarrow$ closed

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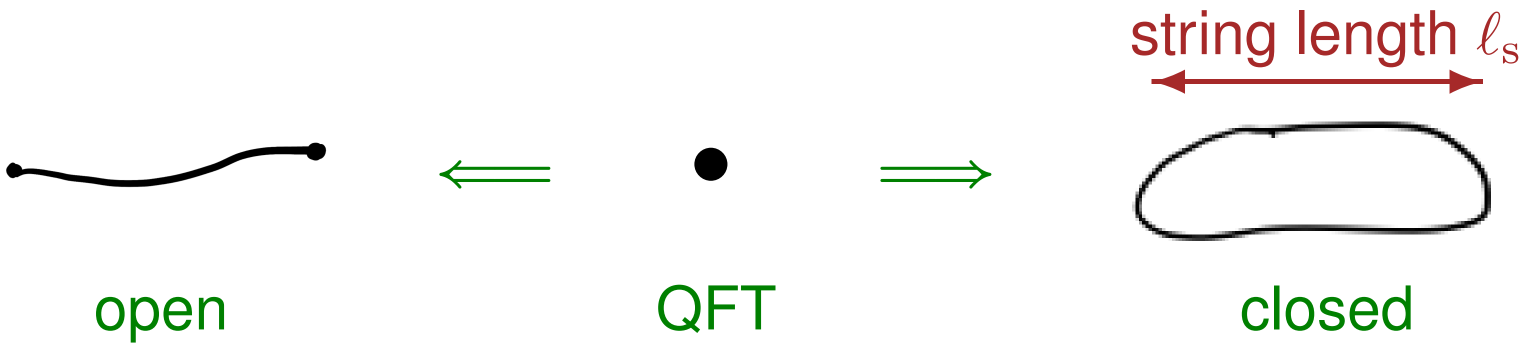
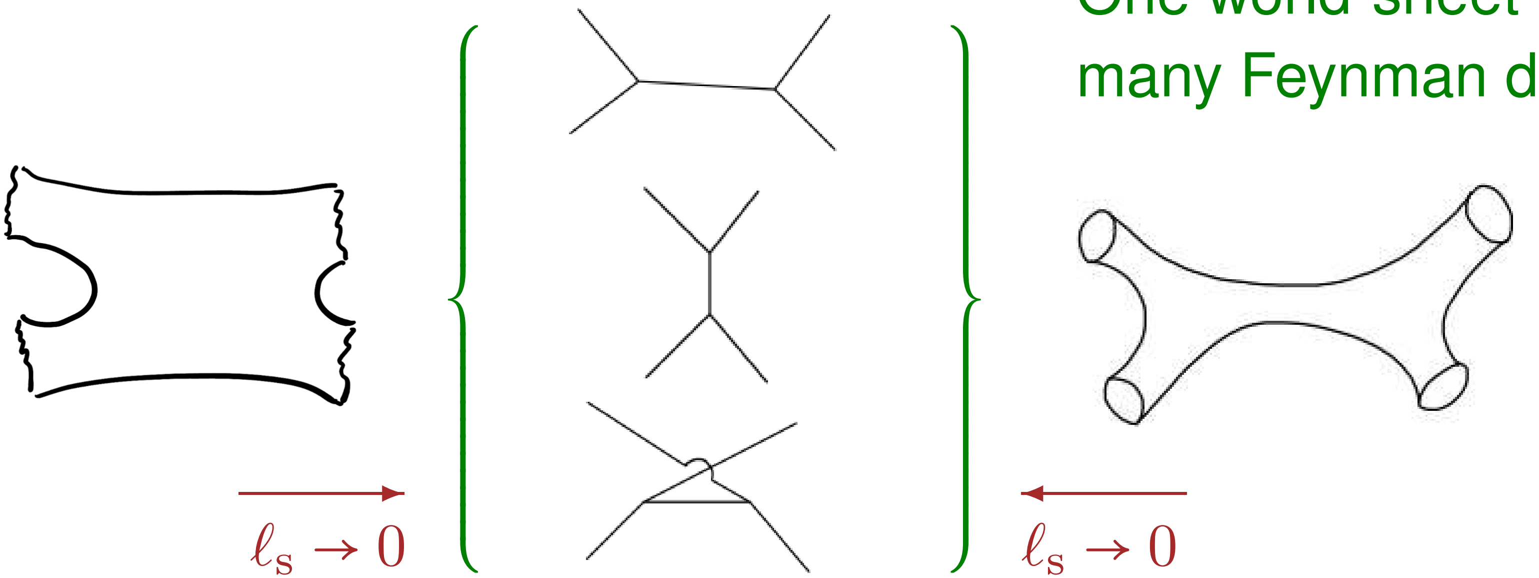


string length ℓ_s

open $\Leftarrow$ QFT $\Rightarrow$ closed

Feynman diagrams get 'fattened', e.g.

One world-sheet type for many Feynman diagrams!



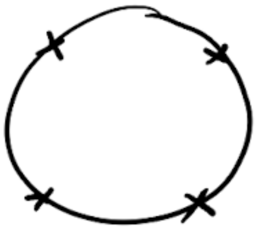
$\ell_s \to 0$ $\ell_s \to 0$

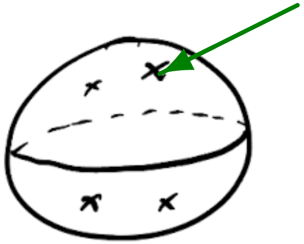
Overview

Redraw diagrams using conformal invariance

tree-level

genus zero



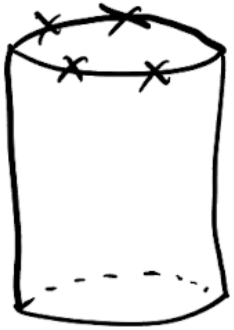


‘punctures’
carry info on
external states

‘vertex operators’

one loop

genus one



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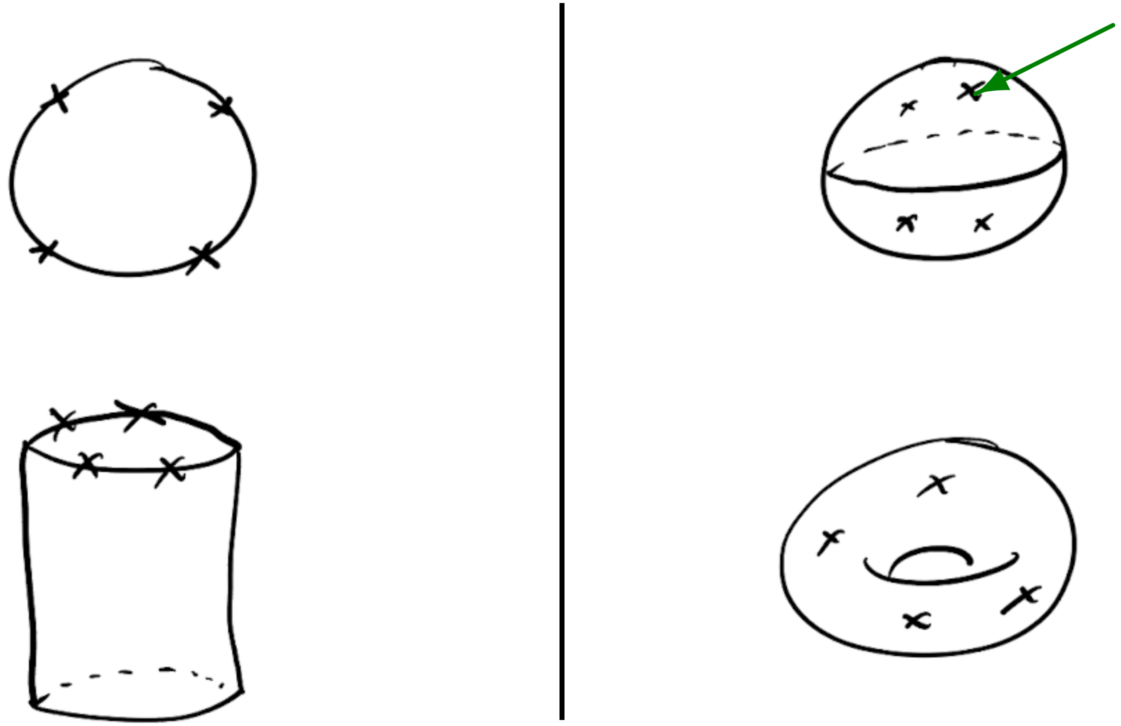

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genus zero

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Punchline: [**Baune, Broedel, Brown, Dupont, Gerken, AK, Mafra, Schlotterer, Schnetz, Stieberger, Taylor, Vanhove, Verbeek, Zerbini, ...**]

sv(open string amplitude) = closed string amplitude

↖ single-valued map

Overview

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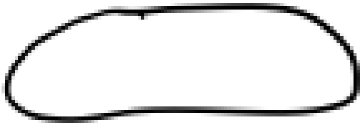
Rough intuition

boundary conditions relate
left-/right-moving excitations

could call coordinate z or $\bar{z}$



(largely) independent
left-/right-moving excitations
given state: specific combination
function of z and $\bar{z}$

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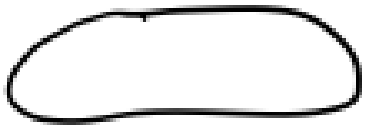

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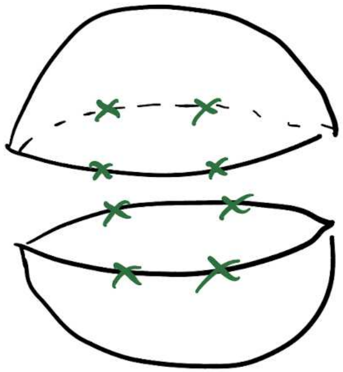
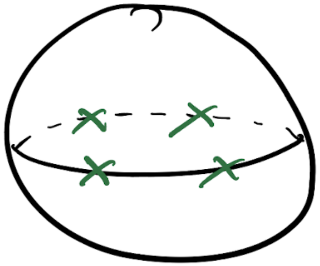


Split sphere into two disks
⇒ closed as specific
combo of two open

$$\mathcal{A}_{\text{op}}\overline{\mathcal{A}_{\text{op}}} \sim \mathcal{M}_{\text{cl}}$$

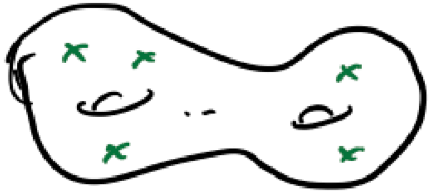
‘double copy’ / KLT



Anatomy of a string amplitude

String scattering amplitude at genus h with n ext. states roughly

$$\mathcal{A}_h(1,\ldots,n) = \int_{\mathcal{M}_h} d\mu_h \int_{\Sigma^n} \prod_{i=1}^{n} d\mu(z_i) \langle V(z_1) \cdots V(z_n) \rangle_\Sigma$$



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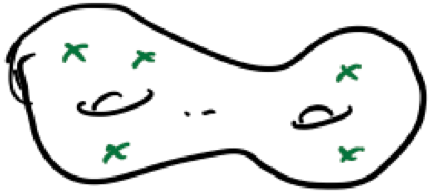
(moduli) space of all surfaces

Σ of genus h (open/closed)



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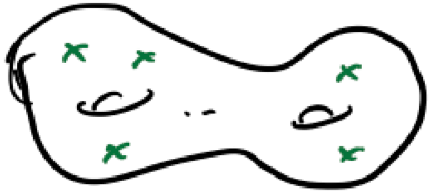
positions of n punctures



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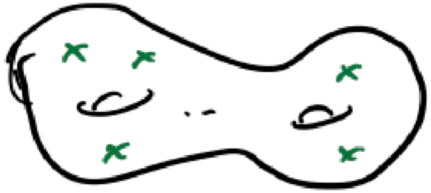

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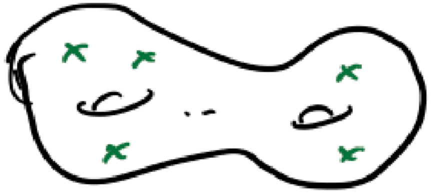
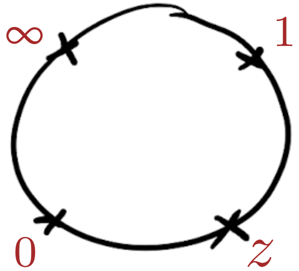


- Subtleties from gauge-fixing and boundaries of moduli space (of Σ and punctures).
  Buzzwords: Riemann–Roch, mapping class group, conformal Killing group, ghost pictures
- Very few results beyond $h = 2$

Example: 4 points at tree level (open)

Only integral $\int d\mu(z)$ over punctures. After removing some overall kinematic factor left with the representative integral

$$Z_4^{\text{op}} = \int_0^1 \frac{dz}{z} z^{s_{12}} (1-z)^{s_{23}} = \frac{\Gamma(s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})}$$



[Veneziano]

Dimensionless Mandelstam variables

$$s_{ij} = 2\alpha' k_i \cdot k_j \,, \qquad \alpha' = \ell_s^2$$

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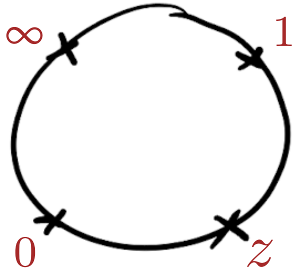

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Dimensionless Mandelstam variables

$$s_{ij} = 2\alpha' k_i \cdot k_j \,, \qquad\qquad \alpha' = \ell_{\text{s}}^2$$

Can obtain low-energy expansion for $s_{ij} \ll 1$

$$\begin{aligned} Z_4^{\text{op}} &= \frac{1}{s_{12}} \exp\left(\sum_{k=2}^{\infty} (-1)^k \frac{\zeta_k}{k} \left(s_{12}^k + s_{23}^k - (s_{12}+s_{23})^k \right) \right) \\ &= \frac{1}{s_{12}} - \zeta_2 s_{23} + \zeta_3 s_{23}(s_{12}+s_{23}) + O(\alpha'^3) \end{aligned}$$

Example: 4 points at tree level (closed)

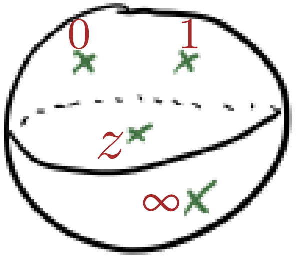
Only integral $\int d\mu(z)$ over punctures. After removing some overall kinematic factor left with the representative integral

$$J_4^{\text{cl}} = \frac{1}{\pi} \int_{\mathbb{C}} \frac{d^2 z}{z\bar{z}(1-\bar{z})} |z|^{2s_{12}} |1-z|^{2s_{23}} = \frac{\Gamma(s_{12})\Gamma(1+s_{23})\Gamma(1+s_{13})}{\Gamma(1-s_{12})\Gamma(1-s_{23})\Gamma(1-s_{13})}$$

$$= \frac{1}{s_{12}} \exp\left(-2 \sum_{k=1}^{\infty} \frac{\zeta_{2k+1}}{2k+1} \left(s_{12}^{2k+1} + s_{23}^{2k+1} - (s_{12}+s_{23})^{2k+1} \right) \right)$$

$$= \frac{1}{s_{12}} + 2\zeta_3 s_{23}(s_{12}+s_{23}) + O(\alpha'^4)$$

[Virasoro Shapiro]



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Comparison of open and closed

$$Z_4^{\mathsf{op}} = \frac{1}{s_{12}} \exp\left(\sum_{k=2}^{\infty} (-1)^k \frac{\zeta_k}{k} \left(s_{12}^k + s_{23}^k - (s_{12} + s_{23})^k \right) \right)$$

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Simplest instance of a single-valued map of periods!

$$\mathsf{sv}(Z_4^{\mathsf{op}}) = J_4^{\mathsf{cl}}$$

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This holds generally at tree level in the α'-expansion...

Interlude

(Single-valued)
multiple polylogarithms

Multiple polylogarithms

Multiple polylogarithms (MPLs) for $z \in \mathbb{C} \setminus (-\infty, 0] \cup [1, \infty)$

[Goncharov]

$$G(a_1, \ldots, a_w; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_w; t)\,, \qquad G(\varnothing; z) = 1$$

with $a_i \in \{0, 1\}$ and multiple zeta values (MZVs) for $n_r \geq 2$

$$\zeta_{n_1, n_2, \ldots, n_r} = (-1)^r G(\underbrace{0, \ldots, 0}_{n_r - 1}, 1, \ldots, \underbrace{0, \ldots, 0}_{n_2 - 1}, 1, \underbrace{0, \ldots, 0}_{n_1 - 1}, 1; z = 1)$$

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Endpoint divergences in integrals shuffle-regularised*, e.g.

$$G(0; z) = \log(z)$$

Differential equation

$$\partial_z G(a_1, \ldots, a_w; z) = \frac{1}{z - a_1} G(a_2, \ldots, a_w; z)$$

---

*MPLs satisfy a shuffle algebra on the letters

Single-valued MPLs

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Can be cancelled systematically by adding complex conjugates and explicit MZV terms [Brown]. E.g.

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The single-valued image $G^{\text{sv}}(\vec{a}; z) = \text{sv}G(\vec{a}, z)$ satisfies the same differential equation in ∂_z

$$\partial_z G^{\text{sv}}(a_1, \ldots, a_w; z) = \frac{1}{z - a_1} G^{\text{sv}}(a_2, \ldots, a_w; z)$$

but not in $\partial_{\bar{z}}$!

The shuffle property is also preserved.

Single-valued MZVs

Special values of svMPLs are single-valued multiple zeta values (svMZVs)

$$\zeta^{\mathsf{sv}}_{n_1,n_2,\ldots,n_r} = (-1)^r G^{\mathsf{sv}}(\underbrace{0,\ldots,0}_{n_r-1},1,\ldots,\underbrace{0,\ldots,0}_{n_2-1},1,\underbrace{0,\ldots,0}_{n_1-1},1;z=1)$$

E.g.

$$\zeta^{\mathsf{sv}}_{2k} = 0\,, \qquad \zeta^{\mathsf{sv}}_{2k+1} = 2\zeta_{2k+1}\,, \qquad \zeta^{\mathsf{sv}}_{3,5} = -10\zeta_3\zeta_5$$

First irreducible svMZV at (transcendental) weight 11: $\zeta^{\mathsf{sv}}_{3,5,3}$

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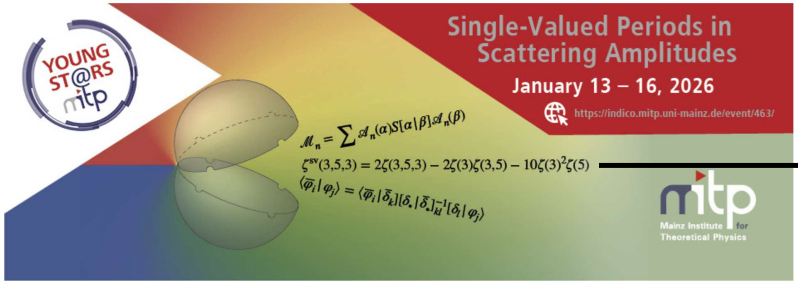
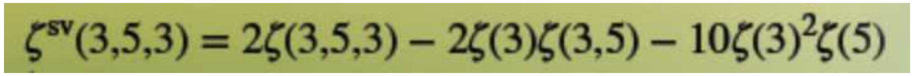


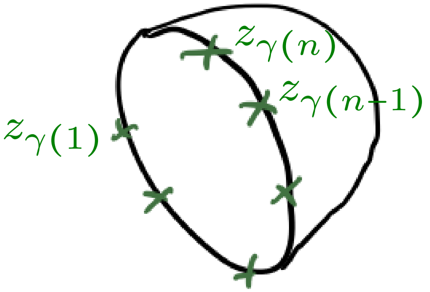


$\zeta^{\mathrm{sv}}(3,5,3) = 2\zeta(3,5,3) - 2\zeta(3)\zeta(3,5) - 10\zeta(3)^2\zeta(5)$

Tree level integrals (open)

Open string:

[Mafra, Schlotterer Stieberger] [Broedel, Schlotterer Stieberger]



$$Z^{\text{op}}(\gamma|\rho) = \int_{\mathfrak{D}(\gamma)} \frac{\prod_{j=1}^{n} dz_j}{\text{vol}\, SL_2(\mathbb{R})} \text{PT}(\rho)\text{KN}_{\text{disk}}$$

with $\gamma, \rho \in S_n$ ($\rightarrow S_{n-3}$), $z_{ij} = z_i - z_j$ and

$$\mathfrak{D}(\gamma) = \{z_j \in \mathbb{R} \,|\, -\infty < z_{\gamma(1)} < \ldots < z_{\gamma(n)} < \infty\}$$

$$\text{PT}(\rho) = \left(z_{\rho(1)\rho(2)} z_{\rho(2)\rho(3)} \cdots z_{\rho(n)\rho(1)}\right)^{-1}$$ [Parke–Taylor factor]

$$\text{KN}_{\text{disk}} = \prod_{1\le i<j}^{n} |z_{ij}|^{s_{ij}} = e^{\sum_{i<j} s_{ij} \underset{\uparrow}{G_{\text{disk}}}(z_i,z_j)}$$ [Koba–Nielsen factor]

world-sheet Green function

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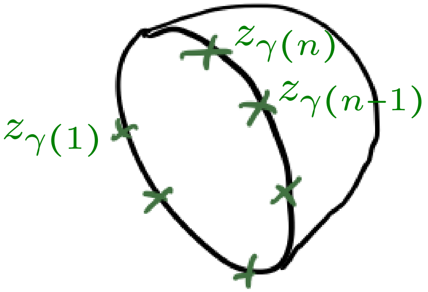

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When α'-expanding in Mandelstam variables can read this as special values of iterated integrals of simple rational functions; actually in the multiple polylogarithm family.

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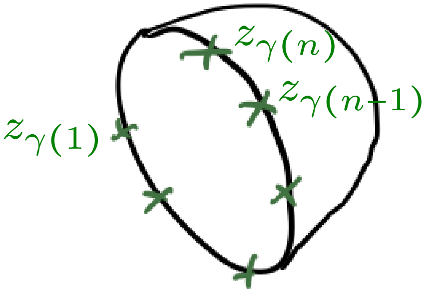

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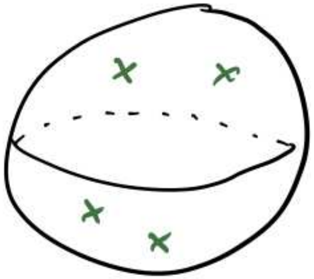
world-sheet Green function

When α'-expanding in Mandelstam variables can read this as special values of iterated integrals of simple rational functions; actually in the multiple polylogarithm family. **[Brown]**

$\Rightarrow$ The coefficients in the α'-expansion are MZVs! **[Terasoma]**

Tree level integrals (closed)

Closed string:



$$J^{\mathsf{cl}}(\gamma|\rho) = \int_{\mathbb{C}^n} \frac{\prod_{j=1}^n d^2 z_j}{\mathrm{vol}\, SL_2(\mathbb{C})} \overline{\mathrm{PT}(\gamma)}\mathrm{PT}(\rho)\mathrm{KN}_{\mathrm{sphere}}$$

with $\gamma, \rho \in S_n$ ($\to S_{n-3}$), $z_{ij} = z_i - z_j$ and

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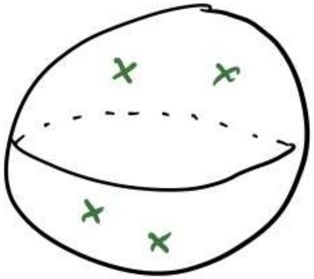

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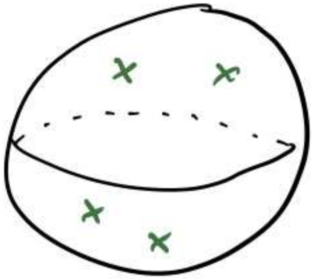

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Remark: Note suggestive form of closed string integral as (absolute) square of open string. Made precise in KLT formalism. [**Kawai Lewellen, Tye**]

Comparison

$$Z^{\mathsf{op}}(\gamma|\rho) = \int_{\mathfrak{D}(\gamma)} \frac{\prod_{j=1}^{n} dz_j}{\mathrm{vol}\, SL_2(\mathbb{R})} \mathrm{PT}(\rho)\mathrm{KN}_{\mathrm{disk}}$$

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An integration domain is mapped to an integrand. Properly:

(homology) cycle $\longleftrightarrow$ differential form (cohomology)

Instance of so-called Betti/de Rham duality.

Comparison

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Key statement

$$\boxed{\mathrm{sv}Z^{\mathrm{op}}(\gamma|\rho) = J^{\mathrm{cl}}(\gamma|\rho)}$$

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Note: [**Brown** **Dupont**] express single-valued periods in general as*

$$\text{sv} \int_{\gamma} \omega = \frac{1}{2\pi i} \int_{X(\mathbb{C})} \nu_\gamma \wedge \bar{\omega} = \sum_{[\alpha],[\beta]} \underset{\uparrow}{\langle \alpha, \beta \rangle} \int_{\alpha} \nu_\gamma \cdot \overline{\int_{\beta} \omega}$$

[**Mizera**] intersection number of homology cycles ~ (inverse) KLT kernel

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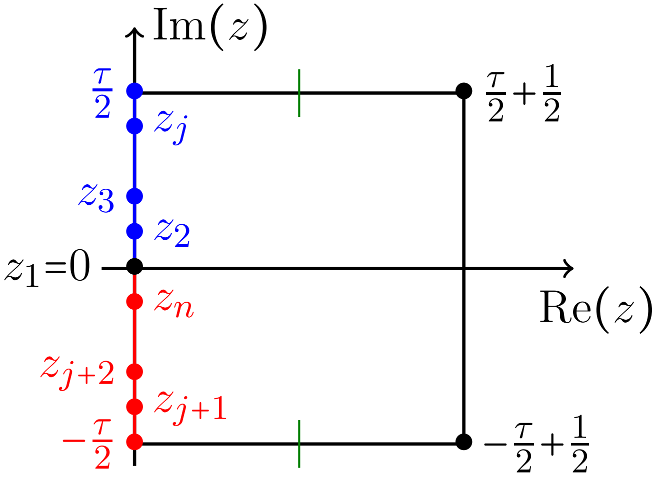
*properly stated using motives

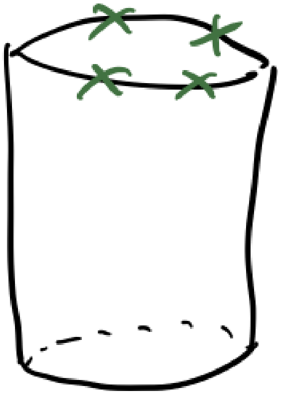
Genus-one amplitudes

Genus-one geometry

Open

$\tau \in i\mathbb{R}$





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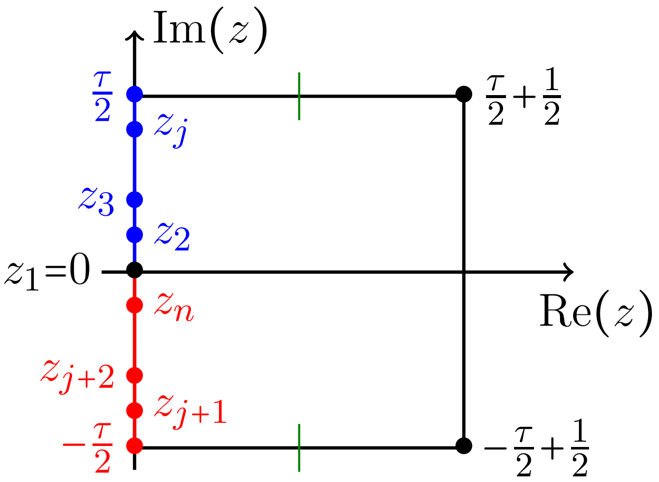
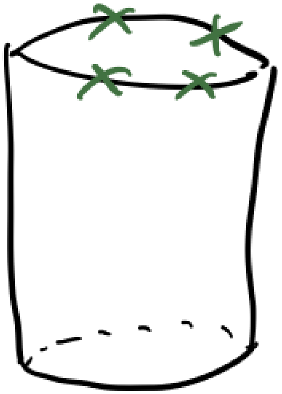
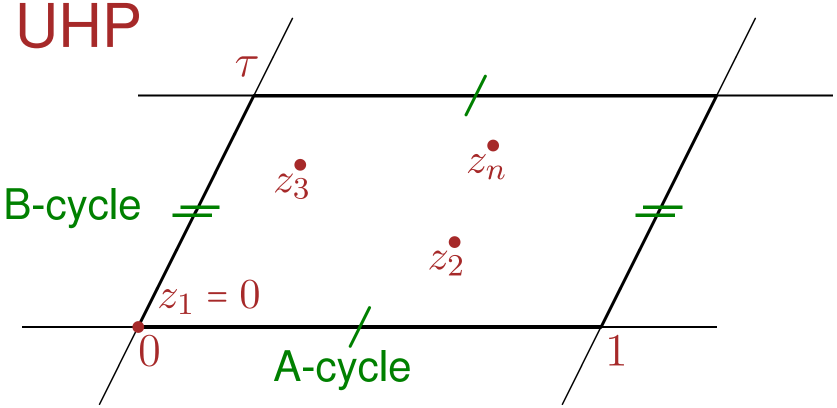
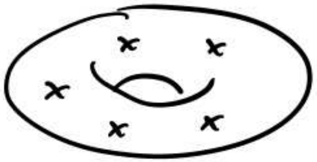




Closed

$\tau \in$ UHP





torus = $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$

inv. under $SL(2,\mathbb{Z})$

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}$$

$\Rightarrow$modular invariance

Think of cylinder as a torus (with $\tau \in i\mathbb{R}$) cut open along the B-cycle.

Genus-one geometry

Functions of the punctures z_i on the torus need to be doubly-periodic, e.g. Jacobi θ functions.

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Important family of such functions generated from Kronecker–Eisenstein series [Brown Levin]

$$\Omega(z,\eta,\tau) = e^{2\pi i\eta \frac{\mathrm{Im} z}{\mathrm{Im}\tau}} \frac{\theta'(0,\tau)\theta(z+\eta,\tau)}{\theta(z,\tau)\theta(\eta,\tau)} = \sum_{a\geq 0} \eta^{a-1} f^{(a)}(z,\tau)$$

Like Fourier transforms of holomorphic propagators

$$f^{(a)}(z,\tau) = \sum_{0\neq p\in\mathbb{Z}+\tau\mathbb{Z}} \frac{e^{i\langle p,z\rangle}}{p^a}$$

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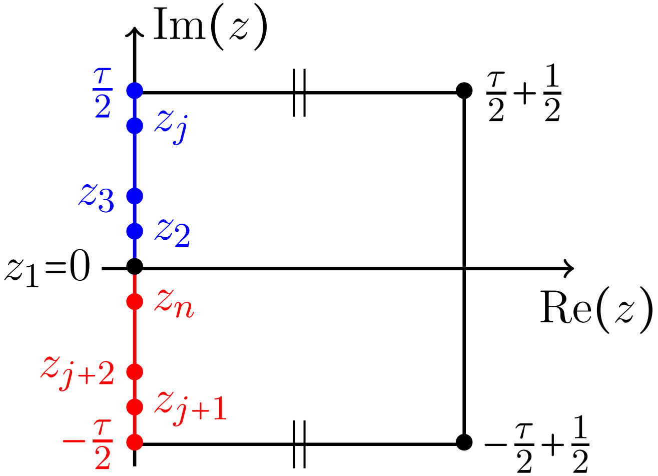
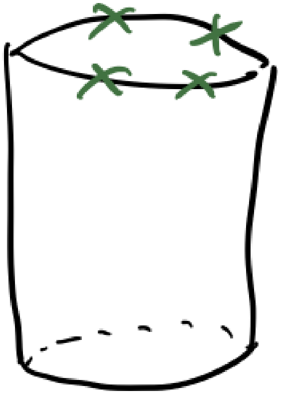
Replace the rational integration kernels seen at genus zero:

MZV $\longrightarrow$ elliptic MZV

One-loop set-up: open string





(Open chain) Kronecker–Eisenstein products

$\eta_{k\ldots n} = \eta_k + \ldots + \eta_n$

$$\varphi^{\tau}_{\vec{\eta}}(1,\ldots,n) := \Omega(z_{12},\eta_{23\ldots n},\tau)\cdots\Omega(z_{n-1,n},\eta_n,\tau)$$

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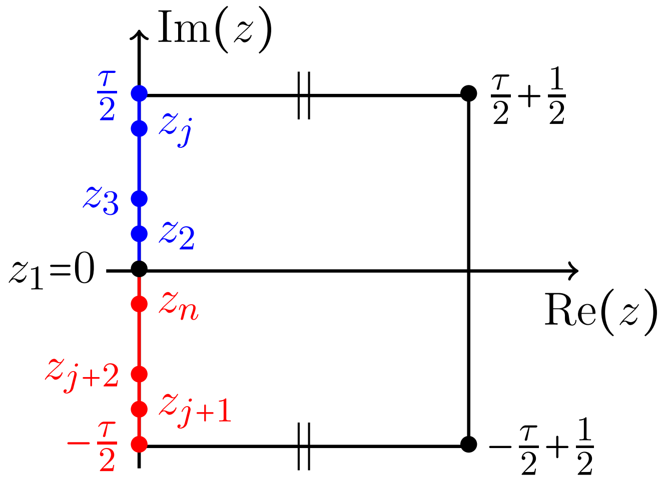

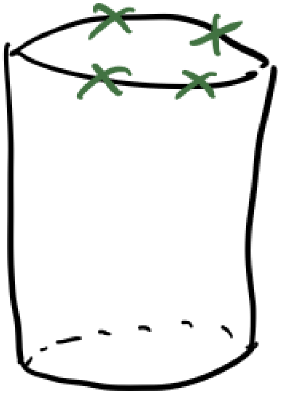

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B-cycle elliptic MZV [Enriquez] generated by ($\tau \in i\mathbb{R}^+$, $\rho,\gamma \in S_{n-1}$)

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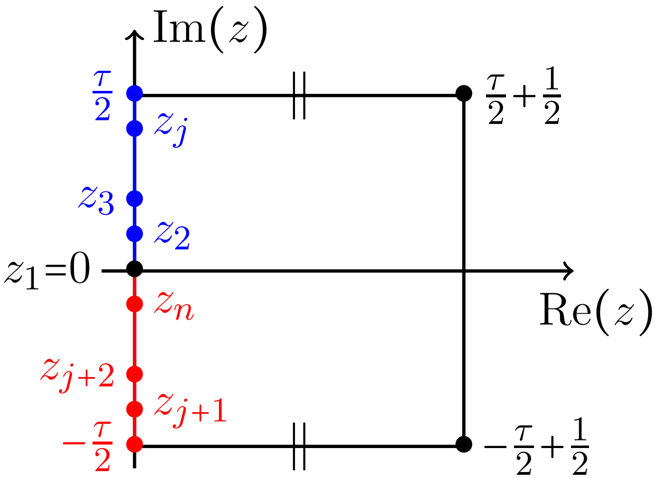

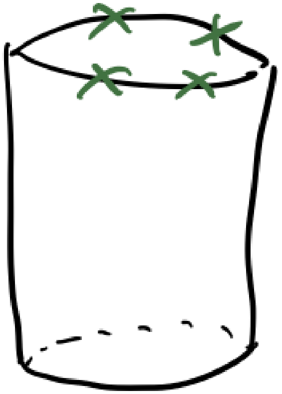

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(sum over) ordering of boundary points

configuration space only!

One-loop set-up: closed string

Consider closed cycle* [Dolan Goddard]

$$\Omega(z_{12},\xi,\tau)\cdots\Omega(z_{n-1,n},\xi,\tau)\Omega(z_{n,1},\xi,\tau)=\xi^{-n}\sum_{w=0}^{\infty}\xi^{w}V_w(1,\ldots,n|\tau)$$

Defines elliptic functions V_w with good modular trm.

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The combinations

$$V(1,\ldots,n|\tau) = \sum_{w=0}^{n-2}\frac{V_w(1,\ldots,n|\tau)}{(2\pi i)^w(n-w-1)!}$$

have residues for approaching points

$$\mathrm{Res}_{z_j=z_{j\pm1}}V(1,\ldots,n|\tau) = \pm\frac{1}{2\pi i}V(1,\ldots j-1,j+1,\ldots,n|\tau)$$

Parallels behaviour of Parke–Taylor factors in open string

$$\mathrm{Res}_{z_j=z_{j\pm1}}\mathrm{PT}(1,\ldots,n|\tau) = \pm\mathrm{PT}(1,\ldots j-1,j+1,\ldots,n)$$

And $V(1,\ldots,n|\tau)$ has right degeneration limit $\tau\to i\infty$.

---

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Open/closed relation

Therefore consider as ‘Betti/de Rham dual’

$$\mathfrak{B}(\gamma) \quad \leftrightarrow \quad V(1,\gamma(2,\ldots,n)|\tau) =: V(\gamma)$$

and define closed string generating series on torus Σ

$$J^{\tau}_{\vec{\eta}}(\gamma|\rho) = (2i)^{n-1} \int_{\Sigma^{n-1}} \Big(\prod_{j=2}^{n} d^2 z_j \Big) \overline{V(\gamma)}\ \varphi^{\tau}_{(\tau-\bar{\tau})\vec{\eta}}(\rho)\mathrm{KN}_{\text{torus}}$$

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Similarly to genus zero want to have

$$\boxed{\mathrm{sv} B^{\tau}_{\vec{\eta}}(\gamma|\rho) = J^{\tau}_{\vec{\eta}}(\gamma|\rho)}$$

Relates elliptic MZV to their single-valued versions:
non-holomorphic modular forms a.k.a. modular graph forms

⟶ Emiel Claasen’s talk

Single-valued at genus one

Differential equations [**Mafra Schlotterer**] [**Gerken, AK Schlotterer**] [**Gerken, AK, Mafra Schlotterer, Verbeek**]

$$2\pi i\partial_\tau B^\tau_{\vec{\eta}}(\gamma|\rho) = \sum_{k=0}^{\infty}(1-k) \quad \tau^{k-2} \qquad \mathrm{G}_k(\tau) \sum_{\alpha\in S_{n-1}} r_{\vec{\eta}}(\epsilon_k)_\rho{}^\alpha \; B^\tau_{\vec{\eta}}(\gamma|\alpha)$$

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$r_{\vec{\eta}}(\epsilon_k)$ and $\mathsf{sv}\, r_{\vec{\eta}}(\epsilon_k)$ explicit $(n-1)! \times (n-1)!$ matrices of differential operators in $\vec{\eta}$.

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2\pi i\partial_\tau J^\tau_{\vec\eta}(\gamma|\rho) = \sum\limits_{k=0}^{\infty}(1-k) & (\tau-\bar\tau)^{k-2} & \mathrm{G}_k(\tau) & \sum\limits_{\alpha\in S_{n-1}} & \text{sv}\, r_{\vec\eta}(\epsilon_k)_\rho{}^\alpha & J^\tau_{\vec\eta}(\gamma|\alpha)
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$\Rightarrow$ Letting sv act as shown maps differential equations!

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⇒ Letting sv act as shown maps differential equations!

[sv$(\pi\tau)$ = sv$\left(\frac{\log q}{2i}\right) = \frac{\log|q|^2}{2i} = \pi(\tau-\bar\tau)$]

Single-valued at genus one (II)

Can also check (in many examples) that the initial conditions of the generating series match.

This establishes [Gerken, AK, Mafra Schlotterer, Verbeek]

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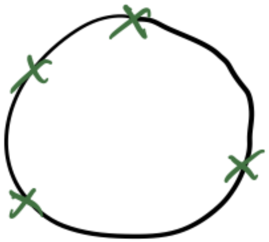
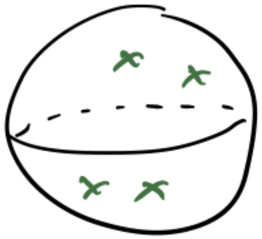
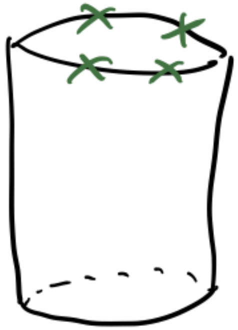
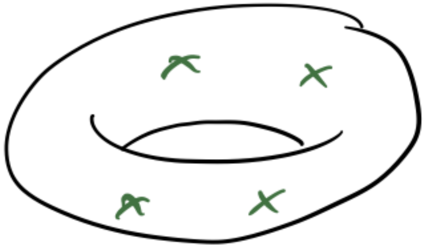
Remarks

- The derivative $\partial_{\bar{\tau}}$ does not agree, cf. $\partial_{\bar{z}}$ at genus zero.
- But the sv-image is fixed by modular invariance! Somewhat replaces single-valuedness here.
- Since ∂_{τ} leads to hol. Eisenstein G_k, can represent solutions using iterated Eisenstein integrals. Studied by [Brown] with single-valued and equivariant combinations.

$\longrightarrow$ Franca Lippert's talk

Genus zero and one summary

String integrals generate interesting periods

	open string	closed string
tree level		
one loop		

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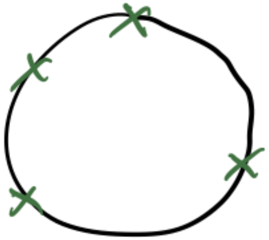
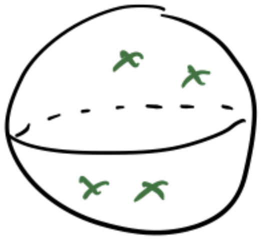
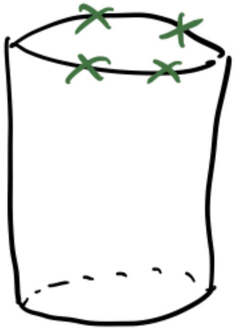
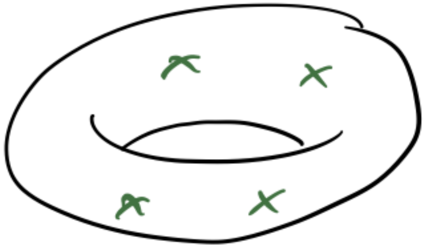
tree level	 multiple zeta values (MZVs)	 single-valued MZVs (svMZVs)
one loop	 elliptic MZVs (eMZVs)	 modular graph forms (MGFs)

Genus zero and one summary

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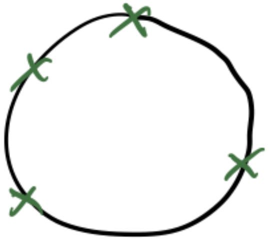
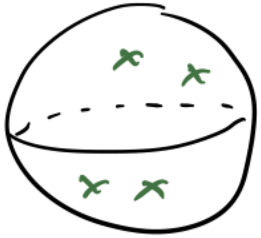
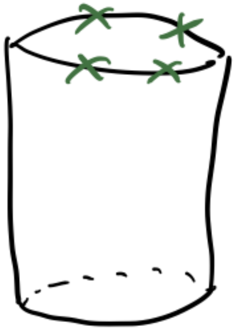
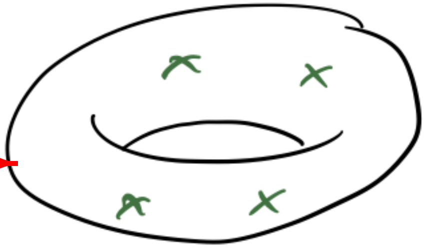
	open string		closed string
tree level	 multiple zeta values (MZVs)	sv →	 single-valued MZVs (svMZVs)
one loop	 elliptic MZVs (eMZVs)	sv →	 modular graph forms (MGFs)

Further topics

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- One-loop KLT formulæ [Stieberger] [Mazloumi Stieberger]

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Thank you for your attention!

Extra slides

A reformulation of svMPLs

Package all MPLs into generating series using non-commuting variables $e_0,\ e_1$

$$\mathbb{G}(e_i;z) = \sum_{w=0}^{\infty} \sum_{a_1,\ldots,a_w=0,1} e_{a_1} e_{a_2} \ldots e_{a_w} G(a_w,\ldots,a_2,a_1;z)$$

$$= \underset{\uparrow}{\mathrm{P\text{-}exp}}\left[-\int_z^0 \mathrm{d}t \left(\frac{e_0}{t} + \frac{e_1}{t-1}\right)\right]$$

path-ordered exponential

Remark: $\mathbb{G}(e_i;z)$ solves Knizhnik–Zamolodchikov equation

$$\partial_z \mathbb{G}(e_i;z) = \mathbb{G}(e_i;z)\left(\frac{e_0}{z} + \frac{e_1}{z-1}\right)$$

with boundary condition $\mathbb{G}(e_i;z) \sim z^{e_0}$ for $z \to 0$.

A reformulation of svMPLs (II)

The Drinfeld associator* is the generating series of MZVs

$$\Phi(e_0,e_1) = \sum_{\text{words } w} \zeta_{\vec{w}}\vec{w} = 1 + \sum_{k\in 2\mathbb{N}+1} \zeta_k \underbrace{g_k(e_0,e_1)}_{\text{canonical (Lie) polynomials}} + \dots$$

With single odd zeta values ζ_k ($k \in 2\mathbb{N}+1$) get zeta generators M_k defined by

$$[M_k, e_0] = 0\,, \qquad [M_k, e_1] = \big[e_1, g_k(e_0,e_1)\big]$$

They normalise the algebra of the e_i.

---

*Drinfeld associator can be seen as $\mathbb{G}(e_i; z = 1)$

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[Levine]

$$[M_k,e_0]=0\,,\qquad [M_k,e_1]=\left[e_1,g_k(e_0,e_1)\right]$$

They normalise the algebra of the e_i. Form the series

$$\mathbb{M}_0^{\text{sv}}=1+\sum_{k\in 2\mathbb{N}+1}\zeta_k^{\text{sv}}M_k+\frac{1}{2}\sum_{k_1,k_2\in 2\mathbb{N}+1}\zeta_{k_1}^{\text{sv}}\zeta_{k_2}^{\text{sv}}M_{k_1}M_{k_2}+\dots$$

with svMZV coefficients (easy in so-called f-alphabet).

---

*Drinfeld associator can be seen as $\mathbb{G}(e_i;z=1)$

A reformulation of svMPLs (III)

Use this to define [**Frost, Hidding, Kamlesh Rodriguez, Schlotterer, Verbeek**]

$$\mathsf{sv}\,\mathbb{G}(e_i;z) = (\mathsf{sv}\,\mathbb{M}_0)^{-1}\underbrace{\overline{\mathbb{G}(e_i;z)^T}}_{\text{complex conjugation and reversal of } e_i \text{ word}}(\mathsf{sv}\,\mathbb{M}_0)\,\mathbb{G}(e_i;z)$$

Can show: removes monodromies around 0 and 1.

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Alternative perspective: Take two different alphabets e_i and e_i' for solving the Knizhnik–Zamoldchikov equation and find a relation a relation between alphabets that cancels monodromies. Original approach by [Brown].

Conjugation above implements change of alphabet / ‘sprinkling’ of MZVs.

Single-valued at genus one

Using a basis of iterated Eisenstein integrals (cf. [Brown]) let

$$\mathbb{I}(\epsilon_k;\tau) = \text{P-exp}\left[\int_\tau^{i\infty} \mathbb{A}(\epsilon_k;\tau_1)\right]$$

with connection

$$\mathbb{A}(\epsilon_k;\tau) = \sum_{k=4}^{\infty}\sum_{j=0}^{k-2}(-1)^j \frac{(k-1)}{j!}(2\pi i)^{1+j-k}\tau^j \mathrm{G}_k(\tau)\mathrm{d}\tau\, \epsilon_k^{(j)}$$

Tsunogai's derivation algebra $\{\epsilon_0,\epsilon_4,\epsilon_6,\ldots\}$

Properties [Tsunogai] [Hain Matsumoto] [Pollack]

- $\epsilon_k^{(j)} = \mathrm{ad}_{\epsilon_0}^j \epsilon_k$ for $k \geq 4$; nilpotency $\epsilon_k^{(k-1)} = 0$
- Pollack relations from holomorphic cusp forms, e.g.

$$[\epsilon_4,\epsilon_{10}] - 3[\epsilon_6,\epsilon_8] = 0$$

Single-valued at genus one (II)

Similar to genus zero M_w have genus-one zeta generators σ_w normalising the 'geometric algebra' $\mathfrak{u} = \langle \epsilon_k^{(j)} \rangle$, i.e. [**Hain Matsumoto**]

$$[\sigma_w, \epsilon_k^{(j)}] \subset \mathfrak{u}$$

[Def. similar to that of M_w]

Decomposition (non-unique) [**Brown**][**Dorigoni, Doroudiani, Drewitt, Hidding AK, Schlotterer, Schneps, Verbeek**]

$$\sigma_w = \underset{\nearrow}{z_w} + \underset{\nwarrow}{\sigma_w^{\text{geo}}}$$

arithmetic part $\notin \mathfrak{u}$ geometric part $\in \mathfrak{u}$

Generating series

$$\mathbb{M}_\sigma^{\text{sv}} = 1 + \sum_{k \in 2\mathbb{N}+1} \zeta_k^{\text{sv}} \sigma_k + \frac{1}{2} \sum_{k_1, k_2 \in 2\mathbb{N}+1} \zeta_{k_1}^{\text{sv}} \zeta_{k_2}^{\text{sv}} \sigma_{k_1} \sigma_{k_2} + \dots$$

Single-valued at genus one (III)

The single-valued iterated Eisenstein integrals are

$$\mathsf{sv}\,\mathbb{I} = (\mathsf{sv}\,\mathbb{M}_\sigma)^{-1}\overline{\mathbb{I}^T}(\mathsf{sv}\mathbb{M}_\sigma)\mathbb{I}$$

Just as genus zero!

Can also be written using the Hopf-algebraic structure: antipode and coaction [Frost, Hidding, Kamlesh Rodriguez, Schlotterer, Verbeek] [AK, Porkert Schlotterer]

Tsunogai derivation algebra

Associated with the once-punctured torus and Lie$[a,b]$ is a distinguished algebra of Tsunogai derivations ϵ_{2i} ($i \in \mathbb{N}_{\geq 0}$) with

$$\epsilon_{2i}(a) = \mathrm{ad}_a^{2i}(b) \qquad\qquad \epsilon_{2i}([a,b]) = 0$$

From genus-one integrals get (conjectured) representations $r_{\vec{\eta}}(\epsilon_k)$ of these algebra on series in parameters $\vec{\eta} = (\eta_2, \eta_3, \ldots, \eta_n)$ as $(n{-}1)! \times (n{-}1)!$ matrix differential operators , e.g. $n = 3$

$$\begin{aligned} r_{\eta_2,\eta_3}(\epsilon_0) = {}& \frac{1}{\eta_{23}^2}\begin{pmatrix} s_{12} & -s_{13} \\ -s_{12} & s_{13} \end{pmatrix} + \frac{1}{\eta_2^2}\begin{pmatrix} 0 & 0 \\ s_{12} & s_{12}{+}s_{23} \end{pmatrix} + \frac{1}{\eta_3^2}\begin{pmatrix} s_{13}{+}s_{23} & s_{13} \\ 0 & 0 \end{pmatrix} \\ & - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big(\frac{1}{2}s_{12}\partial_{\eta_2}^2 + \frac{1}{2}s_{13}\partial_{\eta_3}^2 + \frac{1}{2}s_{23}(\partial_{\eta_2}{-}\partial_{\eta_3})^2\Big) + 2\overset{\text{0 under sv}}{\cancel{\zeta_2}}(s_{12}+s_{13}+s_{23}) \end{aligned}$$

$$r_{\eta_2,\eta_3}(\epsilon_k) = \eta_{23}^{k-2}\begin{pmatrix} s_{12} & -s_{13} \\ -s_{12} & s_{13} \end{pmatrix} + \eta_2^{k-2}\begin{pmatrix} 0 & 0 \\ s_{12} & s_{12}{+}s_{23} \end{pmatrix} + \eta_3^{k-2}\begin{pmatrix} s_{13}{+}s_{23} & s_{13} \\ 0 & 0 \end{pmatrix}$$