

# Unitarity bounds for form factors in $b$ -hadron decays

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Flavor Physics with  $b$ -Baryons and  $B_c$  Mesons, MITP

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Based on arXiv: [2412.04388](#), [2605.26213](#)



# What's in this talk?

Perform **indirect searches** for NP using **flavour physics**



Obtain **precise predictions** for observables  
i.e. predictions for **hadronic form factors (FFs)**



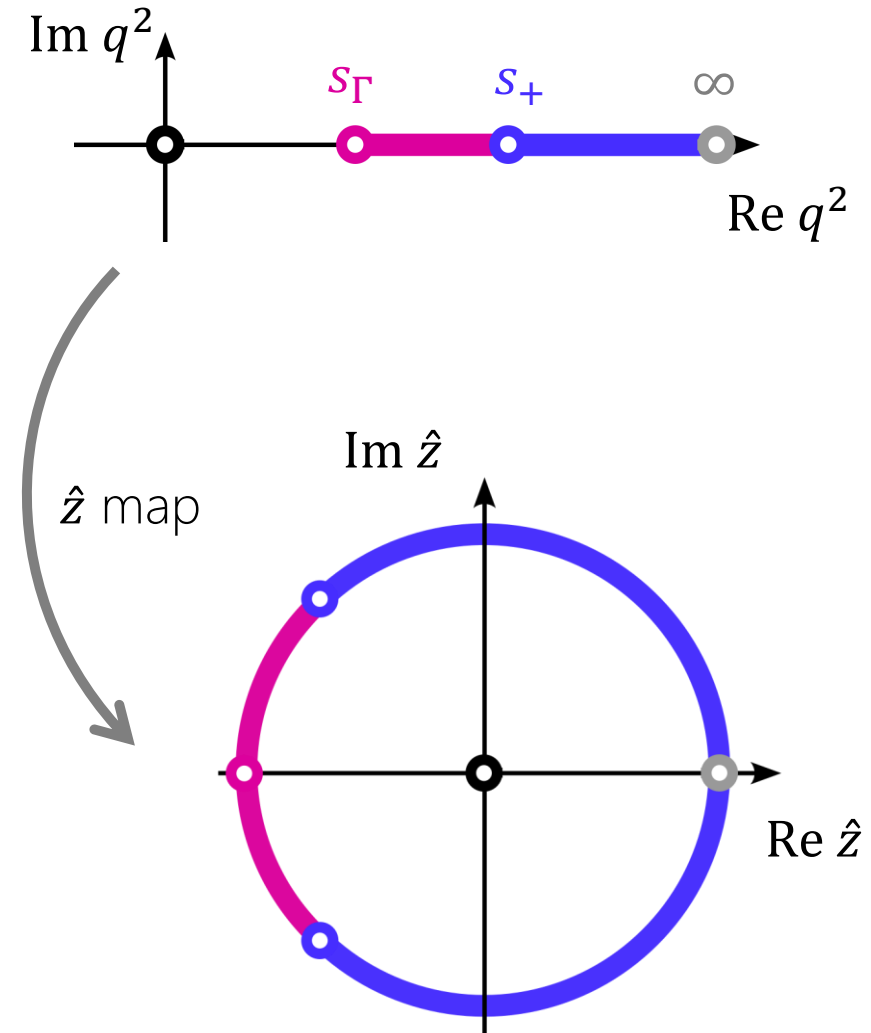
Combine different FF predictions with a **parametrization**



Overcome the limitations of current parametrizations  
by formulating a **new parametrization (GG)**



Perform **new FFs analyses** with GG parametrization



# Introduction

# Precision flavour tests of the SM

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No NP evidence from **direct searches** so far (too heavy?)

⇒ LHC has reached its maximum energy

⇒ Direct NP discovery difficult in coming decades

**Indirect searches** (with flavour)

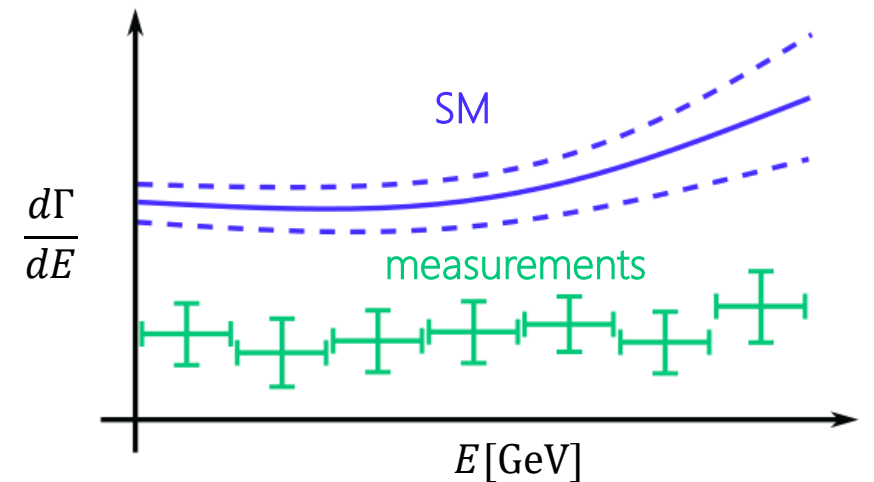
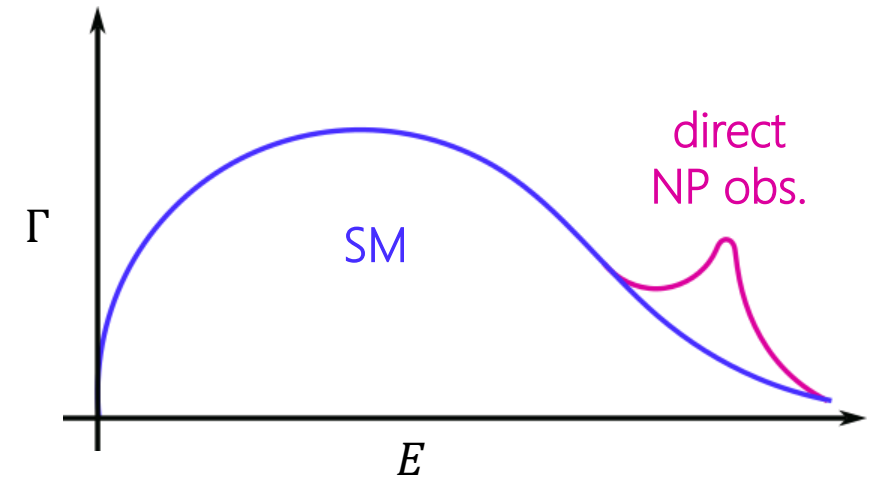
⇒ **Compare** precise measurements and calculations

⇒ Probe the SM at **higher energies** than direct searches

**Extraction SM parameters** (e.g.  $|V_{ub}|$  and  $|V_{cb}|$ )

⇒ 13 out of 19 are flavour parameters

**Need precise SM predictions**



# Predictions and form factors

Indirect searches  $\Rightarrow$  study ***b***-hadron decays to constrain NP

Factorise decay amplitude as e.g. (neglecting QED corrections)

$$\langle D\mu\bar{\nu} | \mathcal{O}_{eff} | \bar{B} \rangle = \langle \mu\bar{\nu} | \mathcal{O}_{lep} | 0 \rangle \langle D | \mathcal{O}_{had} | \bar{B} \rangle$$

Decay amplitudes depend on:

- leptonic matrix elements: perturbative objects, **tiny uncertainties**
- **hadronic matrix elements**: non-perturbative QCD effects, **moderate uncertainties**

Parametrize hadronic matrix elements in terms of **FFs**

$$\langle D(k) | \mathcal{O}_{had} | \bar{B}(k+q) \rangle = \sum_{\lambda} \mathcal{S}_{\lambda}(k, q) \mathcal{F}_{\lambda}(q^2)$$

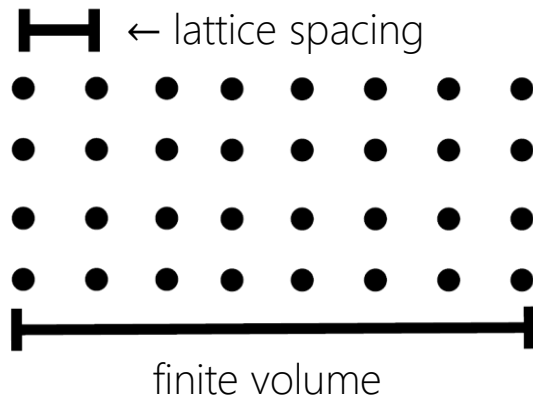
FFs are functions of  $q^2$  (i.e. the momentum transfer squared)

# Methods to compute FFs

Non-perturbative techniques are needed to compute FFs

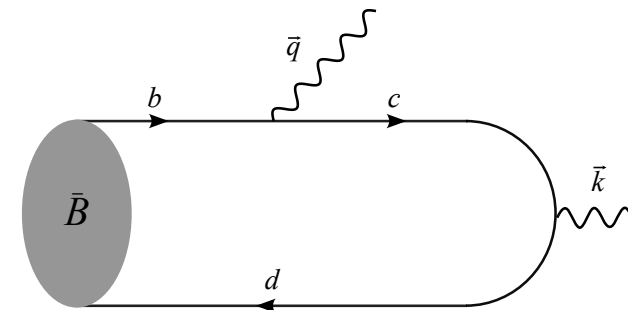
## 1. Lattice QCD (LQCD)

more efficient usually at high  $q^2$



## 2. Light-cone sum rules (LCSRs)

only applicable at low  $q^2$



Complementary approaches to calculate FFs

# Form factors predictions

Need to know  $q^2$  dependence of the FFs

**Problem:** FFs are known (from LCSRs and LQCD) at discrete values of  $q^2$  points

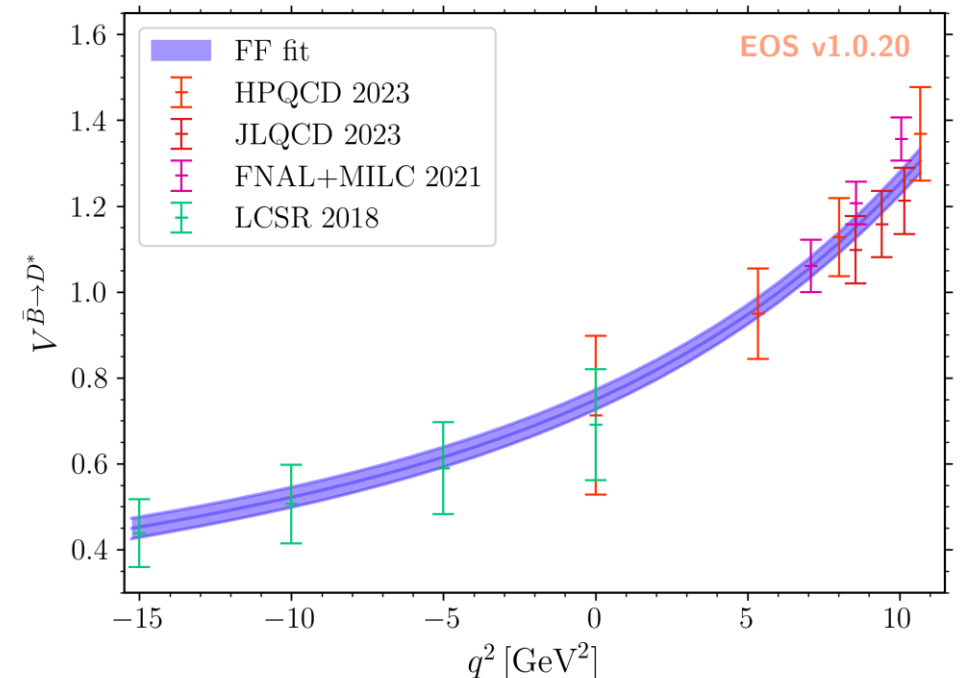
To combine theory inputs and obtain FFs for any  $q^2$  value, **FFs must be parametrized**

A simple  $q^2$  **power expansion is acceptable**

$$\mathcal{F}(q^2) = \sum_{n=0}^N \beta_n (q^2)^n$$

**However**, the series must be truncated  
as only a finite number of coefficients  $\beta_n$  can be fitted

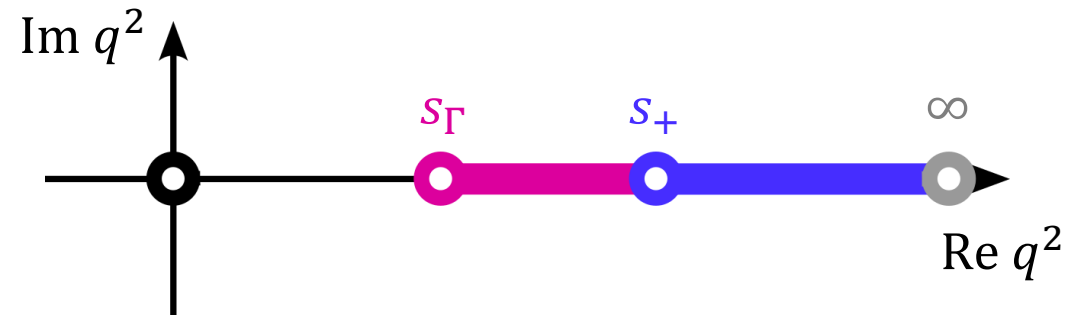
Use **unitarity bounds** to estimate truncation error



FFs parametrization

# Analytic properties of FFs

Study FF analytic structure to find a suitable parametrization. Example  $B \rightarrow K$  FFs



FFs are analytic except for branch cuts (i.e. lines of discontinuity) starting at

$s_+ = (m_B + m_K)^2$ , process threshold

$s_\Gamma = (m_{B_s} + m_\pi)^2 < s_+$ , subthreshold branch cut

Obtain a constrain on the FFs using unitarity (see [Okubo 1971])

$$\int_{s_+}^{\infty} dq^2 |\det J| |\phi(q^2)\mathcal{F}(q^2)|^2 < \chi$$

calculate  $\chi$  perturbatively,  $\phi$  known function

# Traditional approach: BGL

[Boyd/Grinstein/Lebed 1997]

Perform the conformal mapping

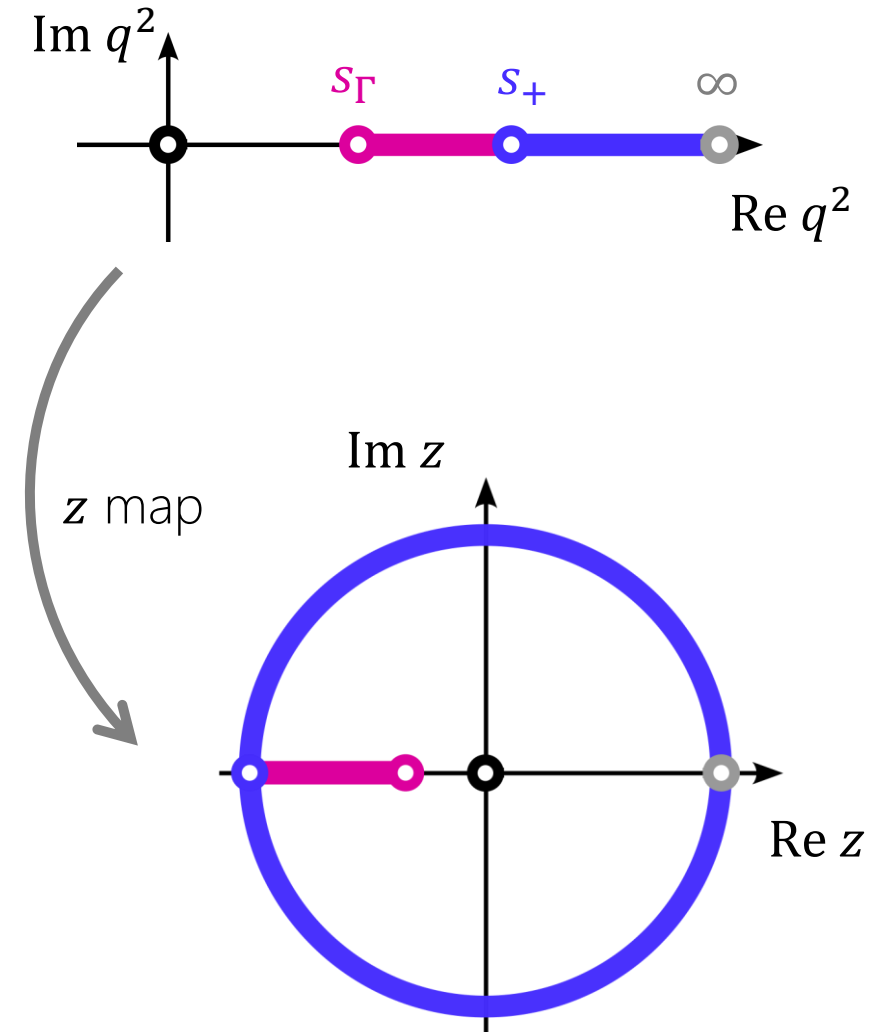
$$z(q^2) = \frac{\sqrt{s_+ - q^2} - \sqrt{s_+}}{\sqrt{s_+ - q^2} + \sqrt{s_+}}$$

expand FFs for  $|z| < 1$  as

$$\mathcal{F}(q^2) = \frac{1}{\phi(q^2)} \sum_{n=0}^{\infty} \beta_n z(q^2)^n$$

obtain a bound on the coefficients

$$\int_{s_+}^{\infty} dq^2 |\det J| |\phi(q^2) \mathcal{F}(q^2)|^2 < \chi \quad \Rightarrow \quad \sum_{n=0}^{\infty} \beta_n^2 < \chi$$



# BGL truncation error

Truncate BGL at order  $N$

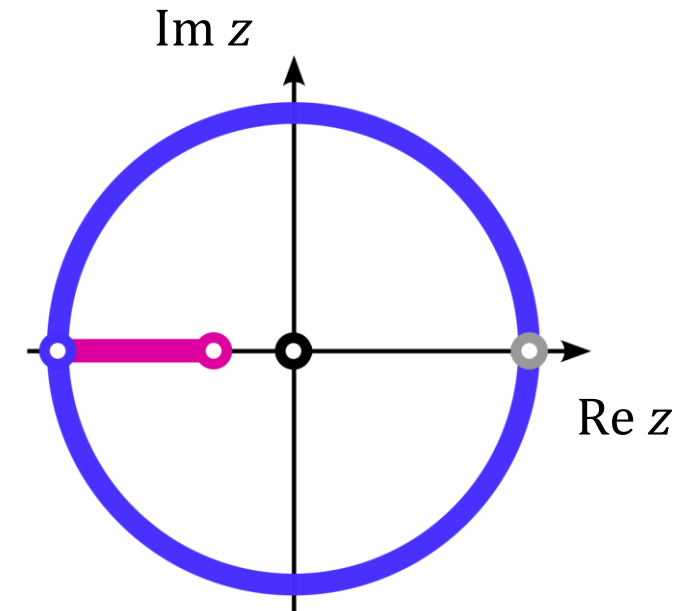
$$\mathcal{F}(q^2) = \frac{1}{\phi(q^2)} \sum_{n=0}^N \beta_n z (q^2)^n$$

The truncation error is given by

$$\left| \sum_{n>N}^{\infty} \beta_n z^n \right| \leq \left( \sum_{n>N}^{\infty} \beta_n^2 \right)^{\frac{1}{2}} \left( \sum_{n>N}^{\infty} |z|^{2n} \right)^{\frac{1}{2}} \leq \frac{|z|^{N+1}}{\sqrt{1 - |z|^2}}$$

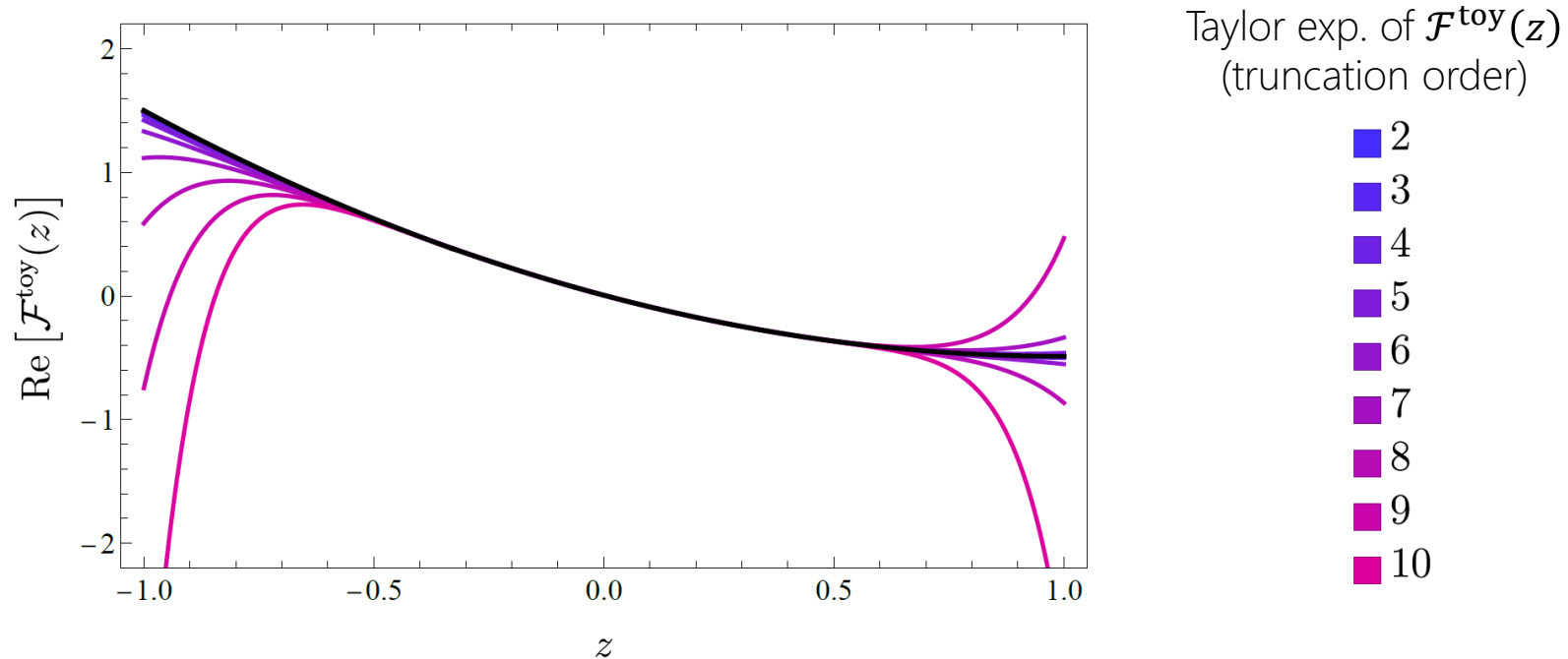
which tends to zero as  $N \rightarrow \infty$  for any  $|z| < 1$  since  $\sum_{n>N} |z|^{2n} \rightarrow 0$

**Problem!** series is divergent due to the **branch cut in  $s_{\Gamma}$**



# Impact of branch cuts in a Taylor expansion

Toy example:  $\mathcal{F}^{\text{toy}}(z) = -z + \frac{1}{2}z^2 + 0.01\sqrt{z+0.3} \Rightarrow$  compare with Taylor expansion



Even if the branch cut is suppressed it generates **divergent coefficients**. Hence:

$$\sum_{n=0}^{\infty} \beta_n^2 \not< \chi$$

# Problems with BGL

Having a branch cut invalidate the expansion for  $|z| < 1$

$$\mathcal{F}(q^2) \neq \frac{1}{\phi(q^2)} \sum_{n=0}^{\infty} \beta_n z(q^2)^n$$

same issue appears for all FFs except  $B \rightarrow \pi$ , e.g.  $B \rightarrow D^{(*)}$ ,  $B_c \rightarrow J/\psi$ ,  $\Lambda_b \rightarrow \Lambda$ , ...

It is crucial to address this issue to accurately estimate uncertainties in  $b$ -hadron decays

Issue discussed in the literature, but solutions are unsatisfactory (model dependence)

Find a way to recover the unitarity bound **to estimate truncation error**

[Boyd/Grinstein/Lebed 1995]

[Caprini/Neubert 1996]

[NG/van Dyk/Virto 2020]

...

$$\sum_{n=0}^{\infty} \alpha_n^2 < \chi$$

# Dispersive matrix interpolation

[Bourelly/Machet/de Rafael 1981]

Based on the **same principles as BGL**, i.e. unitarity and analyticity

Not a parametrization in the strict sense but an **interpolation method** (like Lagrange polynomials)

Assume that  $\mathcal{F}$  is known at points  $q_1^2, \dots, q_N^2 \Rightarrow$  build a matrix

$$\det \begin{pmatrix} \chi & \phi(q^2)\mathcal{F}(q^2) & \phi(q_1^2)\mathcal{F}(q_1^2) & \dots & \phi(q_N^2)\mathcal{F}(q_N^2) \\ \phi(q^2)\mathcal{F}(q^2) & \frac{1}{1-z(q^2)^2} & \frac{1}{1-z(q^2)z(q_1^2)} & \dots & \frac{1}{1-z(q^2)z(q_N^2)} \\ \phi(q_1^2)\mathcal{F}(q_1^2) & \frac{1}{1-z(q^2)z(q_1^2)} & \frac{1}{1-z(q_1^2)^2} & \dots & \frac{1}{1-z(q_1^2)z(q_N^2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi(q_N^2)\mathcal{F}(q_N^2) & \frac{1}{1-z(q^2)z(q_N^2)} & \frac{1}{1-z(q_1^2)z(q_N^2)} & \dots & \frac{1}{1-z(q_N^2)^2} \end{pmatrix} \geq 0$$

By imposing the positivity of this determinant one finds

$$\mathcal{F}_{\text{lo}}^{\text{DM}}(q^2) \leq \mathcal{F}(q^2) \leq \mathcal{F}_{\text{up}}^{\text{DM}}(q^2)$$

where  $\mathcal{F}_{\text{lo}\backslash\text{up}}^{\text{DM}}$  are functions of  $q^2$  and  $\mathcal{F}(q_1^2), \dots, \mathcal{F}(q_N^2)$

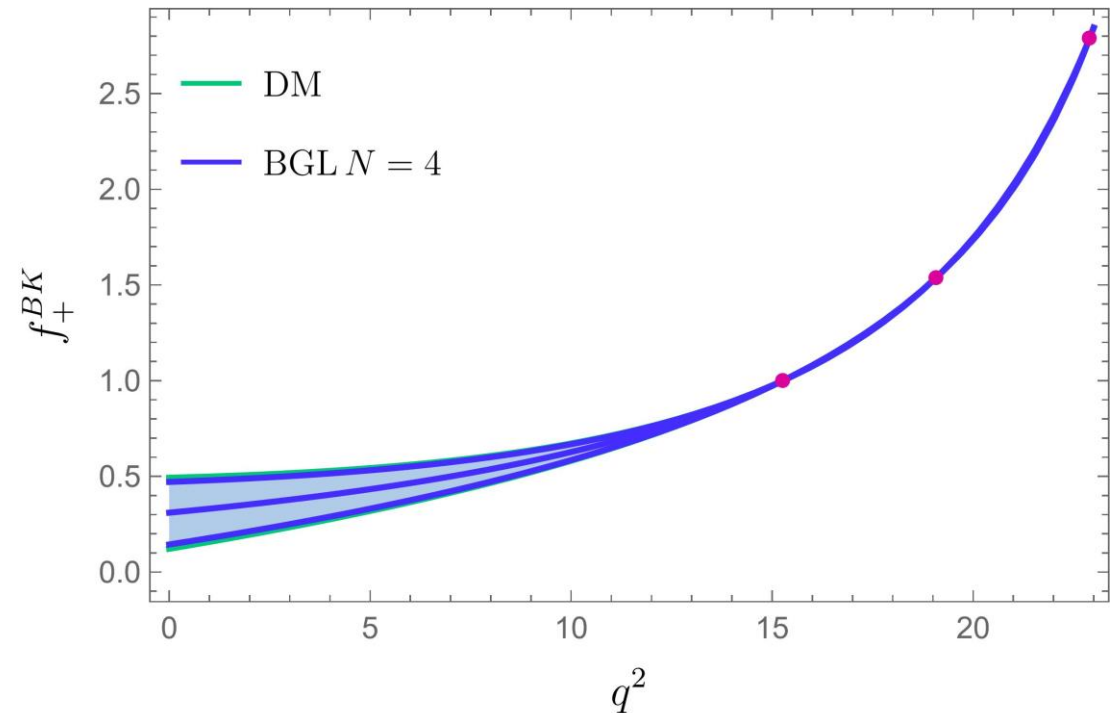
# Compare BGL and DM

**Example:** consider the central values of the FF  $f_+^{BK}$  from HPQCD 2022 at three high- $q^2$  points

1. Truncate BGL at  $N = 4$
2. Extract the first 3 coefficients analytically (solve a linear system of eqs.)
3. Determine the remaining  $N - 3 = 1$  coeffs. by maximizing/minimizing  $f_+^{BK}(\mathbf{0})$  and imposing  $\sum_{n=0}^N \alpha_n^2 < \chi$

Very small difference for  $N = 4$

Negligible difference for  $N > 4$



DM and BGL lead to equivalent results

Prefer the more convenient method  $\Rightarrow$  DM can be challenging for large data sets

# Our approach: GG

[Gopal/Gubernari 2024]

Just a reminder:  $s_+ = (m_B + m_K)^2$ ,  $s_\Gamma = (m_{B_s} + m_\pi)^2$

Modify the conformal mapping ( $s_+ \mapsto s_\Gamma$ )

$$\hat{z}(q^2) = \frac{\sqrt{s_\Gamma - q^2} - \sqrt{s_\Gamma}}{\sqrt{s_\Gamma - q^2} + \sqrt{s_\Gamma}}$$

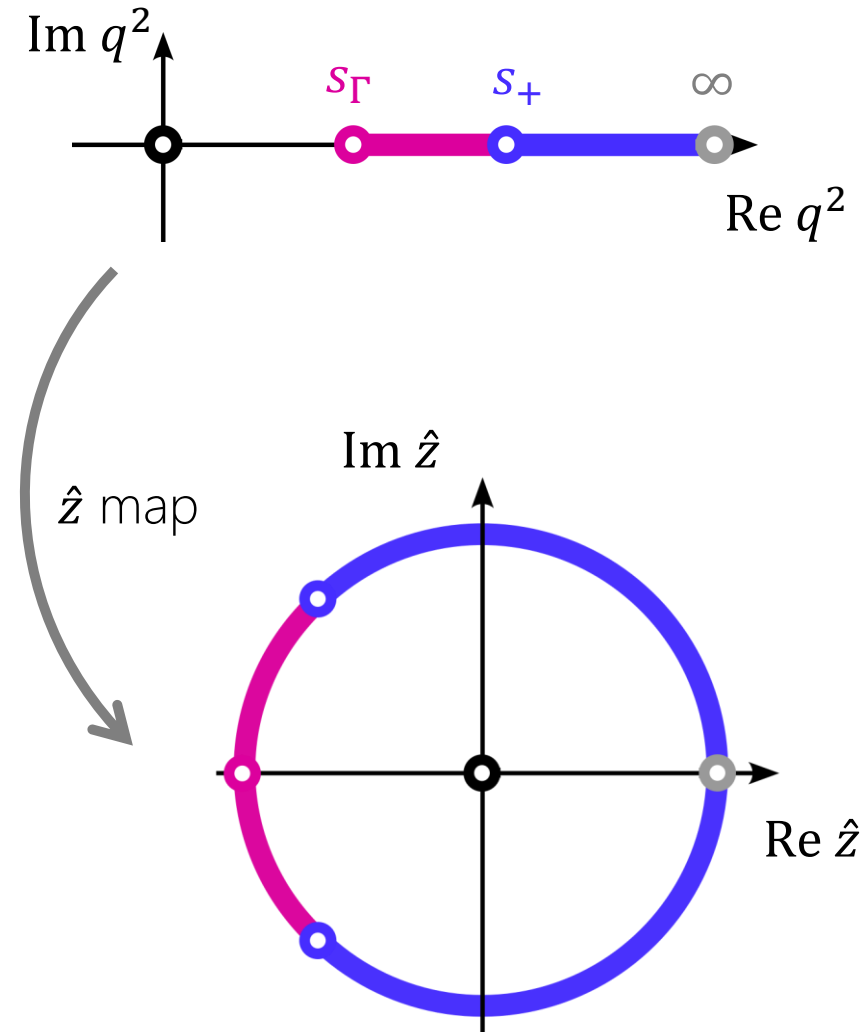
expand FFs for  $|\hat{z}| < 1$  (no singularities now!) as

$$\mathcal{F}(q^2) = \frac{1}{\phi(q^2)} \sum_{n=0}^{\infty} \alpha_n \hat{z}(q^2)^n$$

however

$$\int_{s_+}^{\infty} dq^2 |\det J| |\phi(q^2) \mathcal{F}(q^2)|^2 < \chi \quad \not\Rightarrow \quad \sum_{n=0}^{\infty} \alpha_n^2 < \chi$$

Integral must over the whole circle!



# Our derivation of the unitarity bound

Start from

$$\int_{s_+}^{\infty} dq^2 |\det J| |\phi(q^2)\mathcal{F}(q^2)|^2 < \chi$$

add on both sides

$$\Delta\chi \equiv \int_{s_\Gamma}^{s_+} dq^2 |\det J| |\phi(q^2)\mathcal{F}(q^2)|^2$$

Estimate  $\Delta\chi$  using large  $q^2$  scaling behaviour (for  $B \rightarrow K$  FFs  $\frac{\Delta\chi}{\chi} < 1\%$ )

Obtain the unitarity bound

$$\int_{s_\Gamma}^{\infty} dq^2 |\det J| |\phi(q^2)\mathcal{F}(q^2)|^2 < \chi + \Delta\chi \quad \Rightarrow \quad \sum_{n=0}^{\infty} \alpha_n^2 < \chi + \Delta\chi$$

New parametrization for FFs that allows to calculate the truncation error!

# $\Delta\chi$ estimate

Approximate FFs using an order-of-magnitude estimate

$$|\mathcal{F}(q^2)|^2 \simeq K \left( \frac{s_\Gamma}{q^2} \right)^2$$

According to [Becher/Hill 2005]  $K \sim 1$ , to be conservative I use  $K \sim 100$

In most cases (can be larger for  $B_c$  and  $\Lambda_b$  decays)

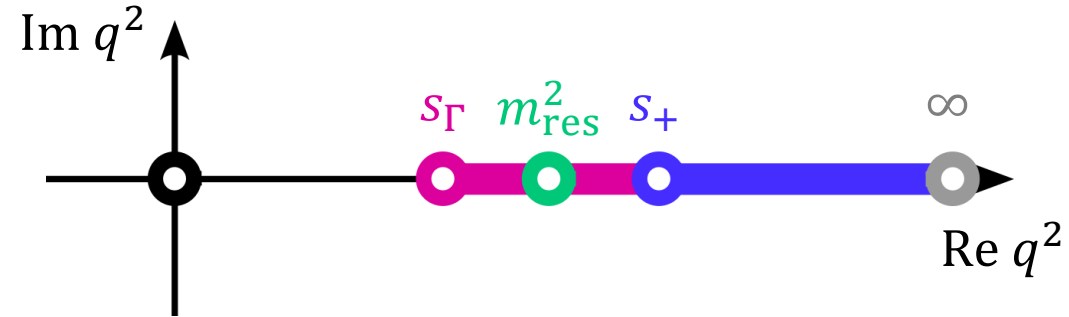
$$\frac{\Delta\chi}{\chi} \equiv \frac{1}{\chi} \int_{s_\Gamma}^{s_+} dq^2 |\det J| |\phi(q^2) \mathcal{F}(q^2)|^2 \ll 1$$

Only upper bound for  $\Delta\chi$  is needed, not a precise calculation!

Overestimating  $\Delta\chi$  weakens the bound  $\Rightarrow$  a more accurate estimate desirable

# Subthreshold resonances

Resonances can appear for  $s_\Gamma < q^2 < s_+$  making  $\Delta\chi$  very large (happens, e.g., for  $f_0^{BK}$  )



Modify the dispersion relation (take more subtractions that needed)

$$\Delta\chi \equiv \int_{s_\Gamma}^{s_+} dq^2 |\det J| \frac{(q^2 - m_{res}^2)^2}{q^2} |\phi(q^2)\mathcal{F}(q^2)|^2 := \int_{s_\Gamma}^{s_+} dq^2 |\det J| |\hat{\phi}(q^2)\mathcal{F}(q^2)|^2$$

The calculation of  $\chi$  must also be adjusted accordingly

$$\hat{\chi} := m_{res}^4 \chi(k+2) - m_{res}^2 \chi(k+1) + \chi(k)$$

where  $k$  is the minimal number of subtractions

# Choosing the right parametrization

## Fit without unitarity bounds

Parametrize FFs with a Simplified Series Expansion  
(see Eq. (3.6) of [NG 2026])

$$\mathcal{F}(q^2) = \frac{1}{1 - \frac{q^2}{m_{\text{res}}^2}} \sum_{n=0}^{\infty} \beta_n \hat{z}(q^2)^n$$

Recommended for presentation of theory calculations (lattice QCD)

$$\hat{z}(q^2, s_0) = \frac{\sqrt{s_{\Gamma} - q^2} - \sqrt{s_{\Gamma} - s_0}}{\sqrt{s_{\Gamma} - q^2} + \sqrt{s_{\Gamma} - s_0}}$$

## Fit with unitarity bounds

Parametrize FFs with GG,  
i.e. improved BGL  
(see Eq. (4.1) of [NG 2026])

$$\mathcal{F}(q^2) = \frac{1}{\mathcal{B}(q^2)\phi(q^2)} \sum_{n=0}^{\infty} \alpha_n \hat{z}(q^2)^n$$

with  $\sum_{n=0}^{\infty} \alpha_n^2 < \chi + \Delta\chi$

with optimal choice  $s_0 = s_{\Gamma} \left( 1 - \sqrt{1 - \frac{s_-}{s_{\Gamma}}} \right)$

Numerical results

# FF analyses

Perform three FF global analyses

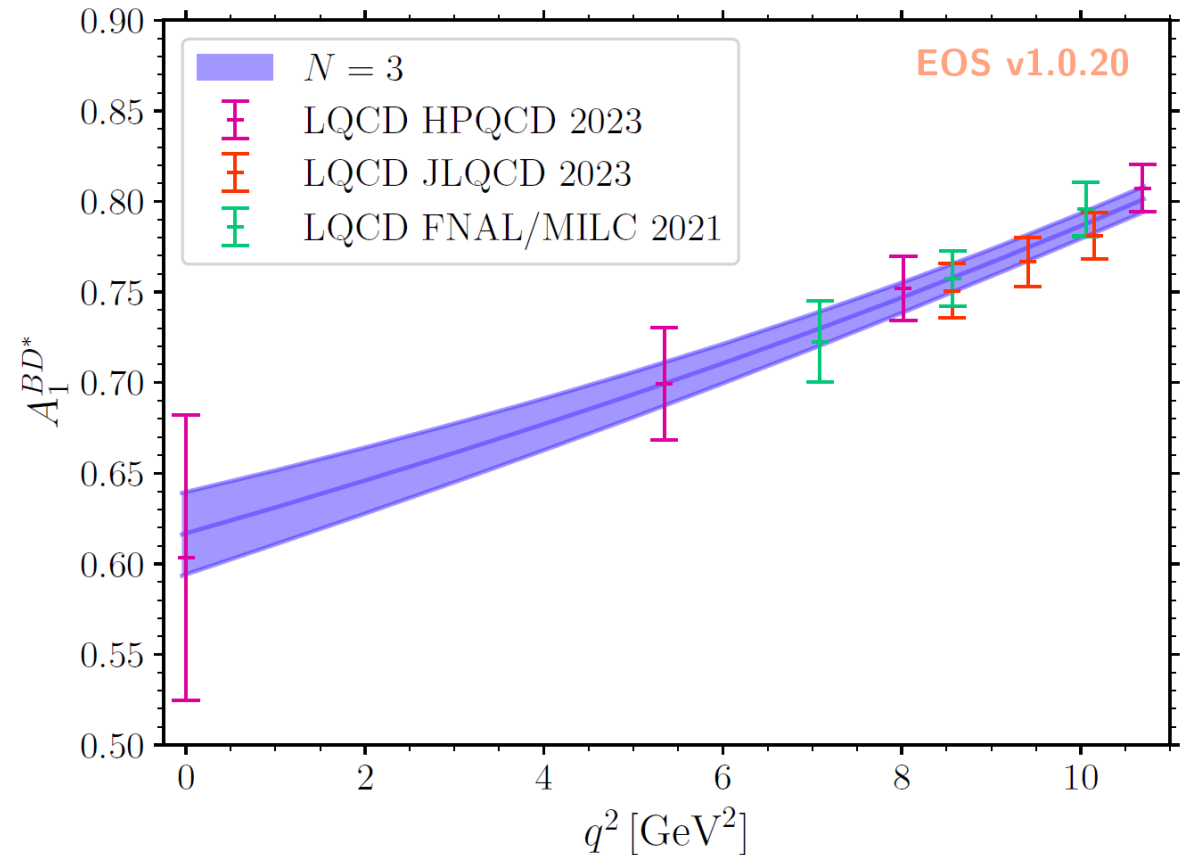
Transition	FFs	Theory calculations	Free fit parameters	$p$ -value
$b \rightarrow u, d$	$B \rightarrow \pi$ $B_S \rightarrow K$ $B_S \rightarrow K^*$	LQCD+LCSR	59	99.99%
$b \rightarrow s$	$B \rightarrow K$ $B \rightarrow K^*$ $B_S \rightarrow \phi$	LQCD+LCSR	76	99.99%
$b \rightarrow c$	$B \rightarrow D$ $B_S \rightarrow D_S$ $B \rightarrow D^*$ $B_S \rightarrow D_S^*$	LQCD	70	49.07%

First global analyses of  $b \rightarrow u, d$  and  $b \rightarrow c$  FFs

Theory calculations are mutually compatible

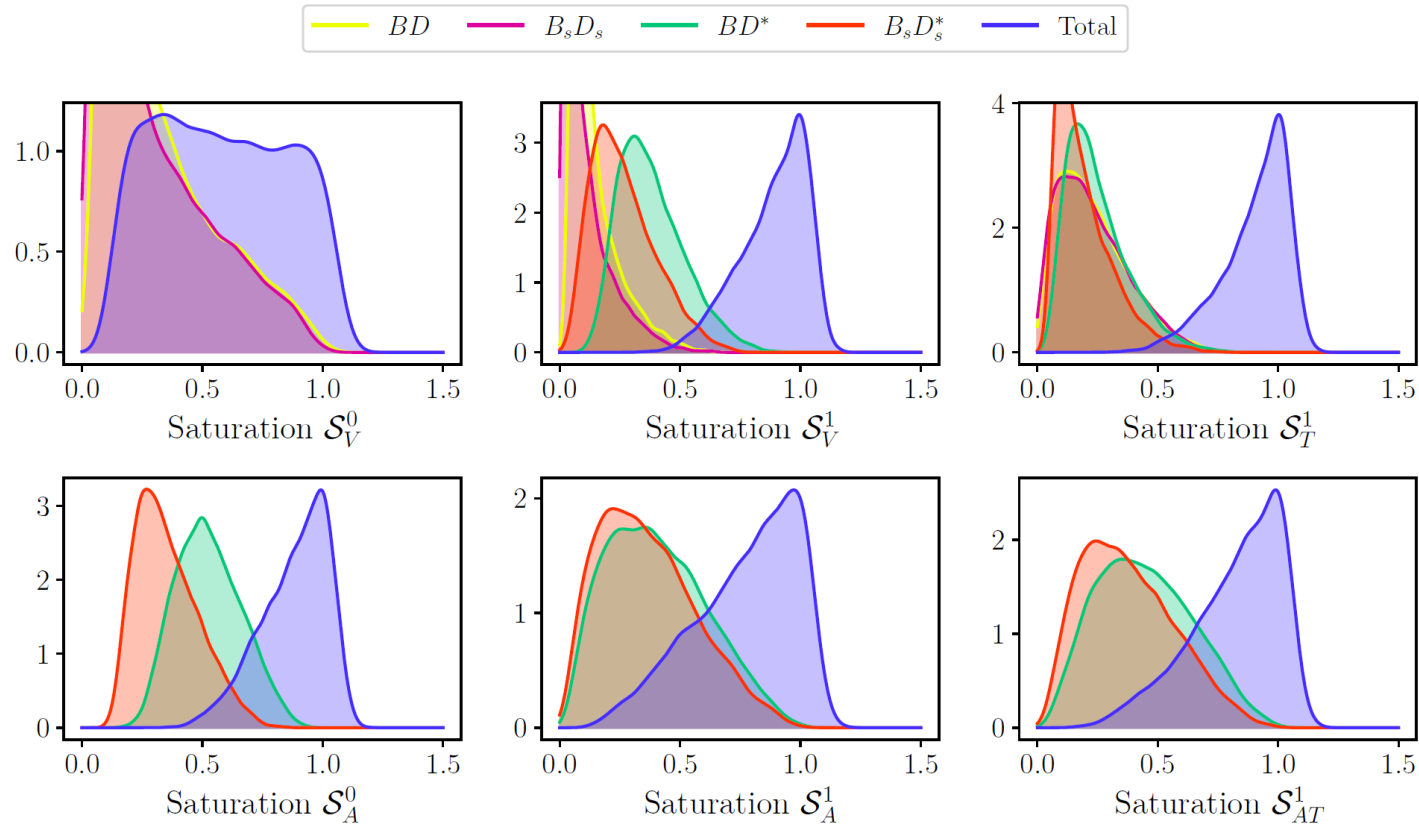
Implemented in EOS with name G2026

FF results in the whole semileptonic region,  
plots, analysis files, and Mathematica  
notebook provided as [supplementary material](#)



# Saturation of unitarity bounds

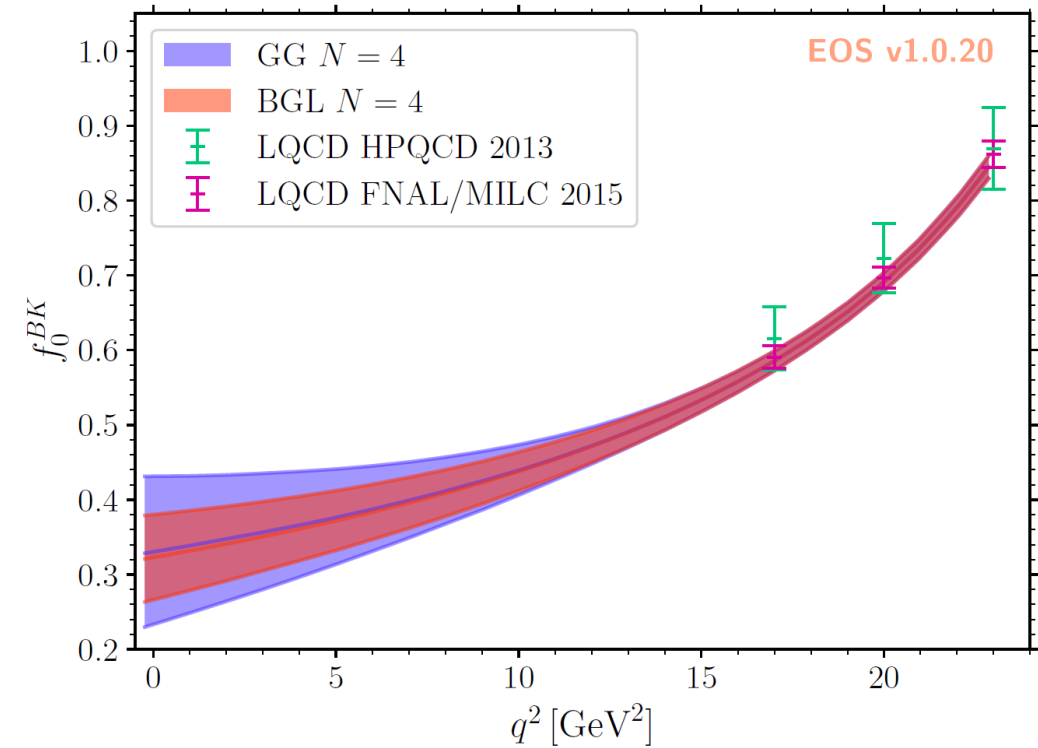
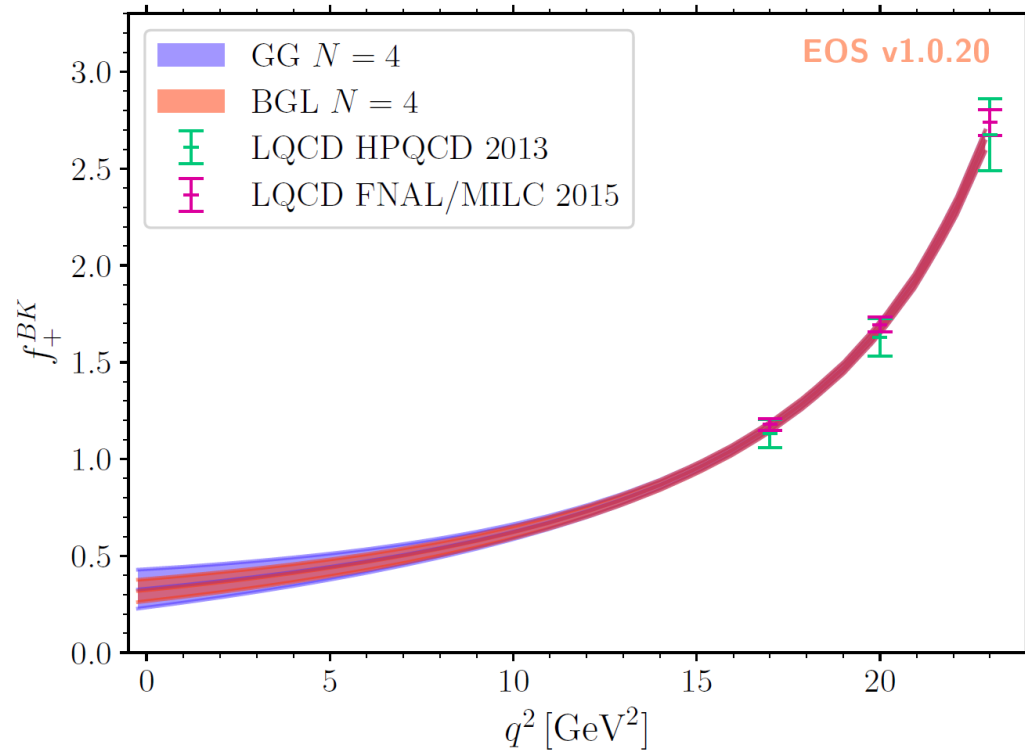
Global analyses have stronger constraining power



$$\mathcal{S}_V^1 = \sum_{n=0}^3 \left| \alpha_n^{f_+^{BD}} \right|^2 + \sum_{n=0}^3 \left| \alpha_n^{f_+^{B_s D_s}} \right|^2 + \sum_{n=0}^3 \left| \alpha_n^{V^{BD^*}} \right|^2 + \sum_{n=0}^3 \left| \alpha_n^{V^{B_s D_s^*}} \right|^2 \leq 1$$

# Compare with BGL

Fit FFs without HPQCD 2022 to make the extrapolation effects visible



BGL might underestimate the uncertainties

Summary and conclusions

# Summary and conclusions

Predictions of hadronic **form factors (FFs) are crucial** for indirect searches for new physics

Different FFs calculations are combined using a **parametrization**

The **truncation error** can be rigorously calculated using unitarity

However, the **traditional (BGL) parametrization neglect subthreshold branch cuts**, which can lead to underestimated uncertainties

I propose a **new easy to implement parametrization (GG)** to solve the issue

I perform **three FF global analyses** ( $b \rightarrow u$ ,  $b \rightarrow s$ ,  $b \rightarrow c$ )

Numerical results, analysis files, and plots provided as [supplementary material](#)

Thank you!

Backup slides

# Problems with cut modelling and polynomials

Model the branch cut and subtract it

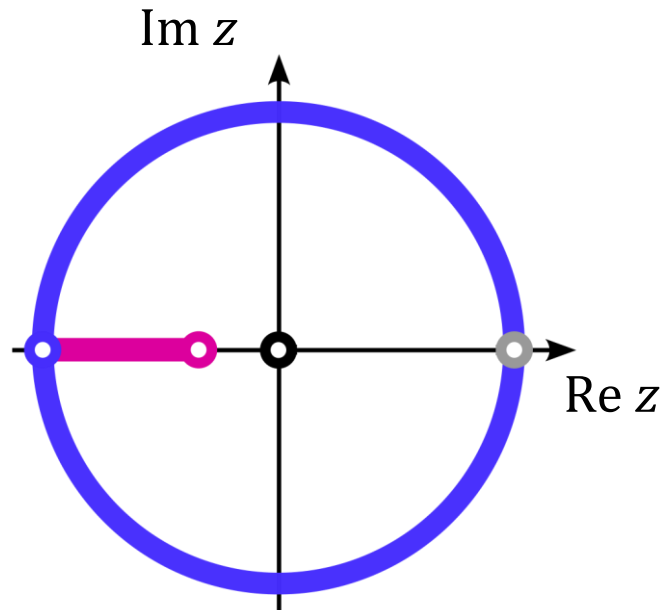
$$\tilde{\mathcal{F}}(z) \equiv \mathcal{F}(z) - \mathcal{F}_{\text{cut}}(z)$$

expand  $\tilde{\mathcal{F}}(z)$

[Boyd/Grinstein/Lebed 1995]  
[Caprini/Neubert 1996]

**Problem:**  $\mathcal{F}_{\text{cut}}(z)$  is not known

$\Rightarrow$  cannot rely on exact numerical  
cancellation of singularities

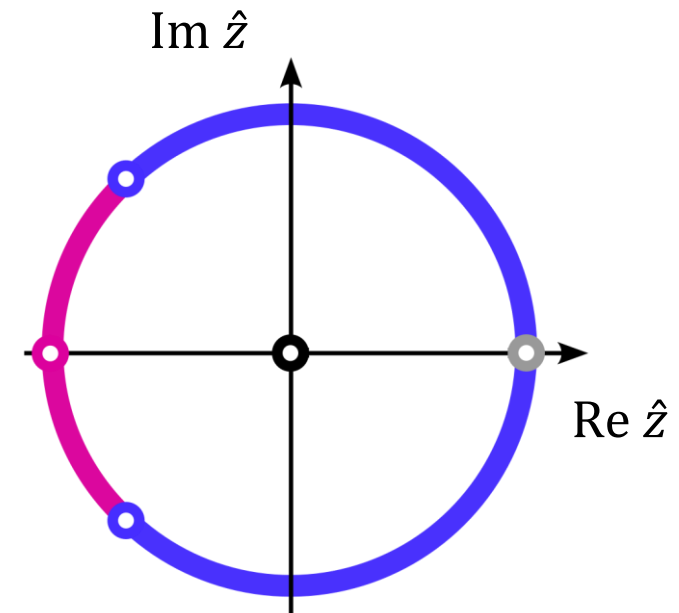


Expand in polynomials orthogonal  
on the blue arc

[NG/van Dyk/Virto 2020]  
[Flynn/Jüttner/Tsang 2023]

$$\mathcal{F}(\hat{z}) = \frac{1}{\phi(\hat{z})} \sum_{n=0}^{\infty} \beta_n p_n(\hat{z})$$

$|p_n(\hat{z})| \rightarrow \infty$  for  $n \rightarrow \infty$  and  
some  $\hat{z}$  in the unit disk



# Polynomial parametrization

polynomial parametrization ( $\hat{z}$  polynomials) [NG/van Dyk/Virto 2020]

$$\mathcal{H}_\lambda(\hat{z}) = \frac{1}{\mathcal{P}(z)\phi(z)} \sum_{n=0}^{\infty} \beta_n p_n(\hat{z}) \quad \sum_{n=0}^{\infty} |\beta_n|^2 < 1$$

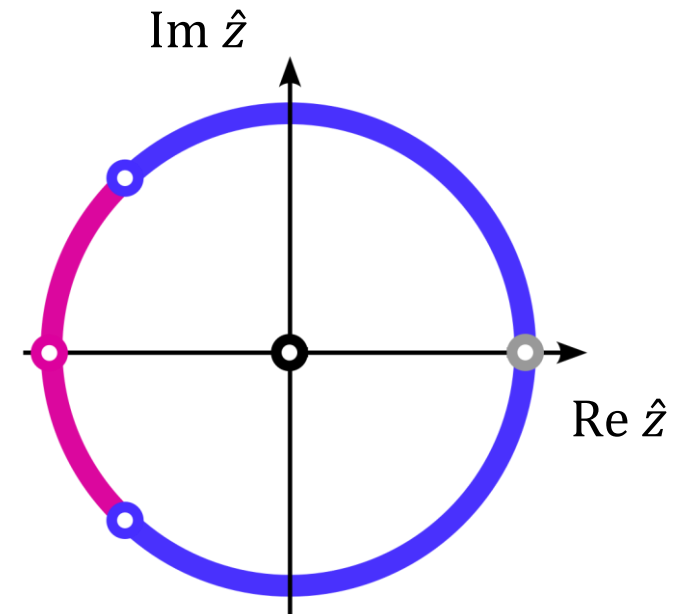
$|p_n(\hat{z})| \rightarrow \infty$  for  $n \rightarrow \infty$  some  $z$  in the unit disk

$$p_0^{B \rightarrow K}(\hat{z}) = \frac{1}{\sqrt{2\alpha_{BK}}}$$

$$p_1^{B \rightarrow K}(\hat{z}) = \left( \hat{z} - \frac{\sin(\alpha_{BK})}{\alpha_{BK}} \right) \sqrt{\frac{\alpha_{BK}}{2\alpha_{BK}^2 + \cos(2\alpha_{BK}) - 1}}$$

$$p_2^{B \rightarrow K}(\hat{z}) = \left( \hat{z}^2 + \frac{\sin(\alpha_{BK})(\sin(2\alpha_{BK}) - 2\alpha_{BK})}{2\alpha_{BK}^2 + \cos(2\alpha_{BK}) - 1} \hat{z} + \frac{2 \sin(\alpha_{BK})(\sin(\alpha_{BK}) - 2\alpha_{BK})}{2\alpha_{BK}^2 + \cos(2\alpha_{BK}) - 1} \right)$$

$$p_3^{B \rightarrow K}(\hat{z}) = \dots$$



# Anomalous branch cuts

Rare decays (e.g.  $B \rightarrow K\mu^+\mu^-$ ) also depend on non-local FFs

$$\langle K\mu^+\mu^- | \mathcal{O}_{eff} | B \rangle = \langle \mu^+\mu^- | \mathcal{O}_{lep} | 0 \rangle \langle K | \mathcal{O}_{had} | B \rangle + \text{non local}$$

Rescattering, e.g.  $B \rightarrow DD_s^* \rightarrow K\mu^+\mu^-$ , generates anomalous branch cuts in the complex plane

Apply the same procedure as for the subthreshold branch cuts, but:

- $\hat{z}$  map is very hard to obtain
- $\Delta\chi$  calculation extremely challenging

[Ciuchini et al. 2022]  
[Mutke et al. 2024]

