Lattice QCD calculation of distribution amplitudes for baryons

M. Chu et al., Phys.Rev.D 111 (2025) 3, 034510
H. Bai et al., Phys.Rev.D 112 (2025) 11, 114515
M. Zhang et al., arxiv: 2606.???? (In preparation)

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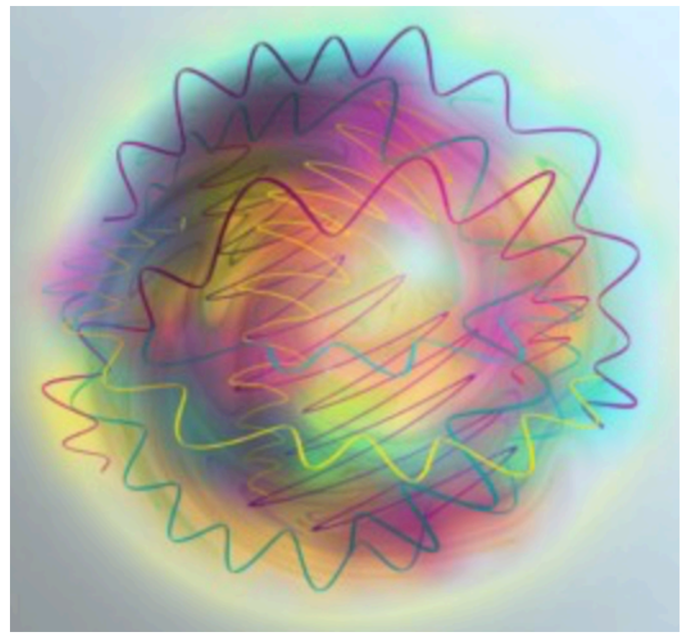
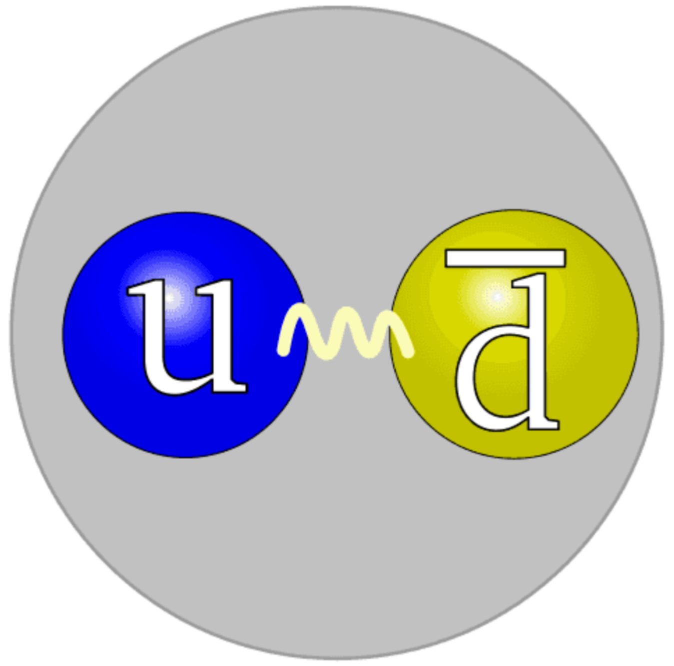
26/5/2026

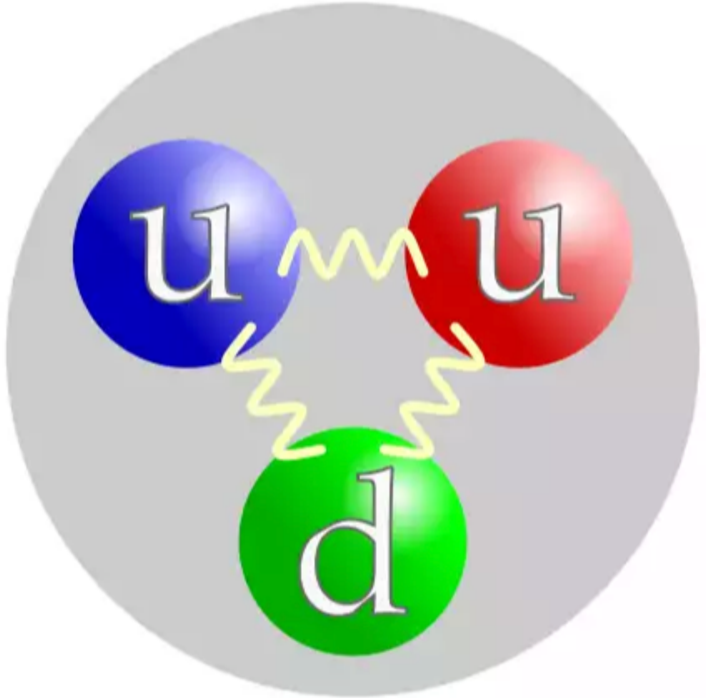


Lattice Parton Collaboration

Outline

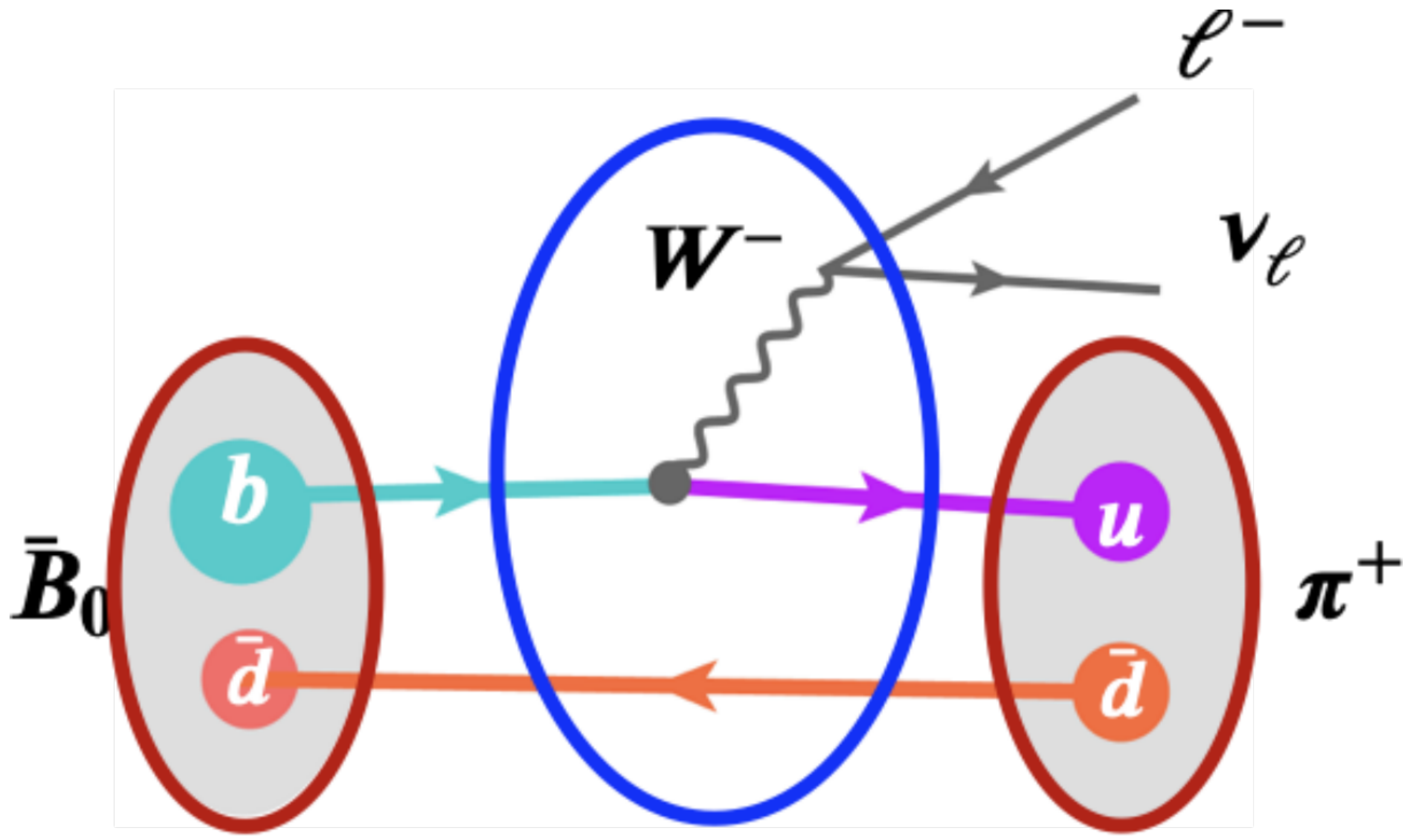
- Distribution Amplitudes
- Lattice QCD and LaMET
- Framework 1.0 and results
- Framework 2.0 and results
- Summary





Distribution Amplitudes

DAs are important inputs in hard exclusive processes. Such as $\bar{B}^0 \to \pi^+ + l^- + \nu_l$



decay width

$$iM = \langle \pi^+ l^- \nu_l | \bar{B}^0 \rangle \sim \int [dk] \mathrm{Tr}\left[L(t) H(k_1, k_2, k_3) \boxed{\Phi_{\bar{B}^0}(x_1) \Phi_{\pi^+}(x_2)} \right]$$

At leading-order, DAs represent the coefficients of hadron state expanded with Fock states.

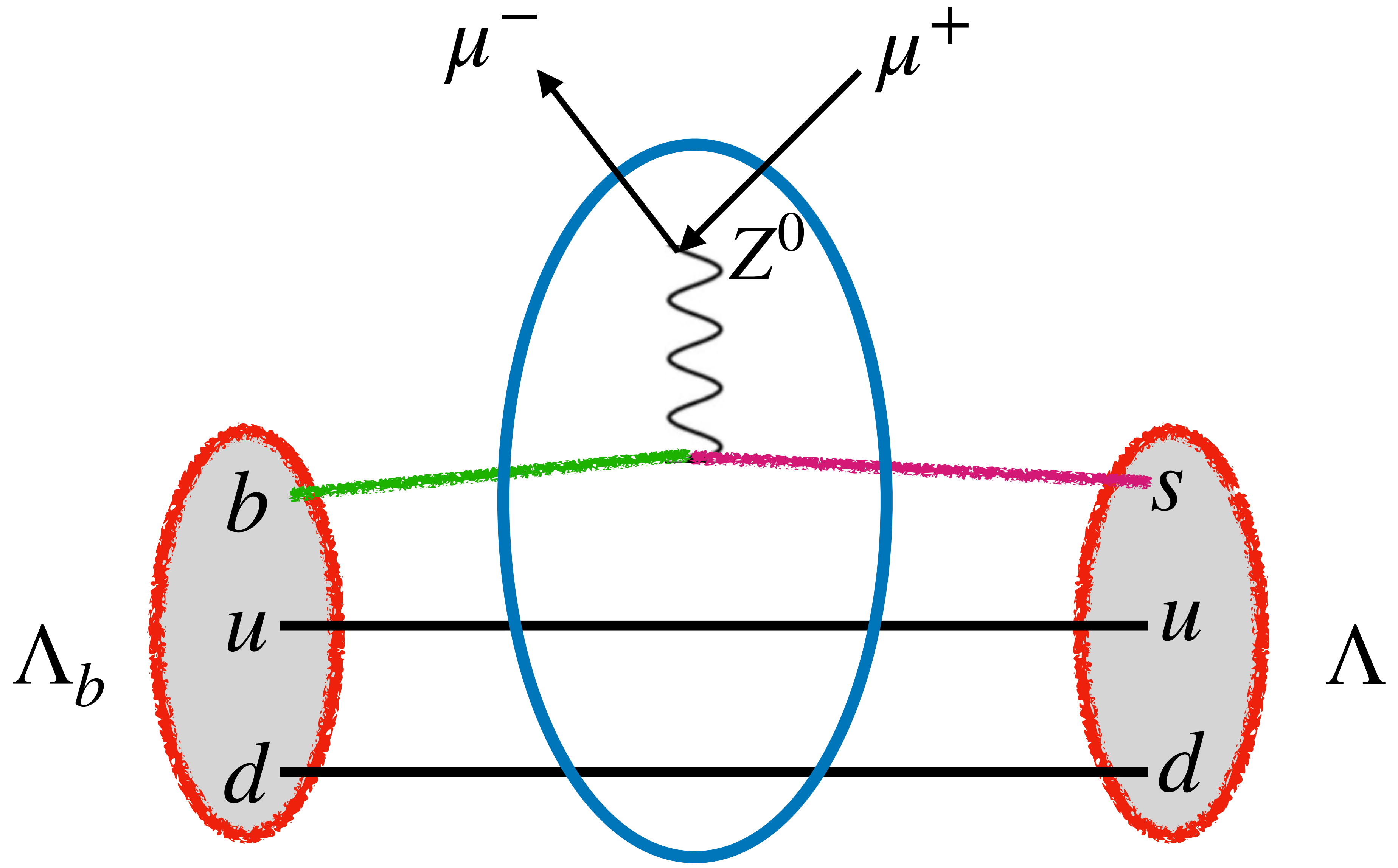
$$|h\rangle = \sum_{n,\lambda_i} \int [dx][dk_\perp] \psi_n(x_i, k_{\perp i}, \lambda_i) \prod_{fermions} \frac{u(x_i, k_{\perp i}, \lambda_i)}{\sqrt{x_i}} \prod_{gluons} \frac{\varepsilon(x_i, k_{\perp i}, \lambda_i)}{\sqrt{x_i}} |n\rangle$$

Among these, the simplest one is the pion DA:

$$\int \frac{d\xi^-}{2\pi} e^{ixp^+\xi^-} \langle 0 | \bar{\psi}_1(0) n \cdot \gamma\gamma_5 U(0, \xi^-) \psi_2(\xi^-) | \pi(p) \rangle = i f_\pi \Phi_\pi(x)$$

Distribution Amplitudes

weak decay: $\Lambda_b \to \Lambda + \mu^+ + \mu^-$



$$iM = \langle \Lambda \mu^+ \mu^- | \Lambda_b \rangle \propto \Phi_{\Lambda_b}(x_1) \boxed{\Phi_\Lambda}$$

large momentum transfer form factor

$$F(Q^2) = \int_0^1 dx \boxed{T_H(x, Q^2)} \boxed{\phi(x, \mu)}$$

hard scattering coefficient

(perturbatively calculated)

distribution amplitude

DAs are important ingredients in hadron physics !

Distribution Amplitudes

light meson ⟶ baryon

lattice calculation for meson LCDAs

few lowest moments

G.S. Bali et al., Phys.Lett.B 774 (2017) 91-97

G.S. Bali et al., JHEP 08 (2019) 065

complete x dependence

J. Zhang et al. Phys.Rev.D 95 (2017) 9, 094514

J. Zhang et al. Nucl.Phys.B 939 (2019) 429-446

J. Hua et al., Phys.Rev.Lett. 127 (2021) 6, 062002

J. Hua et al., Phys.Rev.Lett. 129 (2022) 13, 132001

challenges and necessity of lattice calculation for baryon

1. two directions for Wilson line;
2. hadron operator is complicated
3. no simple models for three valence quark system.
4. big difference between phenomenological methods.

explorations of form factor

W. Detmold et al., Phys.Rev.D 92 (2015) 3, 034503

W. Detmold et al., Phys.Rev.D 93 (2016) 7, 074501

R. Horgan et al., Phys.Rev.D 89 (2014) 9, 094501

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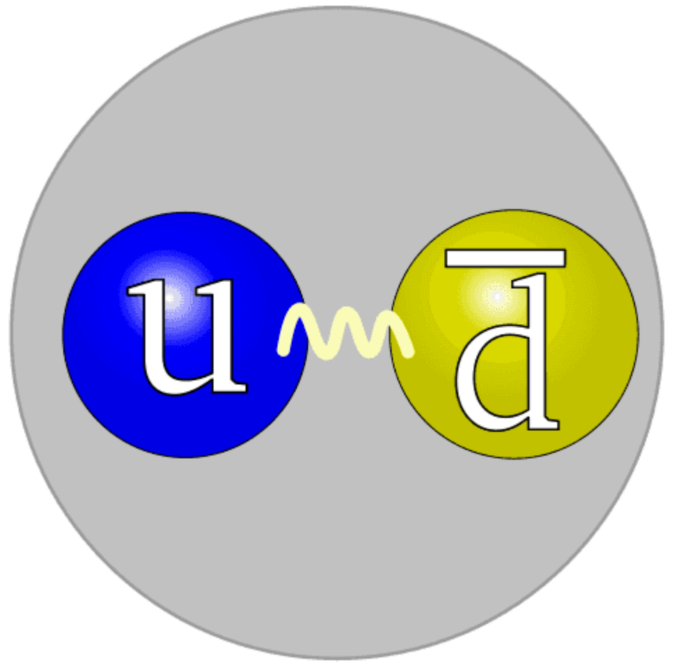
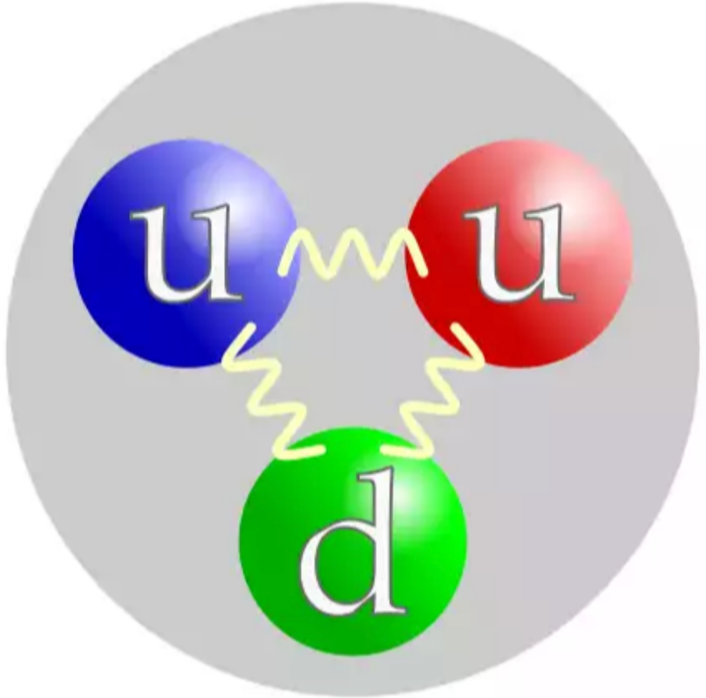




Lattice QCD and LaMET

Discretized 4D Euclidean space-time

Lattice QCD action:

$$S_E^{\mathrm{latt}} = -\underbrace{\sum_{\square}\frac{6}{g^2}\mathrm{Re\,tr}_N\left(U_{\square,\mu\nu}\right)}_{\text{gauge action}} - \underbrace{\sum_q \bar{q}\left(D_\mu^{\mathrm{lat}}\gamma_\mu + am_q\right)q}_{\text{fermion action}}$$

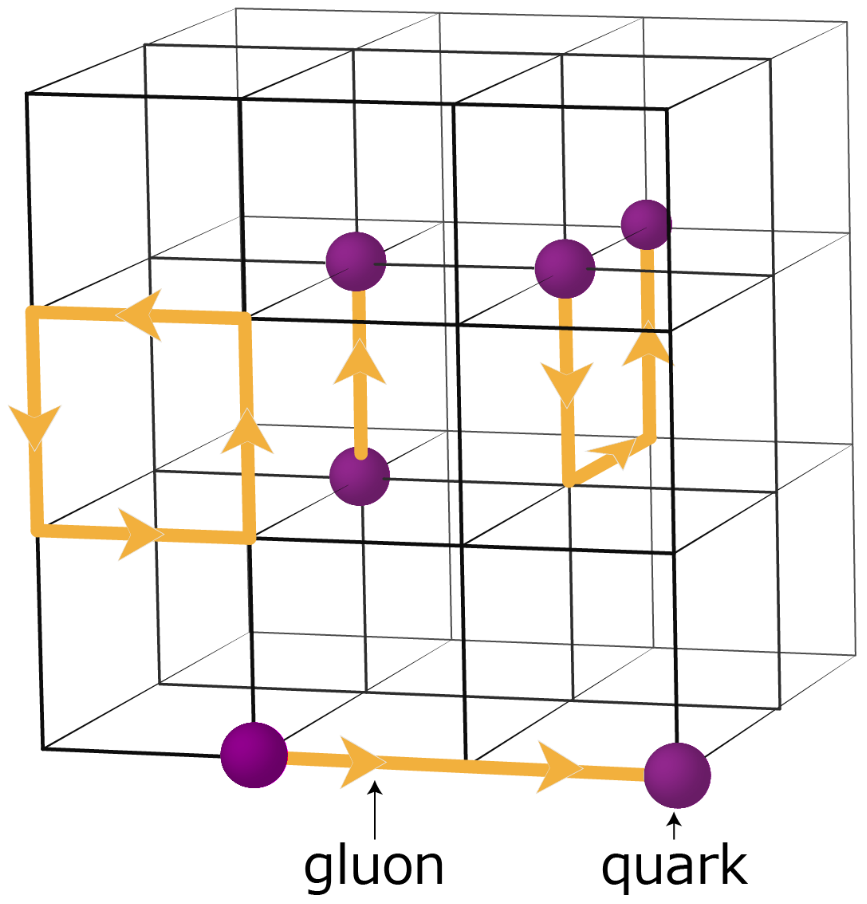
Quantum observables in path integral:

$$\langle 0|\hat{O}|0\rangle = \frac{\int[d\bar\psi][d\psi][dA]\hat{O}e^{-S(\bar\psi,\psi,A)}}{\int[d\bar\psi][d\psi][dA]e^{-S(\bar\psi,\psi,A)}}$$

Similar to partiton functions in thermal physics

↓

The fields distributed with the probability density $e^{-S[\psi,\bar\psi,A]}$



From continuum to lattice

Continuum	Lattice
real time t	Euclidean time $\tau_E = -it$
Gauge field $A_\mu(x)$	Gauge link $U_\mu(n)$
Quark field Ψ	det of Dirac matrix

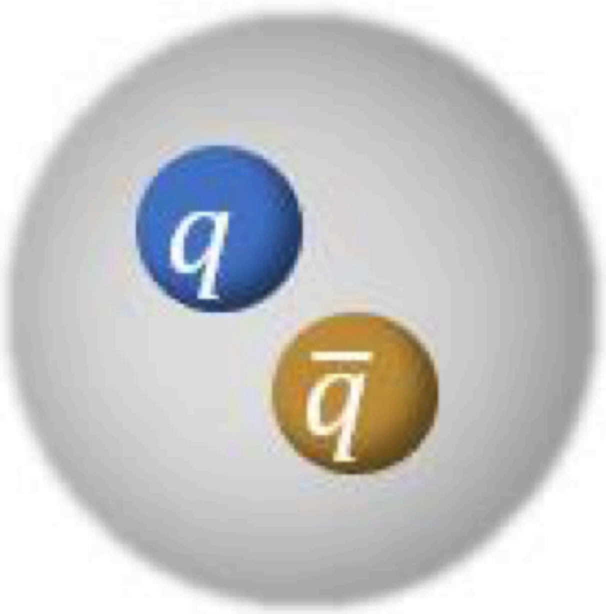
Lattice QCD and LaMET

Observables: hadron spectrum

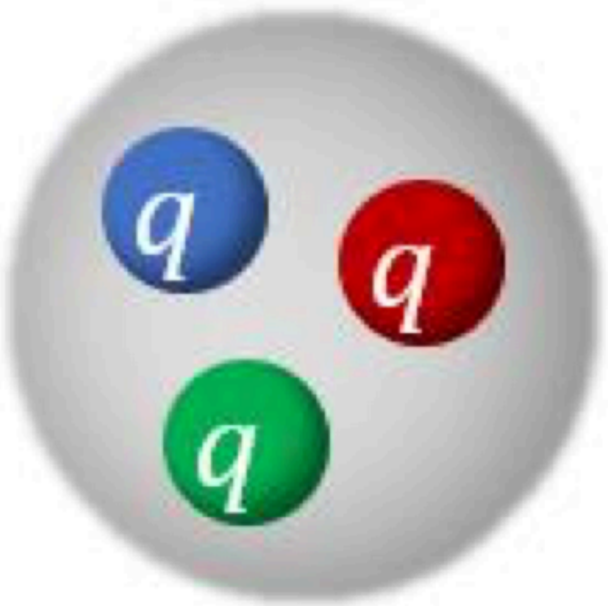
hadron operators

mesons: $\hat{O}_M = \bar{q}\Gamma q$

baryons: $\hat{O}_B = \epsilon_{abc} P_{\pm} q_{1a} (q_{2b}^T \Gamma^B q_{3c})$



meson



baryon

Observables: hadron spectrum

two point correlation functions

$$\langle \hat{O}(t)\hat{\bar{O}}(0)\rangle = \langle 0|e^{-iHt}e^{iHt}\hat{O}(t)e^{-iHt}e^{iHt}|H\rangle\langle H|\hat{\bar{O}}(0)|0\rangle$$
$$= e^{iE_Ht}\langle 0|\hat{O}(0)|H\rangle\langle H|\hat{\bar{O}}(0)|0\rangle$$

from continuum to lattice: $e^{iE_Ht} \to e^{-E_Ht}$

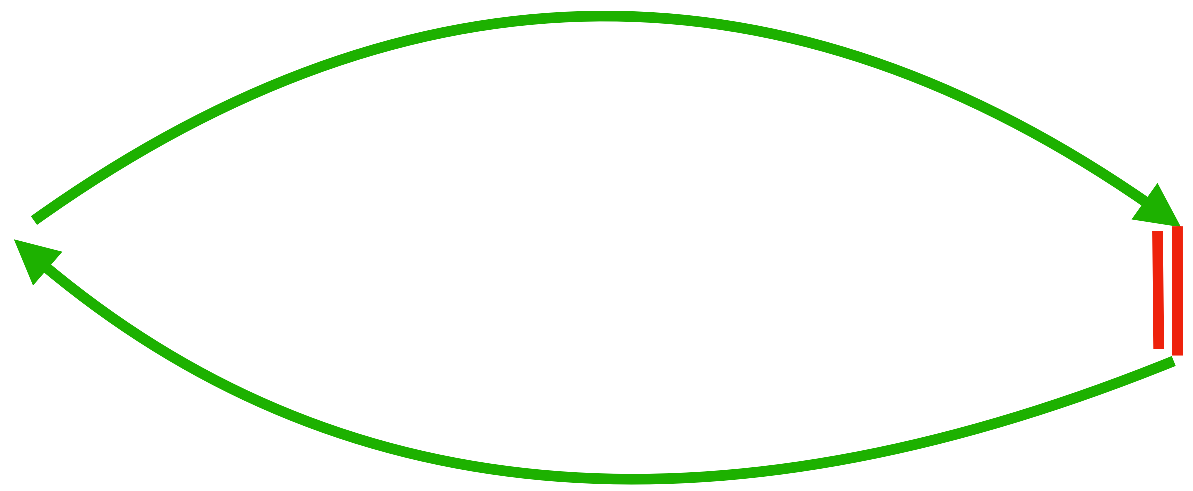
lattice calculation (take pion as e.g.)

$$\langle \hat{O}_\pi(n)\hat{\bar{O}}_\pi(m)\rangle = \langle \bar{d}(n)\gamma_5 u(n)\bar{u}(m)\gamma_5 d(m)\rangle$$
$$= -\,\mathrm{Tr}[\underline{\gamma_5 D_u^{-1}(n|m)\gamma_5 D_d^{-1}(m|n)]}$$

quark propagators

Lattice QCD and LaMET

Observables: hadron structure



reduction formula: from nonlocal 2pt to DA

$$C_2(z,t) = \int d^3x e^{-i\vec{p}\vec{x}} \langle 0 | \hat{O}_H(\vec{x},t;z) \hat{O}_H^\dagger(0,0) | 0 \rangle$$

insert the hadron state

$$= \int d^3x e^{-i\vec{p}\vec{x}} \langle 0 | \hat{O}_H(\vec{x},t;z) \sum_H \int \frac{d^3q}{(2\pi)^3 2E} | \pi(q) \rangle \langle \pi(q) | \hat{O}_H^\dagger(0,0) | 0 \rangle$$

Spatial Translation

$$= \int \frac{d^3q_0}{(2\pi)^3 2E_0} \int d^3x e^{-i(\vec{p}-\vec{q}_0)\vec{x}} \langle 0 | \hat{O}_H(0,t;z) | \pi(q_0) \rangle \langle \pi(q_0) | \hat{O}_H^\dagger(0,0) | 0 \rangle$$

integration

$$= \langle 0 | \hat{O}_H(0,t;z) | \pi(p) \rangle \langle \pi(p) | \hat{O}_H^\dagger(0,0) | 0 \rangle$$

$$= e^{-E_0 t} \phi(z,P^z) \langle \pi(q_0) | \hat{O}_H^\dagger(0,0) | 0 \rangle$$

Pion distribution amplitude: $\phi(z,P^z) = \langle 0 | \bar{\psi}(z) \gamma^\mu \gamma_5 W(z,0) \psi(0) | \pi(P^z) \rangle$

Excited states: $\dfrac{C_2(z,t)}{C_2(0,t)} = \phi(z,P^z) \dfrac{1 + c_0 e^{-\Delta E \tau}}{1 + c_1 e^{-\Delta E \tau}}$

Lattice QCD and LaMET

main idea

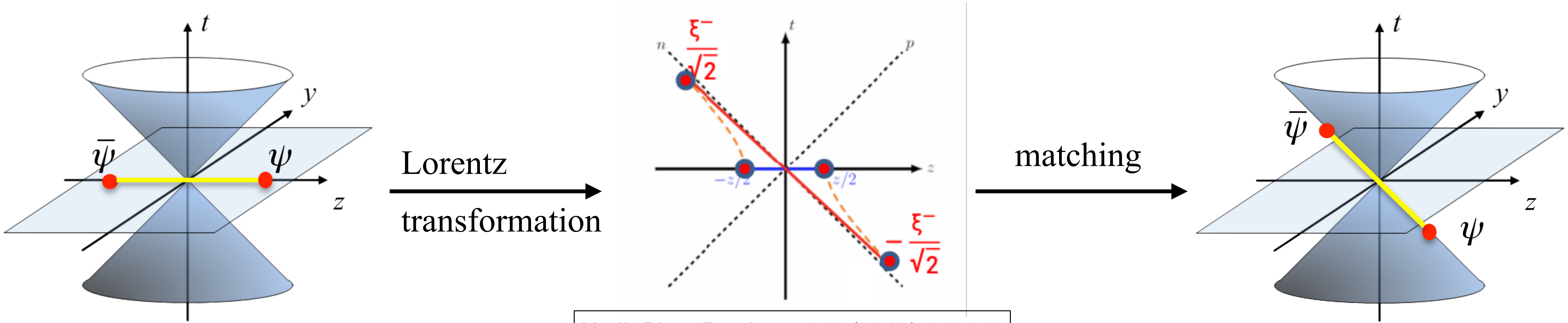
Equal time correlation

$$\tilde{\phi}(z) \sim \langle 0 | \bar{\psi}(\frac{z}{2})\Gamma U(\frac{z}{2}, -\frac{z}{2})\psi(-\frac{z}{2}) | P^z \rangle$$

main idea

Light-cone correlation:

$$\phi(x) \sim \langle 0 | \bar{\psi}(\frac{\xi^+}{\sqrt{2}})\Gamma U(\frac{\xi^+}{\sqrt{2}}, -\frac{\xi^+}{\sqrt{2}})\psi(-\frac{\xi^+}{\sqrt{2}}) | P^+ \rangle$$



Lorentz transformation

matching

X. Ji, Phys.Rev.Lett. 110 (2013) 262002

Due to the IR structure are only based on states, then the difference between $\phi(x)$ and $\tilde{\phi}(x)$ is only UV structure, which can be perturbatively determined.

$$\tilde{\phi}(y) = \int_0^1 dx\, C(x,y)\, \phi(x) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{(yP^z)^2}, \frac{\Lambda_{QCD}^2}{((1-y)P^z)^2}\right)$$

Lattice QCD and LaMET

renormalization

renormalization of non-local operators

$$\underline{\hat{O}^B_\Gamma(z,\Lambda)} = Z_\Gamma(z,\Lambda,\mu)\underline{\hat{O}^R_\Gamma(z)}$$

renormalized operator bare operator

Wilson line renormalization

J. Green, Phys.Rev.Lett. 121 (2018) 2, 022004

J. Green, Phys.Rev.D 101 (2020) 7, 074509

$$\hat{O}^B_\Gamma(z,\Lambda) = Z_{\psi,z}(\Lambda,\mu)e^{\delta m(\Lambda)|z|}\hat{O}^R_\Gamma(z)$$

$$\lim_{T\to\infty} W(L,T) = c(L)e^{-V(L)T}, \;\; V(L) = \sigma L - \frac{\pi}{12L} - 2\delta m$$

RI/MOM renormalization

C. Sturm, et,al Phys.Rev.D 80 (2009) 014501

J. Chen, et,al Phys.Rev.D 97 (2018) 1, 014505

$$\tilde{h}_R(z,P^z,p^z_R,\mu_R) = \lim_{a\to 0} Z^{-1}_{\hat{O}_\Gamma}(z,p^z_R,\mu_R,a)\tilde{h}_B(z,P^z,a)$$

$$\langle q|\hat{O}_B|q\rangle = \mathrm{Tr}[\Lambda^\Gamma_0(z,a,p)\hat{P}], \;\; Z_q Z^{-1}_{\hat{O}_\Gamma}\Lambda^\Gamma_0(z,a,p)|_{p=p_R} = \Lambda^\Gamma_{\mathrm{tree}}(z,a,p)$$

renormalization

hybrid renormalization

short distance $(z < z_s)$: $h(z,a,P_z) = \dfrac{h^B(z,a,P_z)}{\boxed{Z_X(z,a)}}$

RI/MOM factor

✓ P=0 matrix element

vacuum state ME

long distance $(z \geq z_s)$: $h(z,a,P_z) = e^{\left(\frac{m_{-1}}{a}+m_0\right)|z-z_s|}\dfrac{h^B(z,a,P_z)}{Z_X(z_s,a)}$

J. Holligan et al,. J.Phys.G 51 (2024) 6, 065101

X. Han et al,. Phys.Rev.D 111 (2025) 3, 034503

one lattice ensemble

$$\ln\left[\frac{Z_X(z,a)}{Z_X(z+1,a)}\right] \sim \delta m$$

determination of $\delta m = \dfrac{m_{-1}}{a}$ and m_0:

1.long distance $(z \geq z_s)$: fit $Z_X(z,a) \sim Ae^{-\delta m z}$

2.short distance $(z < z_s)$: fit $(m_0 + \delta m)z = \ln\left[\dfrac{C_0^{\mathrm{NLO}}(z,\mu)}{Z_X(z,a)}\right] + I_0$

Lattice QCD and LaMET

Light-Cone distribution amplitude in LaMET

- Equal time correlation on lattice (quasi-DA):

$$\tilde{\Phi}_A\left(z_1, z_2, \mu, P^z\right) P^z f_\Lambda u_\Lambda(P^z) =$$

$$\epsilon_{ijk}\left\langle 0 \left| u^{i,\mathrm{T}}\left(z_1 n_z\right) \tilde{\Gamma} d^j\left(z_2 n_z\right) s^k(z_3 n_z)\right| \Lambda(P^z)\right\rangle .$$

propagators → 2pt → matrix elements

- Non-perturbative renormalization:

$$\tilde{\Phi}_A(z_1, z_2, \mu, P^z \neq 0) = \frac{\tilde{\Phi}^0_A(z_1, z_2, \mu, P^z \neq 0)}{\tilde{\Phi}^0_A(z_1, z_2, \mu, P^z = 0)} .$$

ratio scheme renormalization

- Fourier transformation:

$$\tilde{\phi}(z_1, z_2, \mu) = \int_0^1 dx_1 \int_0^1 dx_2 e^{i(x_1 z_1 P^z + x_2 z_2 P^z)} \tilde{\Phi}(x_1, x_2, \mu)$$

large z extrapolation…

- Matching to the light-cone:

$$\tilde{\phi}(x_1, x_2) = \int_0^1 dy_1 \int_0^{1-y_1} dy_2 C(x_1, x_2, y_1, y_2) \phi(y_1, y_2)$$

$$+\mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{(x_1 P^z)^2}, \frac{\Lambda^2_{\mathrm{QCD}}}{(x_2 P^z)^2}, \frac{\Lambda^2_{\mathrm{QCD}}}{[(1 - x_1 - x_2)P^z]^2}\right)$$

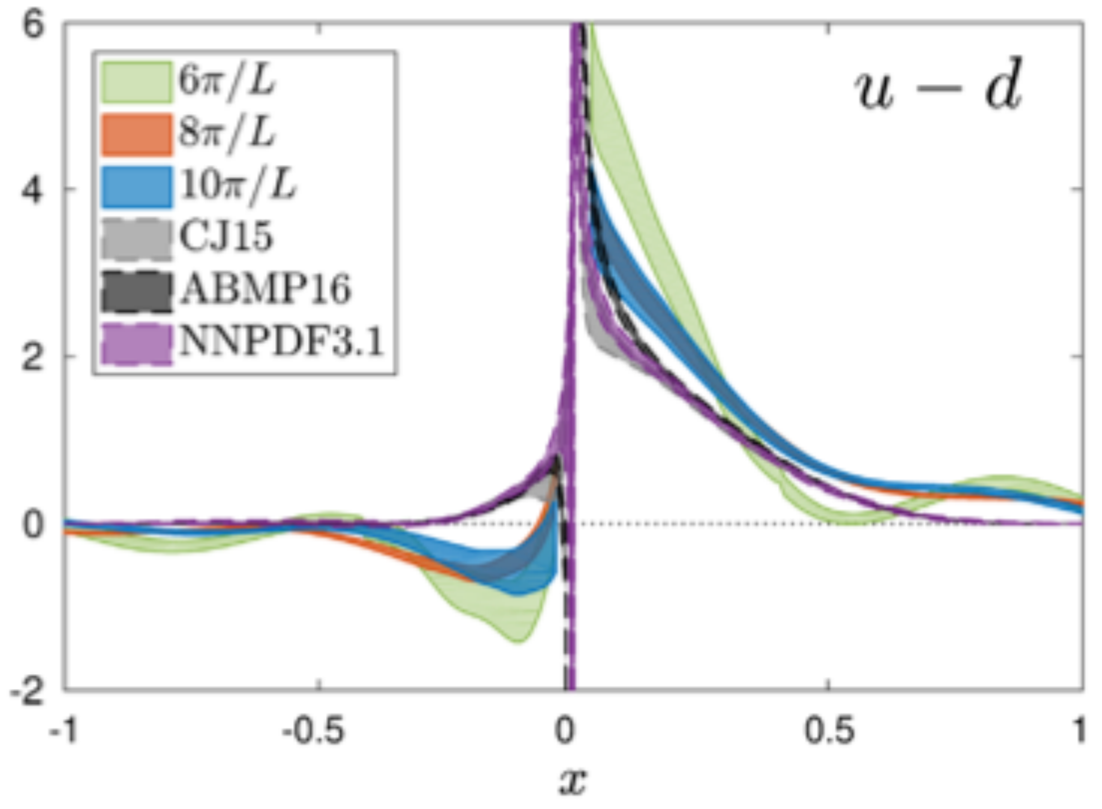
infinite momentum limit…

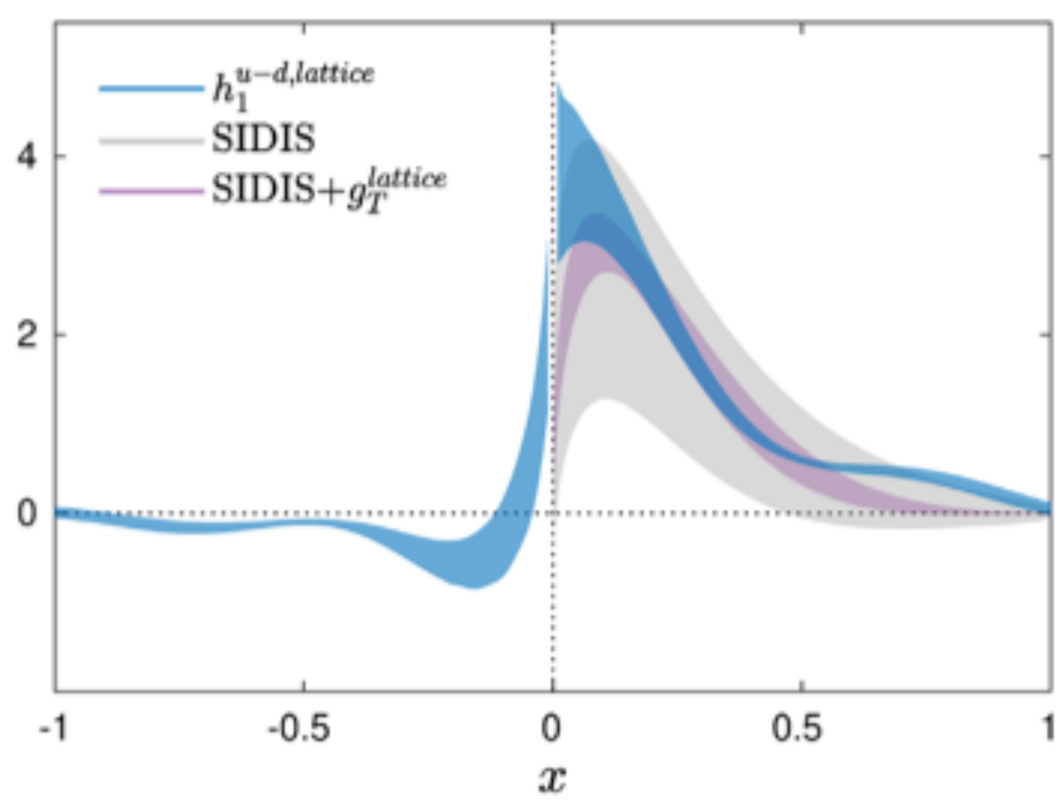
Lattice QCD and LaMET

achievements

Proton unpolarized quark PDF

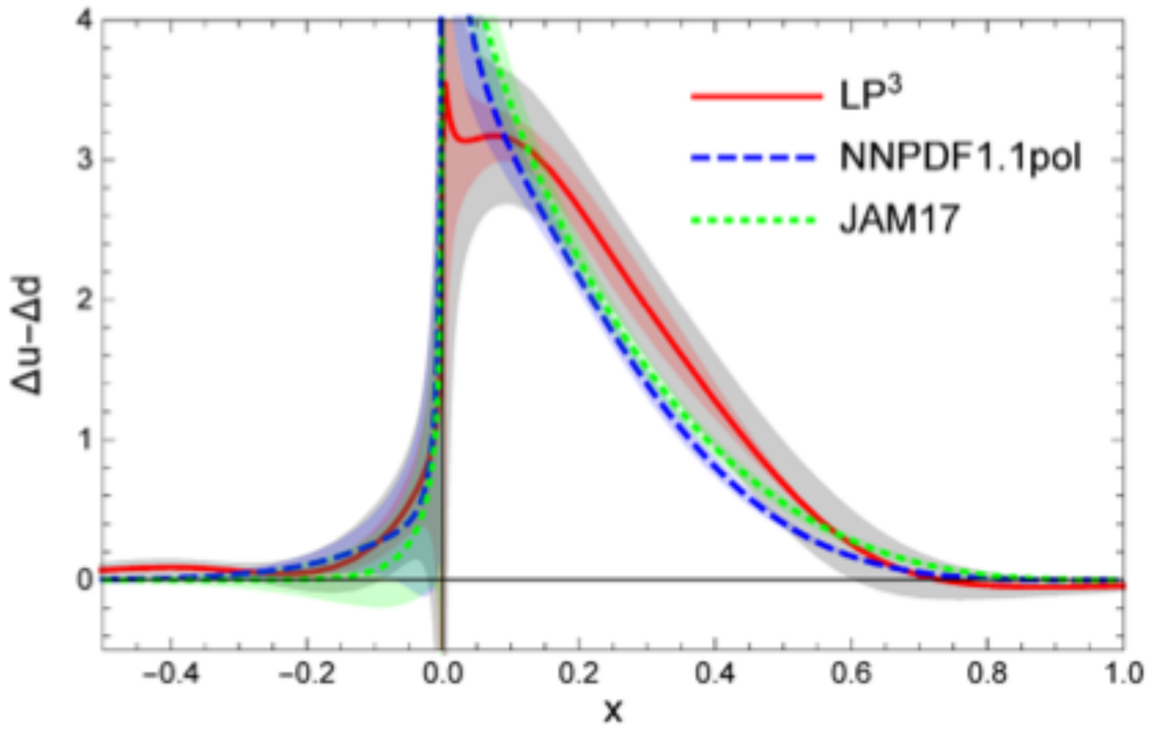
C. Alexandrou, et,al., Phys.Rev.D. 98 (2018) 9, 091503





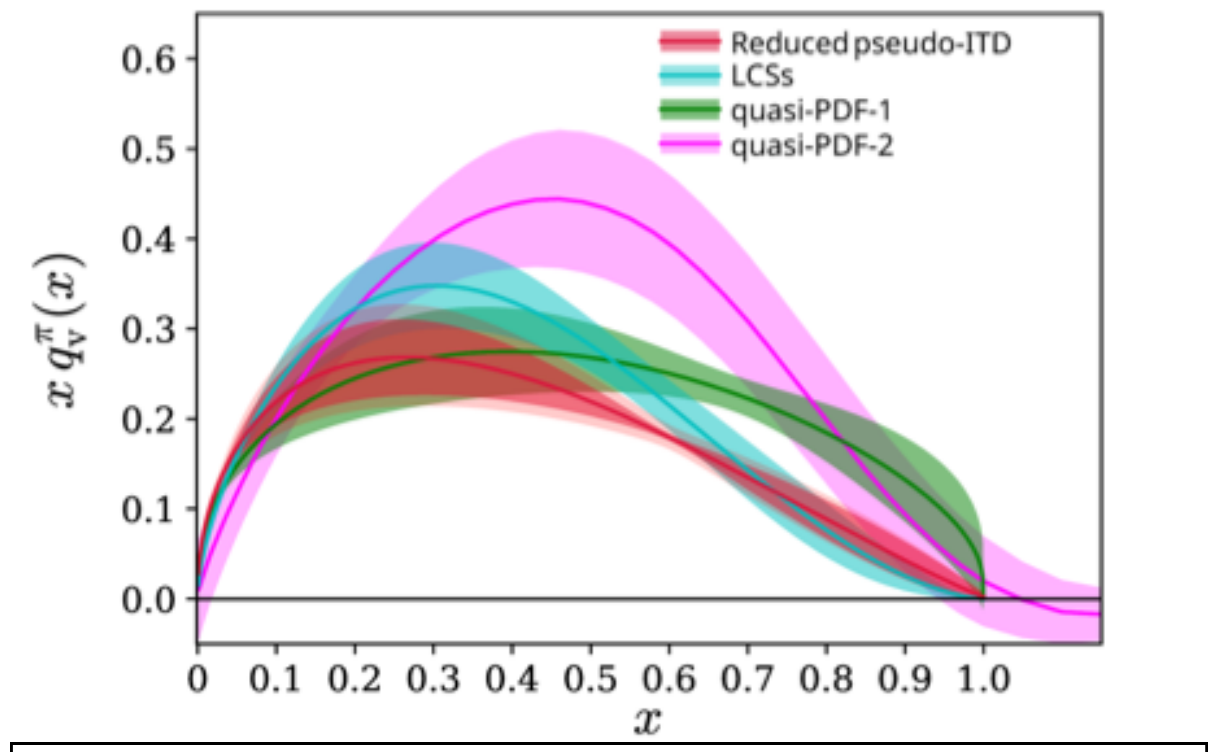
C. Alexandrou, et,al., Phys.Rev.Lett. 121 (2018) 11, 112001

Proton helicity quark PDF



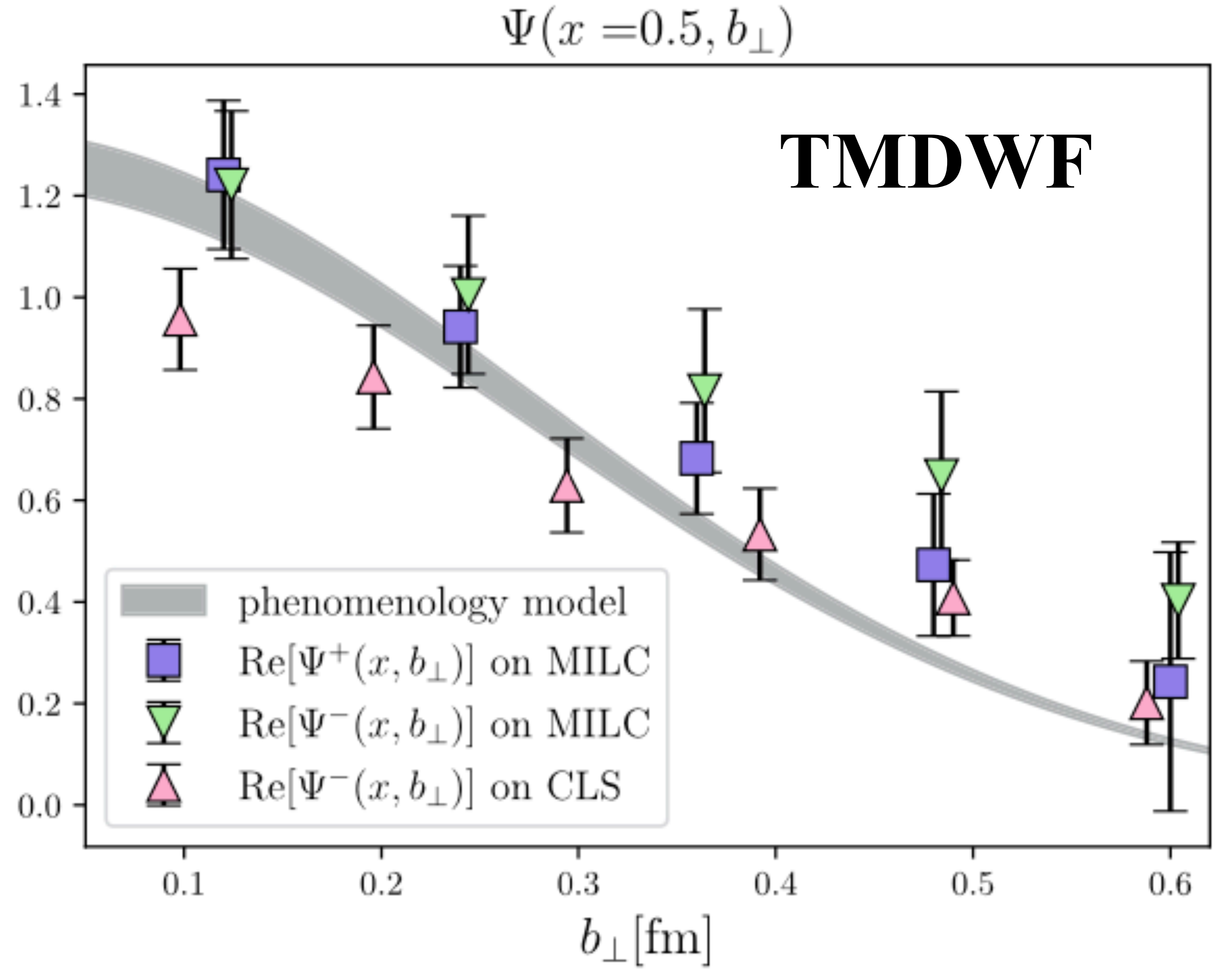
H. Lin, et,al., Phys.Rev.Lett. 121 (2018) 24, 242003

Pion valence quark PDF

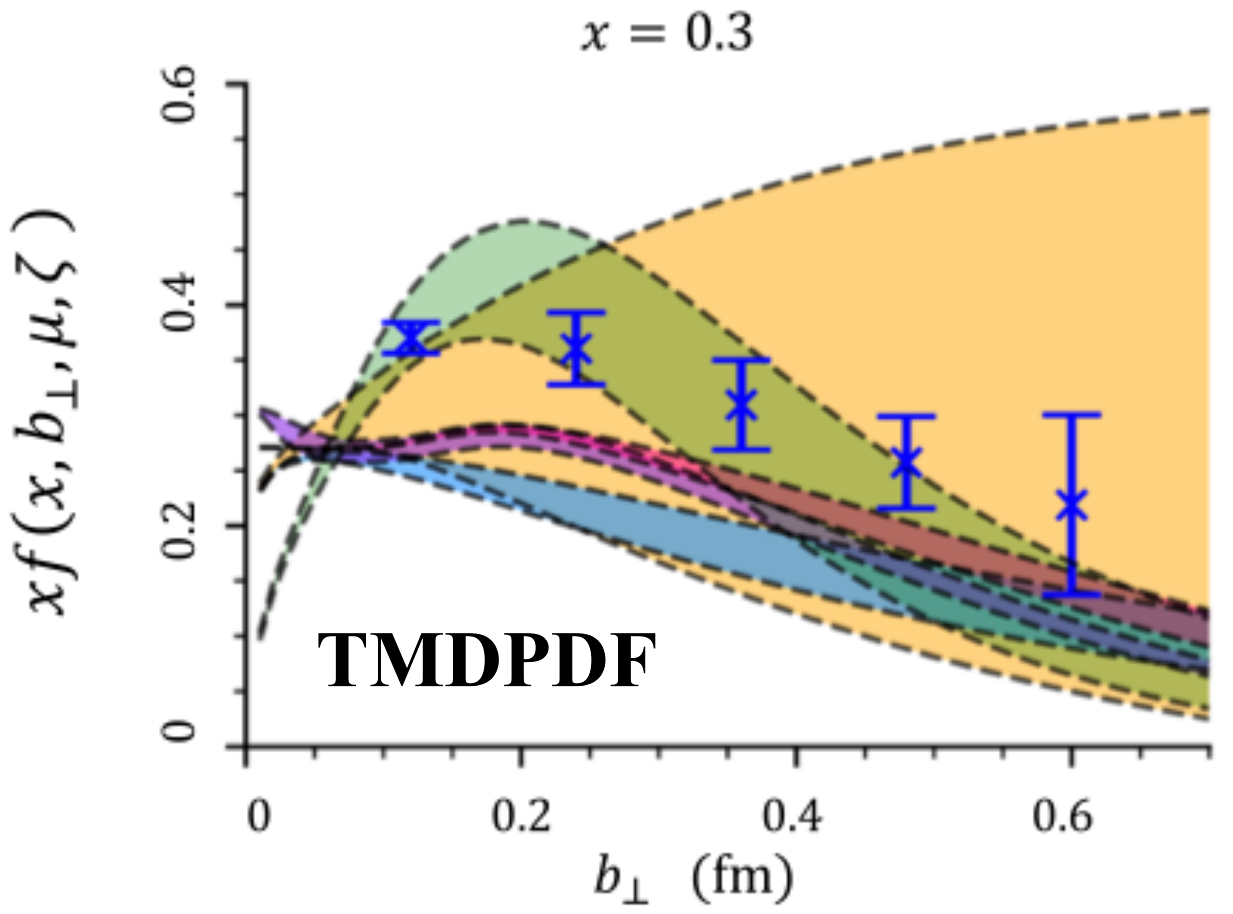


B. Joo, et,al., Phys.Rev.D 100 (2019) 11, 114512

achievements

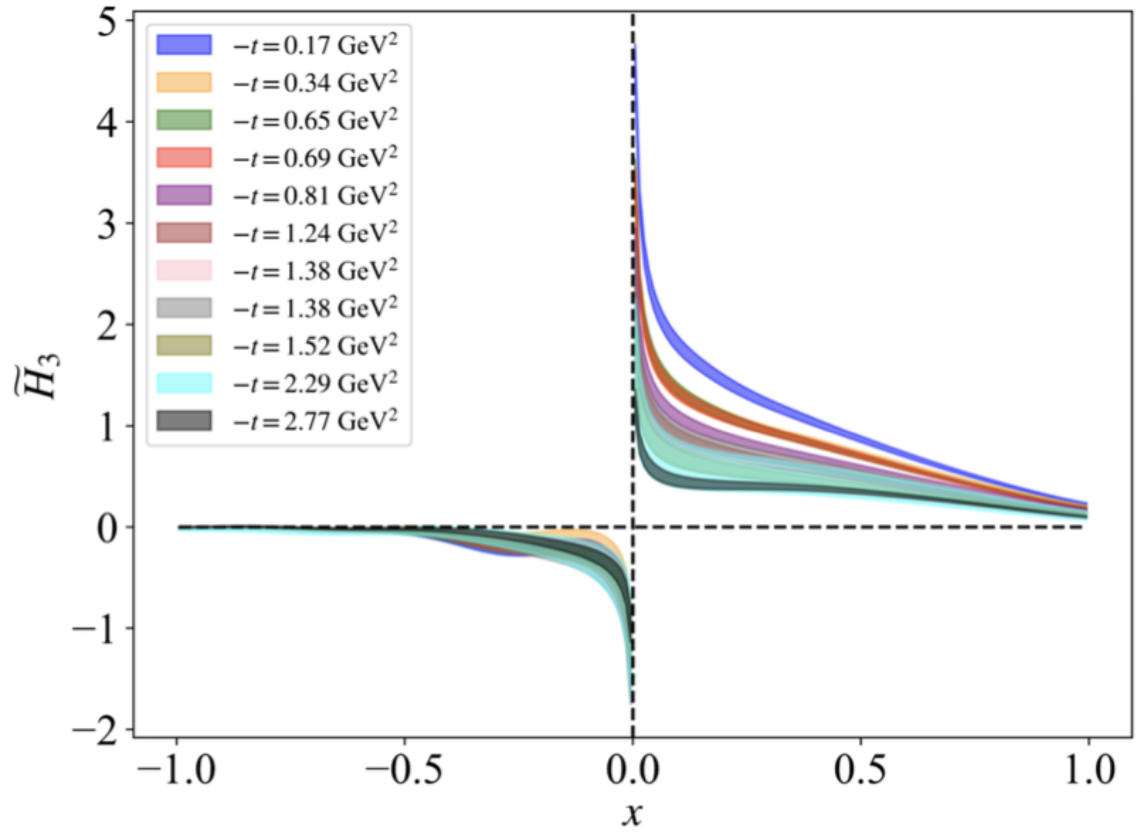


M. Chu, et,al., Phys.Rev.D 109 (2024) 9, L091503



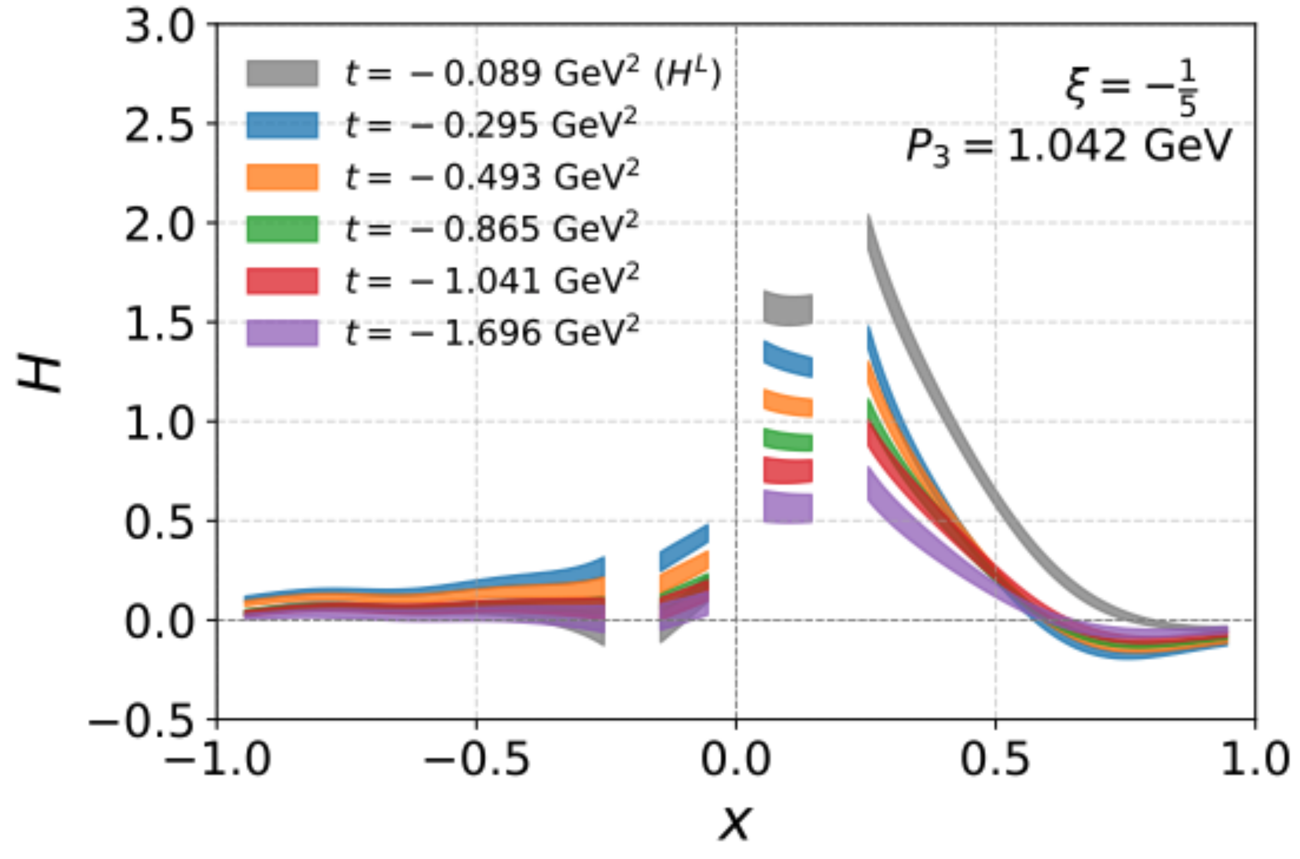
J. He, et,al., Phys.Rev.D 109 (2024) 11, 114513

Proton axial vector GPD



S. Bhattacharya et al., Phys.Rev.D 109 (2024) 3, 034508

Proton unpolarized quark GPD

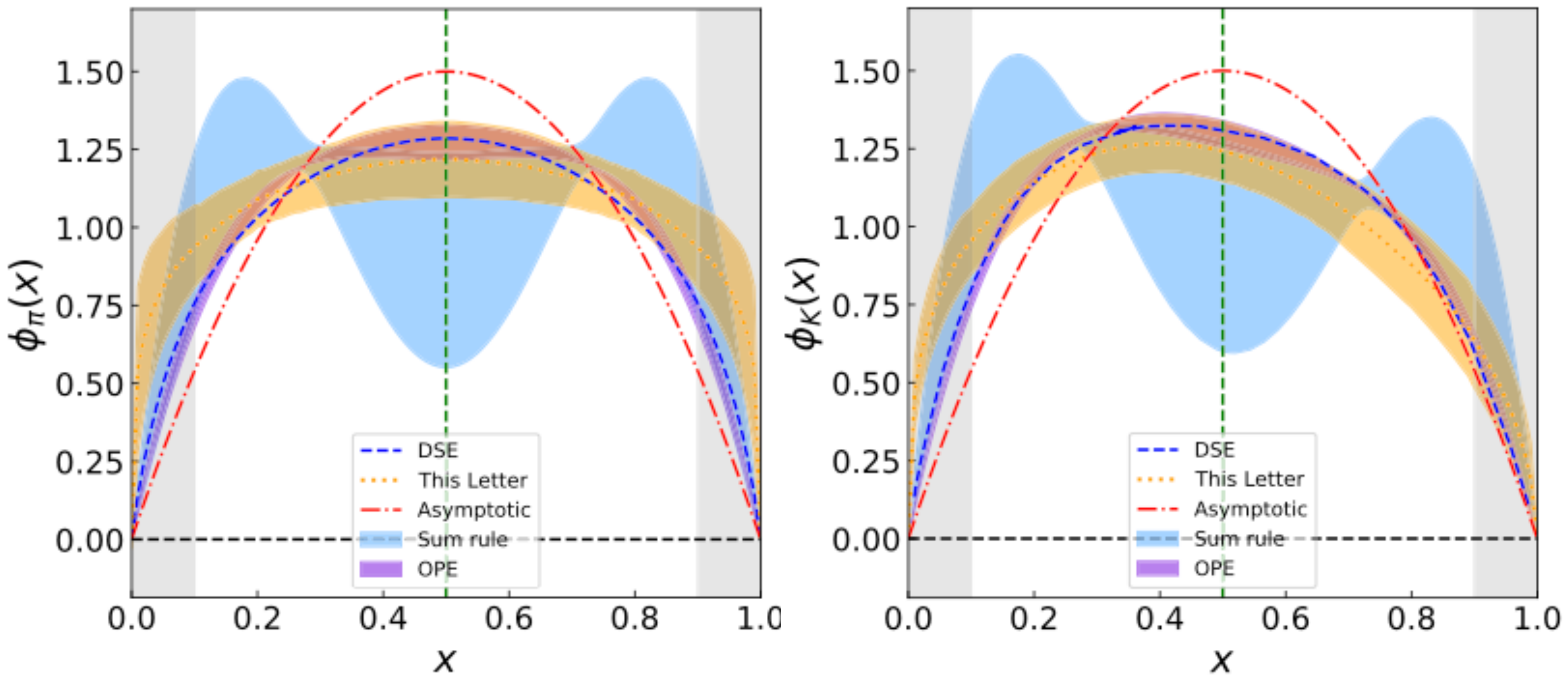


M. Chu, et,al., Phys.Rev.D 112 (2025) 9, 094510

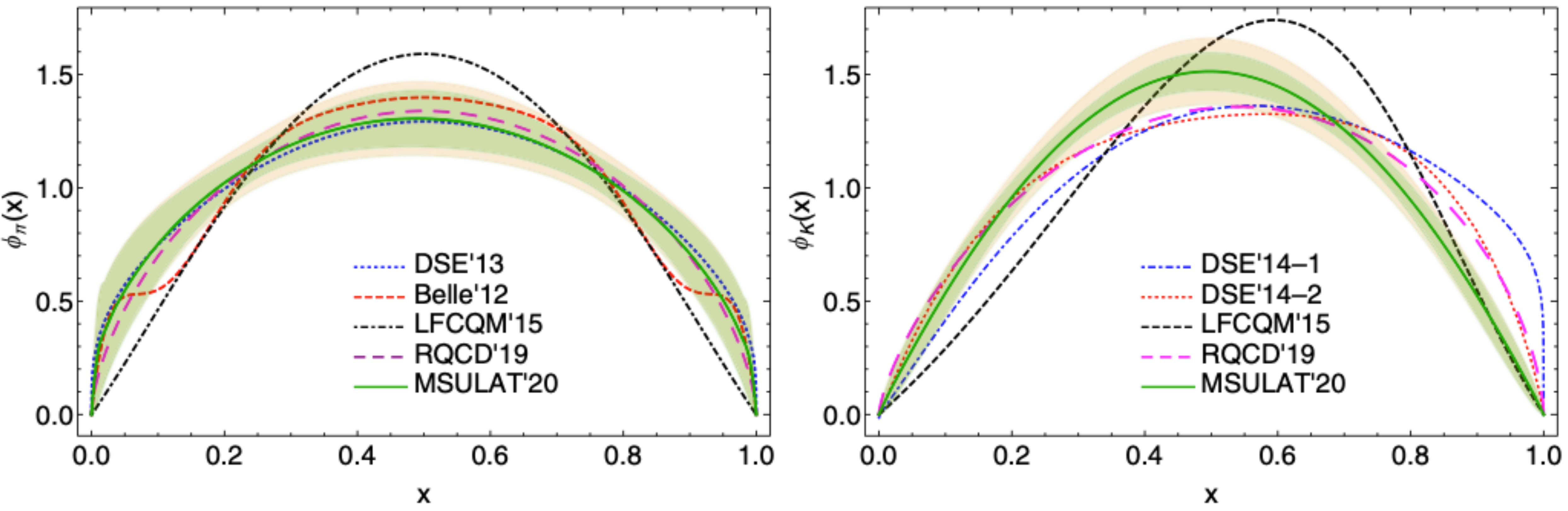
Lattice QCD and LaMET

achievements

Pion and Kaon distribution amplitude



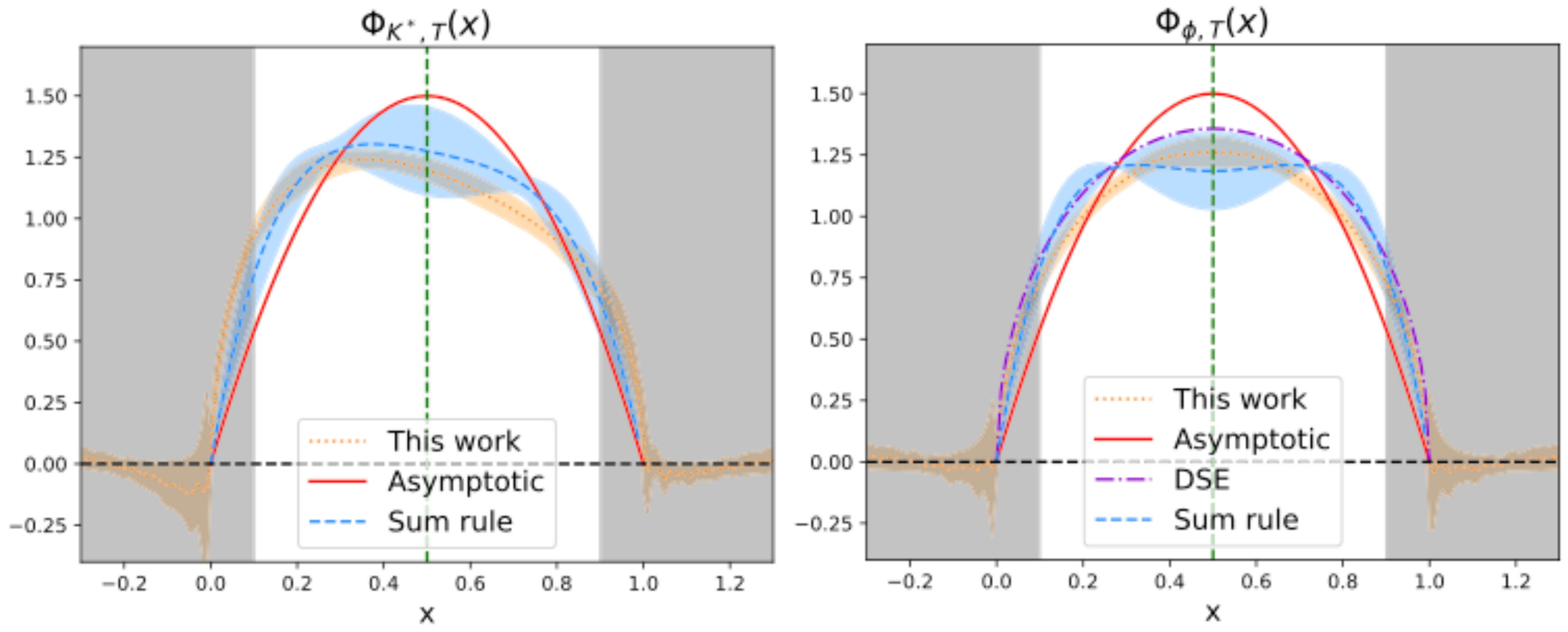
J. Hua, M. Chu et,al., Phys.Rev.Lett. 129 (2022) 13, 132001



R. Zhang et al., Phys.Rev.D 102 (2020) 9, 094519

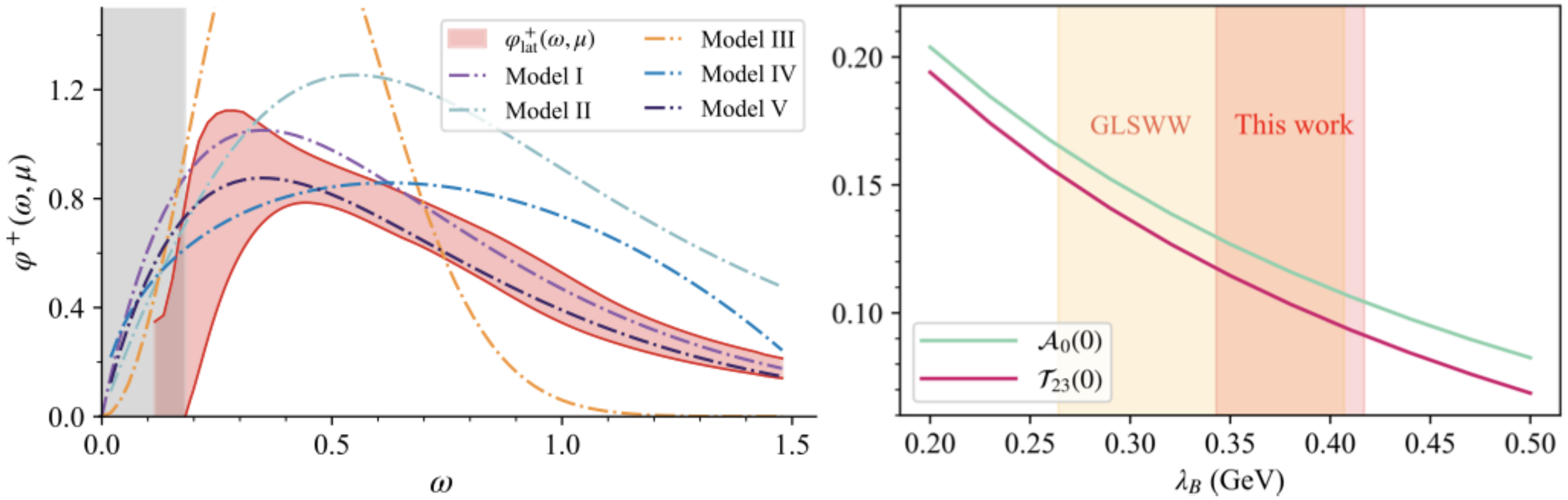
achievements

Vector meson distribution amplitude



J. Hua, M. Chu et,al., Phys.Rev.Lett. 127 (2021) 6, 062002

Heavy meson distribution amplitude



X. Han et,al., Phys.Rev.D 111 (2025) 3, 034503

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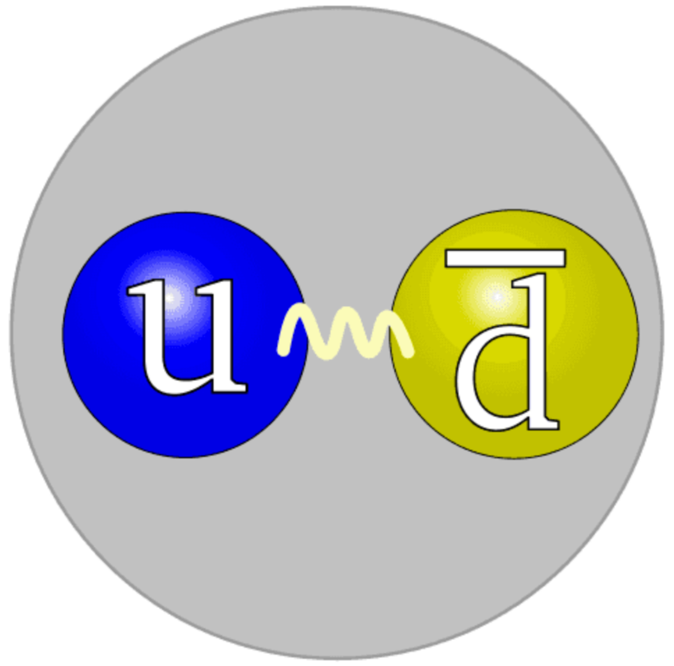
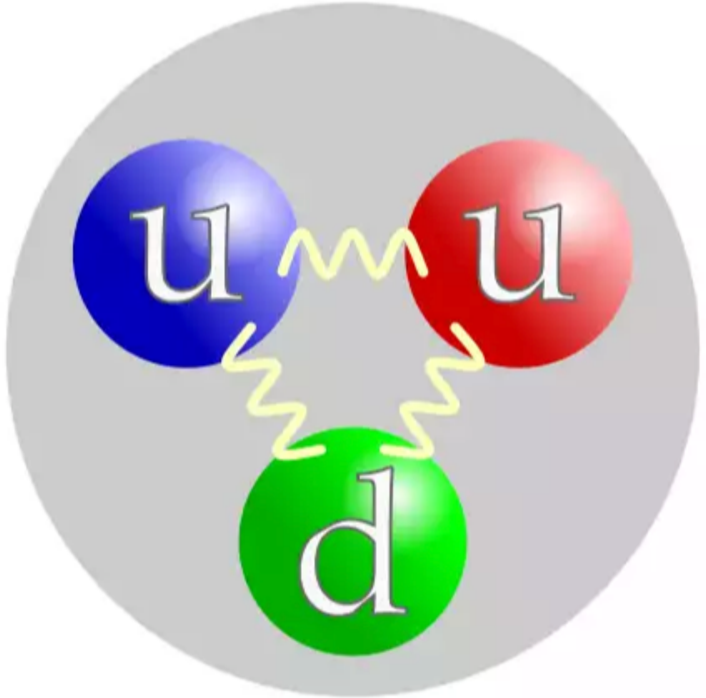




Framework 1.0 and results

operation definitions

one of three leading twists

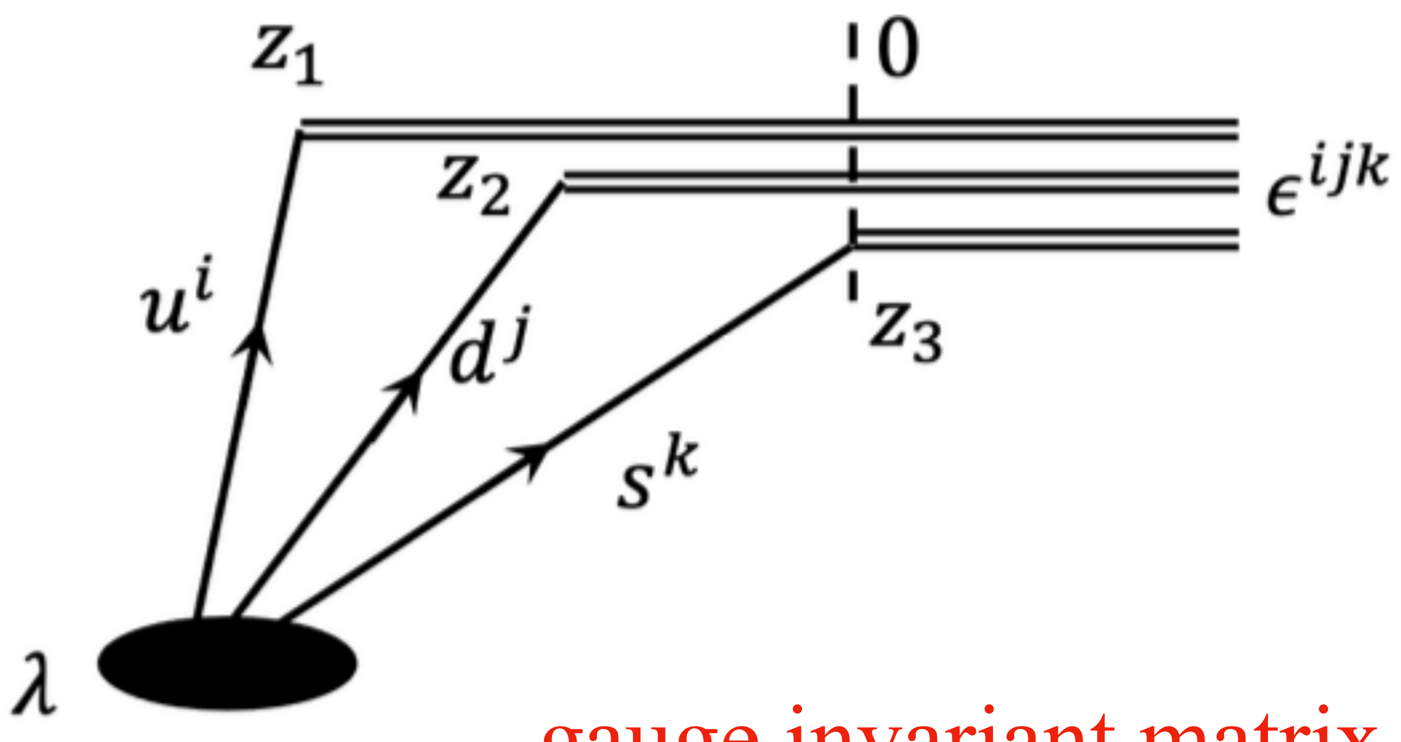
$$
\begin{aligned}
H(z_1, z_2, z_3)_{\alpha\beta\gamma} = \frac{1}{4} f_\Lambda \Big[& (P^\mu \gamma_\mu C)_{\alpha\beta} (\gamma_5 s_\Lambda)_\gamma V(z_i \cdot P \cdot n) \\
& + \boxed{(P^\mu \gamma_\mu \gamma_5 C)_{\alpha\beta} (u_\Lambda)_\gamma A(z_i \cdot P \cdot n)} \\
& + \frac{1}{4} f_\Lambda^T (i\sigma^{\mu\nu} P_\nu C)_{\alpha\beta} (\gamma_\mu \gamma_5 u_\Lambda)_\gamma T(z_i \cdot P \cdot n) \Big]
\end{aligned}
$$

V. Braun et al., Nucl.Phys.B 589 (2000) 381-409

$$
\epsilon_{ijk} \left\langle 0 \left| u^{i,\mathrm{T}} \left(z_1 n\right) \boxed{\Gamma} d^j \left(z_2 n\right) s^k (z_3 n) \right| \Lambda \right\rangle = \Phi \left(z_1, z_2, \mu\right) P^+ f_\Lambda u_\Lambda(P)
$$

$$\Gamma = C\gamma^5$$

operation definitions



Wilson lines

gauge invariant matrix element

$$
\begin{aligned}
H(z_1, z_2, z_3)_{\alpha\beta\gamma} = {} & \varepsilon^{ijk} \langle 0 \,|\, (W^{ii'}\left(z_0 n, z_1 n\right)) u_\alpha^{i'}\left(z_1 n\right) \\
& \times (W^{jj'}\left(z_0 n, z_2 n\right)) d_\beta^{j'}\left(z_2 n\right) (W^{kk'}\left(z_0 n, z_3 n\right)) s_\gamma^{k'}\left(z_3 n\right) \,|\, \Lambda(P, \lambda) \rangle,
\end{aligned}
$$

lattice setup

Ensenble	Lattice spacing	Volume	Valence pion mass	Momentum
F32P30	0.077 fm	$32^3 \times 96$	303 MeV	2.52 GeV 3.02 Gev 3.52 GeV

Framework 1.0 and results

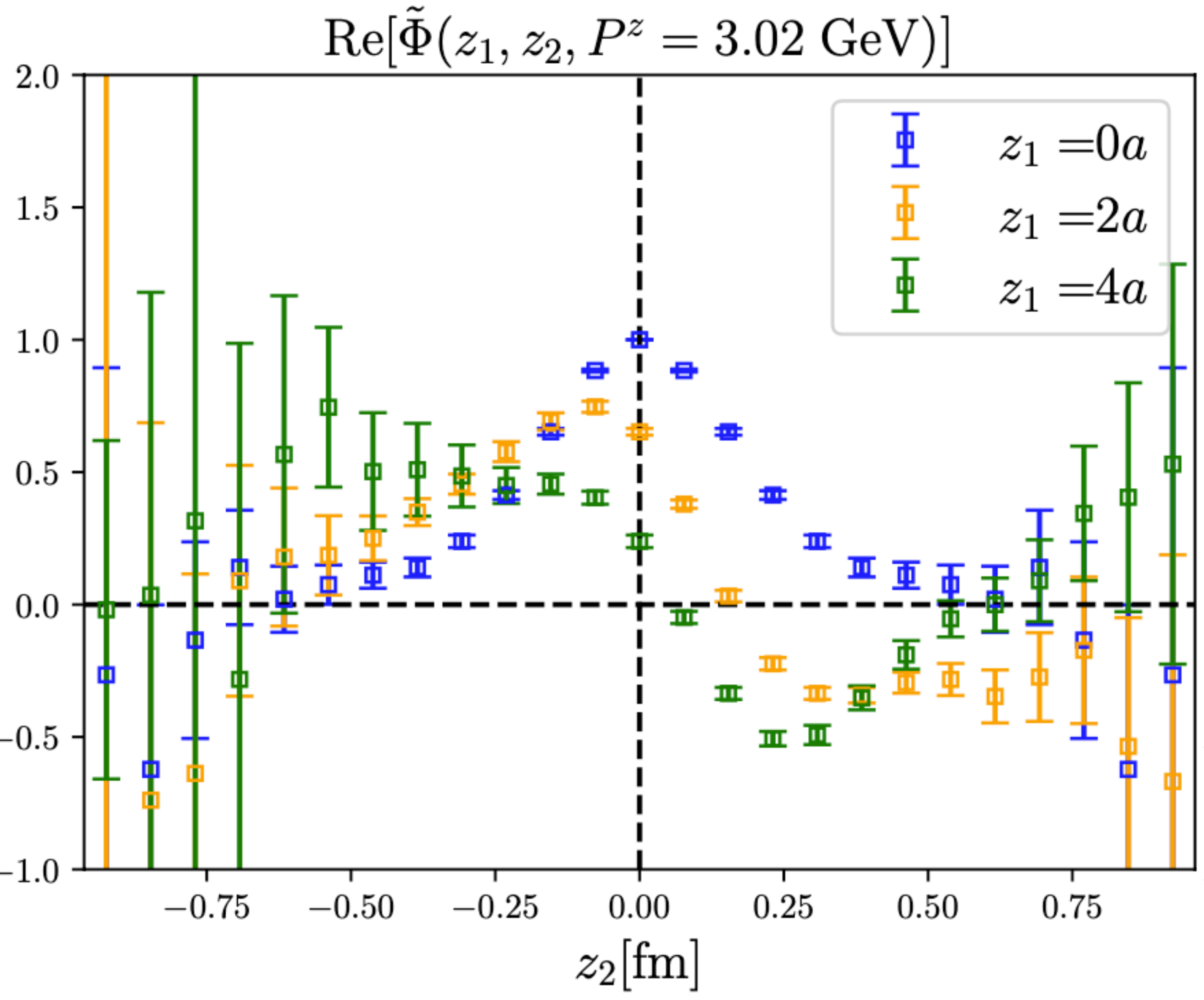
renormalization

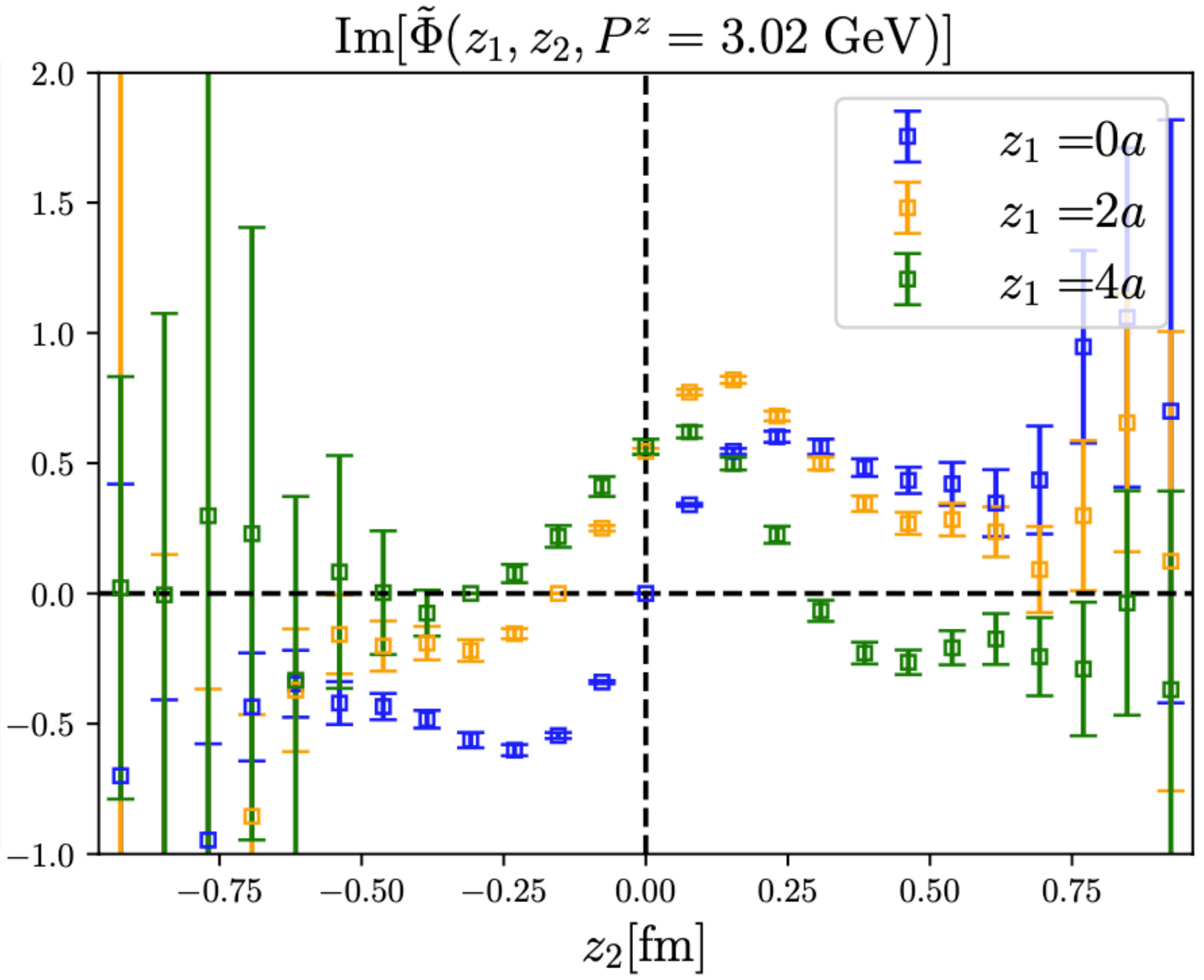
renormalization: ratio scheme (simplified hybrid)

$$\tilde{\Phi}_A(z_1, z_2, \mu, P^z \neq 0) = \frac{\tilde{\Phi}^0_A(z_1, z_2, \mu, P^z \neq 0)}{\tilde{\Phi}^0_A(z_1, z_2, \mu, P^z = 0)}$$

X. Ji et al., Nucl.Phys.B 964 (2021) 115311

quasi-DA in coordinate space





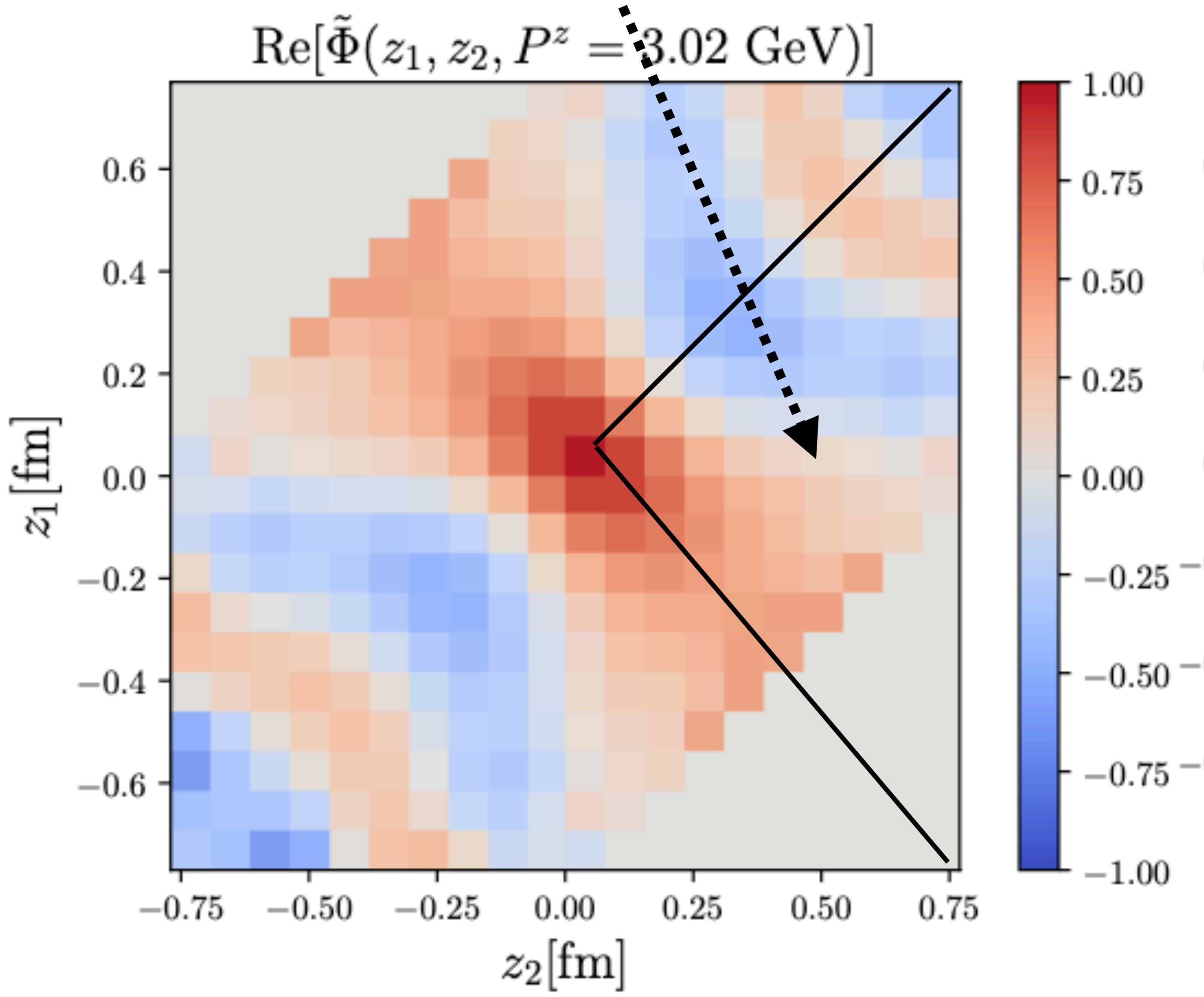
analytic properties

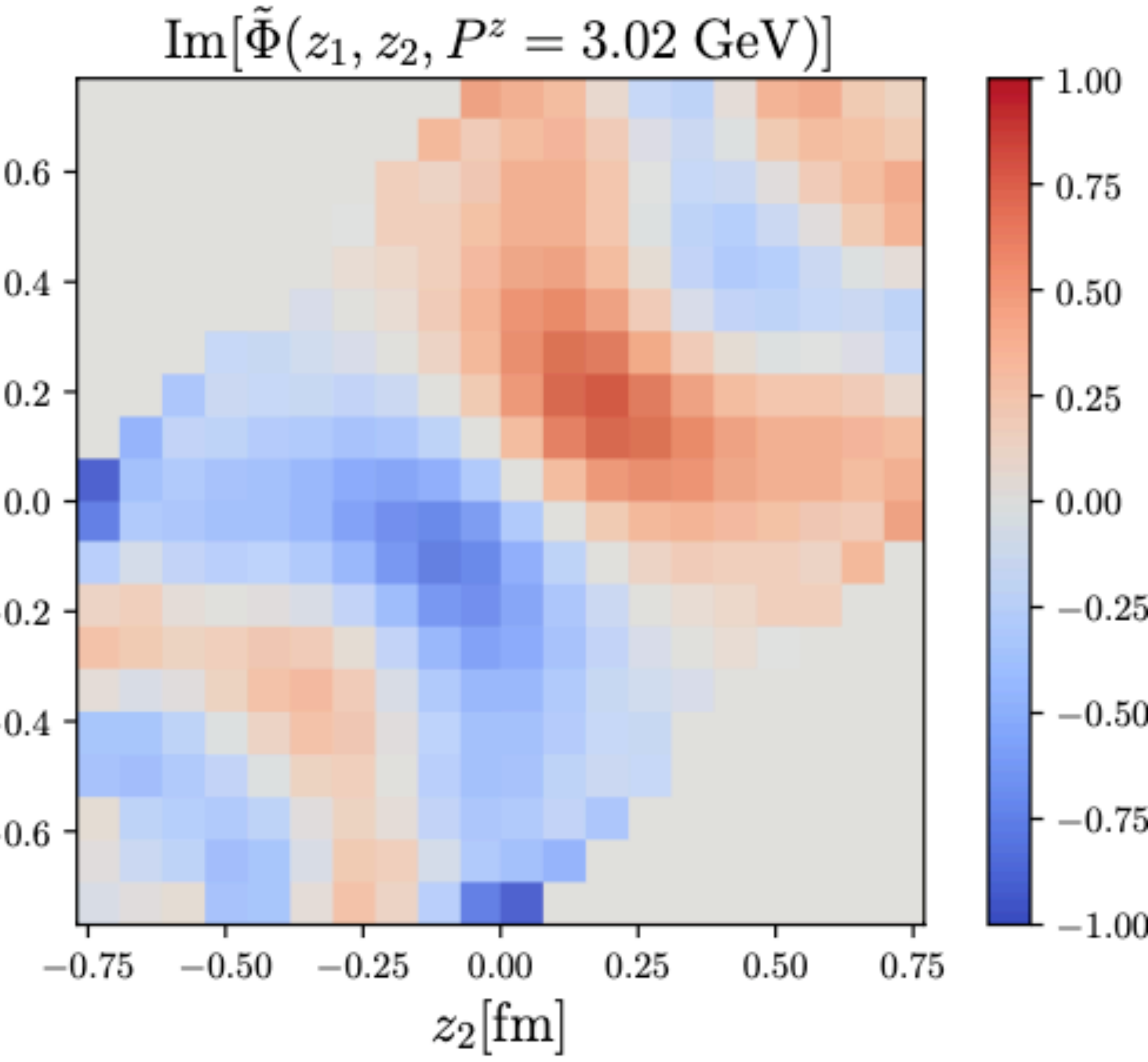
isospin symmetry $\longrightarrow$ $\tilde{\phi}(z_1, z_2) = \tilde{\phi}(z_2, z_1)$

$\tilde{\phi}(x_1, x_2)$ is real by definition $\xrightarrow{\text{FT}}$ $\tilde{\phi}(z_1, z_2) = \tilde{\phi}^*(-z_1, -z_2)$

unique area

3D heat map





Framework 1.0 and results

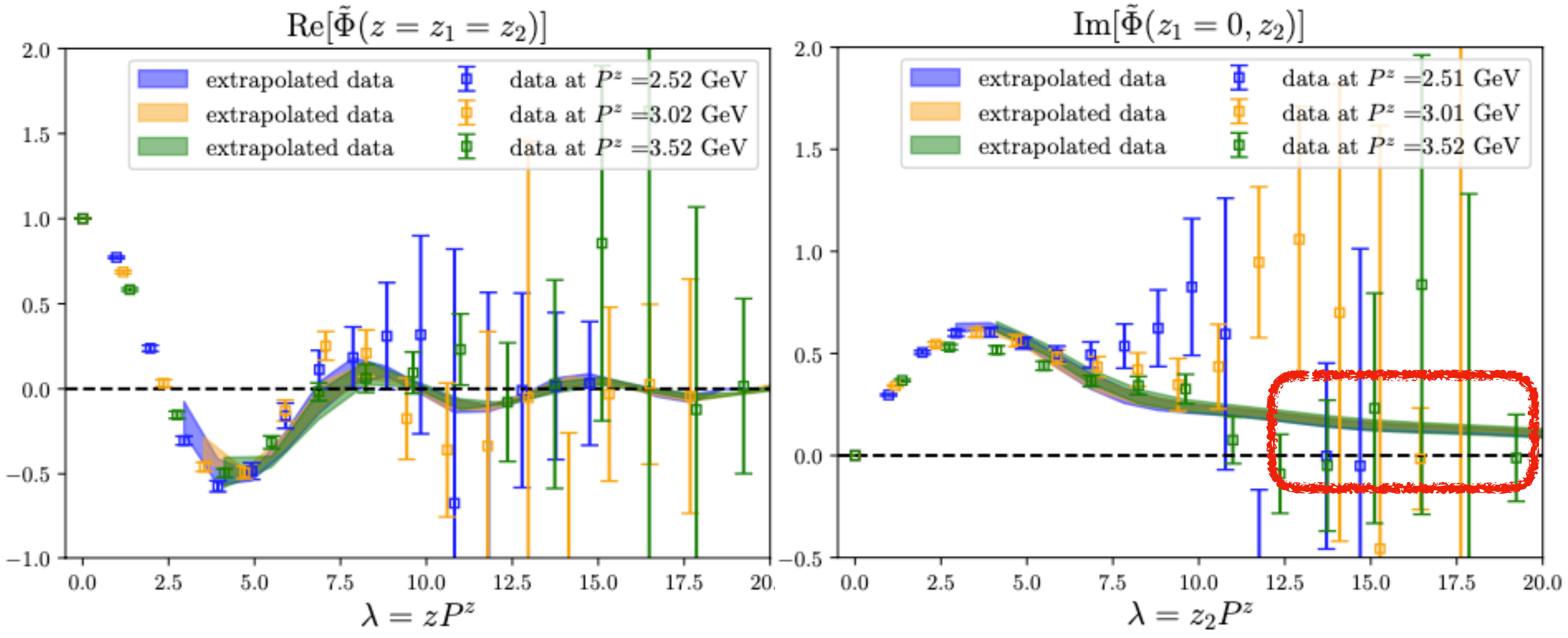
large z extrapolation

2D extrapolation asymtotic behavior:

$$\phi(x_1, x_2) \sim C_0 x_1^{d_1} x_2^{d_1} (1 - x_1 - x_2)^{d_2}$$

$$\downarrow$$

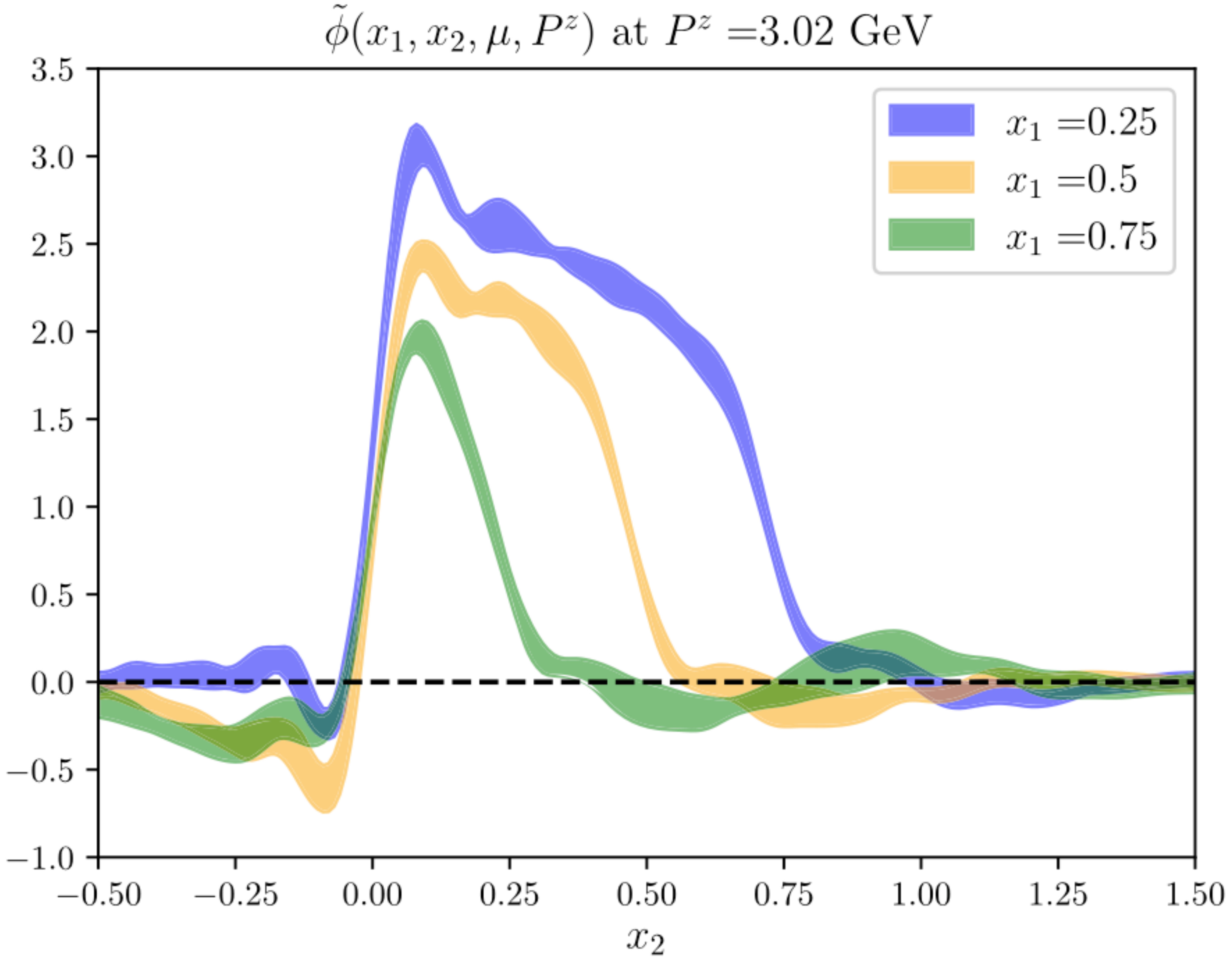
$$\phi(z_1, z_2) \sim \tilde{\phi}(z_1, z_2)$$



quasi DA in momentum space

Fourier transformation including d_1 and d_2

$$\tilde{\Phi}(z_1, z_2; d_1, d_2) = \int_0^1 dx_1 \int_0^1 dx_2\, e^{ix_1 z_1 P^z} e^{ix_2 z_2 P^z} C_0 x_1^{d_1} x_2^{d_1} (1 - x_1 - x_2)^{d_2}$$

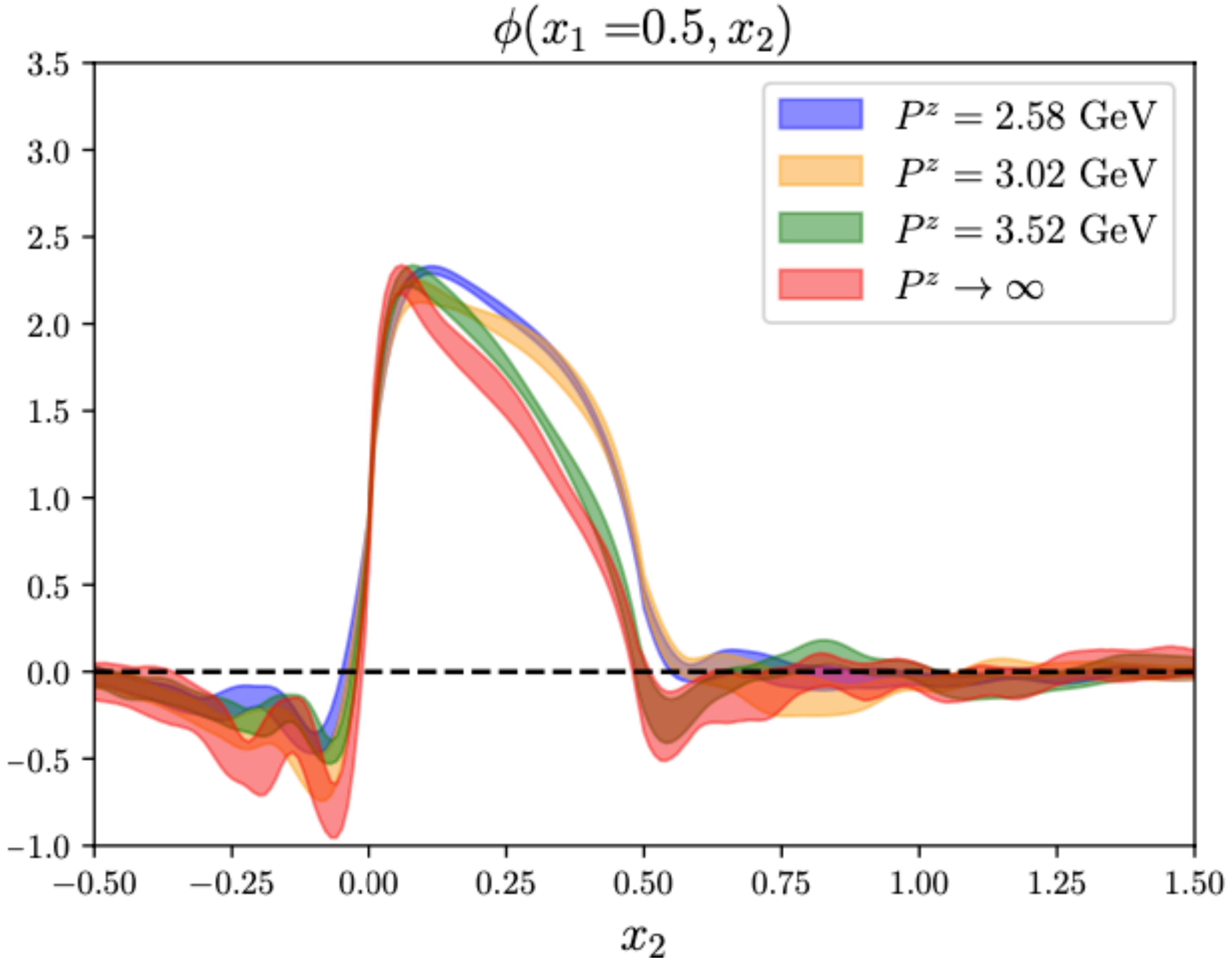


Framework 1.0 and results

LCDA results

infinite momentum limit:

$$\phi(P^z) = \phi(P^z \to \infty) + \frac{c_2}{(P^z)^2} + \frac{c_4}{(P^z)^4}$$

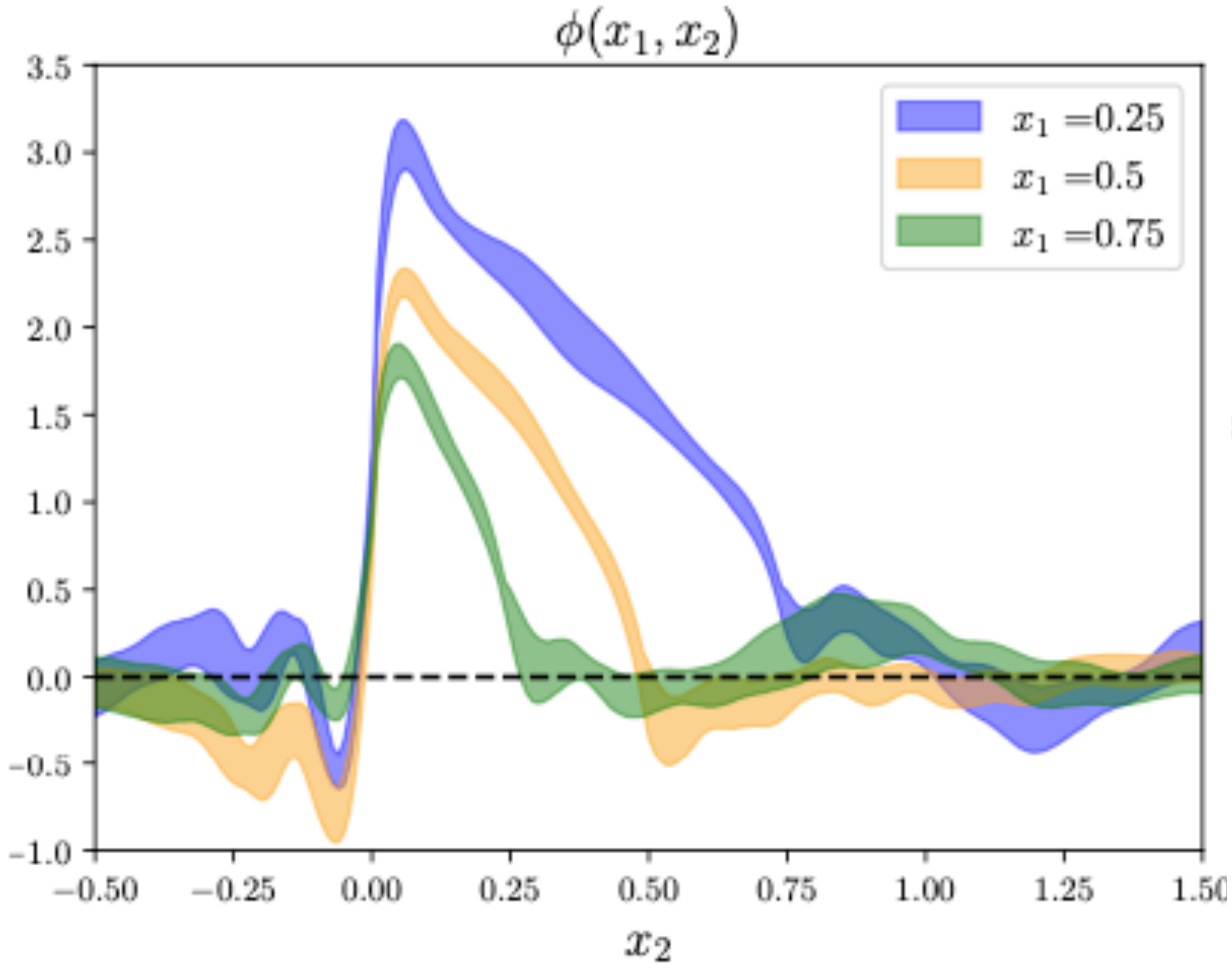


a lot of displacement towards x=0

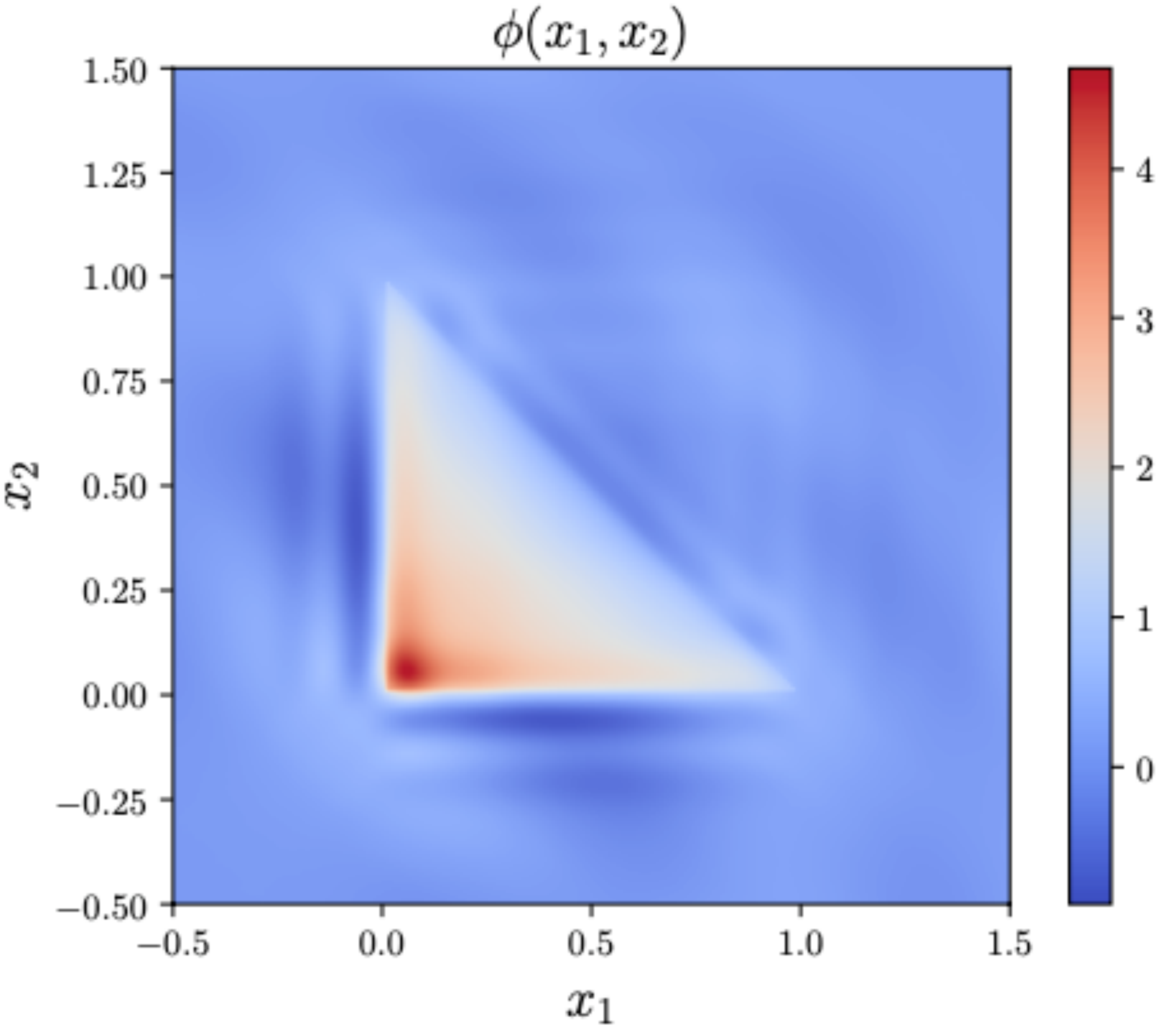
proportion of s quark is large

LCDA results

x_2 dependence when x_1 is fixed



3D heat map



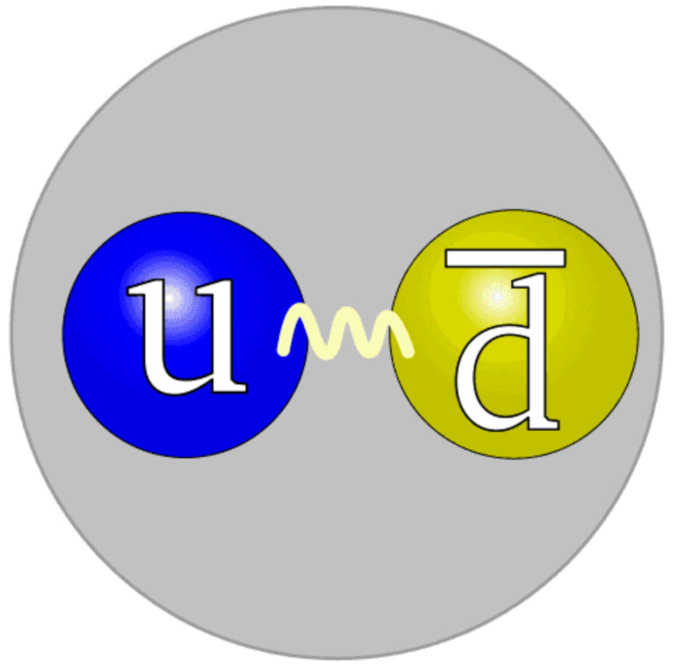
application of form factor

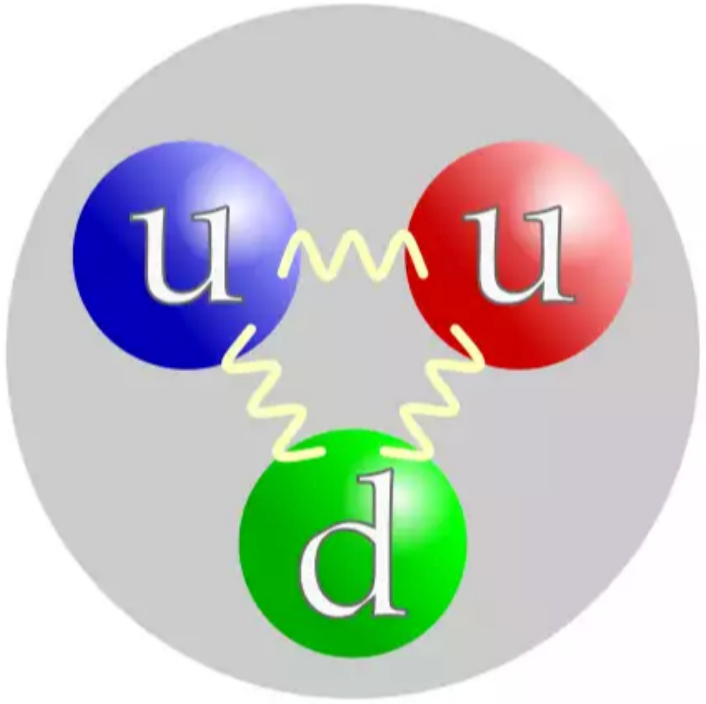
$$\phi(x_1, x_2) \sim C_0 x_1^{d_1} x_2^{d_1} (1 - x_1 - x_2)^{d_2}$$

$\Lambda_b \to \Lambda$ form factor g2	value	difference
$120 x_1 x_2 (1 - x_1 - x_2)$	-0.02364	0
This work	-0.0169(31)	20%

Outline

- Distribution Amplitudes
- Lattice QCD and LaMET
- Framework 1.0 and results
- Framework 2.0 and results
- Summary





Framework 2.0 and results

updates for 2.0

all three leading twists

$$\begin{aligned} H(z_1, z_2, z_3)_{\alpha\beta\gamma} = \frac{1}{4} f_B \Big[& (P^\mu \gamma_\mu C)_{\alpha\beta} (\gamma_5 u_B)_\gamma V(z_i \cdot P \cdot n) \\ & + (P^\mu \gamma_\mu \gamma_5 C)_{\alpha\beta} (u_B)_\gamma A(z_i \cdot P \cdot n) \\ & + \frac{1}{4} f_B^T (i\sigma^{\mu\nu} P_\nu C)_{\alpha\beta} (\gamma_\mu \gamma_5 u_B)_\gamma T(z_i \cdot P \cdot n) \Big] \end{aligned}$$

H. Bai et al., Phys.Rev.D 112 (2025) 11, 114515

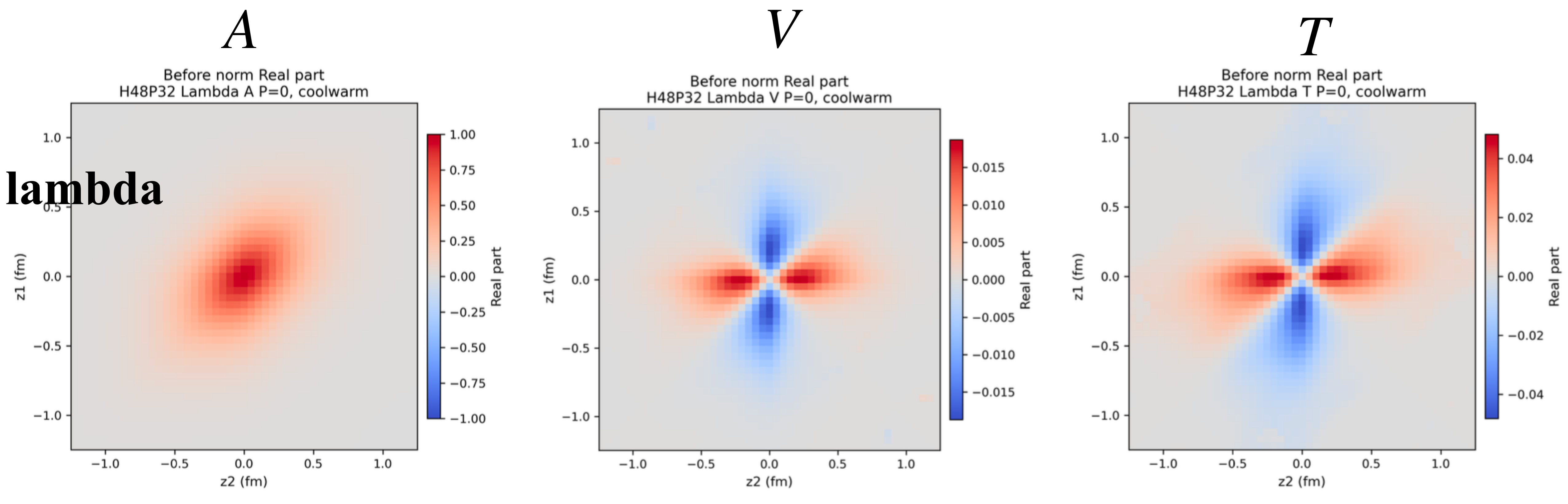
lattice setup

Ensembles	a (fm)	m_π (MeV)	Volume	n_{cfg}	n_{src}	P^z (GeV)
C24P29	0.1052	292.3	24 × 72	864	4 × 9	0, 1.96, 2.45, 2.94
C32P23	0.1052	227.9	32 × 64	954	4 × 8	0, 1.84, 2.21, 2.57, 2.94
C48P14	0.1052	136.4	48 × 96	302	4 × 16	0, 1.96, 2.45, 2.94
F32P30	0.0775	300.4	32 × 96	777	4 × 8	0, 2.00, 2.49, 2.99
F32P21	0.0775	210.3	32 × 64	459	4 × 16	0, 2.00, 2.49, 2.99
G36P29	0.0689	297.2	36 × 108	656	6 × 8	0, 2.00, 2.50, 3.00
H48P32	0.0520	316.6	48 × 144	550	6 × 9	0, 1.98, 2.48, 2.98

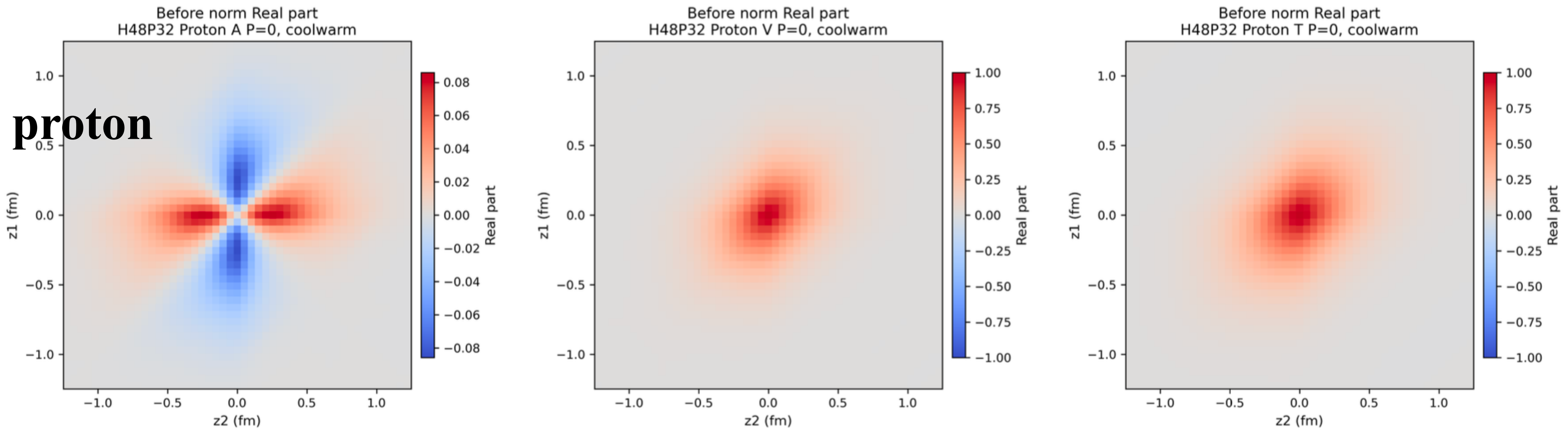
updates for 2.0

symmetries

$$\begin{aligned} V^\Lambda(z_1, z_2) &= -V^\Lambda(z_2, z_1) & V^P(z_1, z_2) &= +V^P(z_2, z_1) \\ A^\Lambda(z_1, z_2) &= +A^\Lambda(z_2, z_1) & A^P(z_1, z_2) &= -A^P(z_2, z_1) \\ T^\Lambda(z_1, z_2) &= -T^\Lambda(z_2, z_1) & T^P(z_1, z_2) &= +T^P(z_2, z_1) \end{aligned}$$



(a) Λ A (b) Λ V (c) Λ T



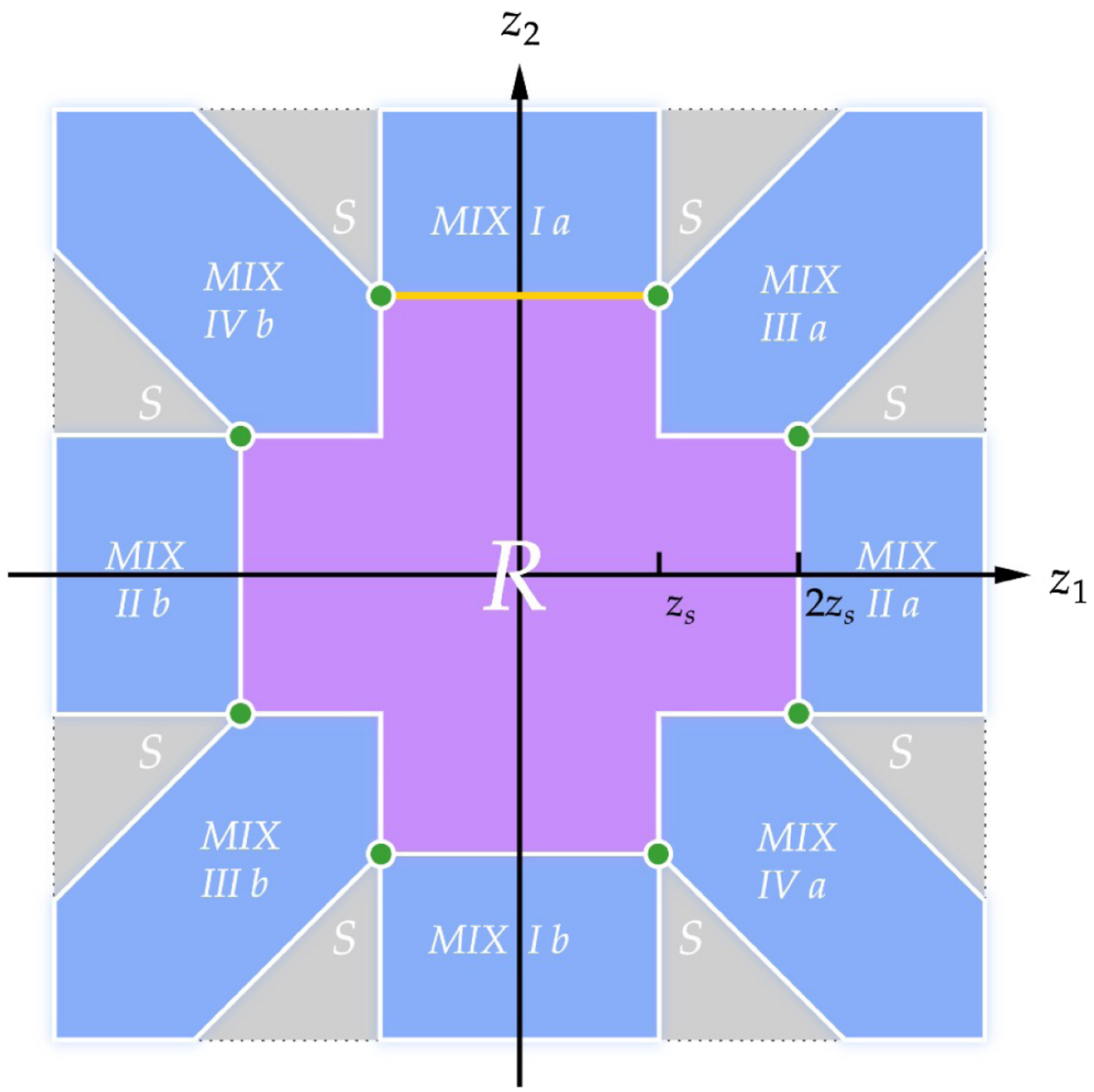
(d) Proton A (e) Proton V (f) Proton T

Framework 2.0 and results

renormalization

H. Bai et al., Phys.Rev.D 112 (2025) 11, 114515

hybrid renormalization: ratio(short) + self(long)



ratio

short-distance region

both z_1 and z_2 are small

ratio

Mixed region

one of z_1 and z_2 is small

self

long-distance region

both of z_1 and z_2 are large

renormalization

H. Bai et al., Phys.Rev.D 112 (2025) 11, 114515

$$Z_R(z_1, z_2, a, \mu) = \exp\left[\left(\frac{k}{a\ln[a\Lambda_{\overline{\text{QCD}}}]} - m_0\right)\tilde{z} + \frac{\gamma_0}{b_0}\ln\left[\frac{\ln[1/(a\Lambda_{\overline{\text{QCD}}})]}{\ln[\mu/\Lambda_{\overline{\text{MS}}}]}\right]\right.$$

$$\left. + \ln\left[1 + \frac{d}{\ln(a\Lambda_{\overline{\text{QCD}}})}\right] + f(z_1, z_2)a^2\right]$$

$$\tilde{z} = \begin{cases} |z_1 - z_2|, & z_1 z_2 < 0 \\ \max(|z_1|, |z_2|), & z_1 z_2 \geq 0 \end{cases}$$

$$M^{rn}(z_1, z_2, P_z, \mu) = \frac{M_0(z_1, z_2, P_z, \mu)}{M_0(z_1, z_2, P_z = 0, \mu)}$$

$$M^{rn}(z_1 - z_2, z_2, P_z, \mu) = \frac{M_0(z_1, z_2, P_z, \mu)}{M_0(z_1 - z_2, z_s, P_z = 0, \mu)}$$

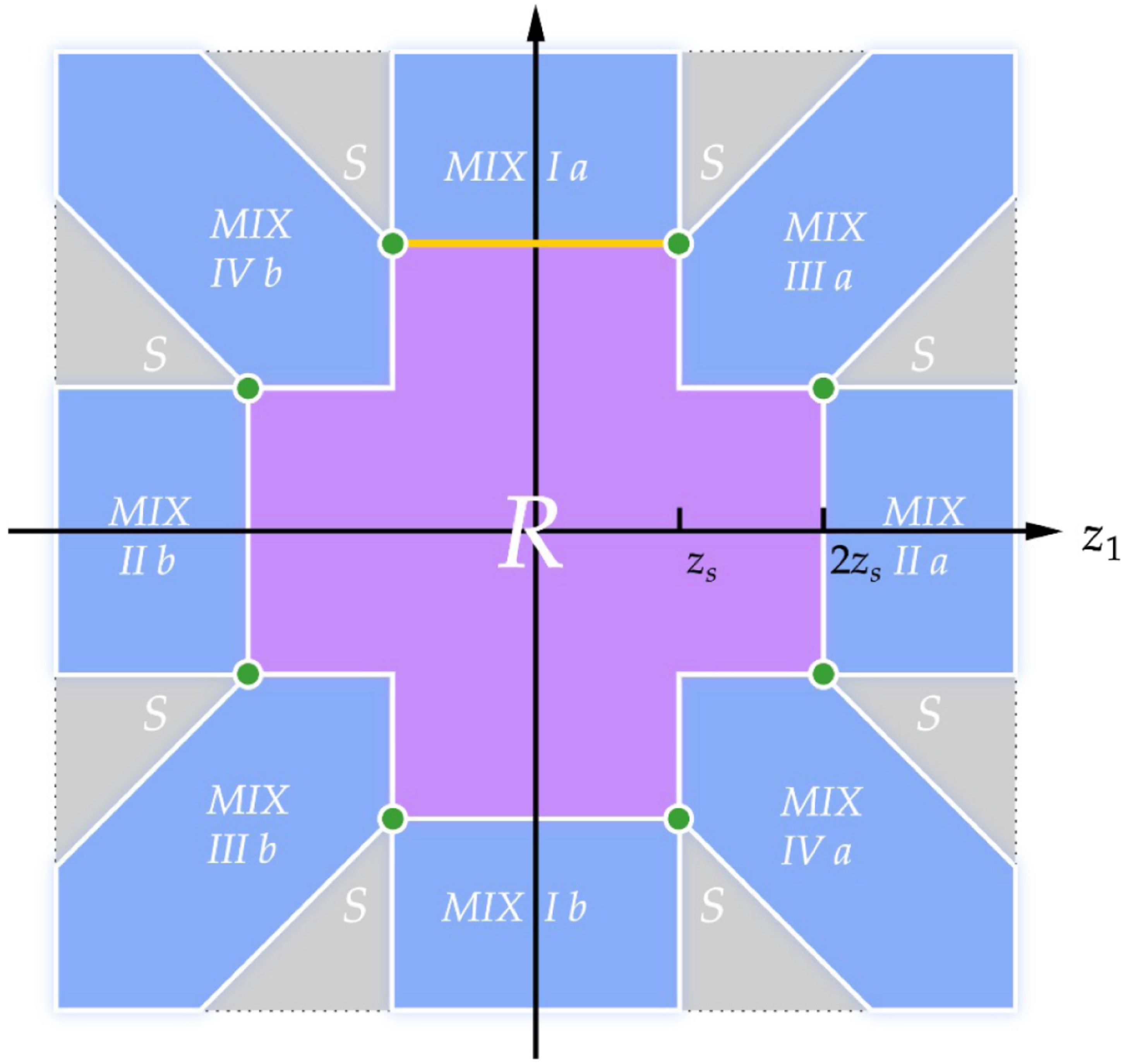
$$M^{rn}(z_1, z_2, P_z, \mu) = \frac{M_0(z_1, z_2, P_z, \mu)}{Z_R(z_1, z_2, \mu)}$$

Framework 2.0 and results

renormalization

H. Bai et al., Phys.Rev.D 112 (2025) 11, 114515

hybrid renormalization: ratio(short) + self(long)



ratio

short-distance region

both z_1 and z_2 are small

ratio

Mixed region

one of z_1 and z_2 is small

self

long-distance region

both of z_1 and z_2 are large

renormalization

H. Bai et al., Phys.Rev.D 112 (2025) 11, 114515

$$Z_R(z_1, z_2, a, \mu) = \exp\Bigg[\underbrace{\left(\frac{k}{a\ln[a\Lambda_{\overline{\text{QCD}}}]} - m_0\right)\tilde{z}}_{\text{fit lattice with per.}} + \underbrace{\frac{\gamma_0}{b_0}\ln\left[\frac{\ln[1/(a\Lambda_{\overline{\text{QCD}}})]}{\ln[\mu/\Lambda_{\overline{\text{MS}}}]}\right]}_{\text{not sensitive to } a}$$

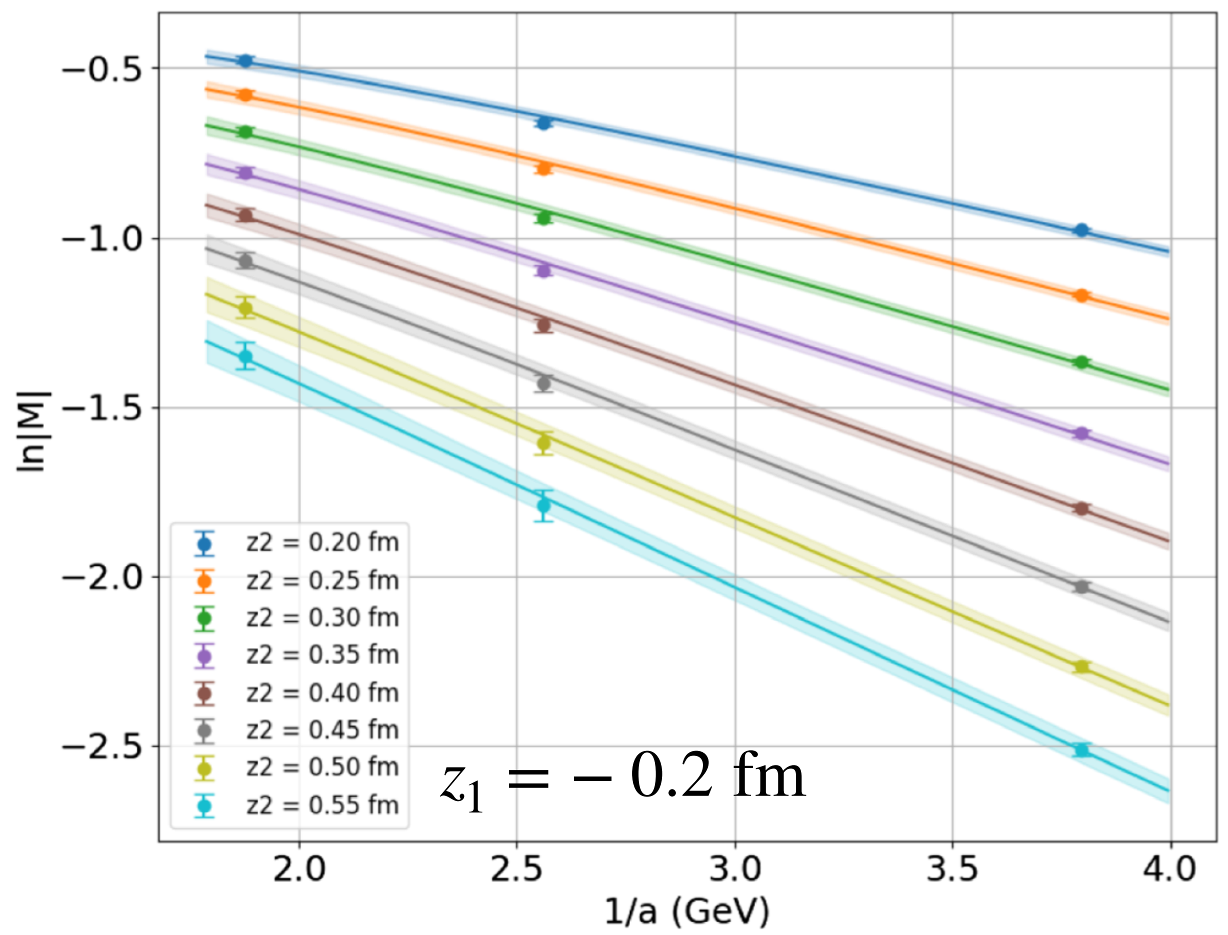
$$+\underbrace{\ln\left[1 + \frac{d}{\ln(a\Lambda_{\overline{\text{QCD}}})}\right]}_{\text{not sensitive to } a} + \underbrace{f(z_1, z_2)a^2}_{\text{fit lattice with per.}}\Bigg]$$

$$\tilde{z} = \begin{cases} |z_1 - z_2|, & z_1 z_2 < 0 \\ \max(|z_1|, |z_2|), & z_1 z_2 \geq 0 \end{cases}$$

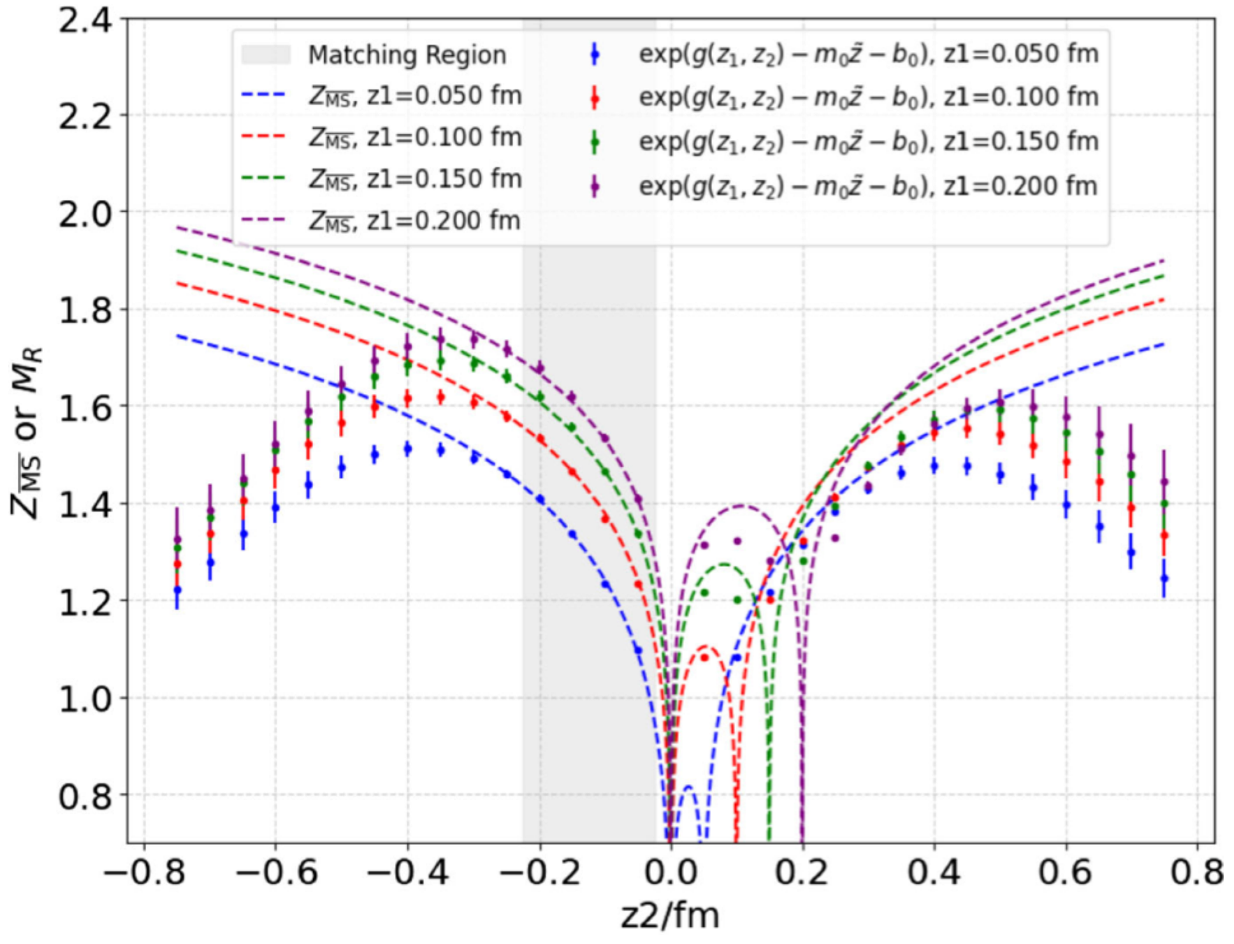
$$\frac{M_0(z_1, z_2, P_z = 0, a, \mu)}{Z_R(z_1, z_2, a, \mu)} = M_{\text{per}}(z_1, z_2, 0, \mu)$$

C. Han et al., JHEP 12 (2023) 044

Step 1: fit a dependent terms: $f(z_1, z_2)$, k



Step 2: fit a independent term: m_0

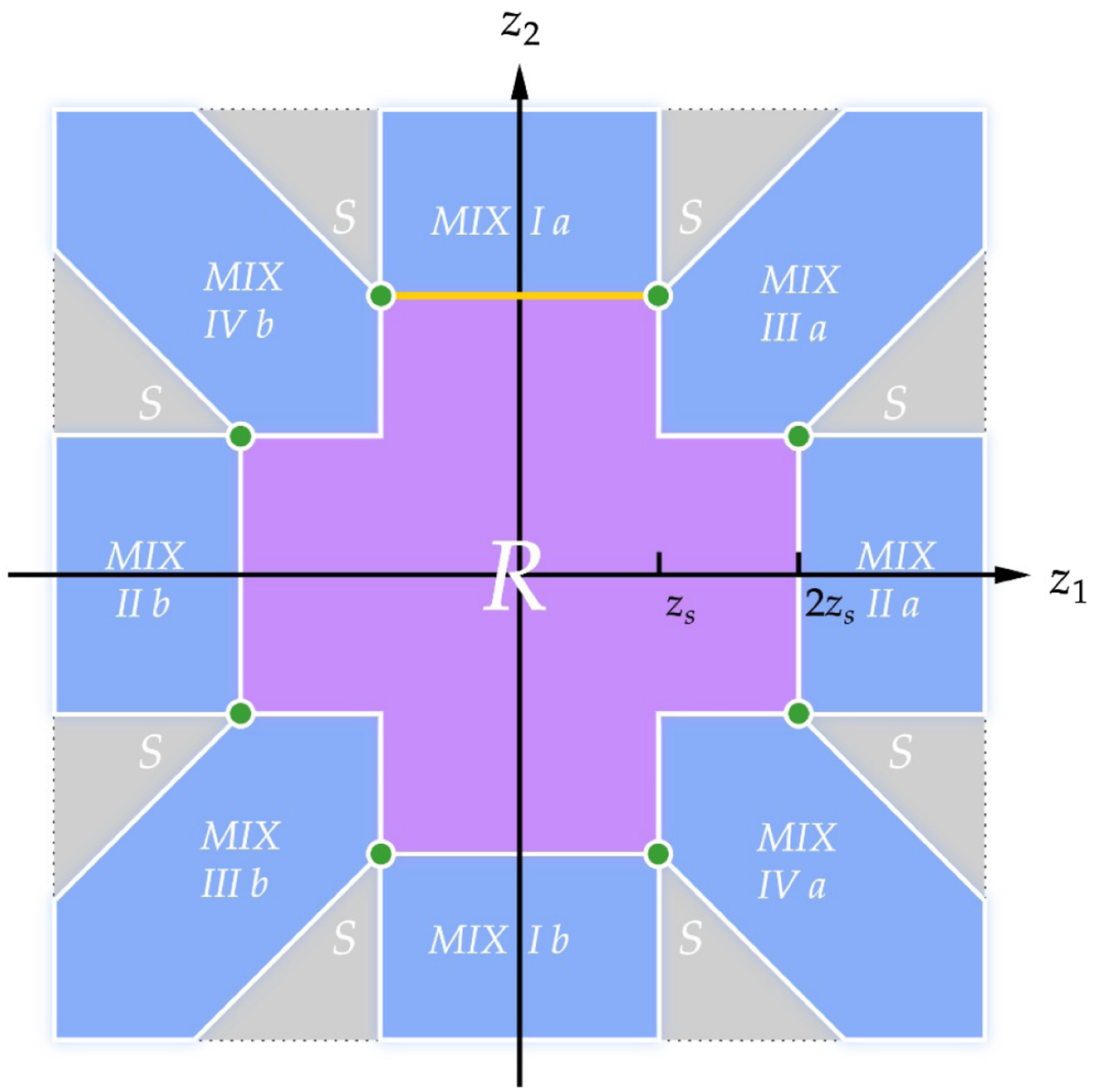


Framework 2.0 and results

renormalization

H. Bai et al., Phys.Rev.D 112 (2025) 11, 114515

hybrid renormalization: ratio(short) + self(long)



ratio

short-distance region

both z_1 and z_2 are small

ratio

Mixed region

one of z_1 and z_2 is small

self

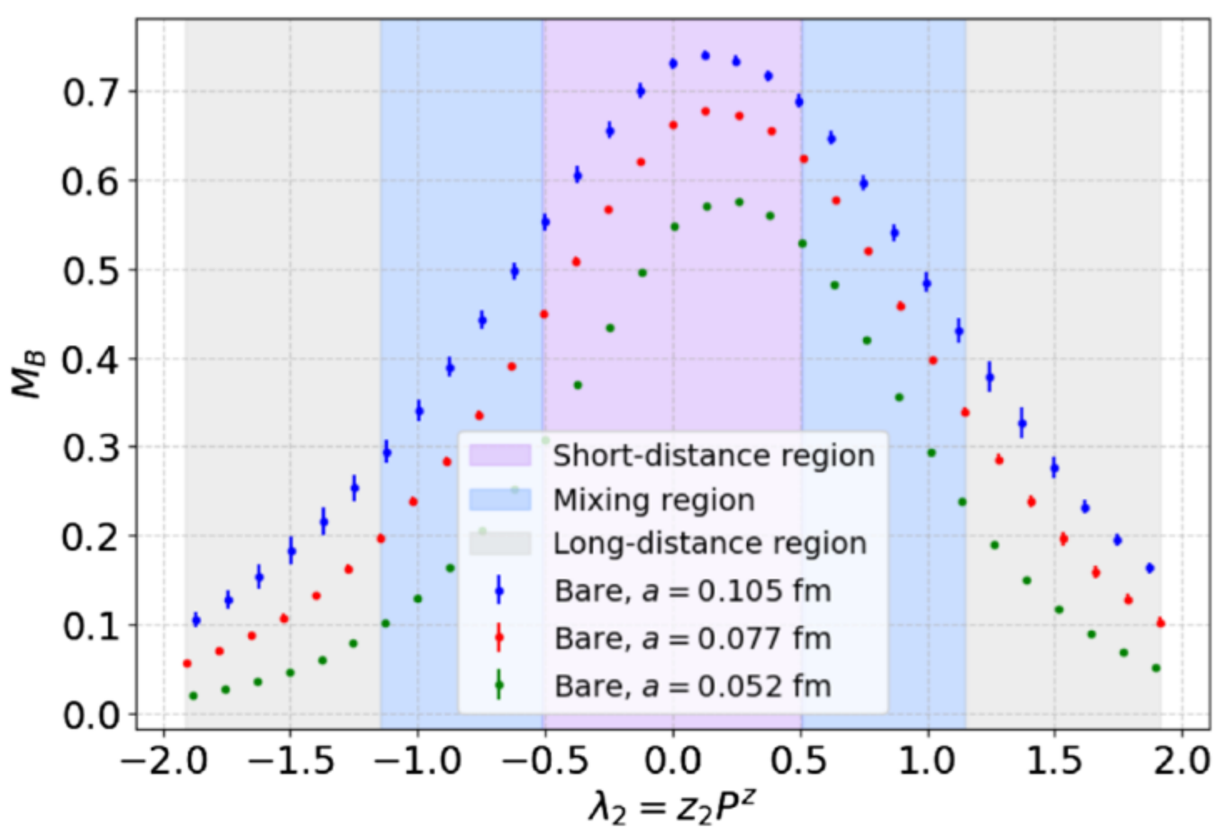
long-distance region

both of z_1 and z_2 are large

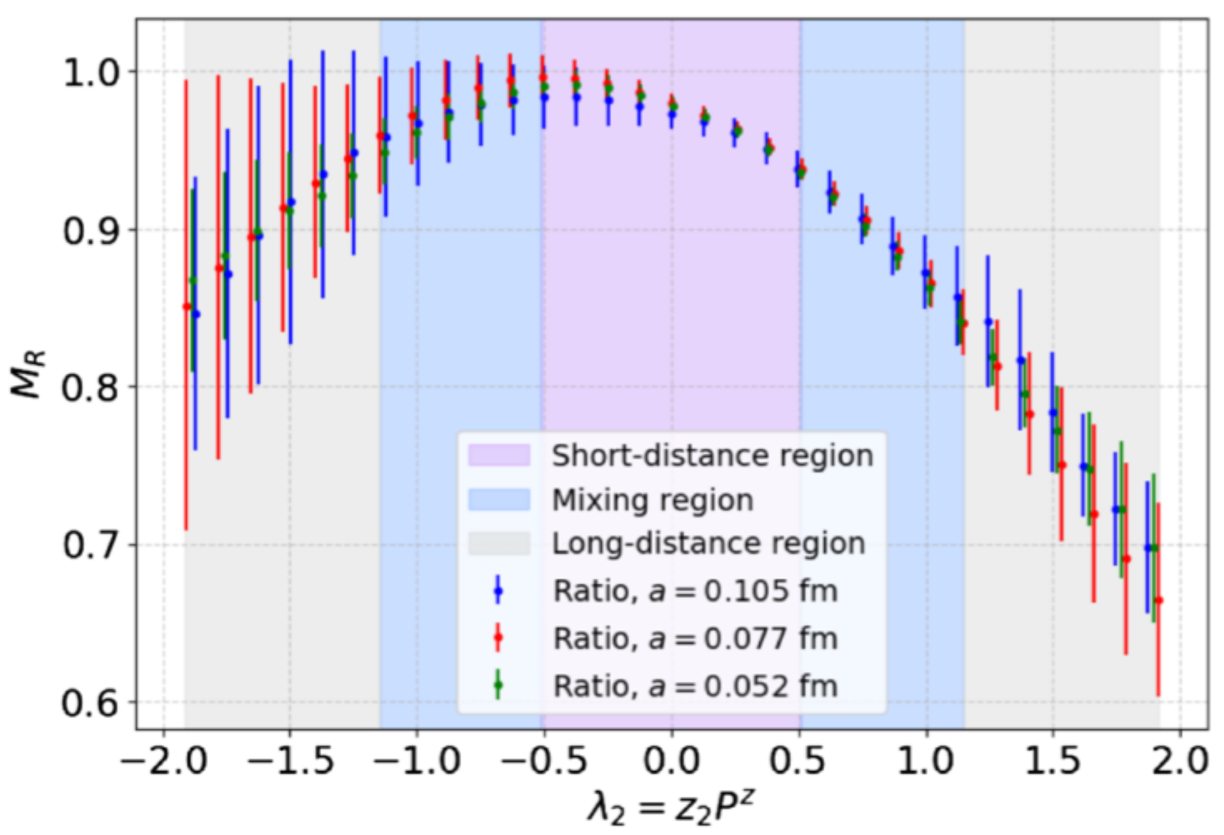
renormalization

$P^z = 0.5$ GeV, $z_1 = 0.25$ fm

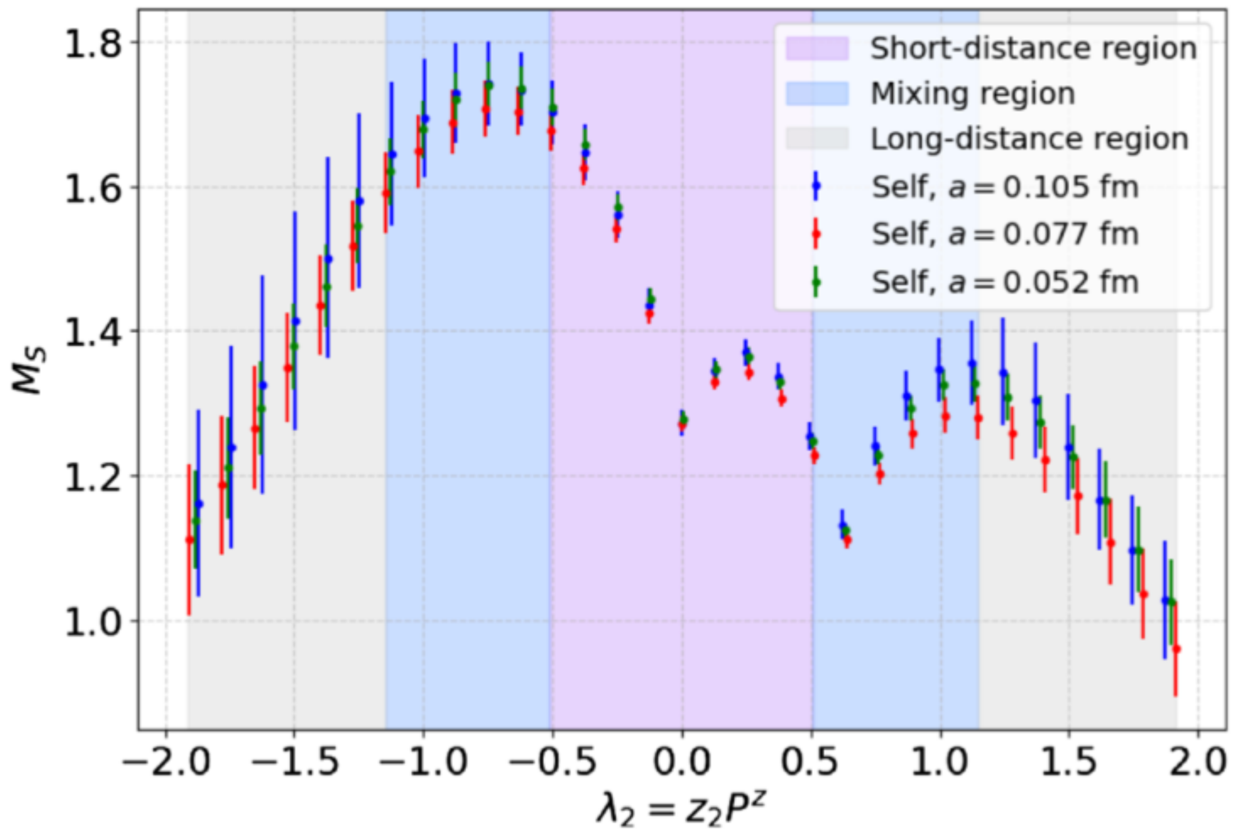
bare matrix elements



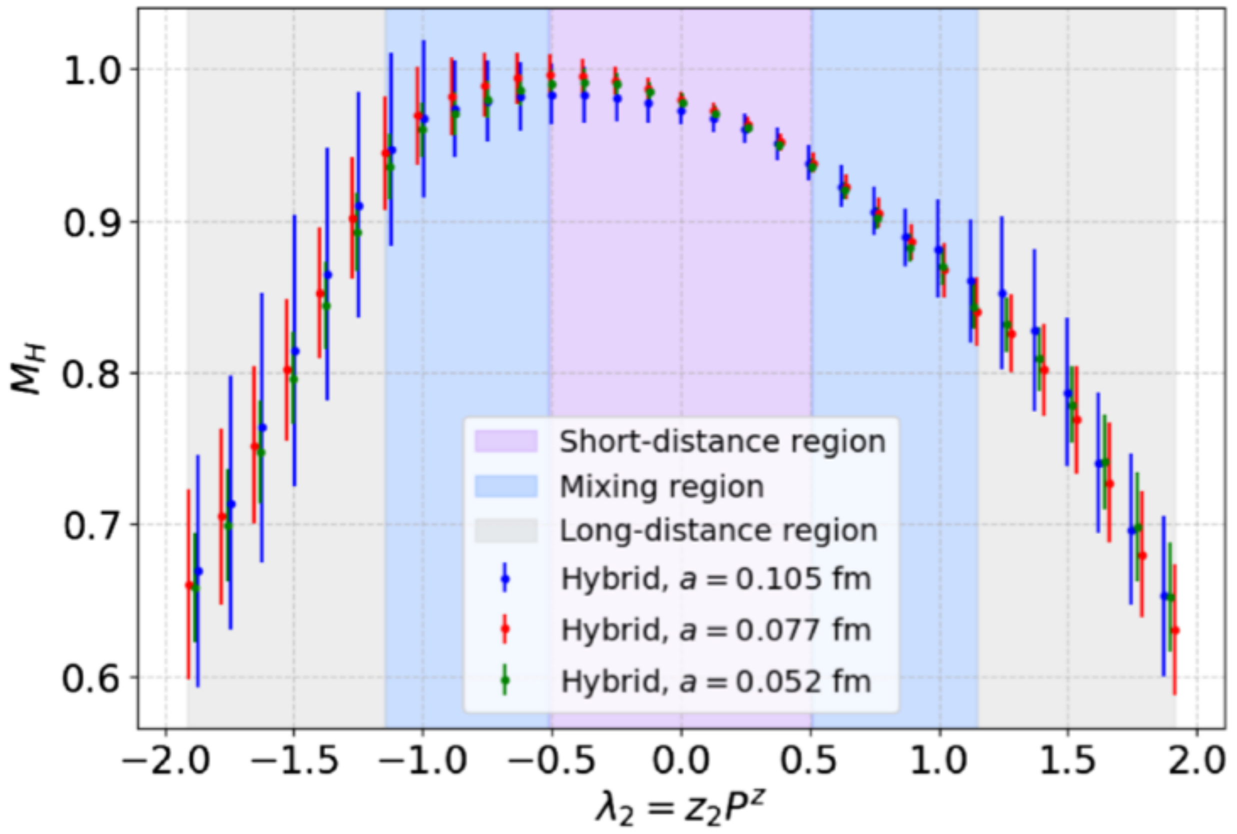
ratio everywhere



self everywhere



hybrid: ratio(short) + self(long)

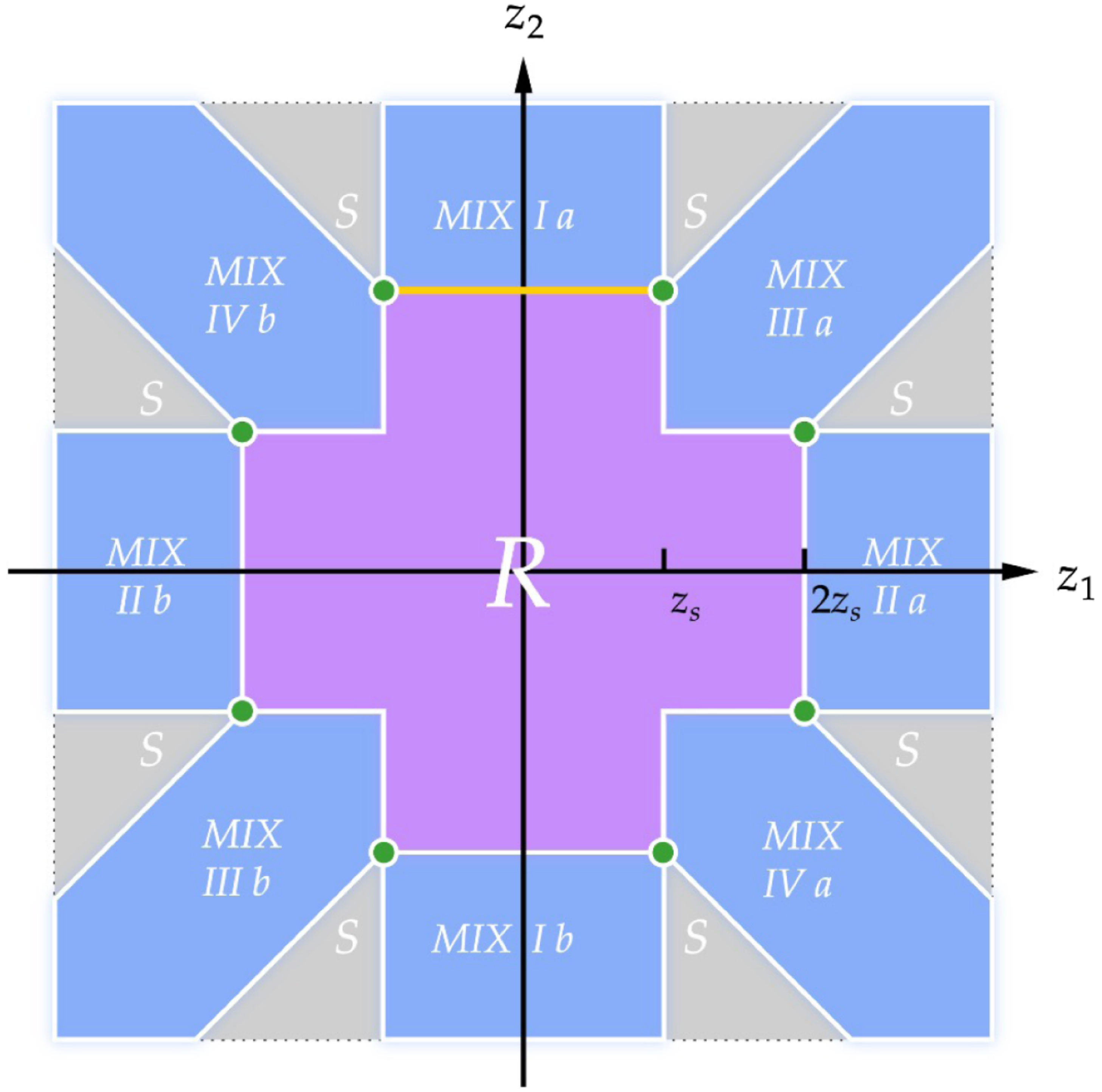


Framework 2.0 and results

renormalization

H. Bai et al., Phys.Rev.D 112 (2025) 11, 114515

hybrid renormalization: ratio(short) + self(long)



ratio

short-distance region

both z_1 and z_2 are small

ratio

Mixed region

one of z_1 and z_2 is small

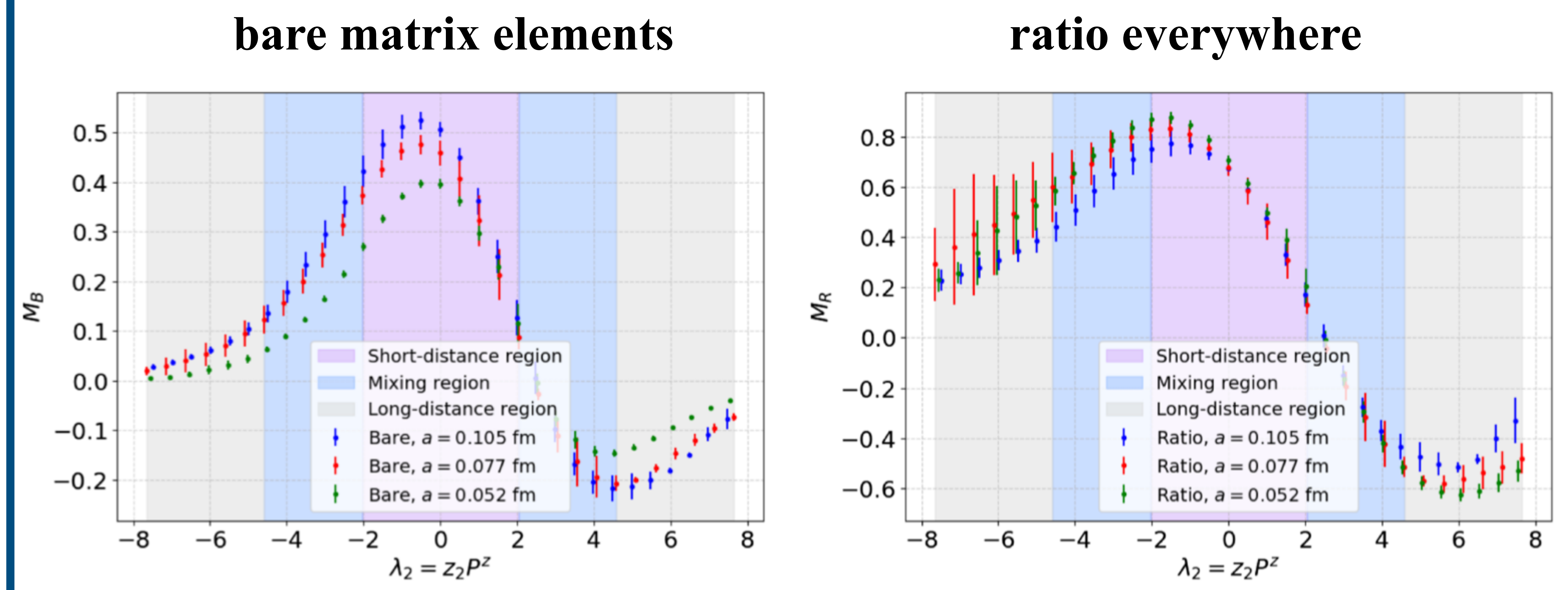
self

long-distance region

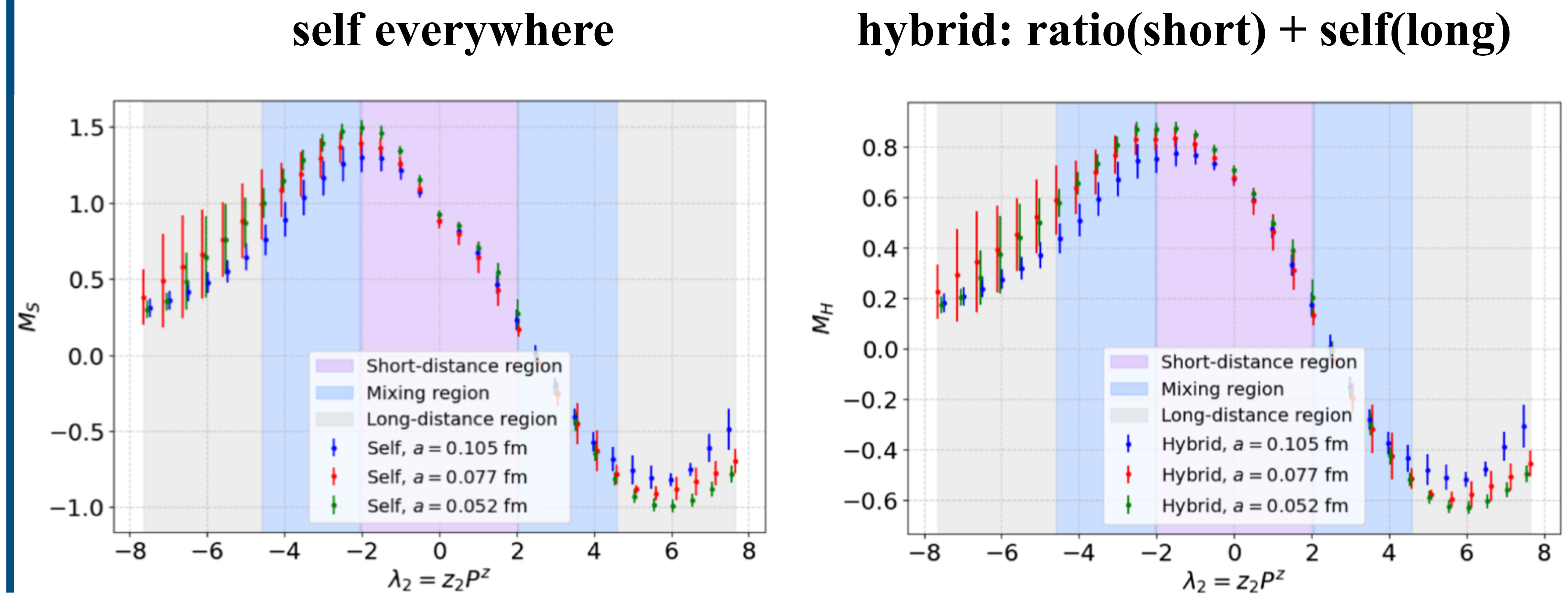
both of z_1 and z_2 are large

renormalization

$P^z = 2$ GeV, $z_1 = 0.25$ fm


bare matrix elements

ratio everywhere


self everywhere

hybrid: ratio(short) + self(long)

Framework 2.0 and results

extrapolation

$$A(z_1,z_2) = e^{iz_1P^z}e^{-\Lambda^{0^-}|z_2|}\left[\overset{\text{LA}}{\underline{P_1(iz_1,i\hat{z}_2,P^z)}} + \overset{\text{NLA}}{\underline{\frac{P_1'(iz_1,i\hat{z}_2,P^z)}{|z_1|}}} + \cdots\right]$$

$$+e^{-\Lambda^{0^-}|z_2|}\left[P_2(iz_1,i\hat{z}_2,P^z) + \frac{P_2'(iz_1,i\hat{z}_2,P^z)}{|z_1|} + \cdots\right]$$

$$+e^{iz_2P^z}e^{-\Lambda^{1/2^-}|z_2|}\left[P_3(iz_1,i\hat{z}_2,P^z) + \frac{P_3'(iz_1,i\hat{z}_2,P^z)}{|z_1|} + \cdots\right]$$

$$+\cdots$$

$|z_2| \to \infty, |z_1 - z_2|$ keep finite

$|z_2| \to \infty, |z_1 - z_2| \to \infty$

$|z_1| \to \infty, |z_2|$ keep finite, $z_1z_2 > 0$

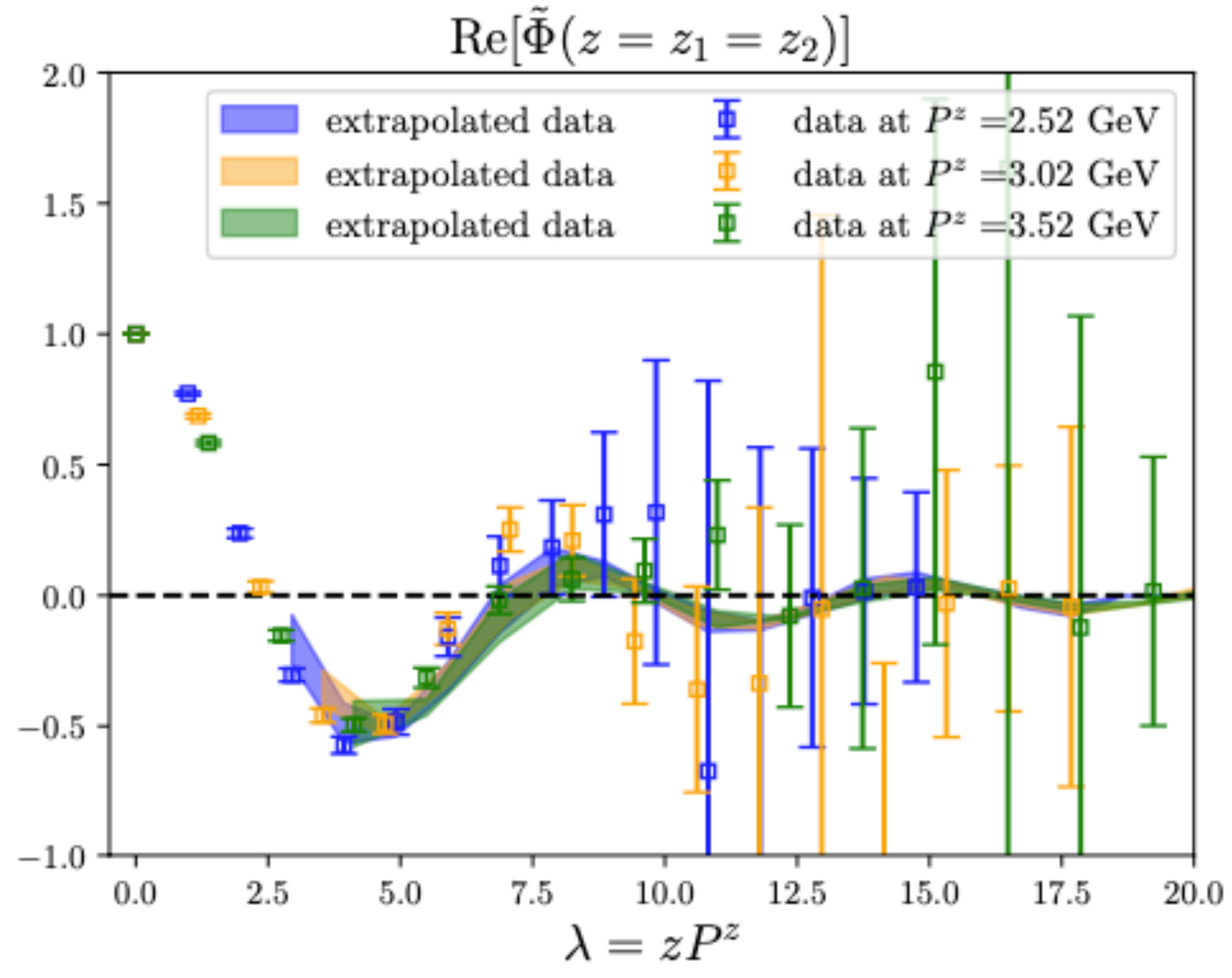
$|z_1| \to \infty, |z_2|$ keep finite, $z_1z_2 < 0$

$|z_2| \to \infty, |z_2| \to \infty$

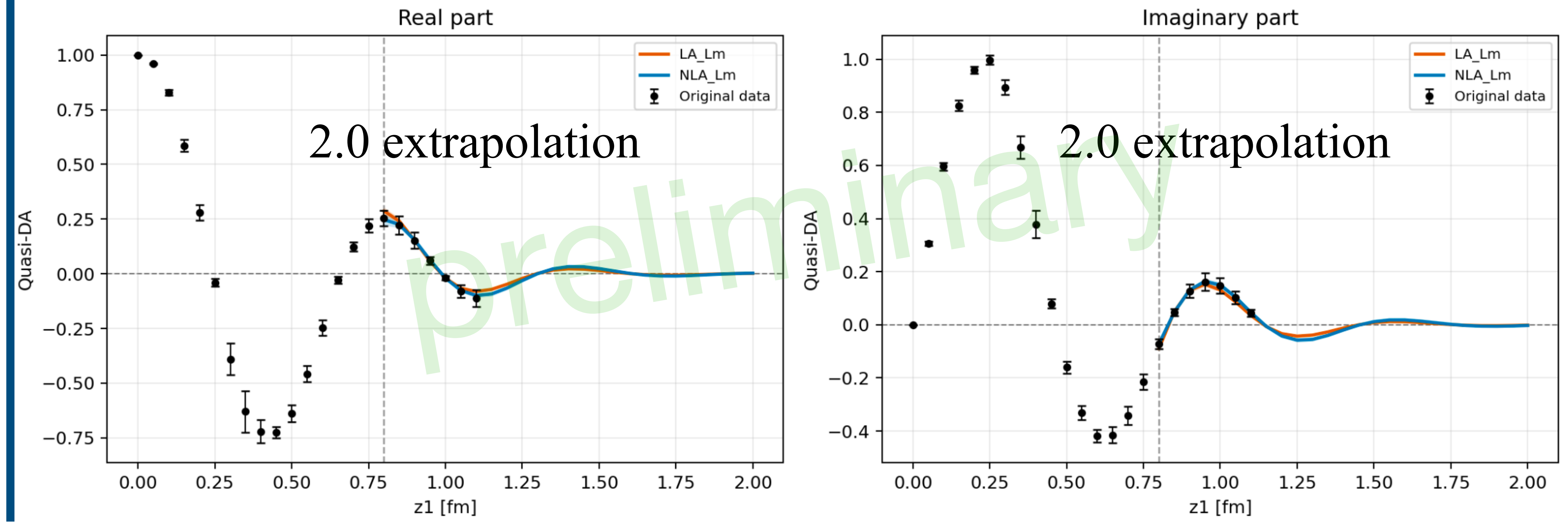
$P^z = 2$ GeV

extrapolation

1.0 extrapolation



$z_1 = z_2,\ \ z_1 + z_2 > 0$



2.0 extrapolation

2.0 extrapolation

Framework 2.0 and results

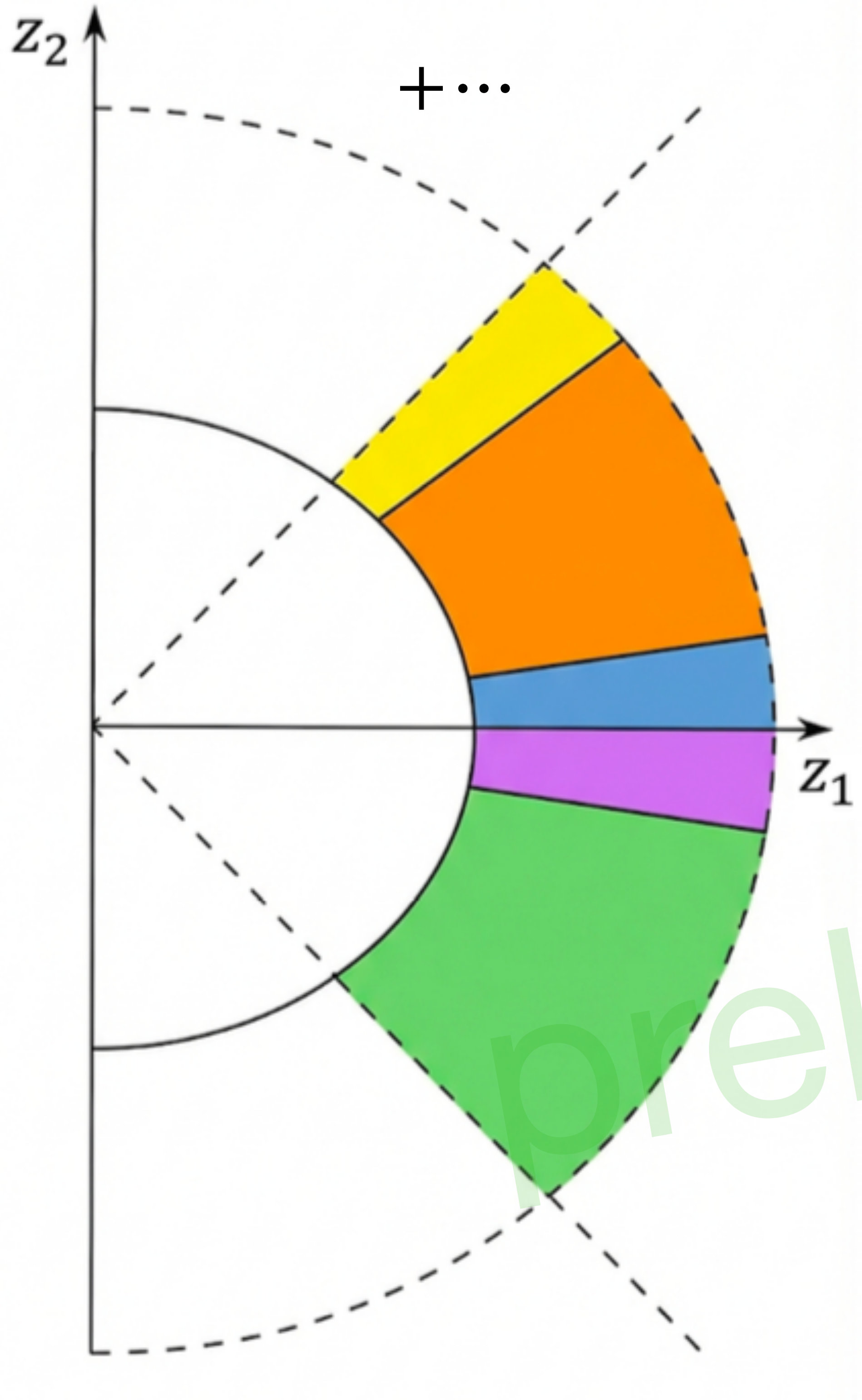
extrapolation

$$A(z_1,z_2)=e^{iz_1P^z}e^{-\Lambda^{0^-}|z_2|}\left[\overset{\text{LA}}{P_1(iz_1,i\hat{z}_2,P^z)}+\overset{\text{NLA}}{\frac{P_1'(iz_1,i\hat{z}_2,P^z)}{|z_1|}}+\cdots\right]$$

$$+e^{-\Lambda^{0^-}|z_2|}\left[P_2(iz_1,i\hat{z}_2,P^z)+\frac{P_2'(iz_1,i\hat{z}_2,P^z)}{|z_1|}+\cdots\right]$$

$$+e^{iz_2P^z}e^{-\Lambda^{1/2^-}|z_2|}\left[P_3(iz_1,i\hat{z}_2,P^z)+\frac{P_3'(iz_1,i\hat{z}_2,P^z)}{|z_1|}+\cdots\right]$$

$$+\cdots$$



$|z_2| \to \infty, |z_1 - z_2|$ keep finite

$|z_2| \to \infty, |z_1 - z_2| \to \infty$

$|z_1| \to \infty, |z_2|$ keep finite, $z_1z_2 > 0$

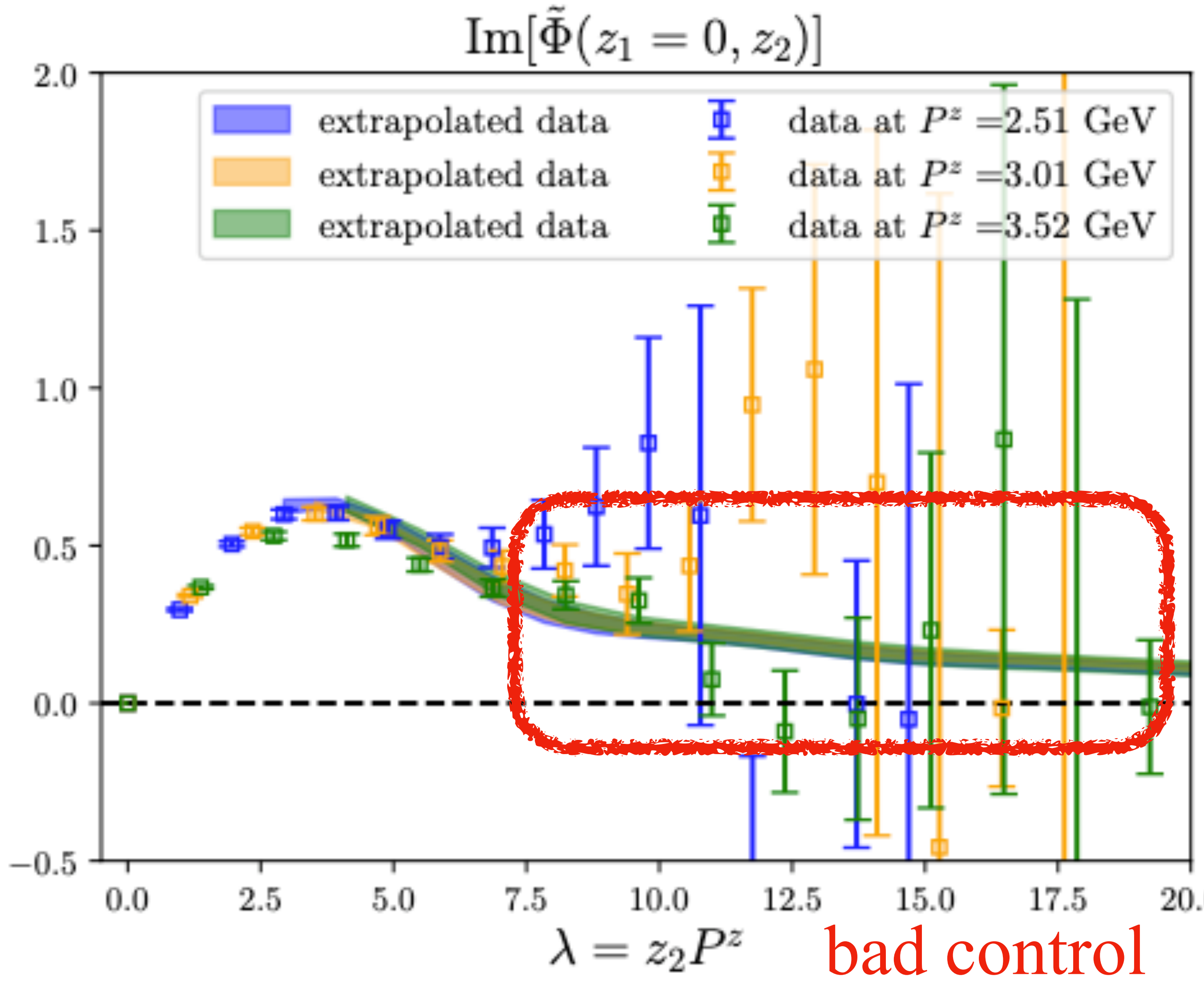
$|z_1| \to \infty, |z_2|$ keep finite, $z_1z_2 < 0$

$|z_2| \to \infty, |z_2| \to \infty$

$P^z = 2$ GeV

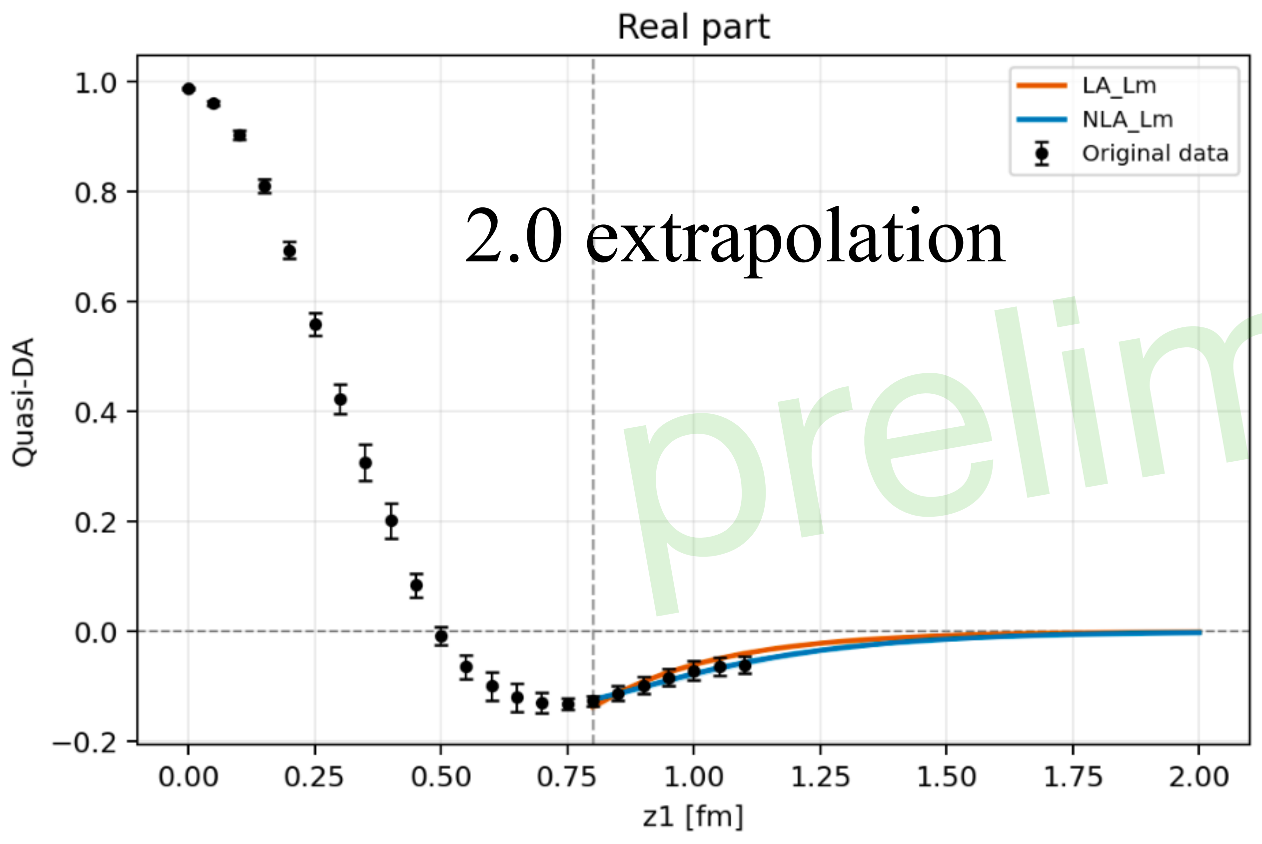
extrapolation

1.0 extrapolation

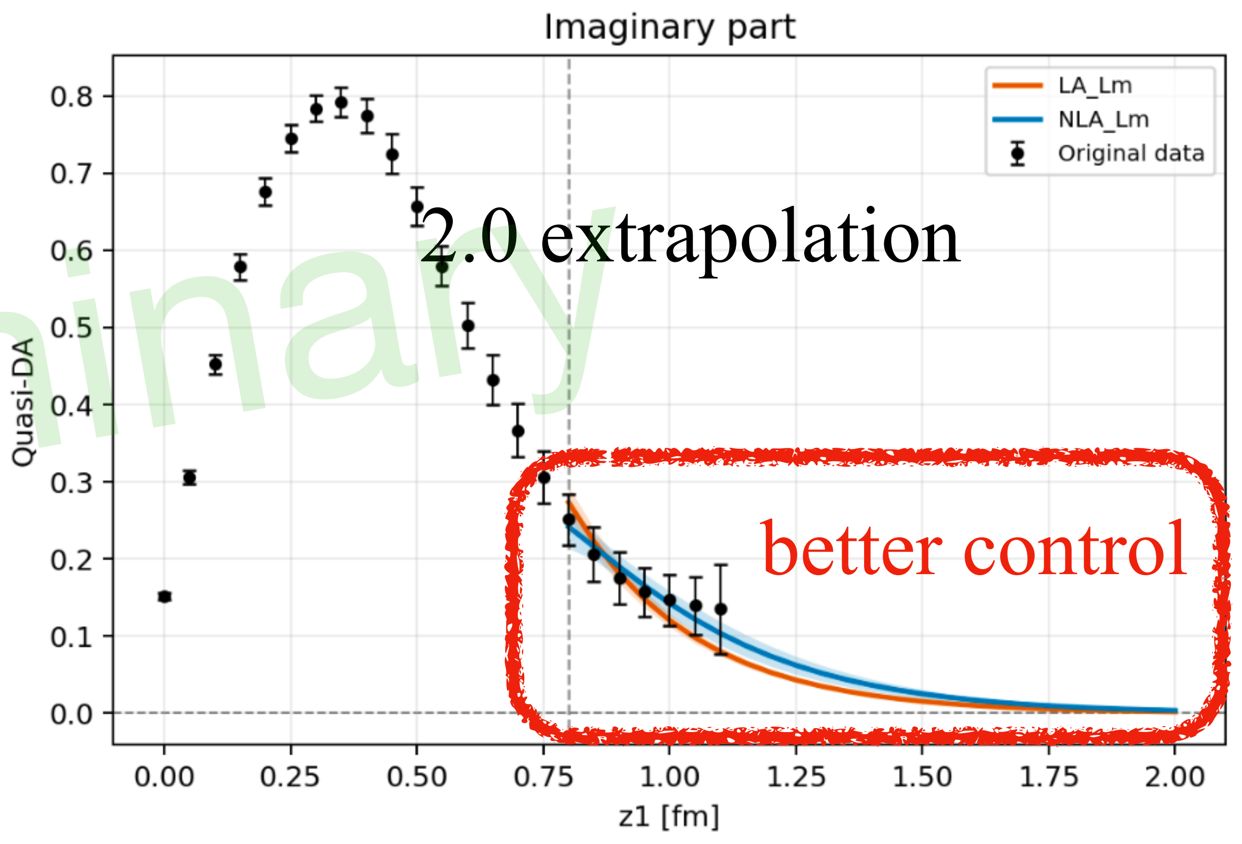


bad control

$z_2 = a,\ \ z_1 > 0$



2.0 extrapolation



2.0 extrapolation

better control

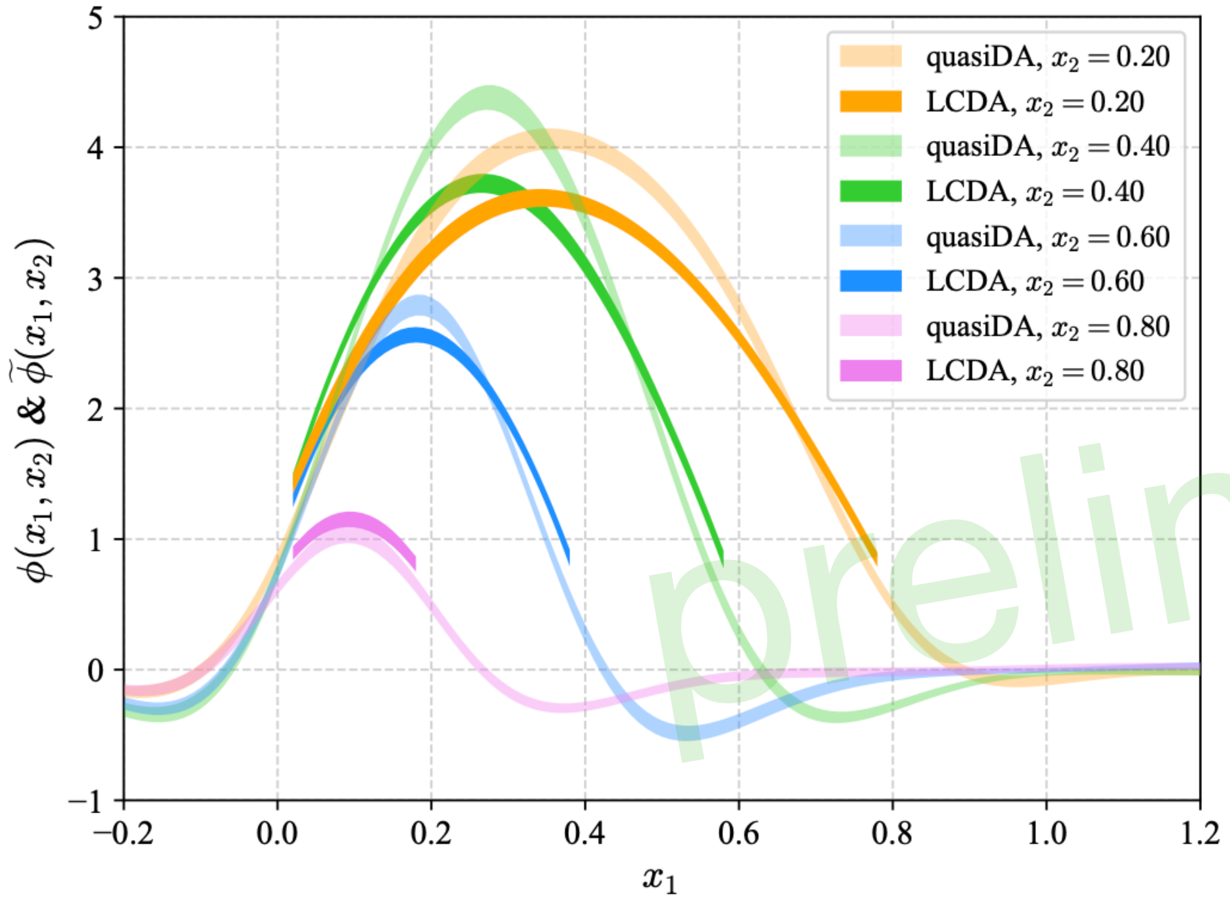
Framework 2.0 and results

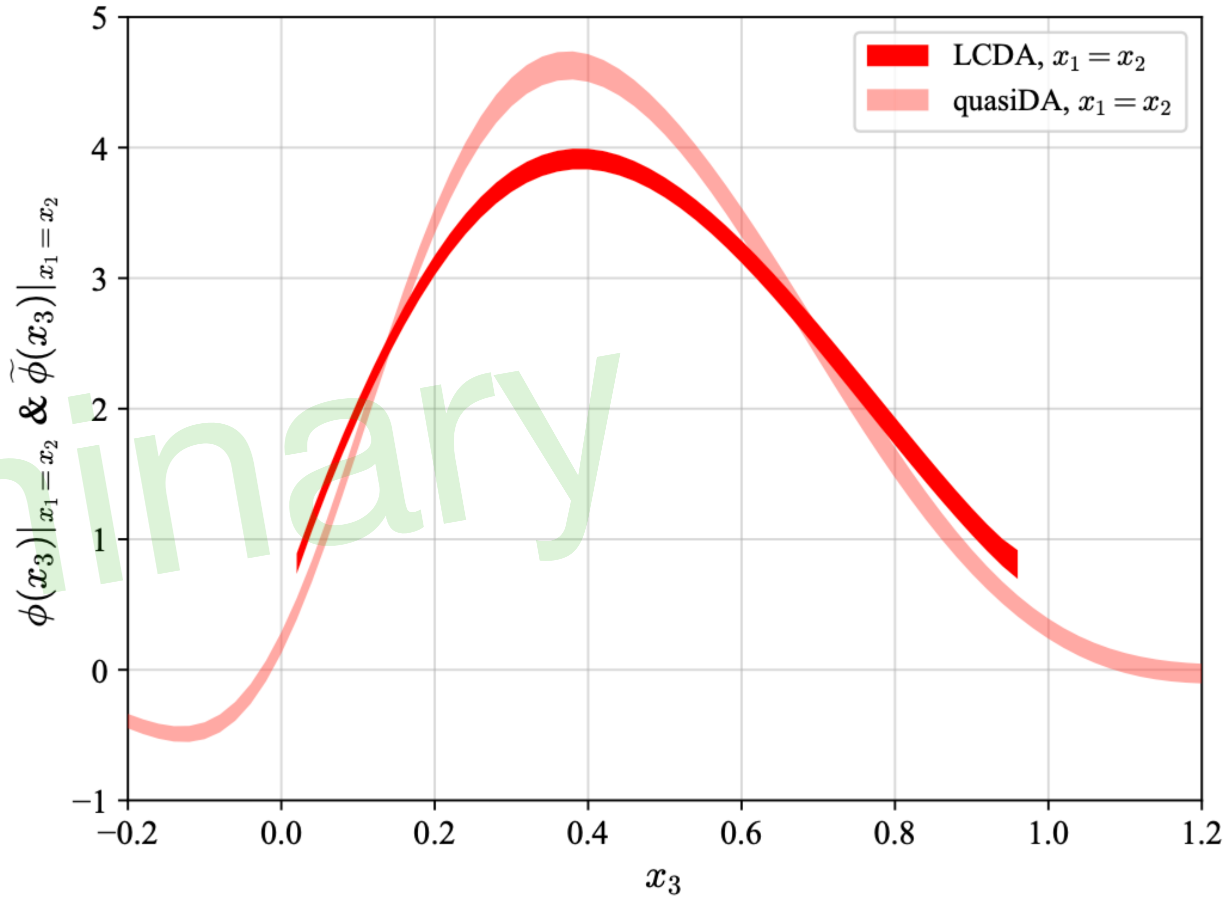
matching

$$\phi(x_1,x_2,\mu) = \tilde{\phi}(x_1,x_2,\mu) + \int dy_1 dy_2 \underline{C^{(1)}(x_1,x_2,y_1,y_2,P^z,\mu)}\tilde{\phi}(x_1,x_2,\mu)$$

$$+\mathcal{O}\left(\frac{\Lambda^2_{QCD}}{(x_1P^z)^2}\right) + \mathcal{O}\left(\frac{\Lambda^2_{QCD}}{(x_2P^z)^2}\right) + \mathcal{O}\left(\frac{\Lambda^2_{QCD}}{([1-x_1-x_2)P^z]^2}\right)$$

$$C^{(1)}(x_1,x_2,y_1,y_2,P^z,\mu) = \left[\underbrace{C_{\overline{\text{MS}}}(x_1,x_2,y_1,y_2,P^z,\mu)}_{\overline{\text{MS}}\text{ matching kernel}} + \underbrace{\delta C_H^{(1)}(x_1,x_2,y_1,y_2,P^z,\mu)}_{\text{Hybrid counter term}}\right]$$

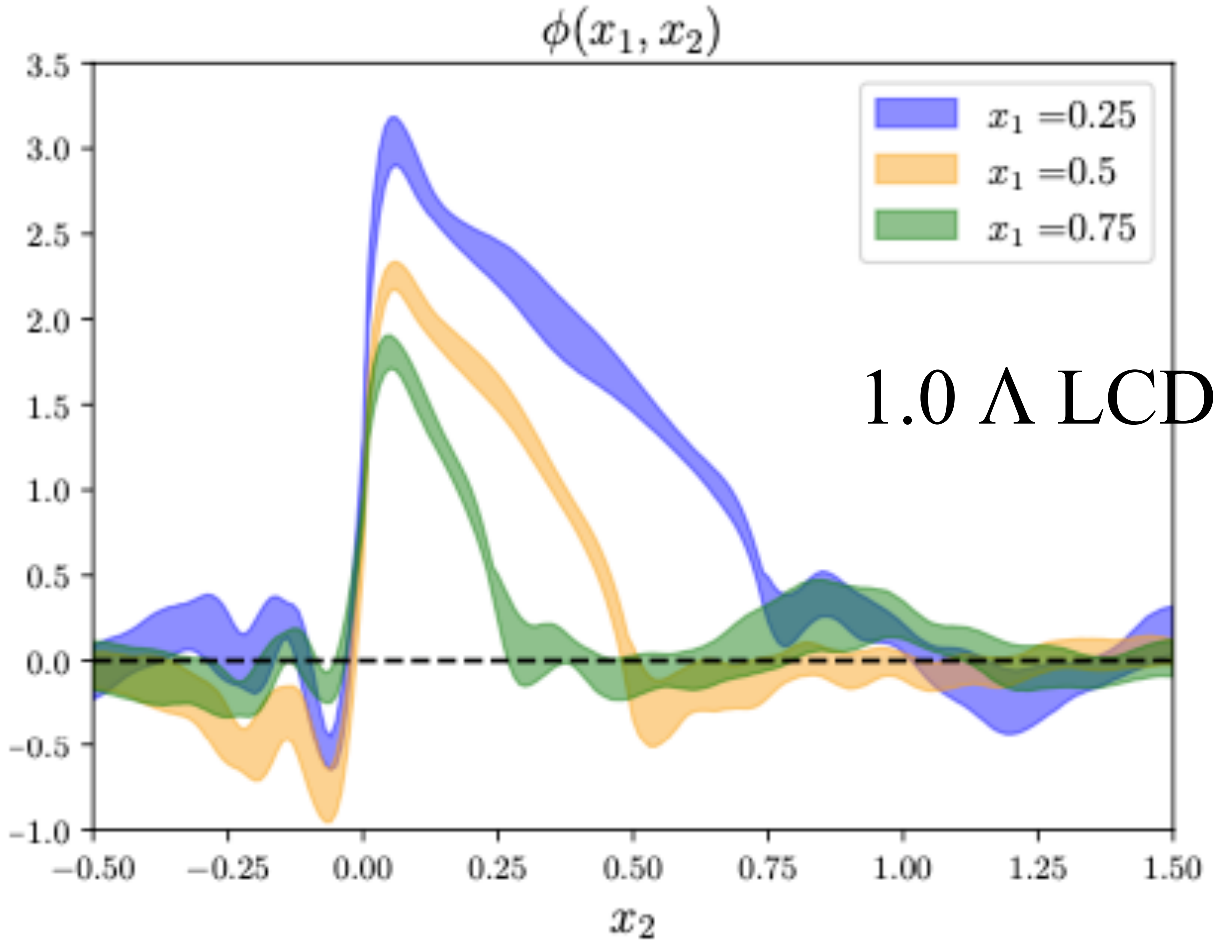


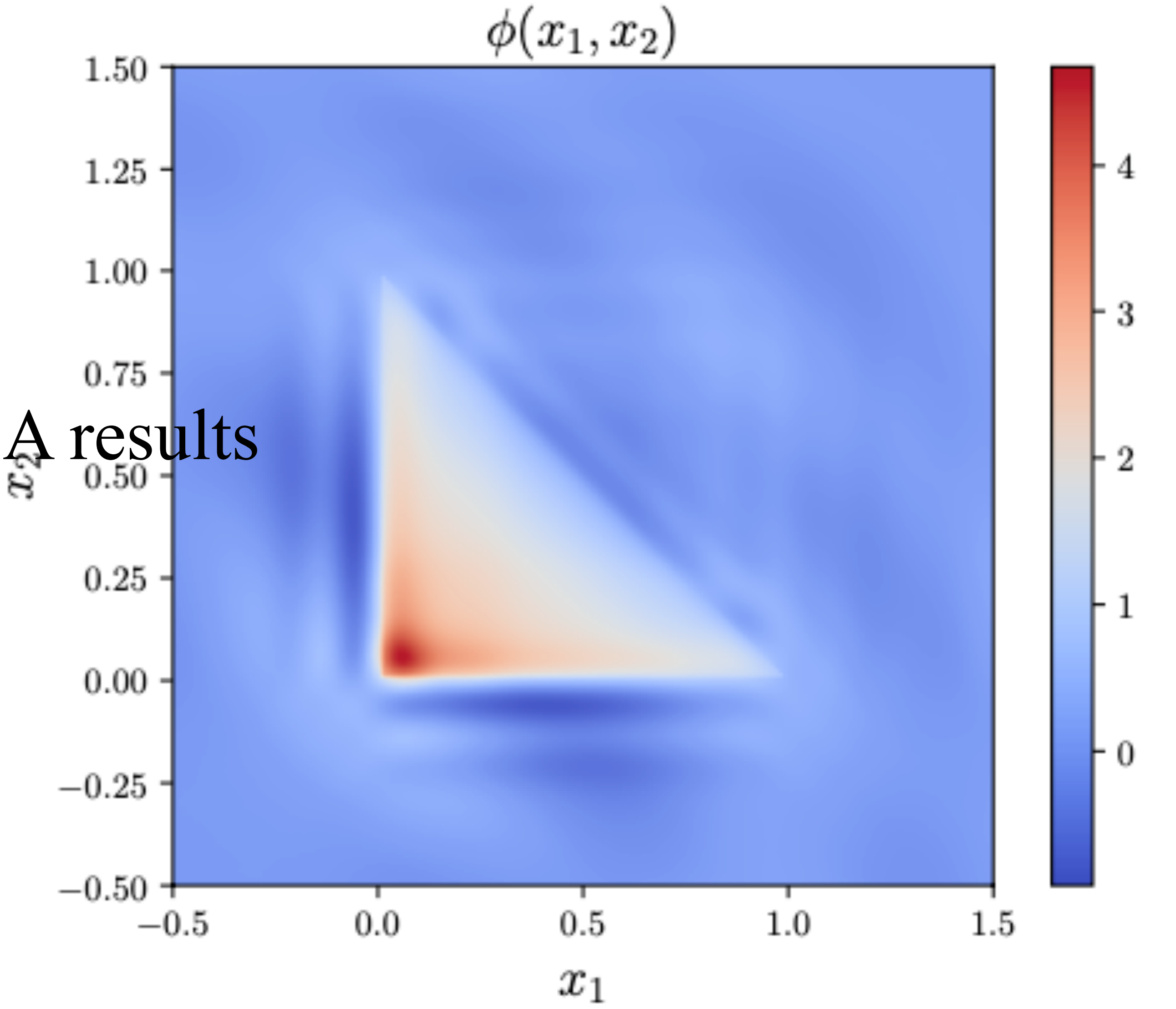


matching

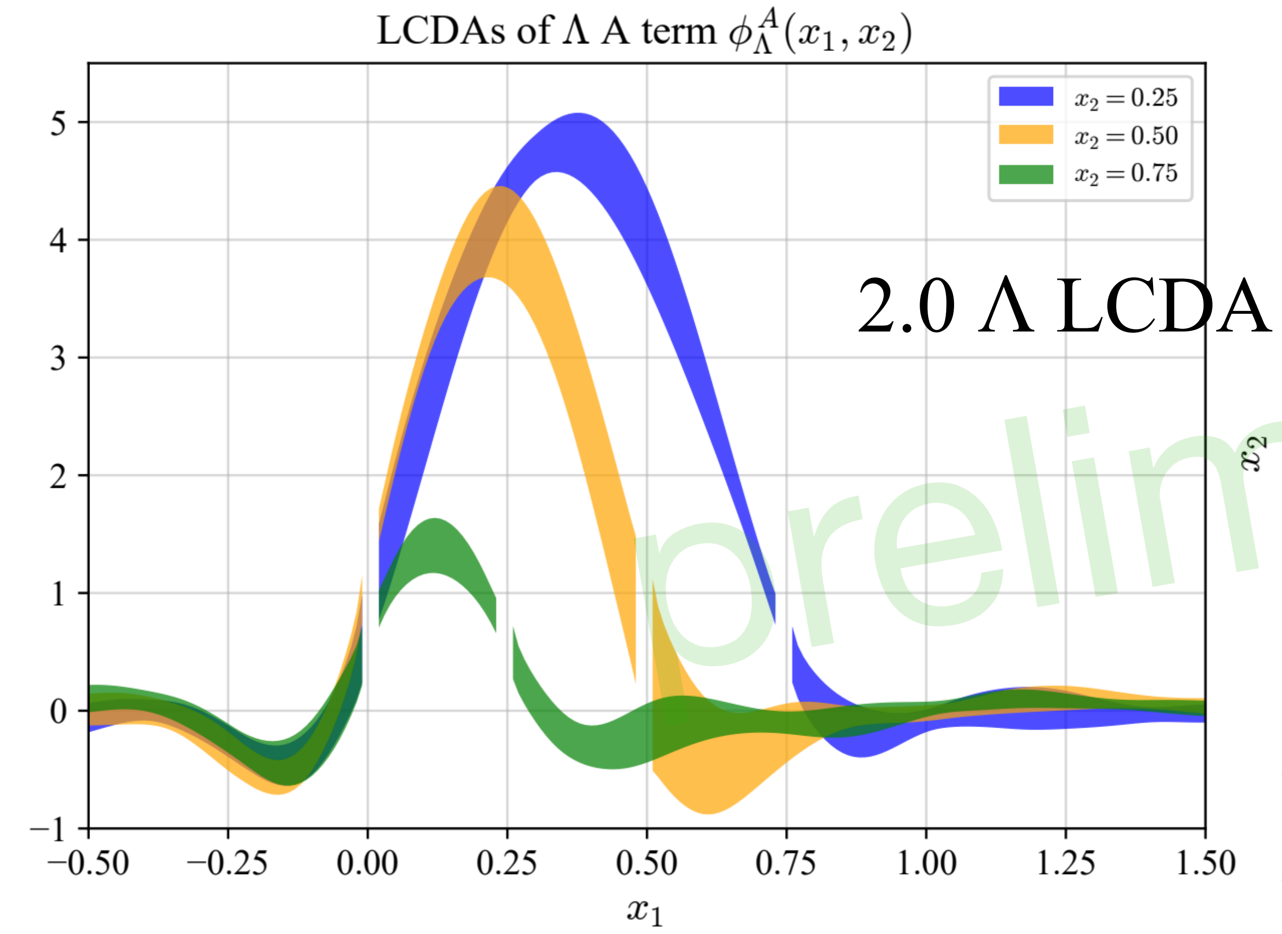
x_2 dependence when x_1 is fixed

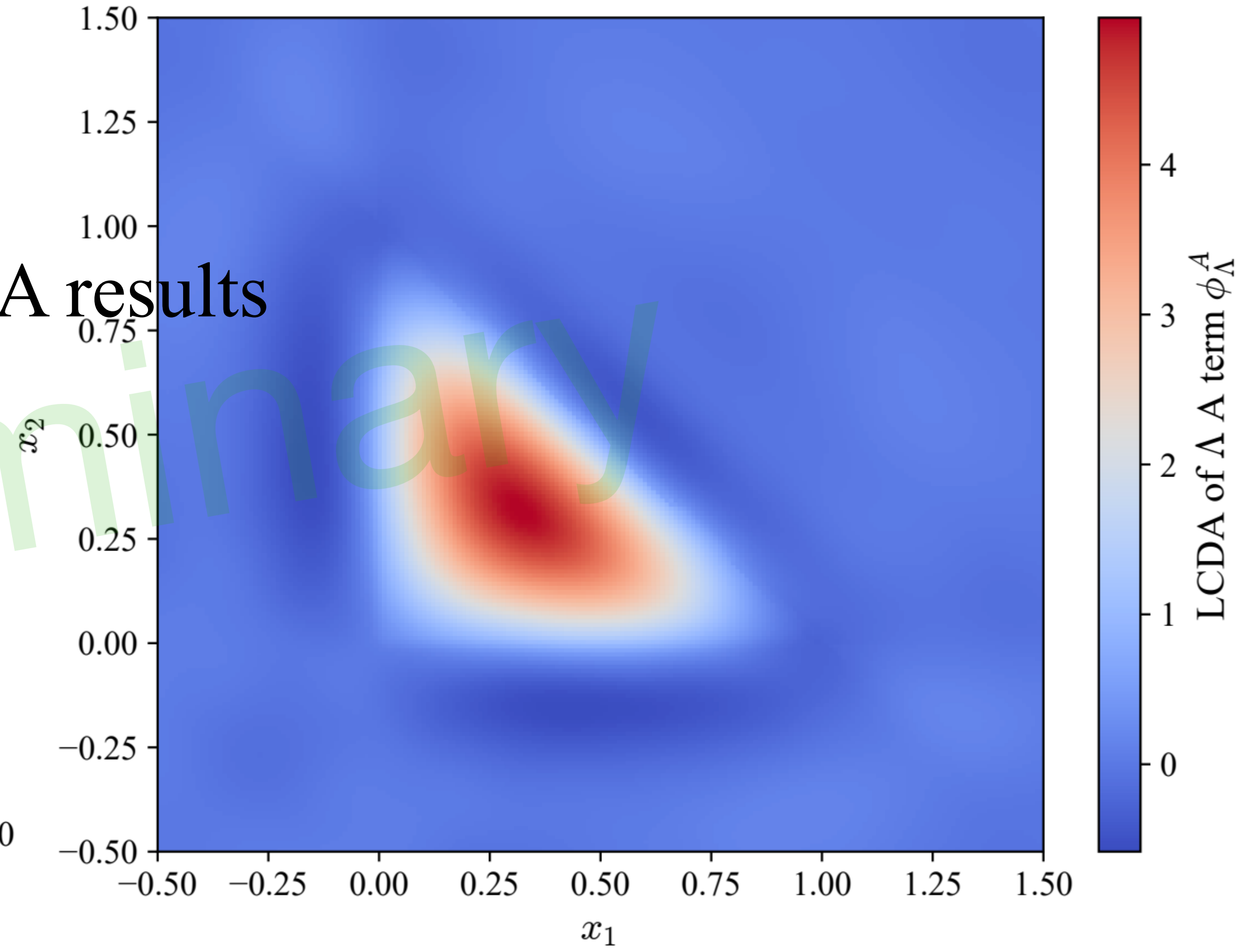
3D heat map





1.0 Λ LCDA results





2.0 Λ LCDA results

Summary

- Understanding baryon LCDAs is crucial for studying different exclusive processes and as key inputs for form factors.

- LaMET and Lattice QCD now enable us to perform ab initio calculations of baryon LCDAs.

- Application to proton LCDA is under investigation.

Thank you!

- Going forward, the investigation of leading renormalon resummation (LRR) and renormalization group resummation (RGR) should bring improvements.