The α_s analyses from hadronic τ decays from OPAL and ALEPH data

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In coll. with O Catà, M Golterman, M Jamin, A Mahdavi, K Maltman, J Osborne, and S Peris

Main references:

1. Phys Rev D **91** (2015)
2. Phys Rev D **85** (2012)
3. Phys Rev D **84** (2011)

Outline

- **Introduction**
- **Analysis with OPAL data**
- **Analysis with ALEPH data**
- **Averaged results, comparison, preliminary combined fits**
- **Final remarks and conclusion**

The main subject of this talk

- Results from analyses based on OPAL and ALEPH data, robustness, averages, quality of fits, combinations etc

$$\alpha_s(m_\tau^2) = 0.303 \pm 0.009 \quad \text{FOPT}$$
$$\alpha_s(m_\tau^2) = 0.319 \pm 0.012 \quad \text{CIPT}$$

$$\alpha_s(m_Z^2) = 0.1165 \pm 0.0012 \; \text{FOPT}$$
$$\alpha_s(m_Z^2) = 0.1185 \pm 0.0015 \; \text{CIPT}$$

DB, Golterman, Maltman, Osborne, Peris 2015
Averaged values from analyses of ALEPH and OPAL data

$$R^\tau_{V+A;ud} = N_c\, S_{\rm EW}\, |V_{ud}|^2 (1 + \delta_{\rm P} + \delta_{\rm NP})$$

$$\delta_{\rm NP} = 0.020 \pm 0.009 \quad \text{FOPT}$$
$$\delta_{\rm NP} = 0.016 \pm 0.010 \quad \text{CIPT}$$

ALEPH based analysis

The main subject of this talk

- Results from analyses based on OPAL and ALEPH data, robustness, averages, quality of fits, combinations etc

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DB, Golterman, Maltman, Osborne, Peris 2015
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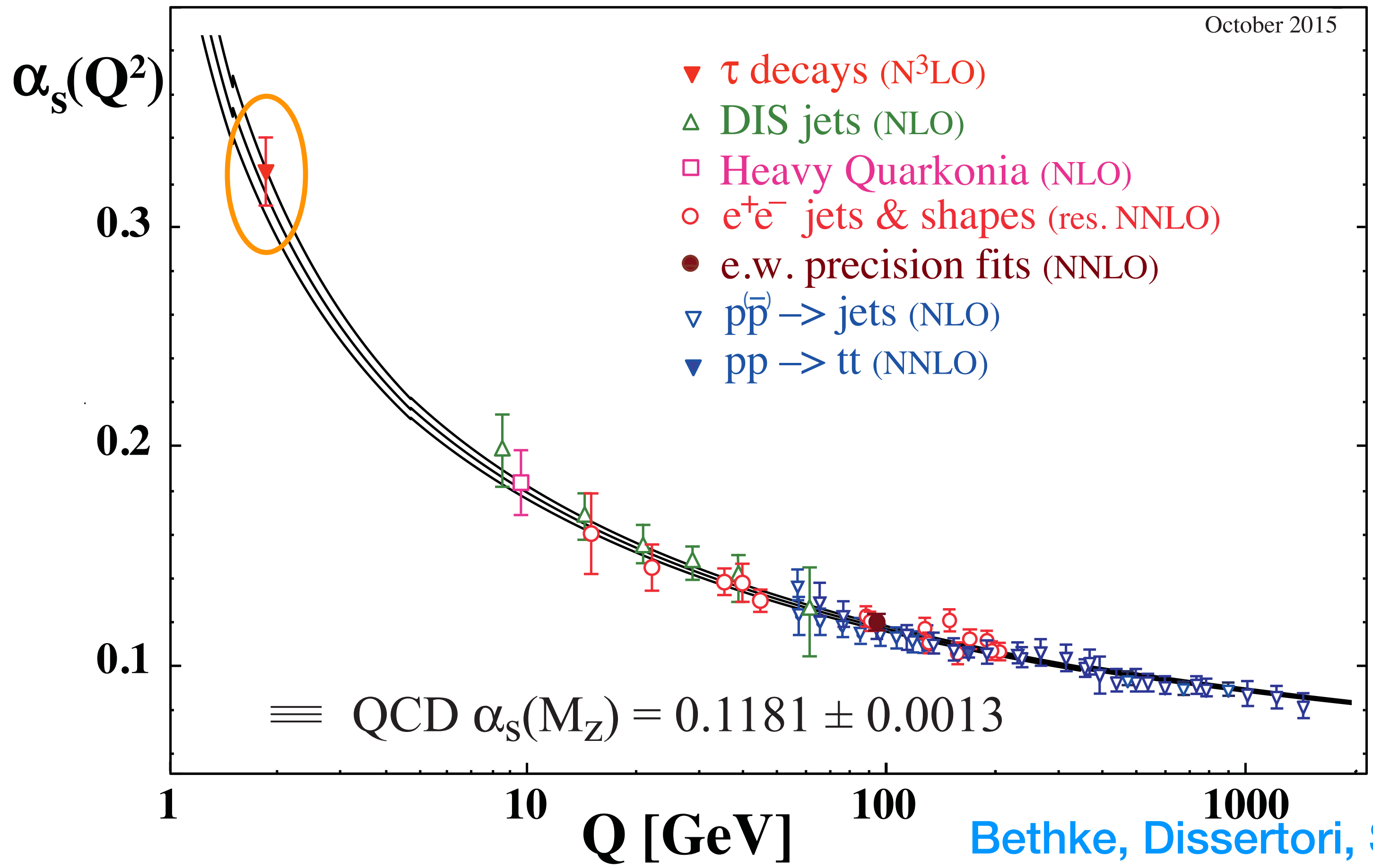
ALEPH based analysis



Bethke, Dissertori, Salam '16 (for the PDG)

Framework

- Uses three weight functions

$$w_1 = 1,\ w_2 = 1 - x^2,\ w_\tau = (1-x)^2(1+2x)$$

  - Only two LO condensate contributions $C_{6,8}$. Good perturbative behaviour.

Beneke, DB, Jamin, '13

- Employs a window of values $s_0^{\rm min} \leq s_0 \leq m_\tau^2$, more data points.

- DV contribution. For $s \geq s_0^{\rm min}$

$$\rho^{\rm DV}_{V/A}(s) = \exp\left(-\delta_{V/A} - \gamma_{V/A} s\right) \sin(\alpha_{V/A} + \beta_{V/A} s)$$

- Analysis done in *V* and/or *A* (necessary)

- Fully self-consistent. No OPE or DV contribution is neglected.

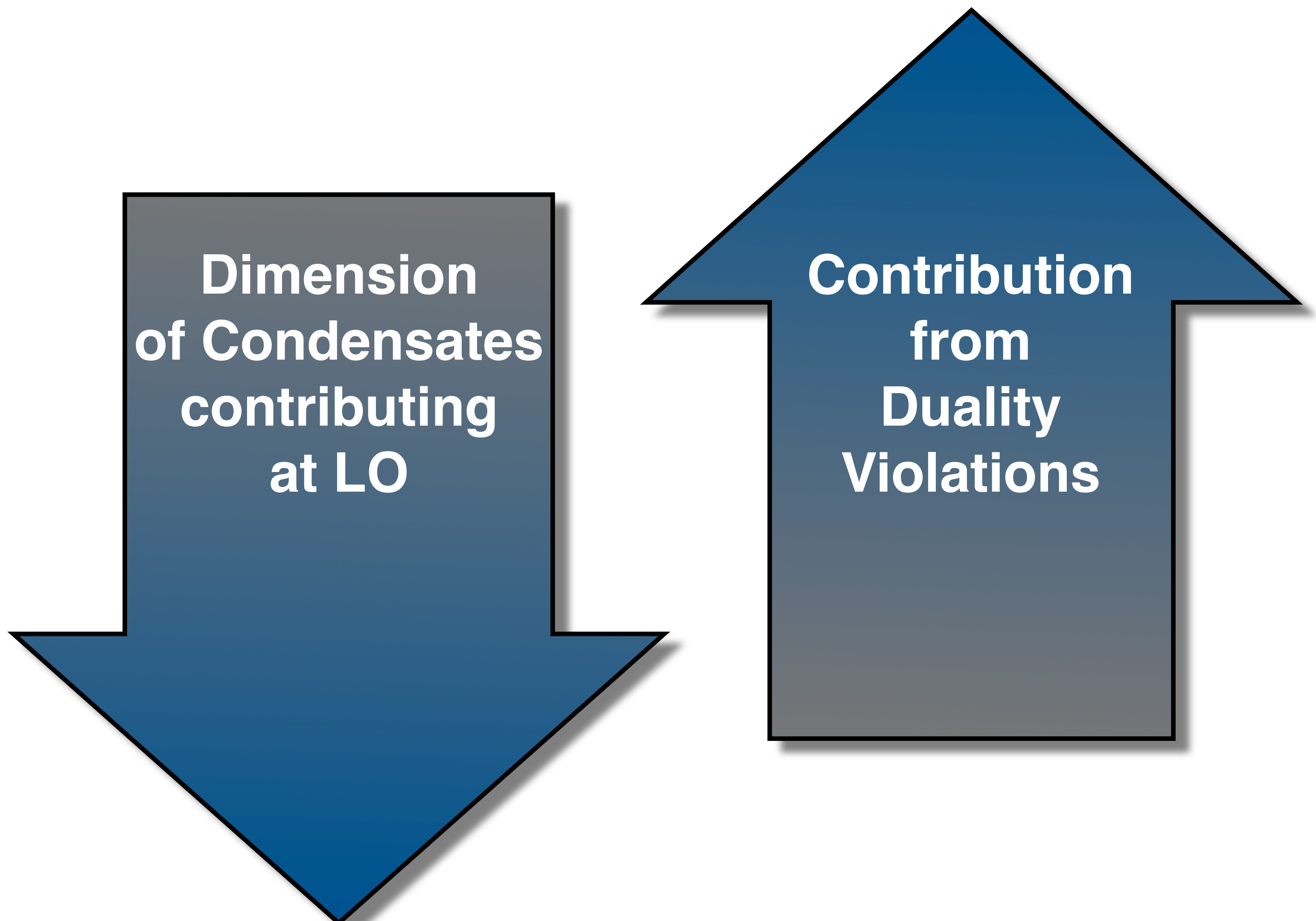
⟶ see Peris' talk

- DVs must be taken into account. (In order to achieve self-consistency)

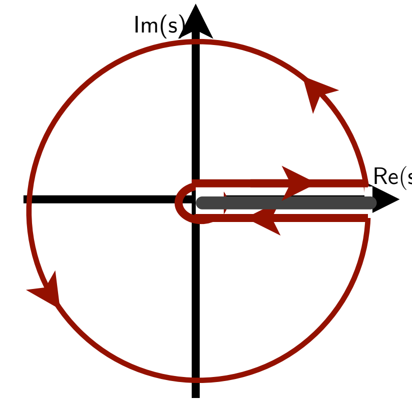
Main message

One cannot suppress Duality Violations and condensates at the same time

⟶ see Peris' talk

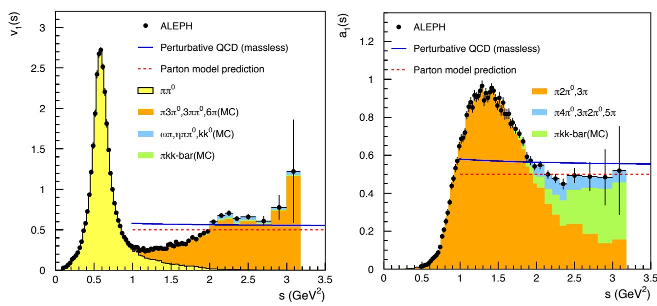


- Inclusion of DVs makes safer the use of lower s_0 and improves self-consistency

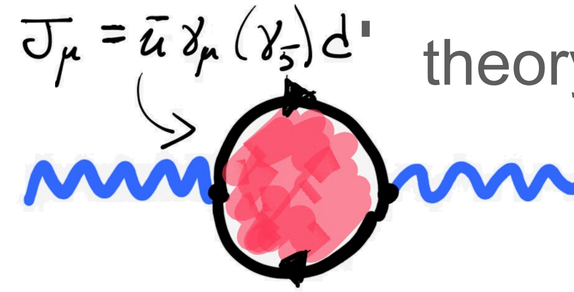


- We **always** analyse **F**inite **E**nergy **S**um **R**ules of the type:

$$\int_0^{s_0} ds\, w\left(\frac{s}{s_0}\right) \frac{1}{\pi} \mathrm{Im}\Pi(s) = \frac{-1}{2\pi i} \oint_{|z|=s_0} dz\, w\left(\frac{z}{s_0}\right) \Pi(z)$$

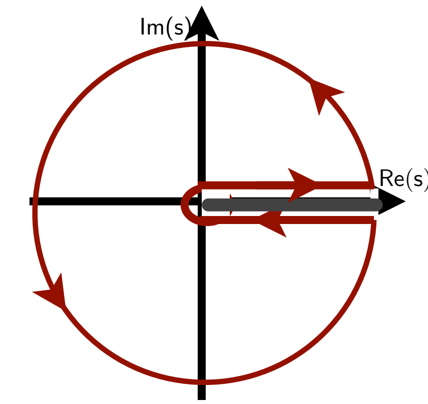


experiment
(OPAL and ALEPH)



theory

Framework: summary



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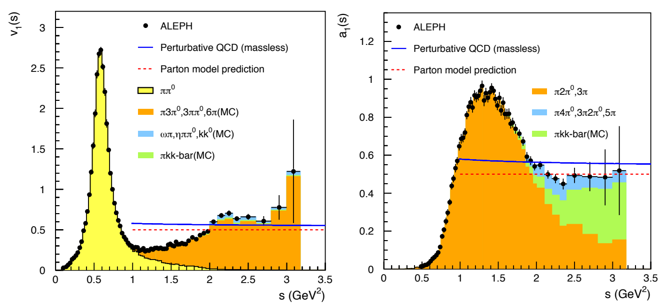
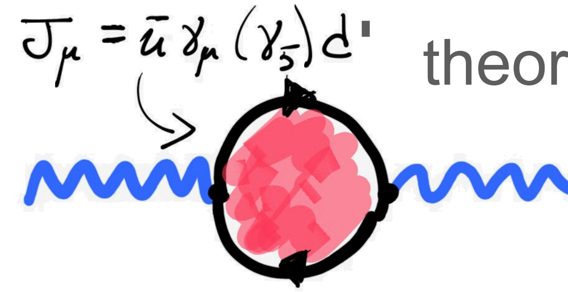


experiment (OPAL and ALEPH)



theory

- $w_1 = 1,\ w_2 = 1 - x^2,\ w_\tau = (1-x)^2(1+2x)$

Framework: summary

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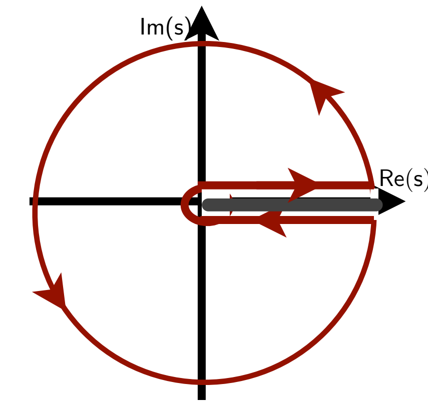
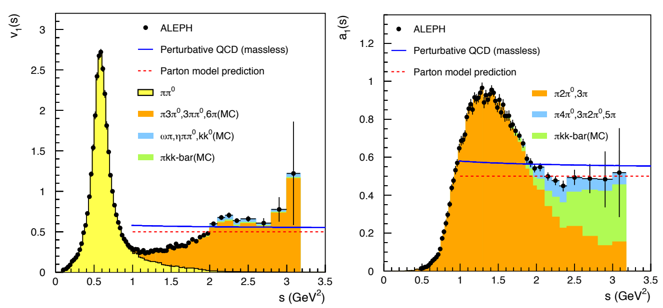
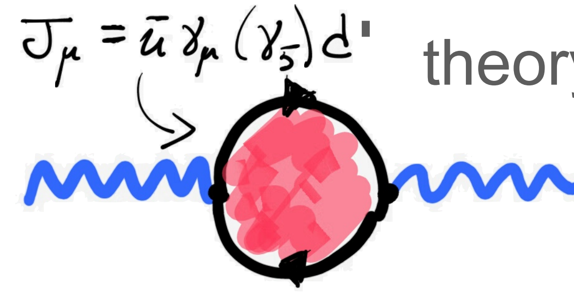


$$\int_0^{s_0} ds\, w\left(\frac{s}{s_0}\right)\frac{1}{\pi}\mathrm{Im}\Pi(s) = \frac{-1}{2\pi i}\oint_{|z|=s_0} dz\, w\left(\frac{z}{s_0}\right)\Pi(z)$$



experiment
(OPAL and ALEPH)

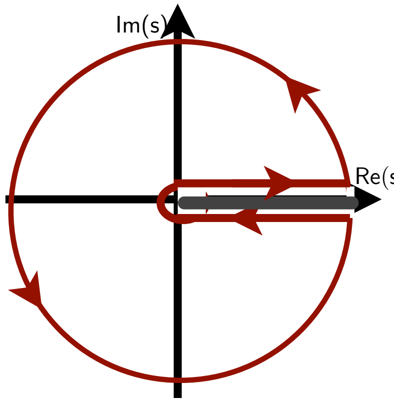


theory

- $w_1 = 1,\ w_2 = 1 - x^2,\ w_\tau = (1-x)^2(1+2x)$

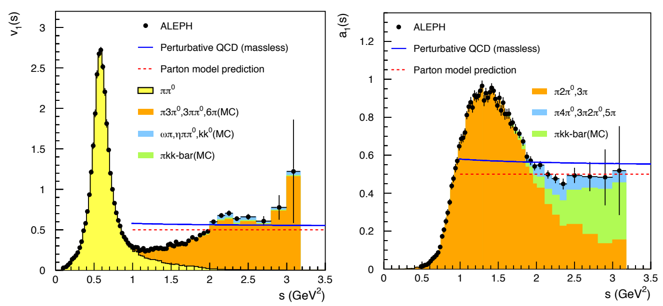
- $s_0^{\rm min} \leq s_0 \leq m_\tau^2$

Framework: summary

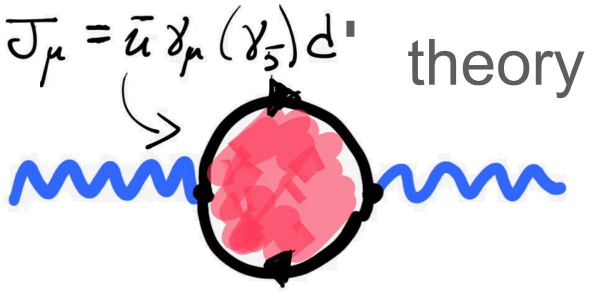


- We **always** analyse **F**inite **E**nergy **S**um **R**ules of the type:

$$\int_0^{s_0} ds\, w\left(\frac{s}{s_0}\right)\frac{1}{\pi}\mathrm{Im}\Pi(s) = \frac{-1}{2\pi i}\oint_{|z|=s_0} dz\, w\left(\frac{z}{s_0}\right)\Pi(z)$$



experiment (OPAL and ALEPH)



theory

- $w_1 = 1,\ w_2 = 1 - x^2,\ w_\tau = (1-x)^2(1+2x)$
- $s_0^{\mathrm{min}} \leq s_0 \leq m_\tau^2$
- Includes the DV contribution. Do not neglect any LO condensate contri.

⟶ see Peris' talk

- Analysis done in *V* and/or *A* (necessary)

Analysis with OPAL data

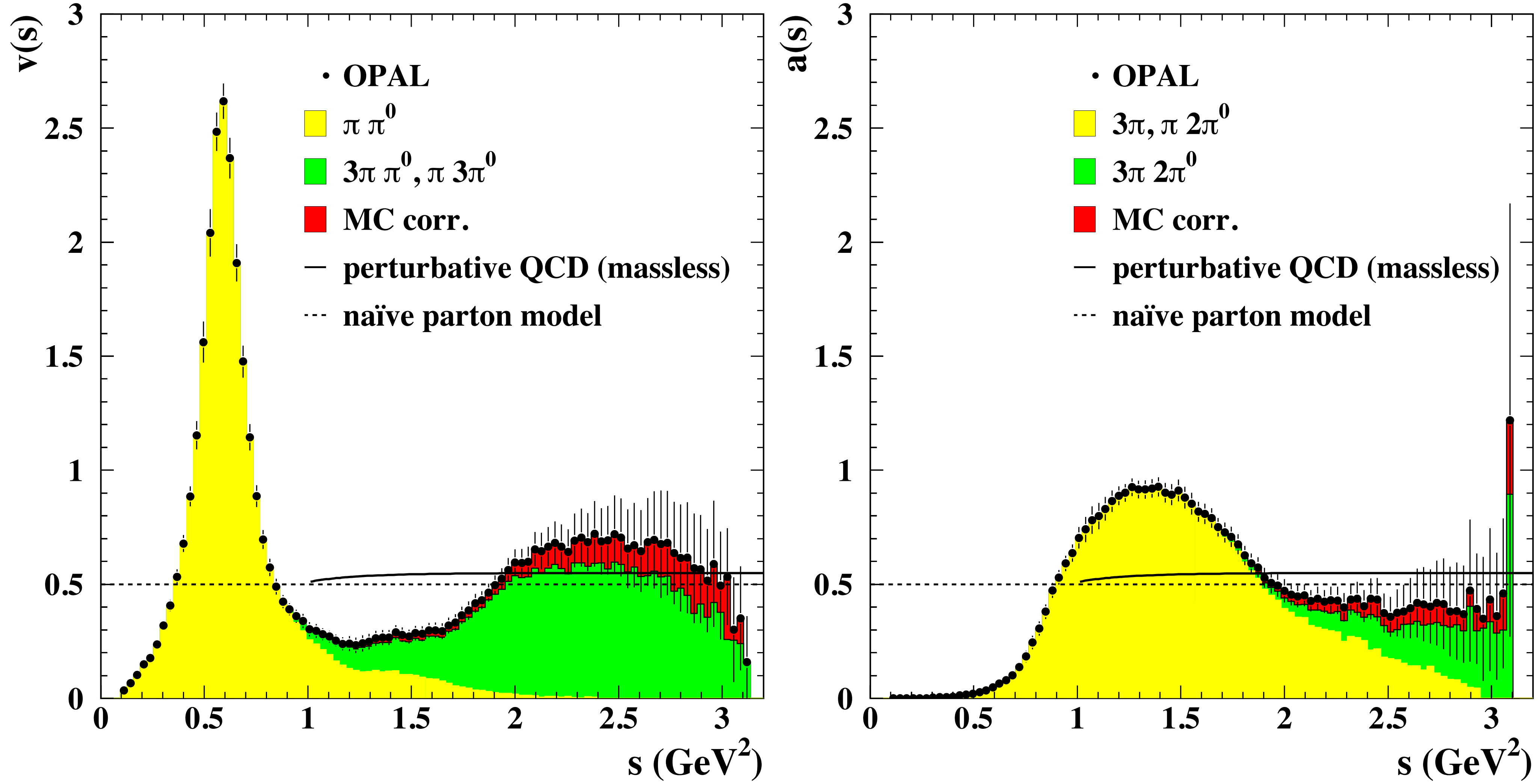
DB, M Golterman, M Jamin, A. Mahdavi, K Maltman, J Osborne, S Peris, *Phys Rev D* **85** (2012)

The OPAL data

- Why use OPAL data with larger errors than ALEPH's?
  - Correlations due to unfolding not properly account for in the 05/08 ALEPH spec. functions.

DB, Catà, Golterman, Jamin, Maltman, Osborne and Peris, '11

OPAL coll., EPJC 7 (1999)



In 2012 this was the only reliable data set

- **Fits to updated OPAL data:**

  - V channel, $w_0 = 1$,
  - V and A channel $w_0 = 1$,
  - V channel $w_0 = 1$, $w_2 = 1 - x^2$, $w_3 = (1-x)^2(1+2x)$,
  - V and A channel $w_0 = 1$, $w_2 = 1 - x^2$, $w_3 = (1-x)^2(1+2x)$,

  and other combinations, all **with consistent results**

- Weinberg sum rules satisfied for all (*V* and *A*) fits

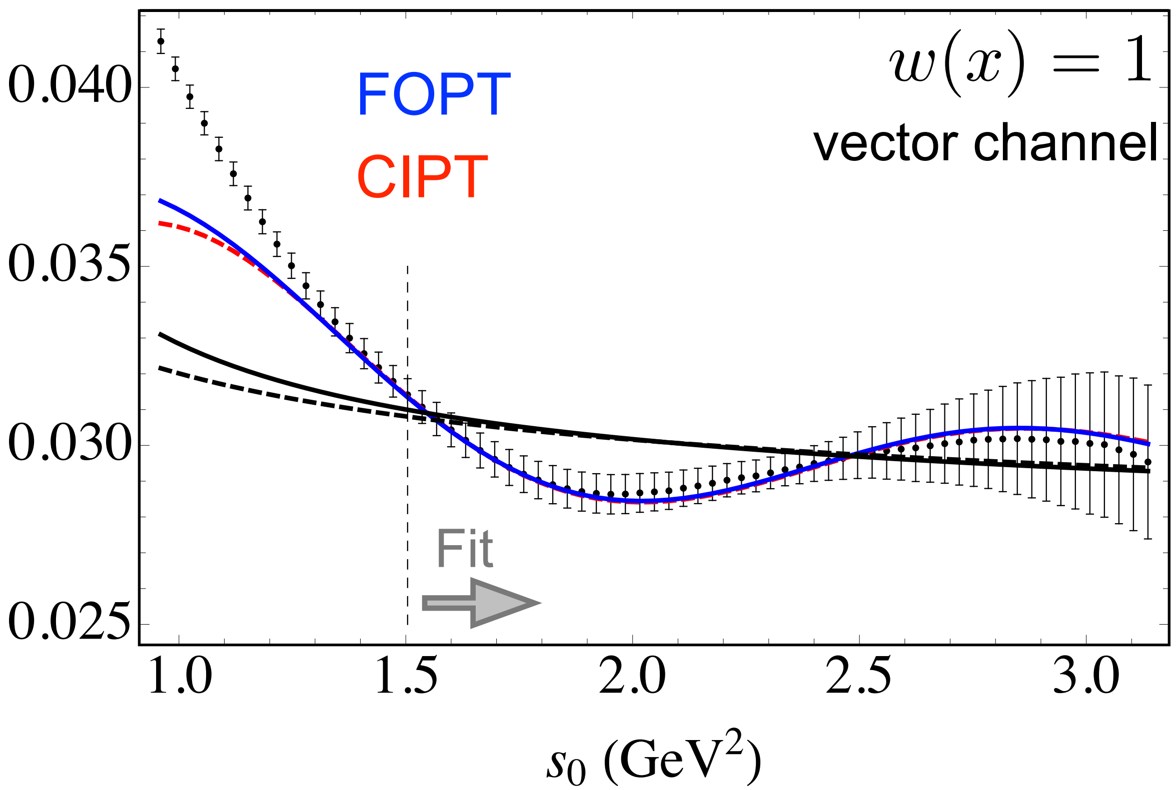
  ⟶ see Peris' talk

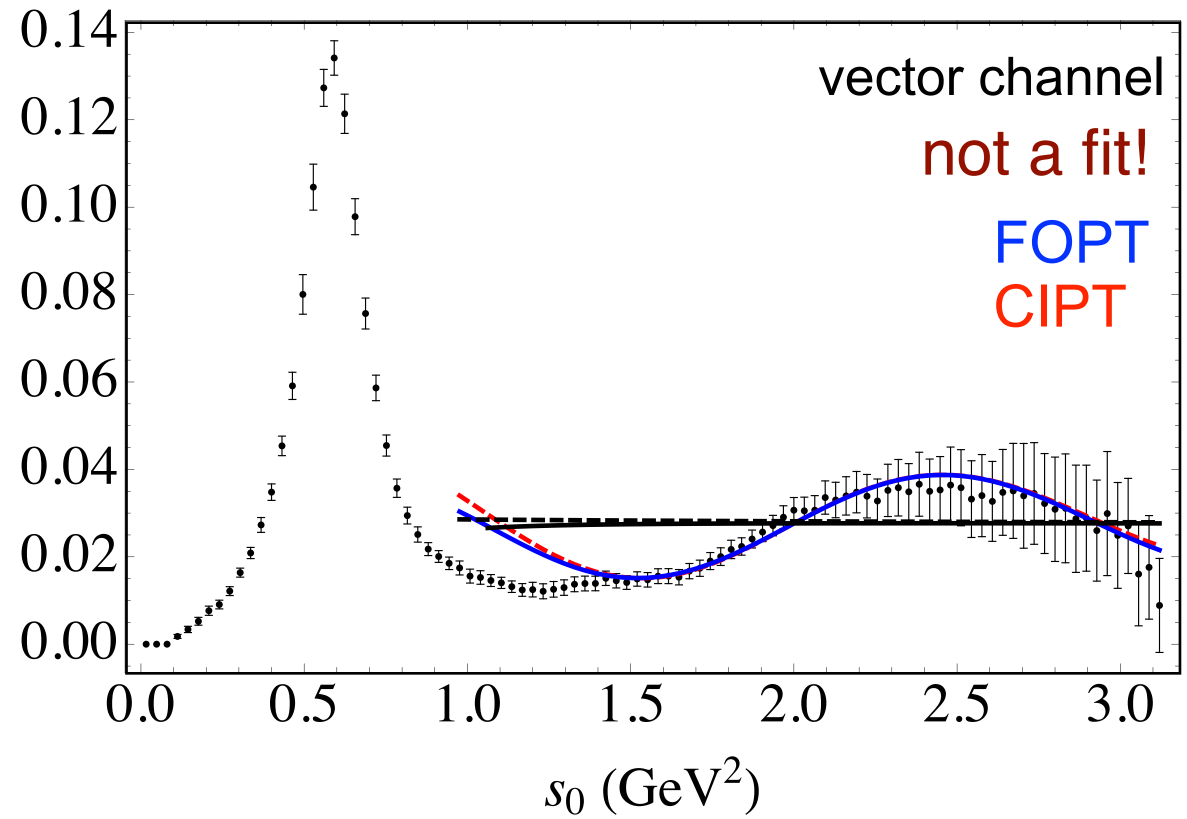
- Markov-chain Monte Carlo study of the chi^2 function

- Extracted values for α_s, OPE condensates, and DV params.
- Calculate from the results the total non-perturbative contribution to R_τ

Vector channel, $w_1 = 1$, $s_0 = 1.55$ GeV^2

5 parms., 47 dof





- $\chi^2/dof = 0.34$

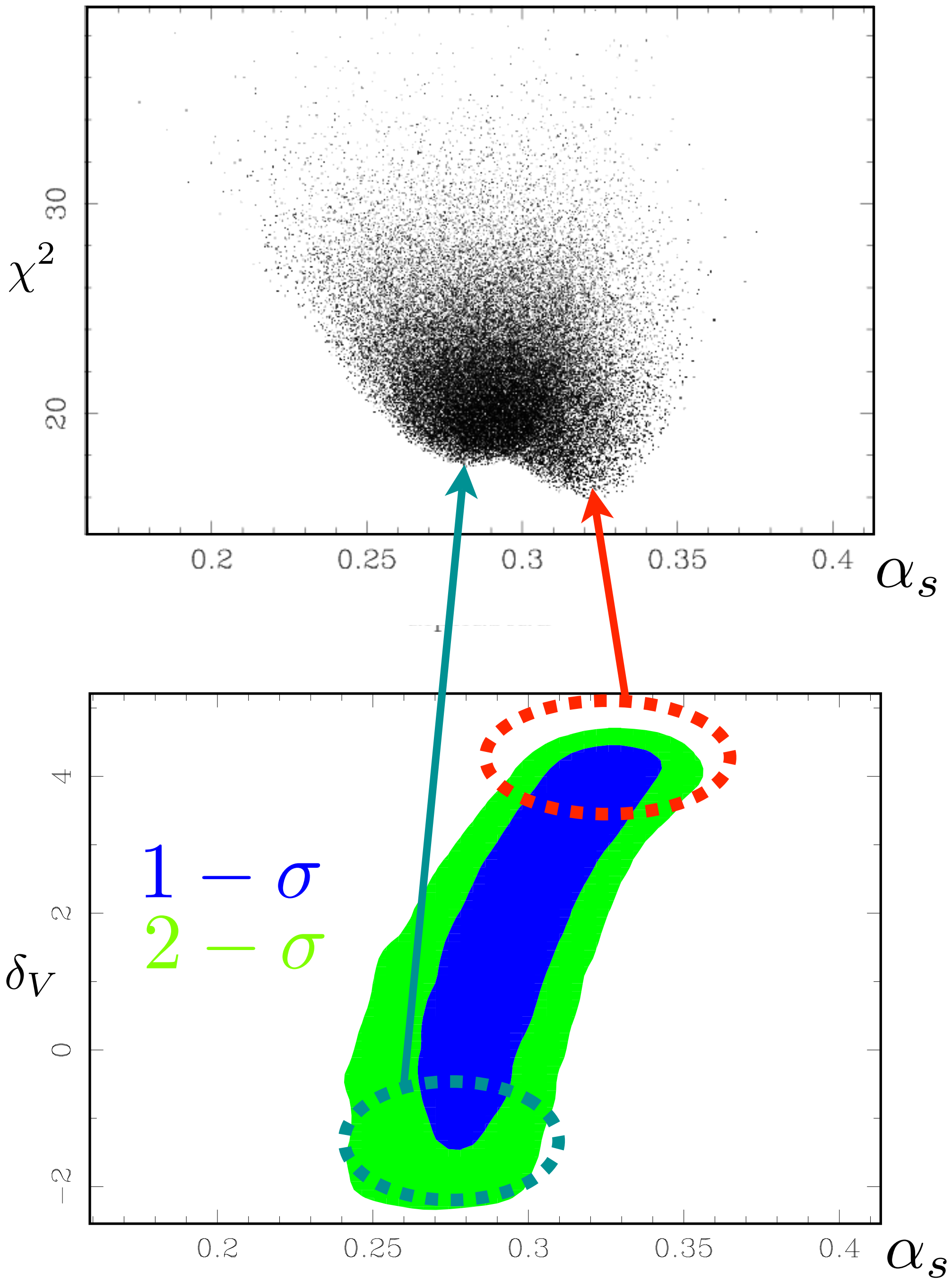
- Results for the strong coupling:

$$\alpha_s(m_\tau^2) = 0.325 \pm 0.018 \ (\overline{\text{MS}}, n_f = 3, \text{FOPT})$$

$$\alpha_s(m_\tau^2) = 0.347 \pm 0.025 \ (\overline{\text{MS}}, n_f = 3, \text{CIPT})$$

McMC checks

- Markov-chain Monte Carlo analysis of the χ^2 distribution.

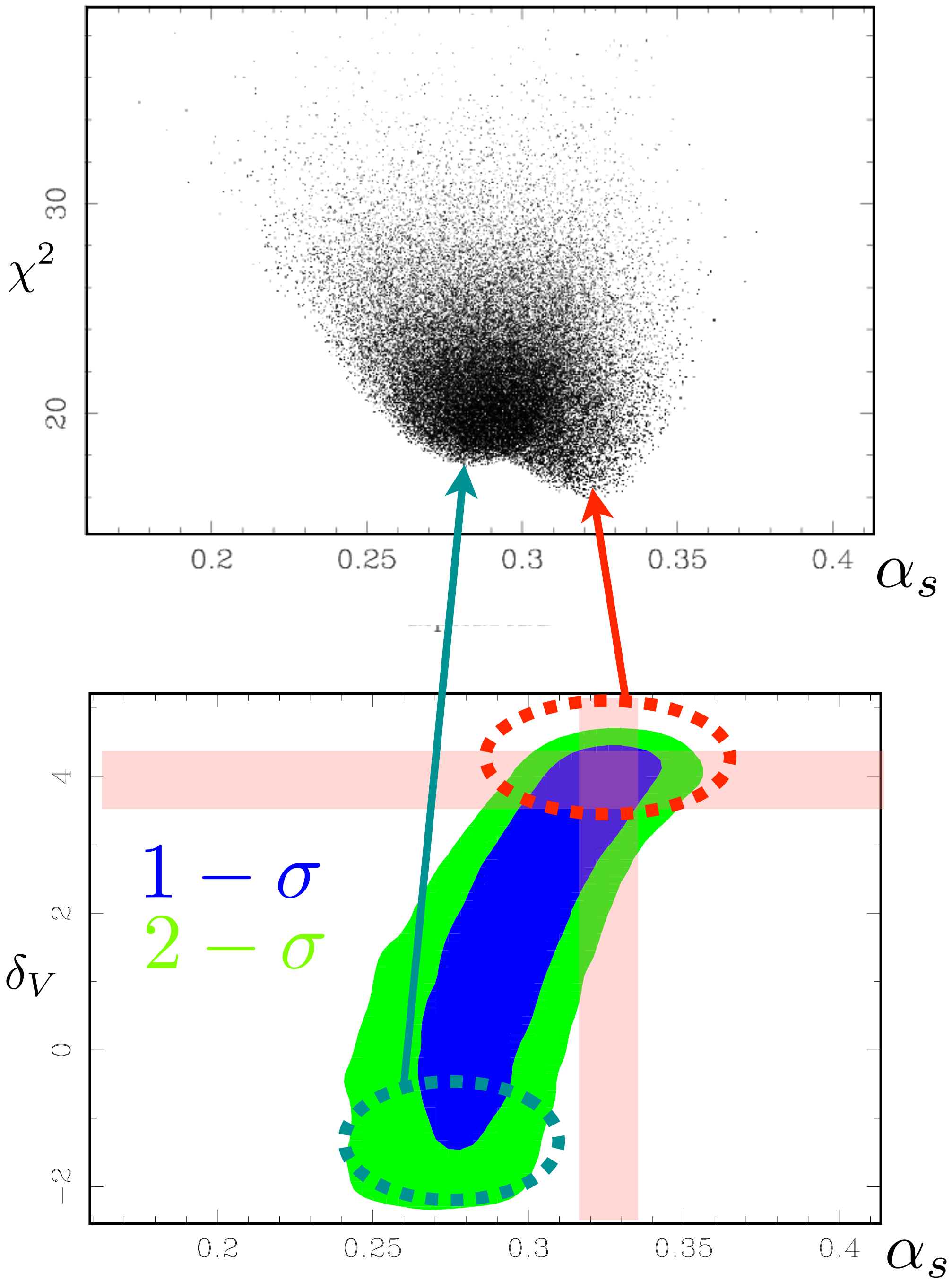


$w(x) = 1$, vector channel

- Tool to understand instabilities of the fits
- Projections from 5d to 2d plots
- χ^2 has two minima
- Physical solution: minimum with higher α_s (which is also the absolute minimum)

McMC checks

- Markov-chain Monte Carlo analysis of the χ^2 distribution.



$w(x) = 1$, vector channel

- Tool to understand instabilities of the fits
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- χ^2 has two minima
- Physical solution: minimum with higher α_s (which is also the absolute minimum)

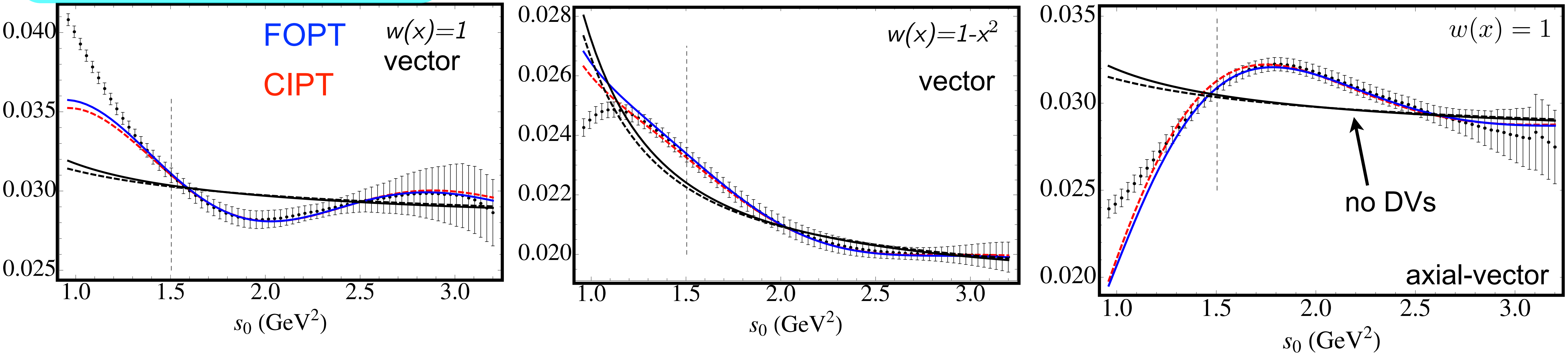
$V\&A$ channels,

$w_1 = 1$, $w_2 = 1 - x^2$, and w_τ, $s_0 = 1.55$ GeV^2

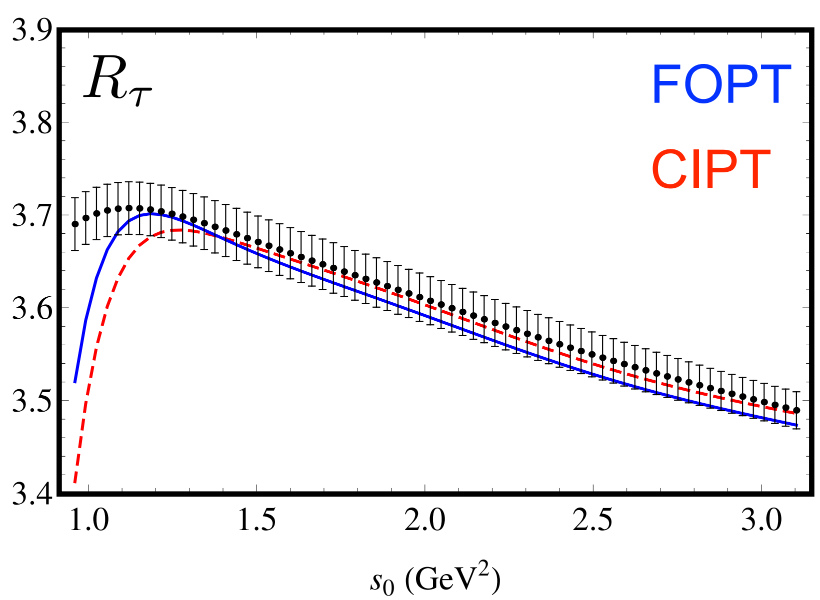
11 parms., 92 dof

$\alpha_s(m_\tau^2) = 0.311 \pm 0.013$ ($\overline{\mathrm{MS}}, n_f = 3$, FOPT)

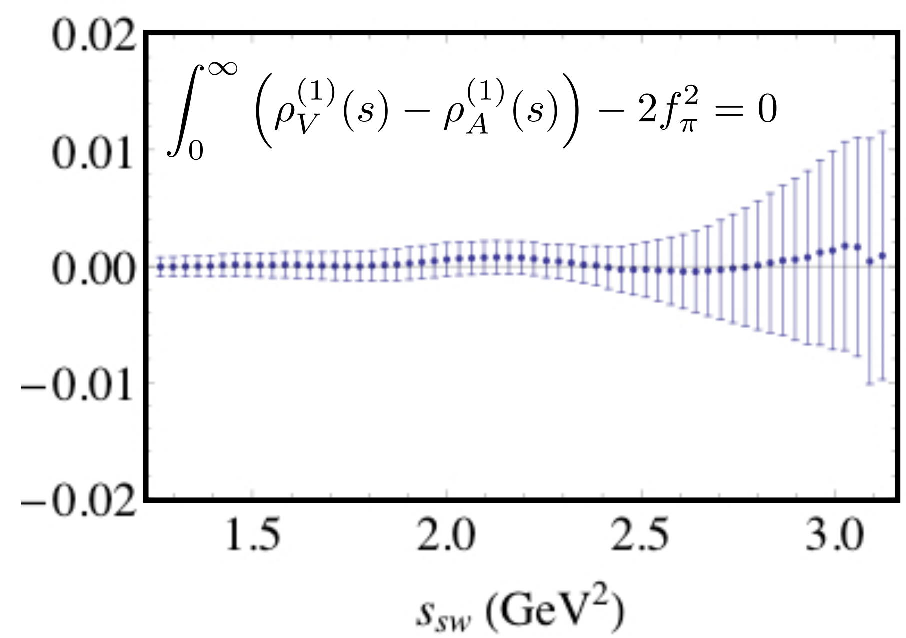
$\alpha_s(m_\tau^2) = 0.337 \pm 0.018$ ($\overline{\mathrm{MS}}, n_f = 3$, CIPT)



Some consistency checks:



First Weinberg sum rule



Final numbers based on OPAL data

- Fits are good and stable within their somewhat large uncertainties

V-channel, w=1

$$\alpha_s(m_Z^2) = 0.1191 \pm 0.0022 \; (\overline{\mathrm{MS}}, n_f = 5, \text{FOPT})$$
$$\alpha_s(m_Z^2) = 0.1218 \pm 0.0027 \; (\overline{\mathrm{MS}}, n_f = 5, \text{CIPT})$$

V&A channels, w=1, w=1-x^2, w_tau

$$\alpha_s(m_Z^2) = 0.1175 \pm 0.0017 \; (\overline{\mathrm{MS}}, n_f = 5, \text{FOPT})$$
$$\alpha_s(m_Z^2) = 0.1206 \pm 0.0020 \; (\overline{\mathrm{MS}}, n_f = 5, \text{CIPT})$$

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$$\alpha_s(m_Z^2) = 0.1218 \pm 0.0027 \; (\overline{\text{MS}}, n_f = 5, \text{CIPT})$$

Both fits are stable against s_0 variations
Choose the chi^2-type fit (McMC)

V&A channels, w=1, w=1-x^2, w_tau

$$\alpha_s(m_Z^2) = 0.1175 \pm 0.0017 \; (\overline{\text{MS}}, n_f = 5, \text{FOPT})$$
$$\alpha_s(m_Z^2) = 0.1206 \pm 0.0020 \; (\overline{\text{MS}}, n_f = 5, \text{CIPT})$$

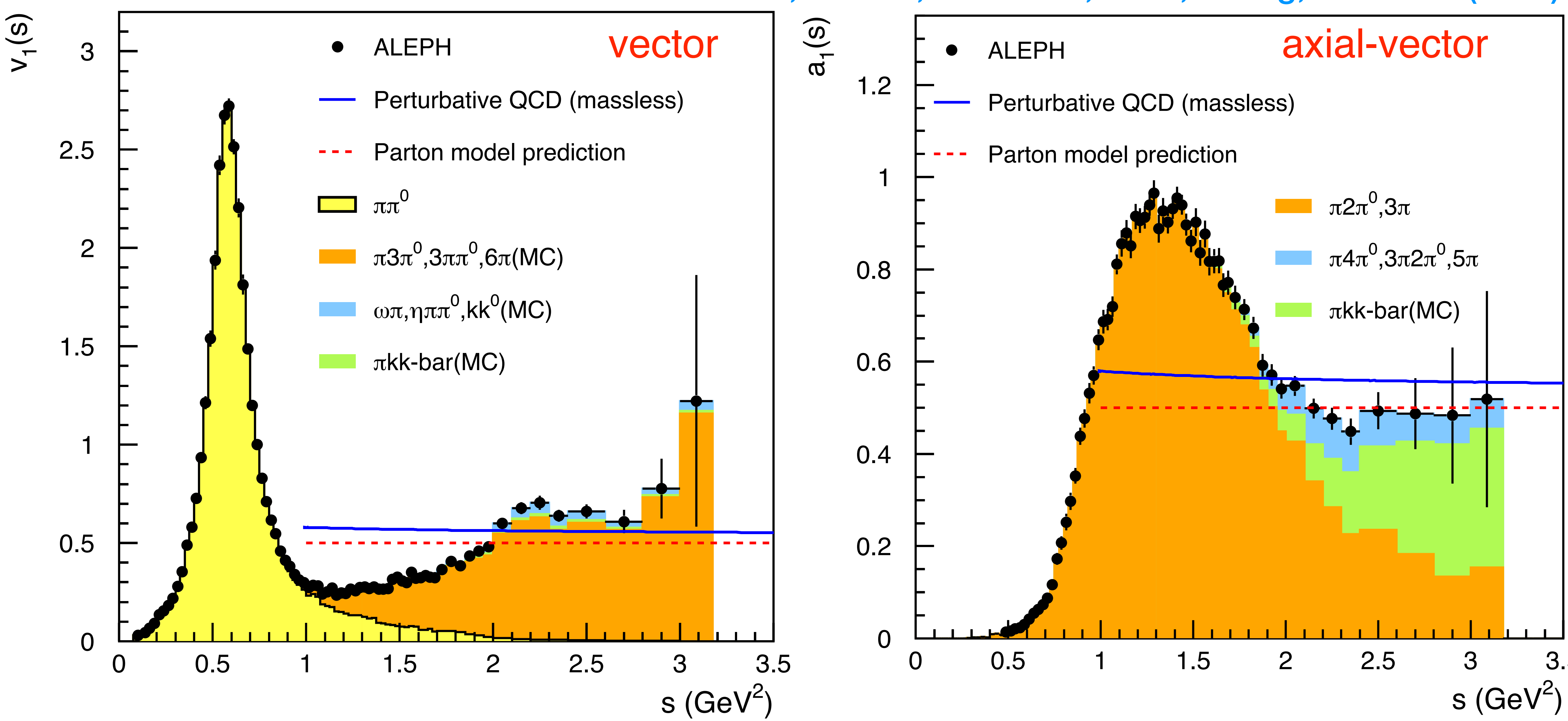
Analysis with ALEPH data

DB, M Golterman, K Maltman, J Osborne, S Peris, *Phys Rev D* **91** (2015)

ALEPH data as of January 2014

- New ALEPH spectral functions with new unfolding, new binning, and corrected correlations.

Davier, Höcker, Malaescu, Yuan, Zhang, EPJC 74 (2014)



Best data set. Only available since January 2014.

- **Fits to revised ALEPH data:**

  - V channel, $w_0 = 1$,
  - V and A channel $w_0 = 1$,
  - V channel $w_0 = 1$, $w_2 = 1 - x^2$, $w_3 = (1-x)^2(1+2x)$,
  - V and A channel $w_0 = 1$, $w_2 = 1 - x^2$, $w_3 = (1-x)^2(1+2x)$,

  and other combinations, all **with consistent results**

- Weinberg sum rules satisfied for all (*V* and *A*) fits

  ⟶ see Peris' talk

- Markov-chain Monte Carlo study of the chi^2 function

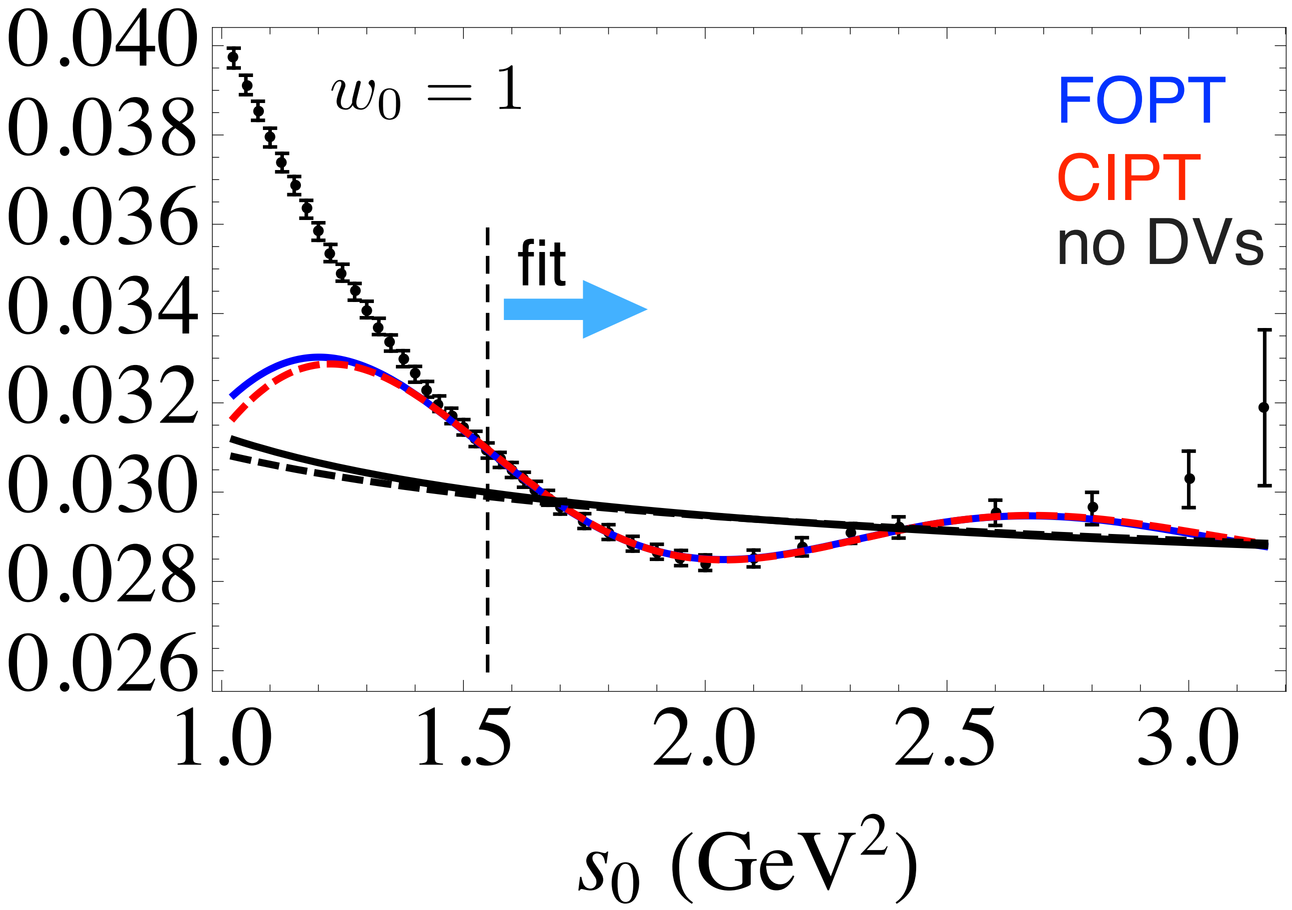
- Extracted values for α_s, OPE condensates, and DV params.

- Calculate from the results the total non-perturbative contribution to R_τ

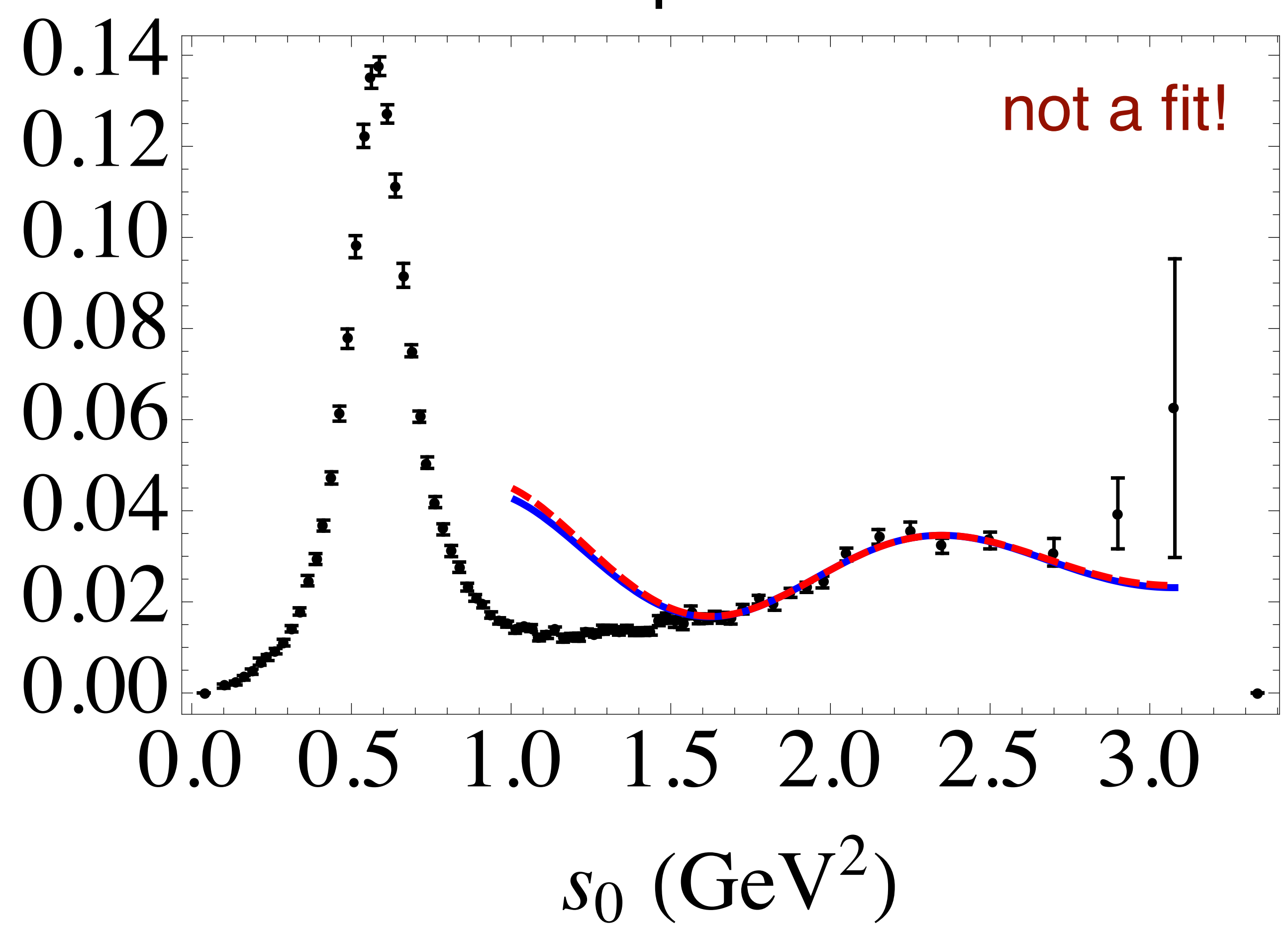
V channel, $w_1 = 1$

$$\chi^2 = 24.5/16,\ s_0^{\min} = 1.55\ \mathrm{GeV}$$

5 parms., 16 dof

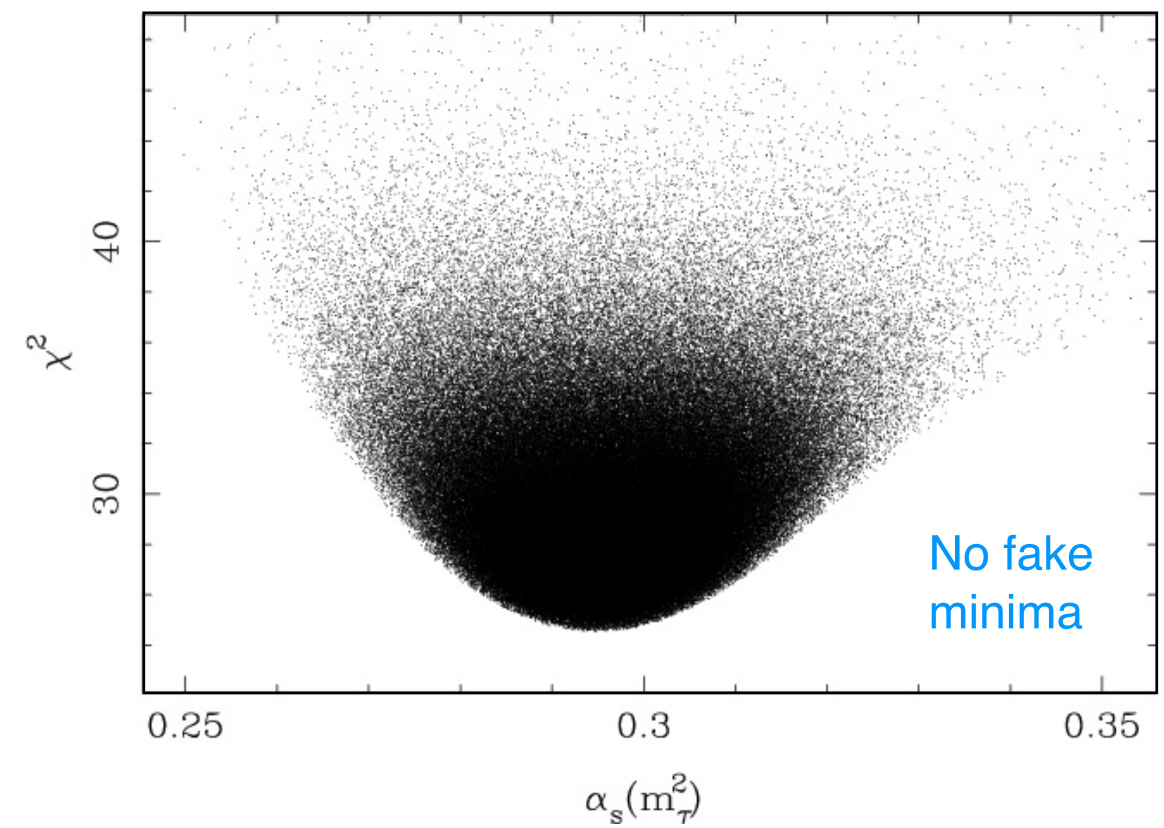


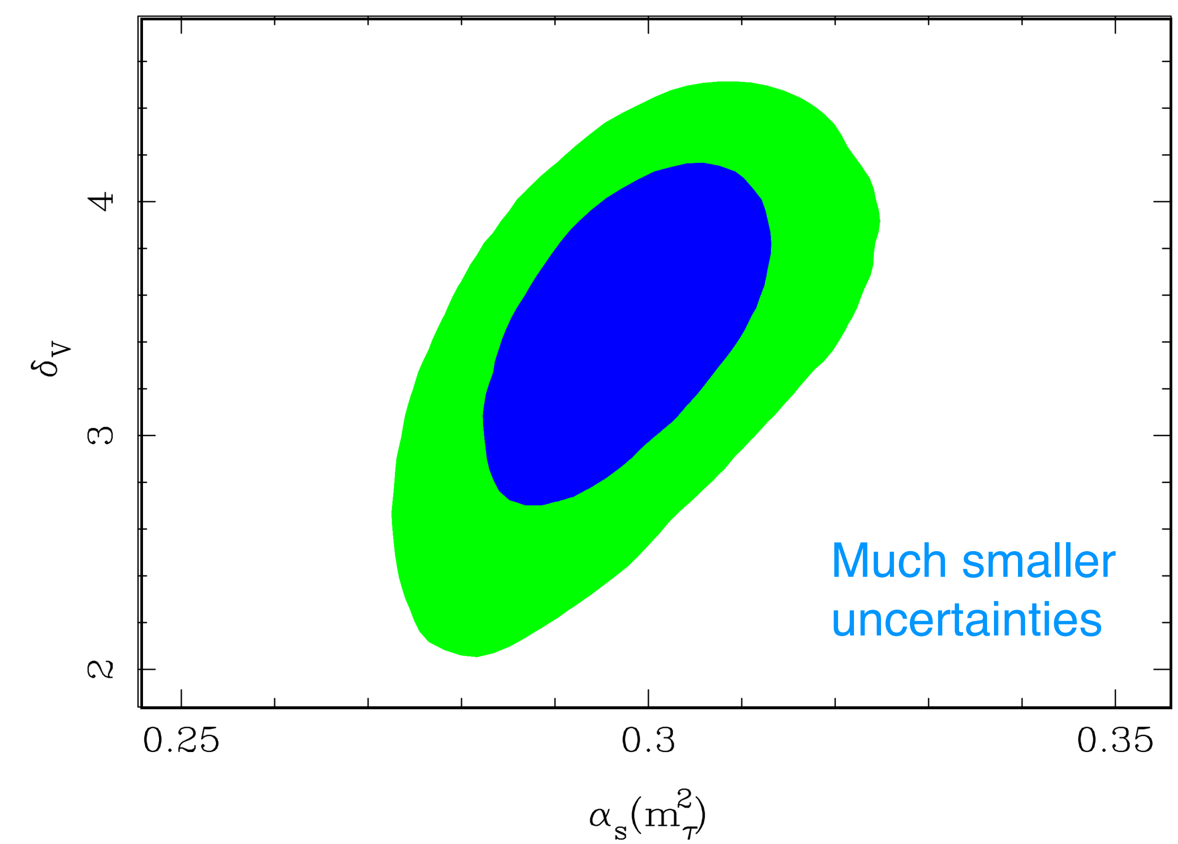
Check of the spectral function



McMC for ALEPH data

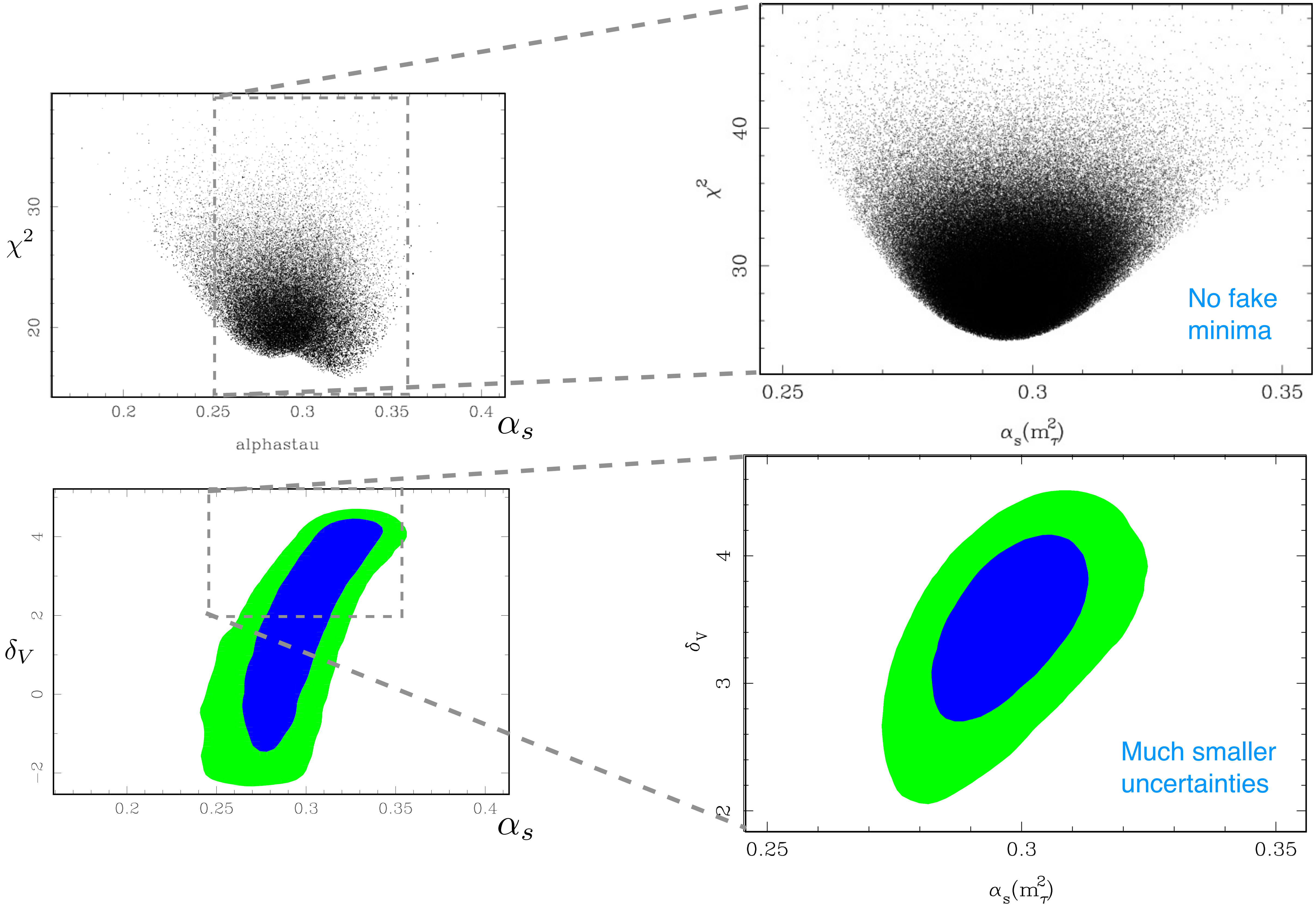
ALEPH data





McMC for ALEPH data

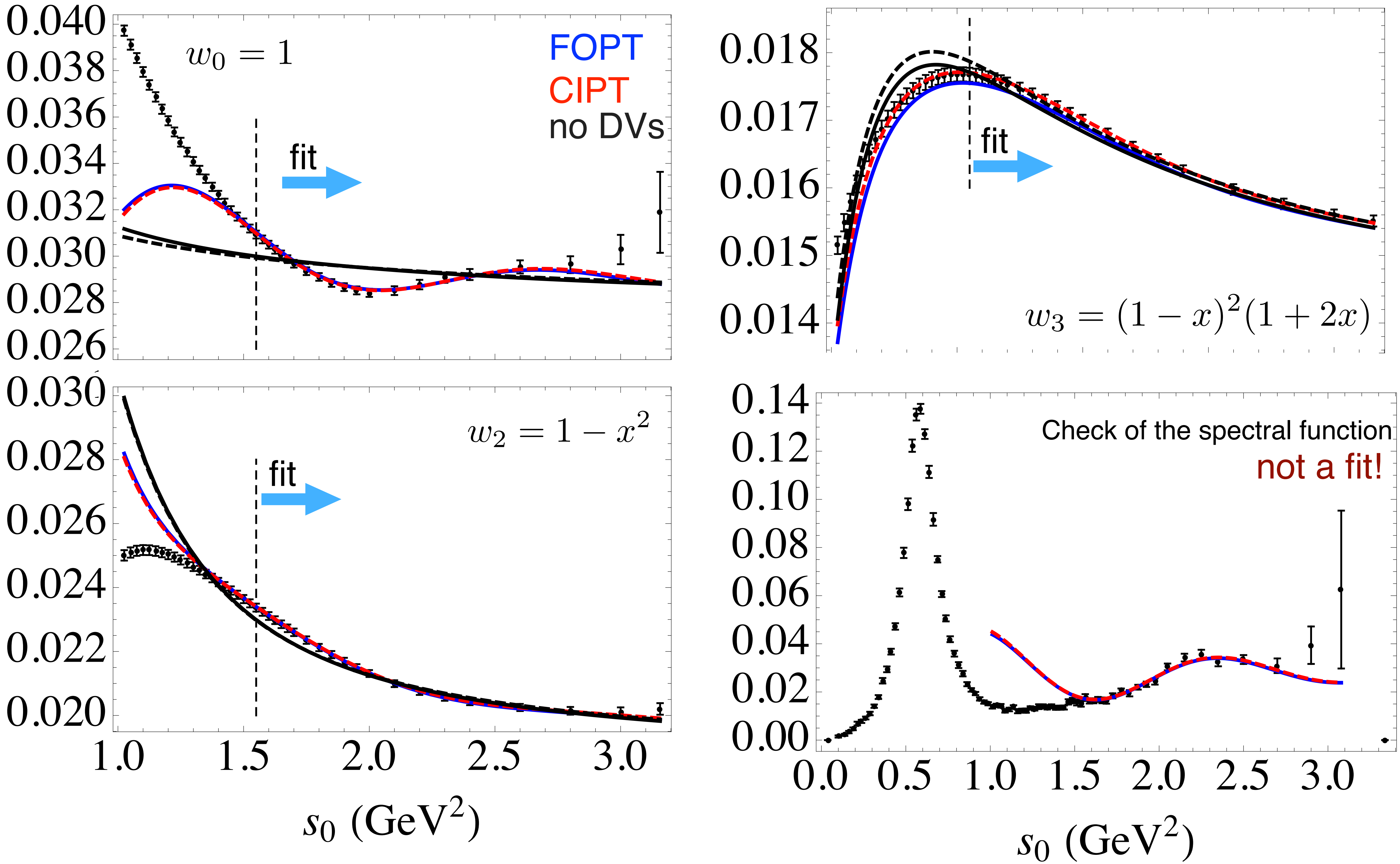
OPAL data **ALEPH data**



V channel, $w_1 = 1$, $w_2 = 1 - x^2$, w_τ

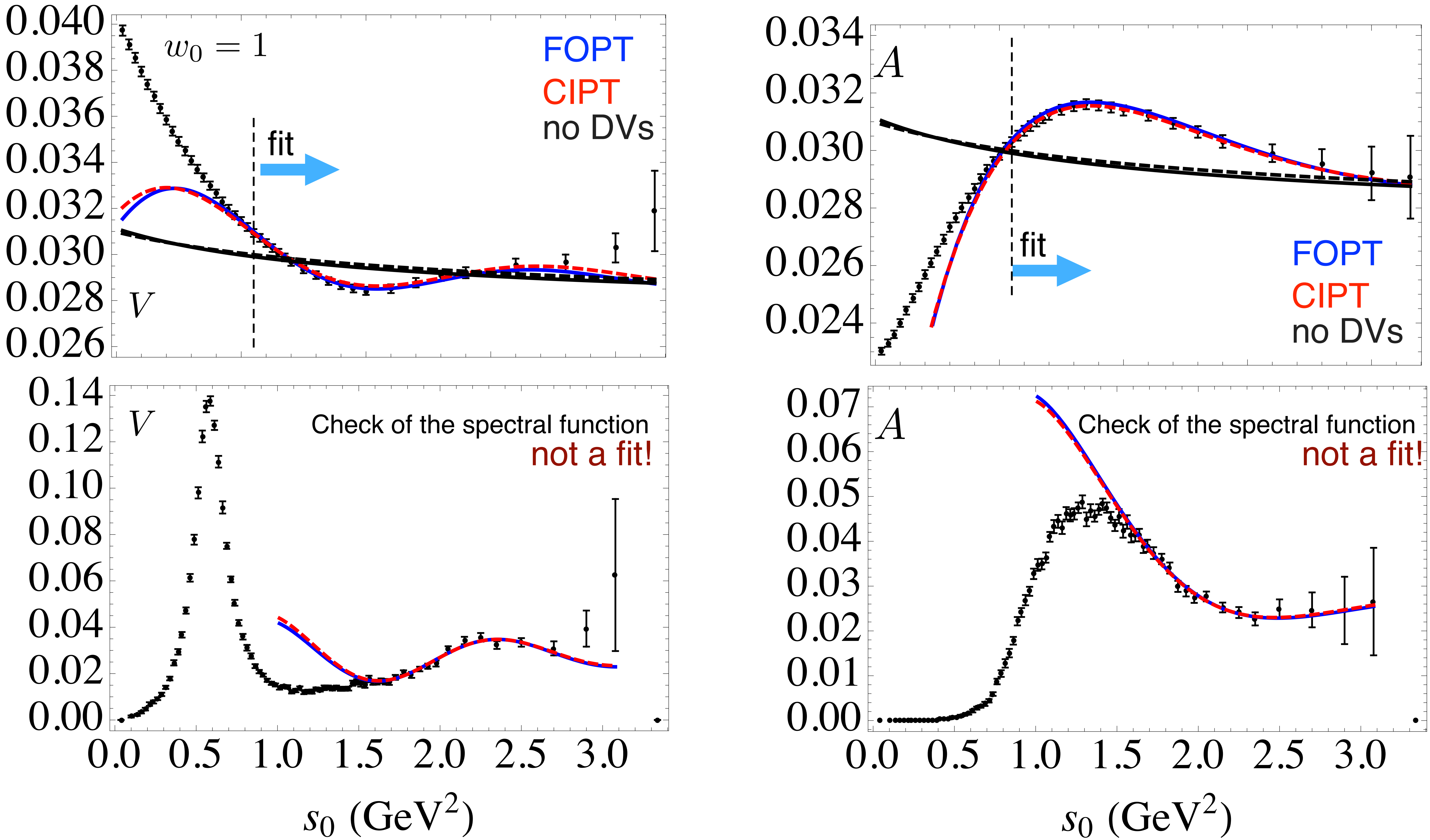
$\text{block} - \text{diagonal}, \ s_0^{\min} = 1.55 \ \text{GeV}$

7 parms., 61 dof



V and A, $w_1 = 1$, $w_2 = 1 - x^2$, w_τ

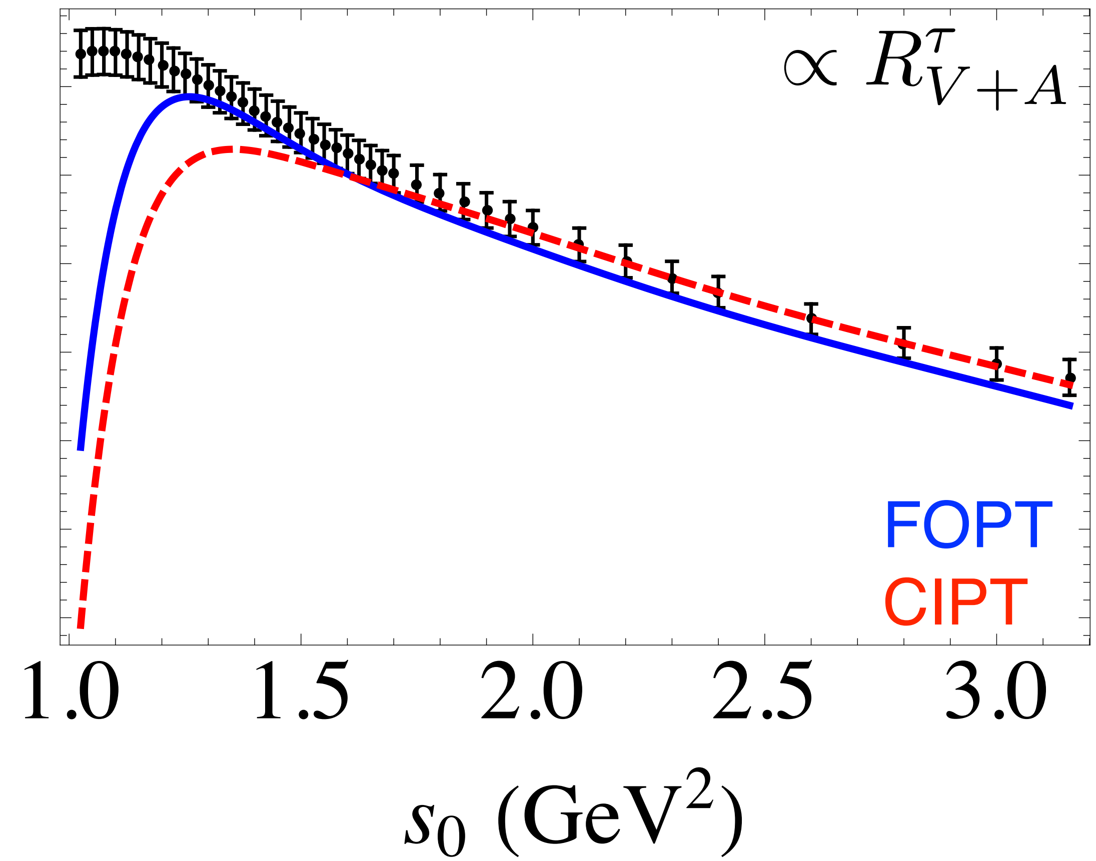
$\text{block} - \text{diagonal}, \ s_0^{\text{min}} = 1.55 \ \text{GeV}$



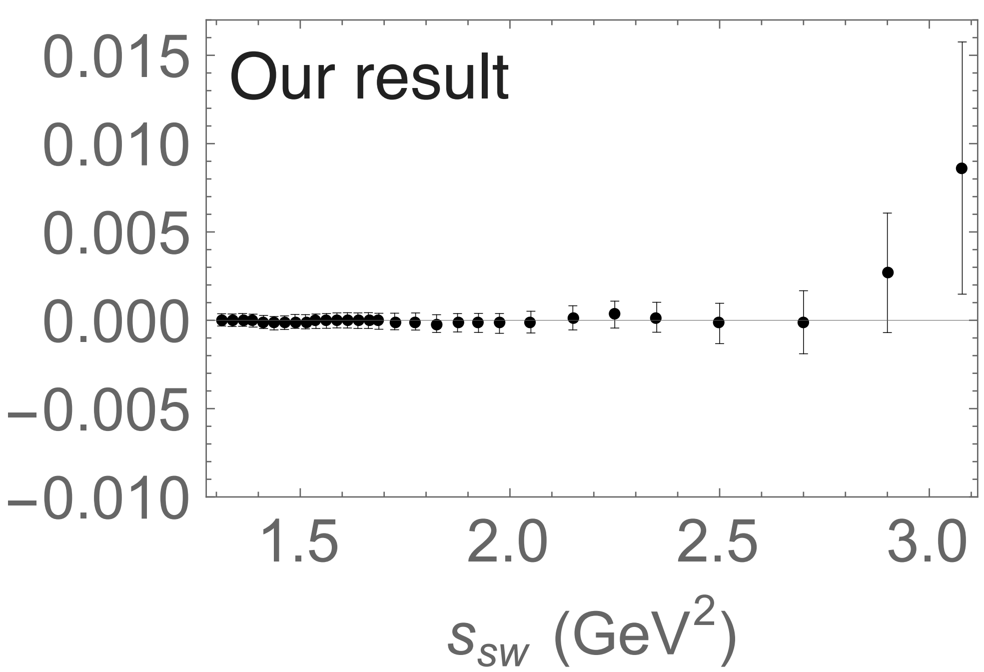
Consistency checks

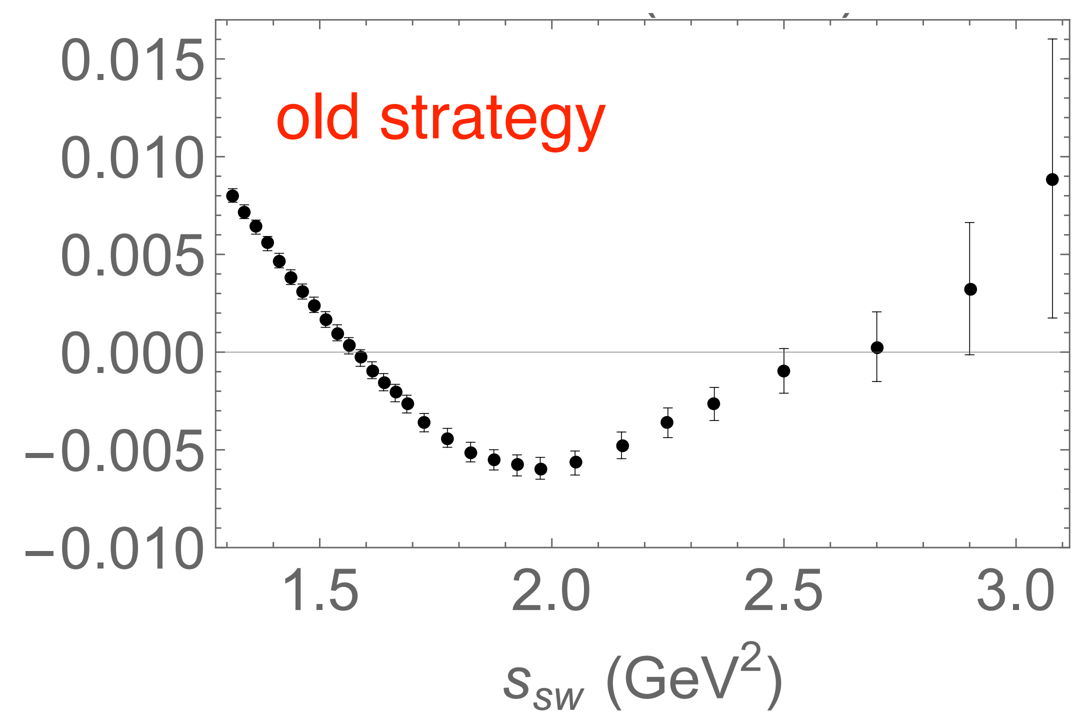
Tests of the fits

- $R^{\tau}_{V+A}(s_0)$



- 1st Weinberg sum-rule: $\int_0^{\infty} \left(\rho_V^{(1)}(s) - \rho_A^{(1)}(s) \right) - 2 f_\pi^2 = 0$





Final numbers based on ALEPH data

- Fits are good, consistent, and stable, no sign of fake minima in the chi^2

V-channel, w=1, s_0

$$\alpha_s(m_Z^2) = 0.1155 \pm 0.0015 \, (\overline{\rm MS}\,, n_f = 5, \text{ FOPT})$$
$$\alpha_s(m_Z^2) = 0.1173 \pm 0.0018 \, (\overline{\rm MS}\,, n_f = 5, \text{ CIPT})$$

V channel, w=1, w=1-x^2, w_tau

$$\alpha_s(m_Z^2) = 0.1155 \pm 0.0014 \quad (\overline{\rm MS}\,, n_f = 5, \text{ FOPT})$$
$$\alpha_s(m_Z^2) = 0.1174 \pm 0.0019 \quad (\overline{\rm MS}\,, n_f = 5, \text{ CIPT})$$

Final numbers based on ALEPH data

- Fits are good, consistent, and stable, no sign of fake minima in the chi^2

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V channel, w=1, w=1-x^2, w_tau

$$\alpha_s(m_Z^2) = 0.1155 \pm 0.0014 \quad (\overline{\rm MS}\,, n_f = 5,\ \rm FOPT)$$
$$\alpha_s(m_Z^2) = 0.1174 \pm 0.0019 \quad (\overline{\rm MS}\,, n_f = 5,\ \rm CIPT)$$

Improved stability against s_0 variations

Analysis at a single point $s_0 = m_\tau^2$,
neglecting higher C_Ds, assuming no DVs

Davier, Höcker, Malaescu, Yuan, Zhang, '14

$$R^\tau_{V+A;ud} = N_c\, S_{\rm EW}\, |V_{ud}|^2 (1 + \delta_{\rm P} + \delta_{\rm NP})$$

$\delta_{\rm NP} = 0.020 \pm 0.009$ FOPT

$\delta_{\rm NP} = 0.016 \pm 0.010$ CIPT

$$\alpha_s : (+10\%\ \text{larger}) \pm (\text{half errors})$$

$\delta_{\rm NP}^{\rm [Davier\ et\ al]} = -0.0064(13)$ (CIPT)

Averaged results, comparison, preliminary combined fits

Values extracted from the
revised ALEPH data
(*V* fit with three weight functions)

$$\alpha_s(m_Z^2) = 0.1155 \pm 0.0014 \quad (\overline{\rm MS}\,, n_f = 5, \text{ FOPT})$$
$$\alpha_s(m_Z^2) = 0.1174 \pm 0.0019 \quad (\overline{\rm MS}\,, n_f = 5, \text{ CIPT})$$

Values extracted from the
updated OPAL data
(*V* fit with w=1)

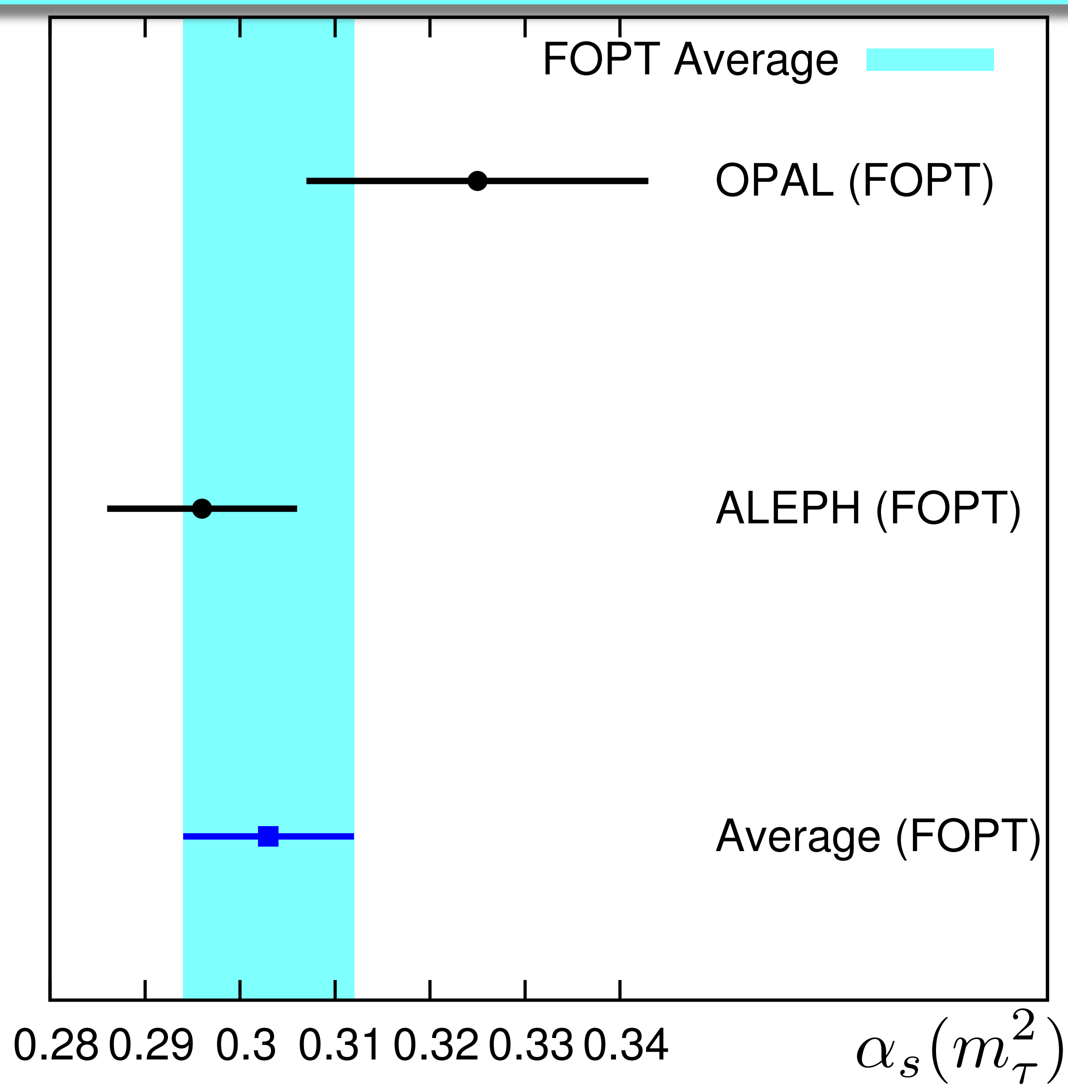
$$\alpha_s(m_Z^2) = 0.1191 \pm 0.0022 \; (\overline{\rm MS}, n_f = 5, \text{FOPT})$$
$$\alpha_s(m_Z^2) = 0.1218 \pm 0.0027 \; (\overline{\rm MS}, n_f = 5, \text{CIPT})$$

Weighted **average** between
ALEPH- and OPAL-based analyses

$$\alpha_s(m_Z^2) = 0.1165 \pm 0.0012 \quad (\overline{\rm MS}\,, n_f = 5, \text{ FOPT})$$
$$\alpha_s(m_Z^2) = 0.1185 \pm 0.0015 \quad (\overline{\rm MS}\,, n_f = 5, \text{ CIPT})$$

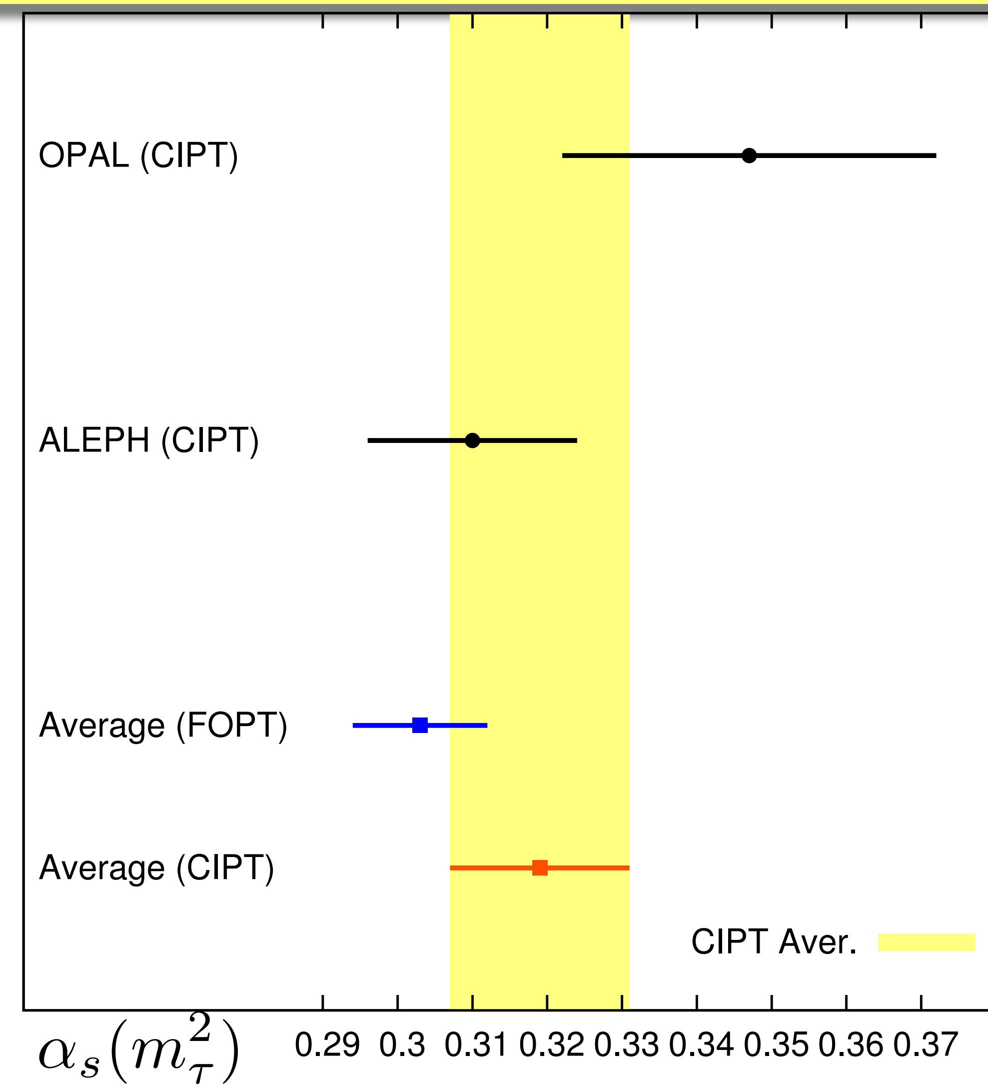
FOPT

$$\alpha_s(m_\tau^2) = 0.303 \pm 0.009 \quad (\overline{\rm MS}\,, n_f = 3,\ {\rm FOPT})$$



CIPT

$$\alpha_s(m_\tau^2) = 0.319 \pm 0.012 \quad (\overline{\rm MS}\,, n_f = 3,\ {\rm CIPT})$$

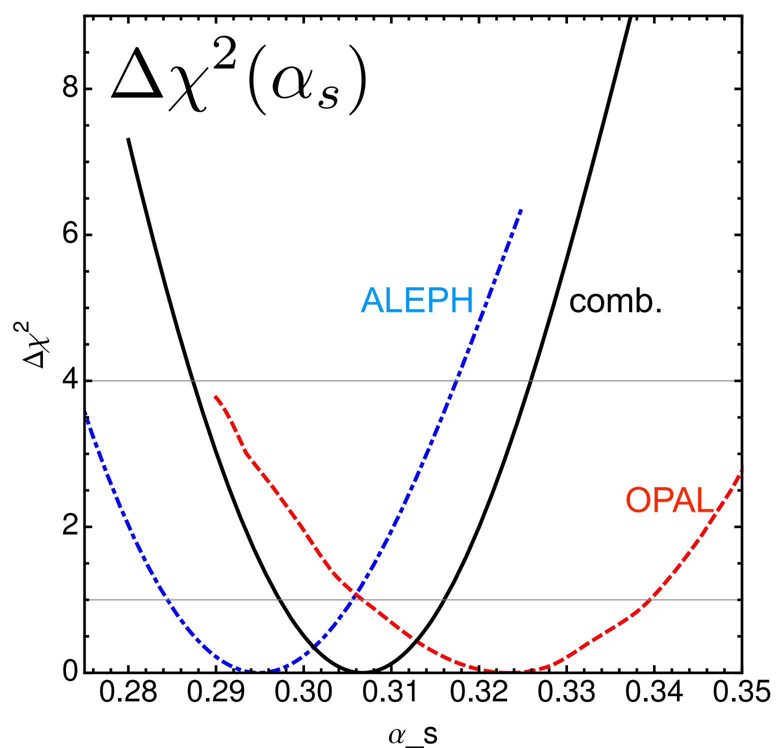


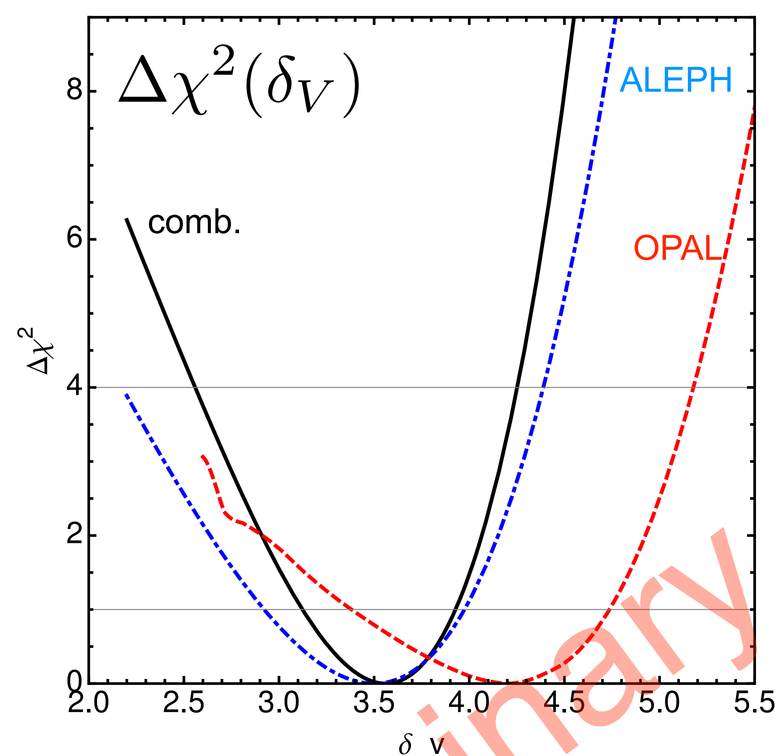
Preliminary results of a combined analysis

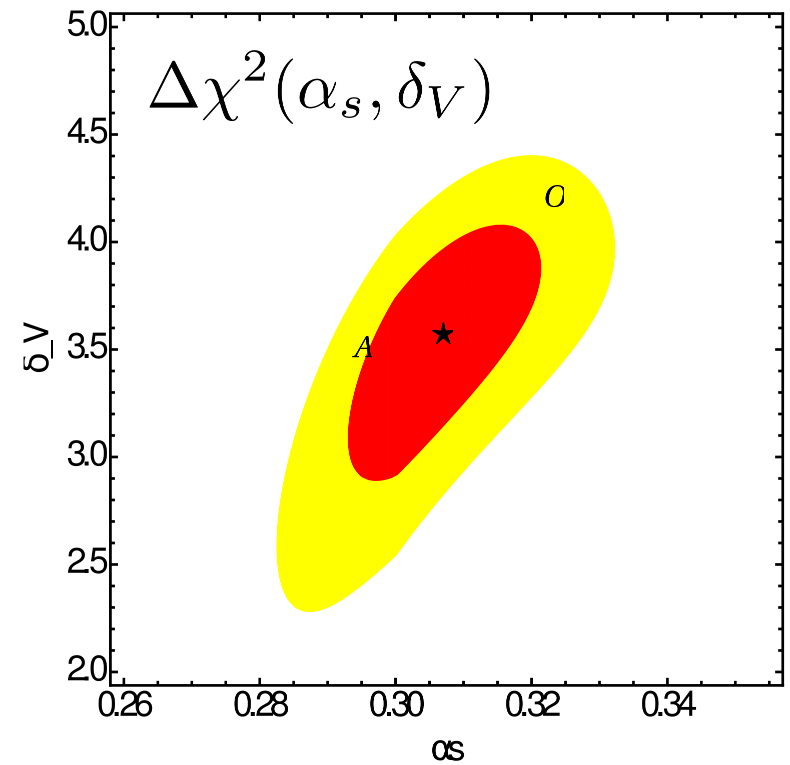
Preliminary combined analysis of $w(x) = 1$ moments

- Assuming ALEPH and OPAL to be uncorrelated

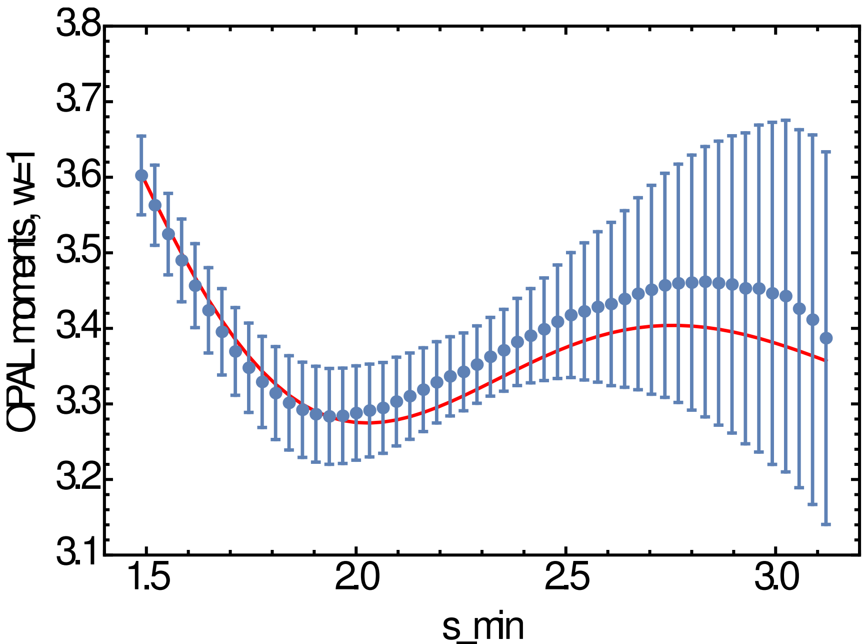
$$\chi^2 = \chi^2_{\text{ALEPH}} + \chi^2_{\text{OPAL}}$$

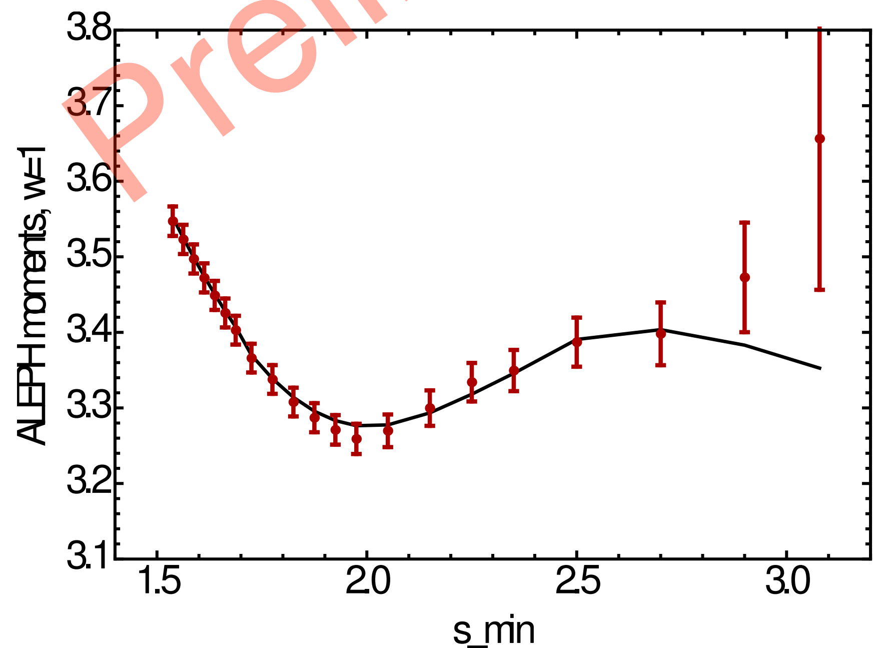






Main gain: improved stability with s_0 variations





$s^{\text{OP}}_{\text{min}}$	$s^{\text{AL}}_{\text{min}}$	χ^2/dof	$\alpha_s(m_\tau)$
1.5	1.50	46.4/70	0.308(10)
1.5	1.55	43.9/73	0.307(09)
1.6	1.60	41.0/68	0.308(10)

FOPT

$$\alpha_s^{\text{comb}}(m_Z^2) = 0.1170 \pm 0.0013$$

An exercise: a fit to the spectral function

Can one drop the FESR and fit only the spectral function?

- As an exercise, one can try to drop the use of FESR and simply fit the spectral functions
- Question: is this equivalent to fit the FESR with $w(x) = 1$? (The answer is **NO**)

Can one drop the FESR and fit only the spectral function?

- As an exercise, one can try to drop the use of FESR and simply fit the spectral functions
- Question: is this equivalent to fit the FESR with *w(x) = 1?* (The answer is **NO**)

$$\int_0^{s_0} ds\, w(s)\, \frac{1}{\pi}\mathrm{Im}\tilde{\Pi}(s) = \frac{-1}{2\pi i}\oint_{|z|=s_0} dz\, w(z)\,\tilde{\Pi}(z)$$

Fit to ALEPH spectral function

Sample of results:

s_0 (GeV^2)	dof	χ^2/dof	$\alpha_s(m_\tau)$	δ_V	
1.525	17	1.72	0.382(86)	3.9(1.8)	...
1.550	16	1.80	0.35(22)	3.5(1.9)	...
1.575	15	1.62	0.3(2.3)	3.5(2.6)	...

Sample of correlations:

	δ_V	γ_V	
α_s	0.96	-0.95	...

Preliminary

Huge uncertainties (~50%), huge correlations (almost ~100%), very unstable results.

Fit to V-channel, w(x)=1, ALEPH data

Sample of results:

$s_{\rm min}$ (GeV^2)	χ^2/dof	α_s	δ_V	
1.525	29.0/17	0.302(11)	3.37(43)	...
1.550	24.5/16	0.295(10)	3.50(50)	...
1.575	23.5/15	0.298(11)	3.50(47)	...

Sample of correlations:

	δ_V	γ_V	
α_s	0.600	-0.606	...

Much **smaller uncertainties** (~7%), **smaller correlations** (~65%), **stable results**.

Conclusions

Conclusions

- We presented a **consistent** analysis.
- Results based on fits that are sound (good statistics), and that pass theory tests (WSRs, energy variation, etc)
- Corrected ALEPH data resolved the issues present in the OPAL analysis.
- Further progress will require better understanding of DVs.
- CIPT or FOPT?

  ⟶ see Jamin's talk
- Better data (Belle?) would also be instrumental.

$$\alpha_s(m_Z^2) = 0.1165 \pm 0.0012 \text{ FOPT}$$
$$\alpha_s(m_Z^2) = 0.1185 \pm 0.0015 \text{ CIPT}$$

avrg. between ALEPH & OPAL

$$R^\tau_{V+A;ud} = N_c\, S_{\rm EW}\, |V_{ud}|^2 (1 + \delta_{\rm P} + \delta_{\rm NP})$$

$\delta_{\rm NP} = 0.020 \pm 0.009$ FOPT

$\delta_{\rm NP} = 0.016 \pm 0.010$ CIPT

ALEPH based analysis

Acknowledgments

Alexander von Humboldt
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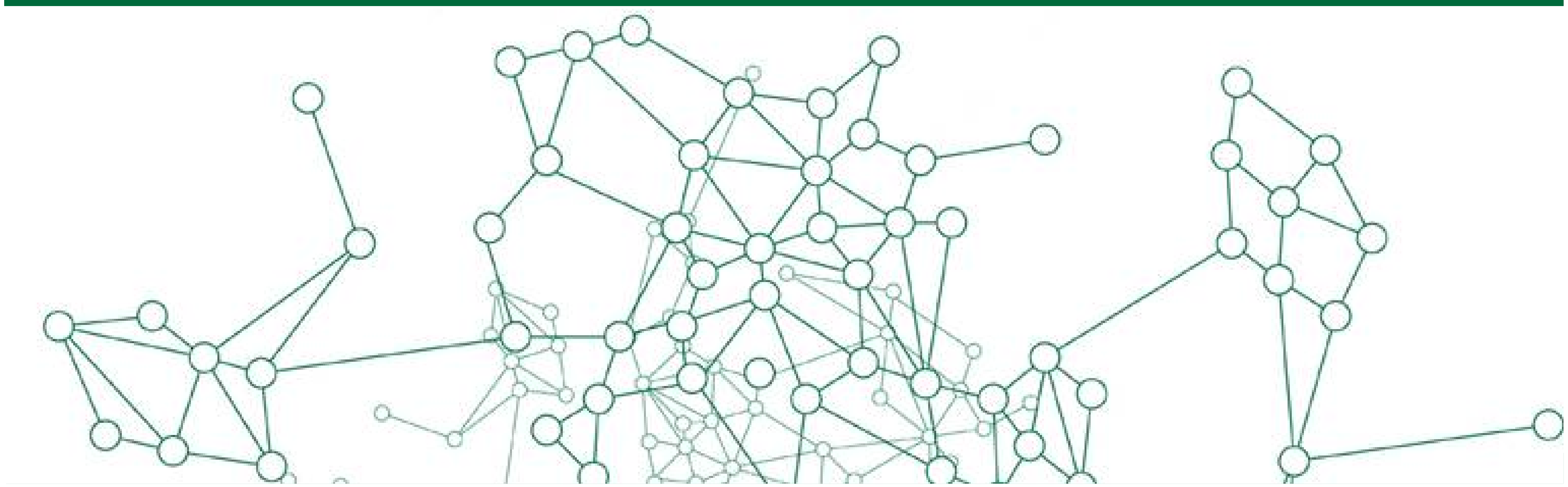
http://www.humboldt-foundation.de



Back-up material

Statistics: two types of fits

- All fits have strong correlations.

- Fits using a single weight function are standard chi^2 fits:

$$\chi^2(\boldsymbol{\theta}) = (\boldsymbol{y} - \boldsymbol{\mu}(\boldsymbol{\theta}))^T C^{-1} (\boldsymbol{y} - \boldsymbol{\mu}(\boldsymbol{\theta}))$$

- Markov-Chain MC scan of the chi^2

- Combining weight functions in a fit introduces strong correlations (tiny eigenvalues in the covariance matrix)

Common difficulty with strong correlations

- Solution:

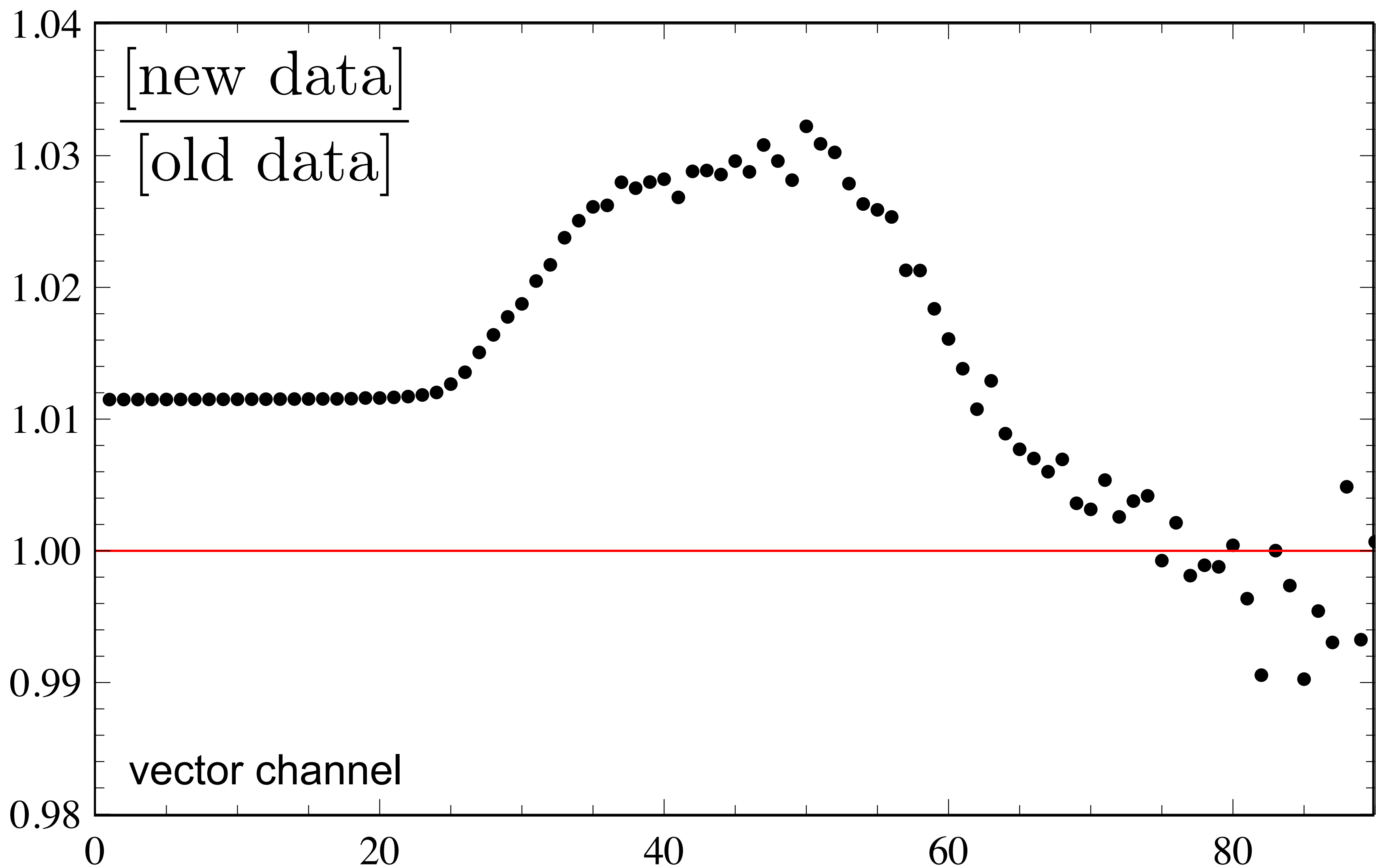
$$\mathcal{Q}^2(\boldsymbol{\theta}) = (\boldsymbol{y} - \boldsymbol{\mu}(\boldsymbol{\theta}))^T \tilde{C}^{-1} (\boldsymbol{y} - \boldsymbol{\mu}(\boldsymbol{\theta}))$$

- $\tilde{C}$ is a **block-diagonal** matrix: includes correlations between a single weight function but does not include correlations between moments of two different weights

- The full covariance matrix is used in the error propagations

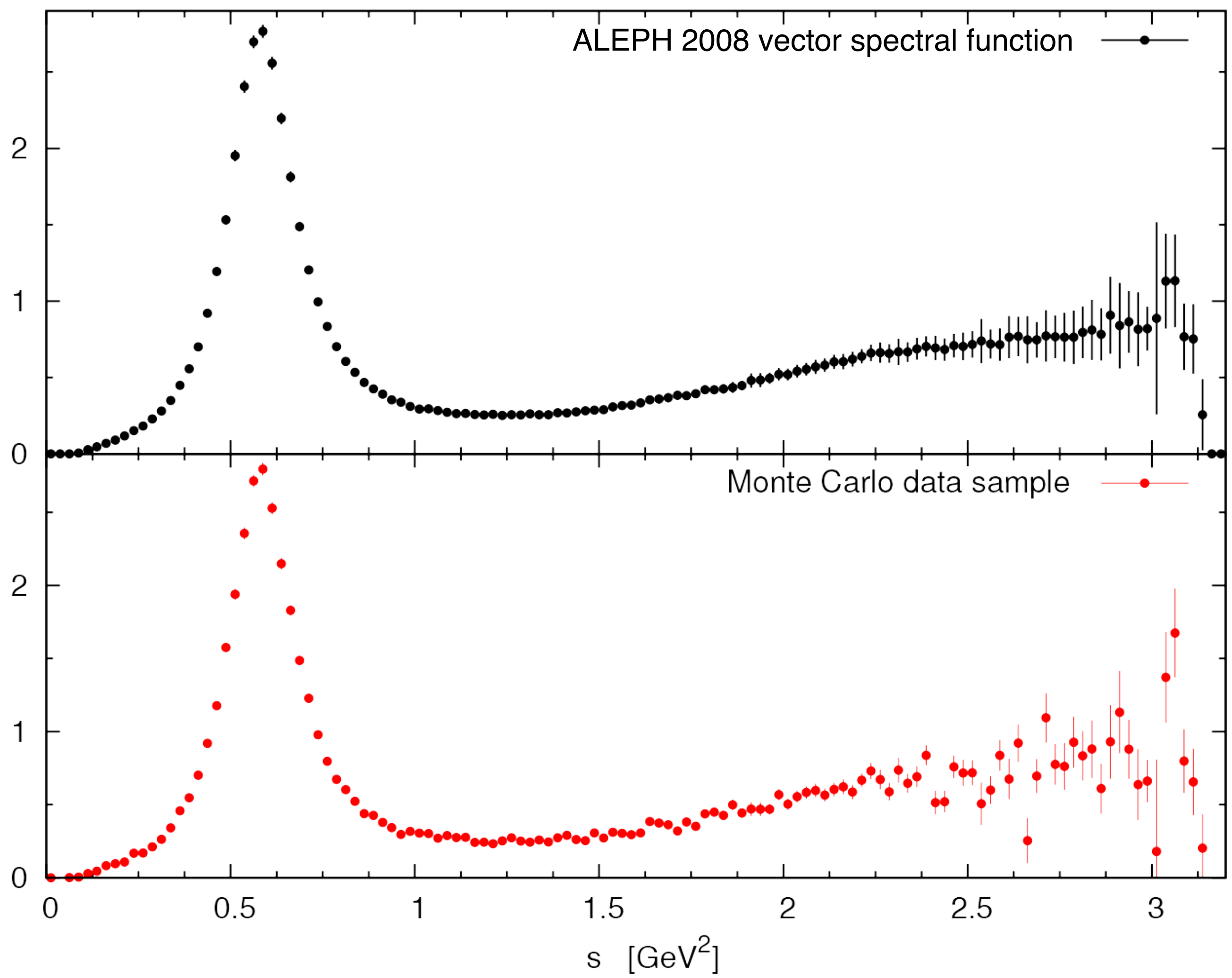
Rescaled OPAL data

- Update of OPAL spectral functions
  - The '98 OPAL spectral functions are constructed as sums over exclusive modes.
  - Normalizations fixed to the 1998 PDG values.
  - We have rescaled the distributions to build updated spectral functions (we employ recent unitarity-constrained HFAG values)

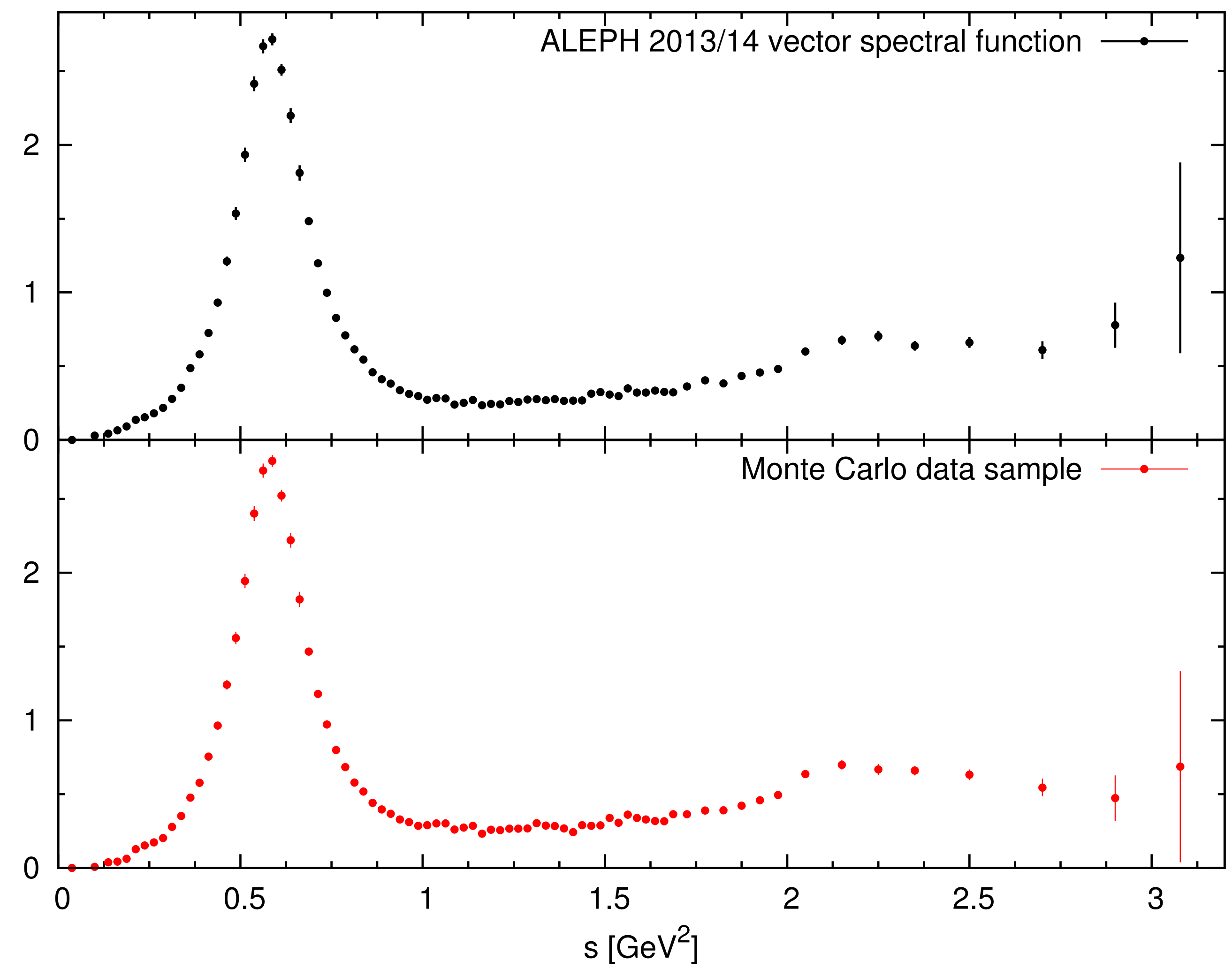


An illustration of the problem with the correlations in the 2005/08 ALEPH spectral functions

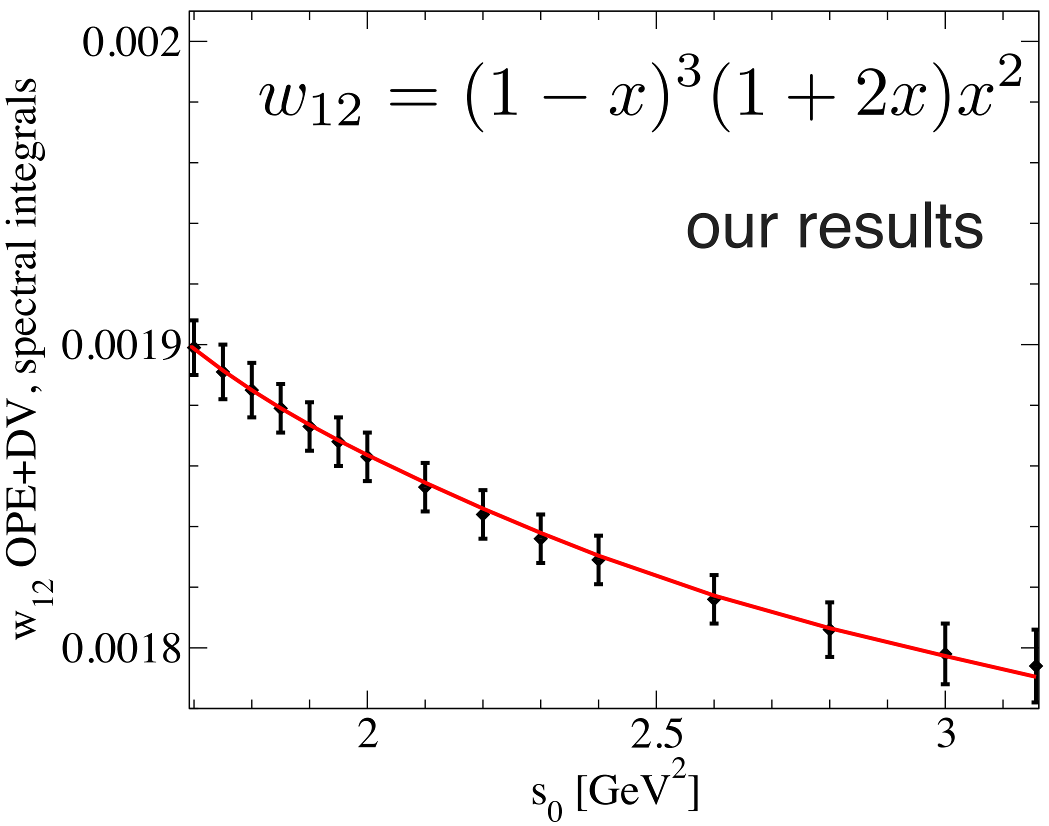
2005/08

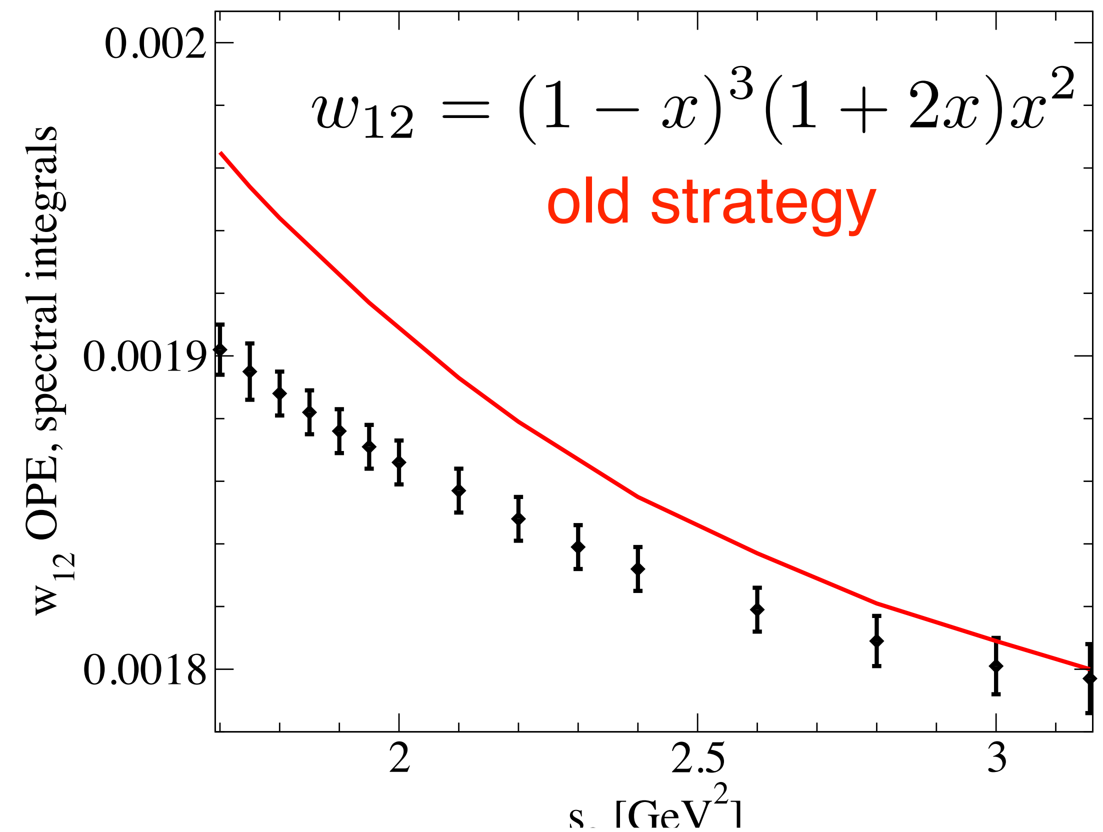


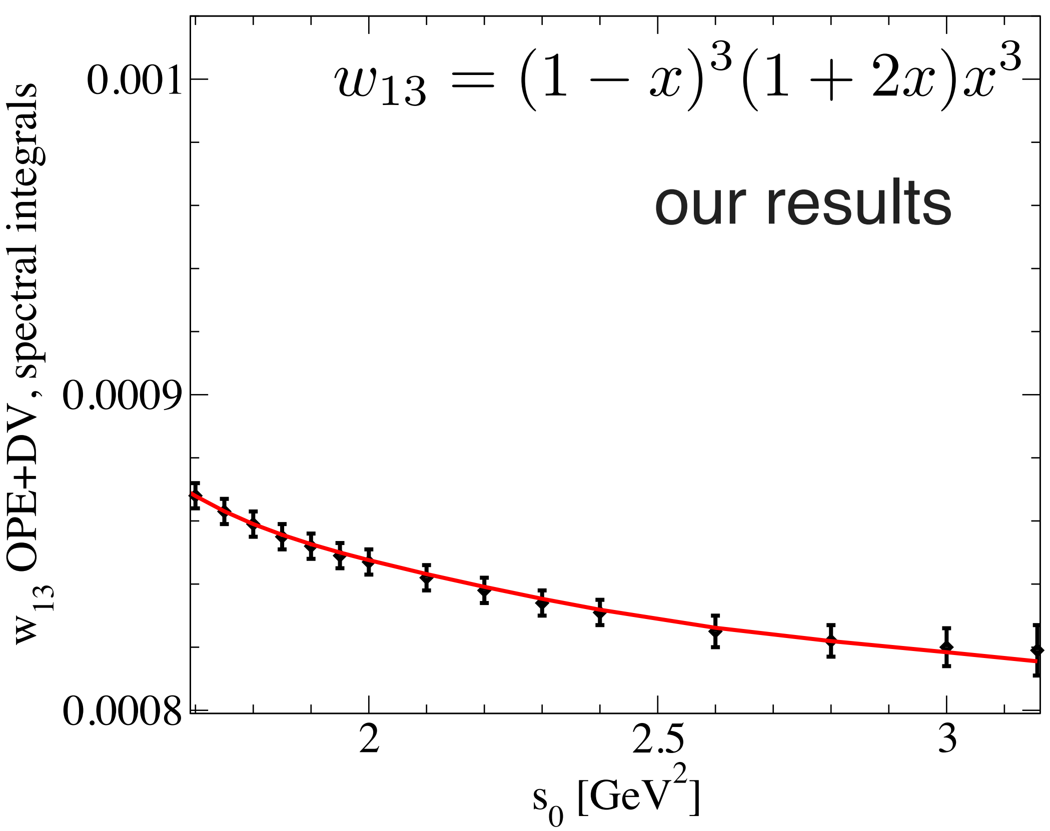
2013/14

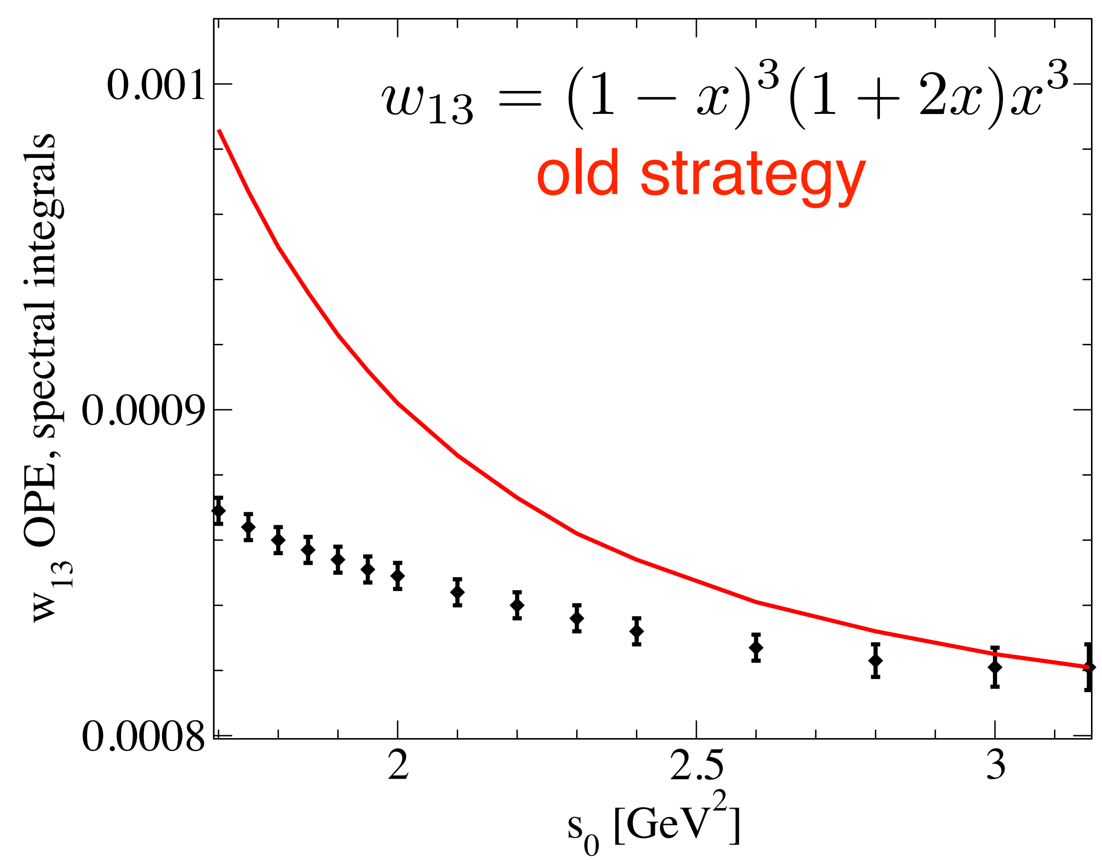


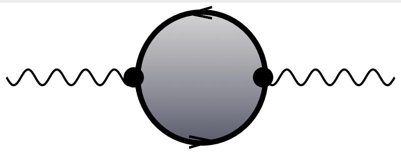
Inclusion of condensates with $D > 8$ is crucial in obtaining this agreement.









- Description in terms of the Adler function (derivative of )

$$D^{(1+0)}_{\rm pert}(s) = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{n,k} \left(\log\frac{-s}{\mu^2}\right)^{k-1} \qquad a_\mu = \alpha(\mu)/\pi$$

- only $c_{n,1}$ are independent (known up to $c_{4,1}$). $c_{n,k}$ depend on $c_{n,1}$ and β_m .
- Prescriptions for the RG improvement

FOPT
$\mu = s_0$

$$\delta^{(0)}_{{\rm FO},w_i} = \sum_{n=1}^{\infty} a(s_0)^n \sum_{k=1}^{n} k\, c_{n,k}\, J^{{\rm FO},w_i}_{k-1} \qquad J^{{\rm FO},w_i}_n \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x} W_i(x)\log^n(-x)$$

CIPT
$\mu = -s_0 x$

$$\delta^{(0)}_{{\rm CI},w_i} = \sum_{n=1}^{\infty} c_{n,1}\, J^{{\rm CI},w_i}_n(s_0) \qquad J^{{\rm CI},w_i}_n(s_0) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x} W_i(x)\, a^n(-s_0 x)$$

Le Diberder and Pich '92