Phenomenological applications of rational approximants

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Determination of Fundamental Parameters in QCD

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Institut de Física d'Altes Energies IFAE

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Outline

1. **Rational approximants**
   - Padé Approximants
   - Quadratic Approximants
2. **Physical applications**
   - Pseudoscalar Transition Form Factors
   - V_{ub} determination from $B \to \pi \ell \nu_\ell$
   - Pion Form Factor $(e^+ e^- \to \pi\pi)$
3. **Outlook**

Given a function $f(z)$

What are Padé approximants to $f(z)$?

$$P_M^N(z) = \frac{\sum_{n=0}^{N} r_n z^n}{\sum_{m=0}^{M} q_m z^m} = \frac{r_0 + r_1 z + \cdots + r_N z^N}{1 + q_1 z + \cdots + q_M z^M}$$

Same Taylor expansion than $f(z)$ up to order $\mathcal{O}(z)^{M+N+1}$

$$f(z) = a_0 + a_1 z + a_2 z^2 + \mathcal{O}(z^3)$$

$$P_M^N(z) = r_0 + (r_1 - r_0 q_1) z + (r_2 - r_1 q_1 + r_0 q_1^2 - r_0 q_2) z^2 + \mathcal{O}(z^3)$$

examples:

$$P_1^0(z) = \frac{a_0}{1 - \frac{a_1}{a_0} z} \quad , \quad P_1^1(z) = \frac{a_0 + \frac{a_1^2 - a_0 a_2}{a_1}}{1 - \frac{a_2}{a_1} z}$$

Convergence of $P_M^N(z)$ to $f(z)$ known for meromorphic and Stieltjes functions

Stieltjes functions

$$f(x) = \int_0^\infty dt \frac{\rho(t)}{1+xt} \quad (\rho(t) > 0) \quad \sim \quad f(x) = \frac{1}{\pi}\int_{s_0}^\infty dt \frac{Imf(t)}{t+x}$$

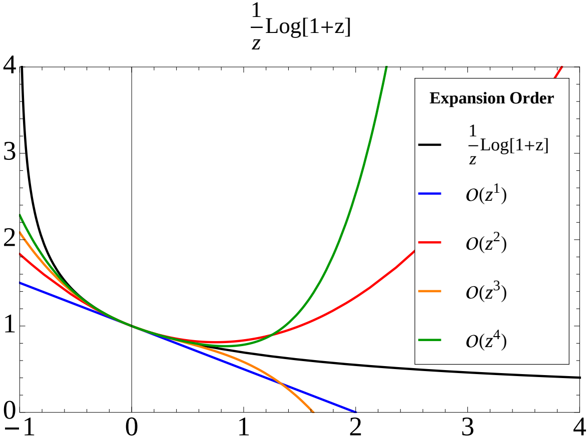
Example:

$$f(z) = \frac{1}{z} Log(1+z) = \frac{1}{\pi}\int_{-1}^{-\infty} dt \frac{Imf(t)}{z-t} = \ldots = \int_0^1 \frac{dt}{1+zt}$$

$$f(z) = \sum_{n=0} \frac{(-z)^n}{n+1} = 1 - \frac{z}{2} + \frac{z^2}{3} - \frac{z^3}{4} + \frac{z^4}{5} + \ldots$$

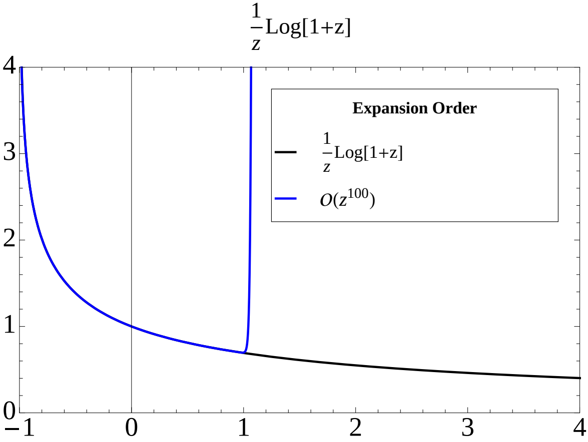
Taylor expansion

$$f(z) = \frac{1}{z} Log(1+z) ; \quad f(z) = 1 - \frac{z}{2} + \frac{z^2}{3} - \frac{z^3}{4} + \frac{z^4}{5}$$



Taylor expansion

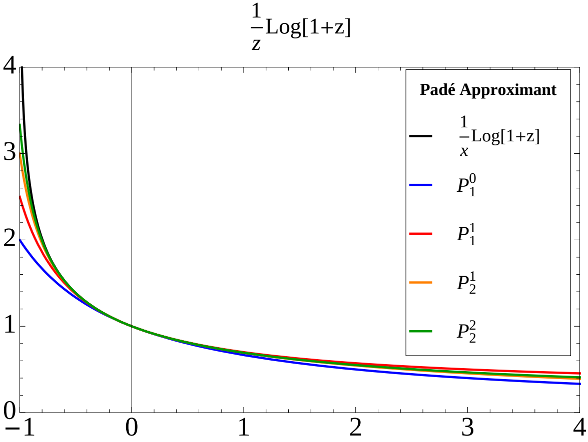
$$f(z) = \frac{1}{z} Log(1+z)\,; \quad f(z) = 1 - \frac{z}{2} + \ldots - \frac{z^{99}}{100} + \frac{z^{100}}{101}$$



$$\text{Converges: } \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} < 1 \Rightarrow |z| < \lim_{n\to\infty} \frac{n+2}{n+1} = 1\,, \quad \text{Diverges: } z > 1$$

Padé Approximants: $f(z) = \frac{1}{z} Log(1+z)$

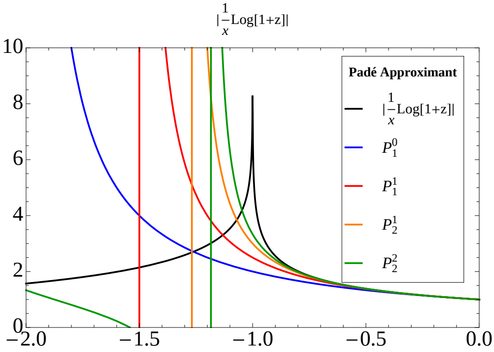
$$P_1^0 = \frac{1}{1+\frac{z}{2}}, \quad P_1^1 = \frac{1+\frac{z}{6}}{1+\frac{2z}{3}}, \quad P_2^1 = \frac{1+\frac{z}{2}}{1+z+\frac{z^2}{6}}, \quad P_2^2 = \frac{1+\frac{7z}{10}+\frac{z^2}{30}}{1+\frac{6z}{5}+\frac{3z^2}{10}}$$



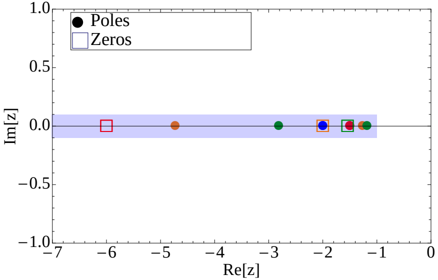
Excellent approach away from the branch cut

Padé Approximants: $f(z) = \frac{1}{z} Log(1+z)$

$$P_1^0 = \frac{1}{1+\frac{z}{2}}, \quad P_1^1 = \frac{1+\frac{z}{6}}{1+\frac{2z}{3}}, \quad P_2^1 = \frac{1+\frac{z}{2}}{1+z+\frac{z^2}{6}}, \quad P_2^2 = \frac{1+\frac{7z}{10}+\frac{z^2}{30}}{1+\frac{6z}{5}+\frac{3z^2}{10}}$$



Fails on the branch cut



Poles and Zeros "mimic" the cut

Quadratic Approximants

$$\left(f(z) - P_M^N(s)\right)^2 = \mathcal{O}\left(z^{M+N+1}\right)^2$$

$$Q(z)f^2(z) + 2R(z)f(z) + S(z) = \mathcal{O}(z^{q+r+s+2})$$

$R(z), S(z), Q(z)$ are polynomials

$$\mathcal{QA}_{q,r,s}(z) = \frac{-R(z) \pm \sqrt{R^2(z) - Q(z)S(z)}}{Q(z)}$$

Branch cut generated

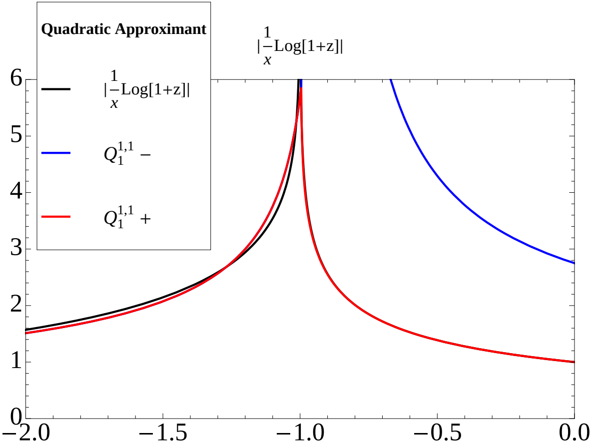
Example: $\frac{1}{z} Log(1+z)$

$$(1 + q_1 z)f^2(z) + 2(r_0 + r_1 z)f(z) + (s_0 + s_1 z) = \mathcal{O}(z^5)$$

$$\mathcal{QA}^{1,1,1}(z) = \frac{-\frac{7}{8} - \frac{5}{12}z \pm \frac{1}{8}\sqrt{14z^2 + 240z + 225}}{1 + \frac{13}{12}z}$$

Quadratic Approximants

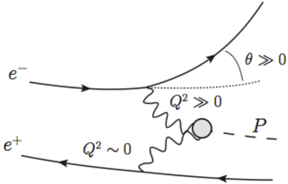
$$\mathcal{QA}^{1,1,1}(z) = \frac{-\frac{7}{8} - \frac{5}{12}z \pm \frac{1}{8}\sqrt{14z^2 + 240z + 225}}{1 + \frac{13}{12}z}$$



Good approach

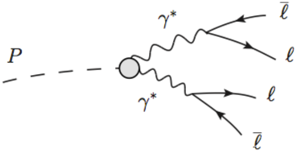
even along the cut!

Physical applications

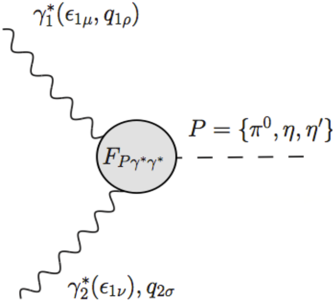


$e^+e^- \to e^+e^-P$

$P\gamma^*\gamma^{(*)}$ vertex

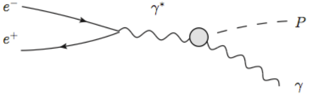


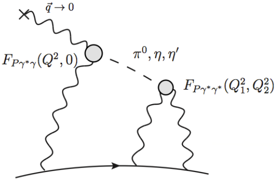
$P \to \ell\ell$, $P \to \ell\ell\gamma$, $P \to \ell\ell\ell\ell$



$(g-2)_\mu$

$e^+e^- \to P\gamma$





Transition Form Factor

e.g. $\gamma\gamma^* \rightarrow \pi^0$

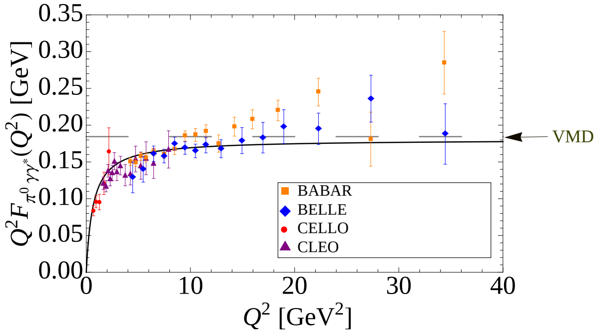
- High Energies (pertubartive QCD) $\Rightarrow F_{\gamma^*\gamma\pi^0}(\infty) = 2F_\pi Q^{-2}$
- Low Energies (χPT) $\Rightarrow F_{\gamma\gamma\pi^0}(0) = (4\pi^2 F_\pi)^{-1}$

Vector Meson Dominance (VMD)

$$F_{\gamma\gamma^*\pi^0}(Q^2) = \frac{F_{\gamma\gamma\pi^0}(0)}{1 + \frac{Q^2}{\Lambda^2}}$$

Transition Form Factor

e.g. $\gamma\gamma^* \to \pi^0$



- Intermediate and high-energy data do not adjusted well
- b_π from fit to low-energy data (CELLO&CLEO) is $\sim 1.4\sigma$ away from fit to all data

$$F_{\gamma\gamma^*\pi^0}(Q^2) = F_{\gamma\gamma\pi^0}(0)\left(1 + b_\pi \frac{Q^2}{M_\pi^2} + c_\pi \frac{Q^4}{M_\pi^4} + \mathcal{O}(Q^6)\right)$$

- How to improve the quality of the fit to stabilize the results?

How to do physics involving $F_{P\gamma^*\gamma}$ in between low-and high-energies realms?

Our proposal: Padé Approximants

- VMD model as the first step of the Padé sequence, $P_1^0(Q^2)$

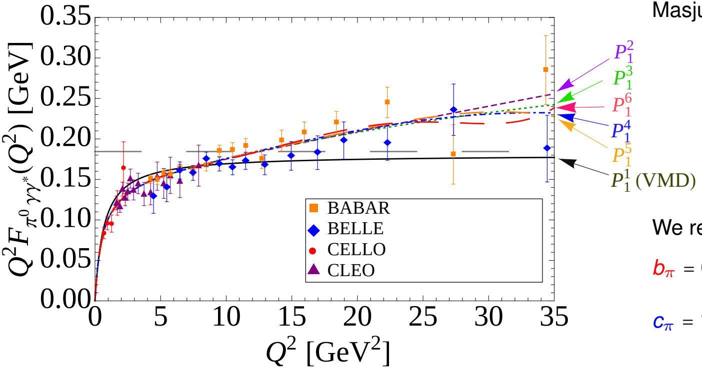
$$F^{VMD}_{\gamma\gamma^*\pi^0}(Q^2) = \frac{F_{\gamma\gamma\pi^0}(0)}{1+\frac{Q^2}{\Lambda^2}} \equiv P_1^0(Q^2)$$

- The length of the sequence is limited by data

Message: Data driven approach

Fit results

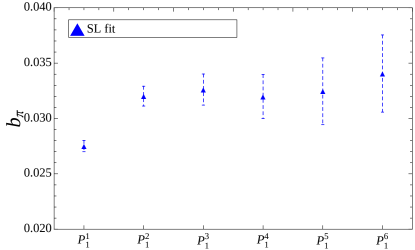
Masjuan PRD86 (2012) 094021

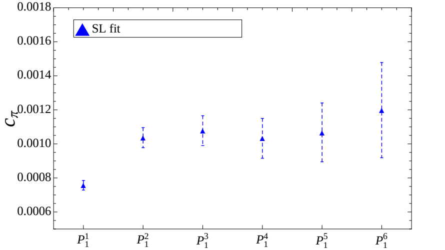


We reach P_1^6

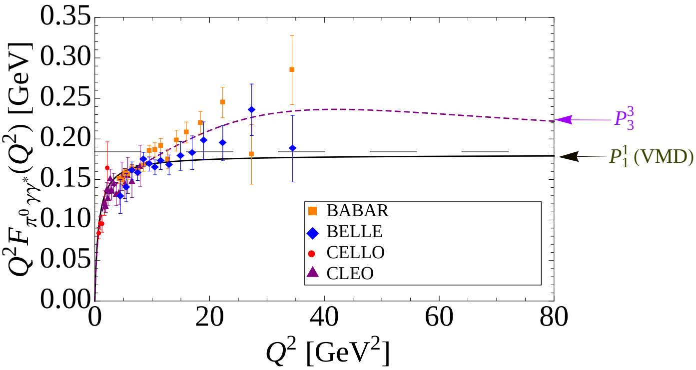
$b_\pi = 0.0340(35)_{stat}(19)_{sys}$

$c_\pi = 1.20(28)_{stat}(25)_{sys} \cdot 10^{-3}$





Fit results



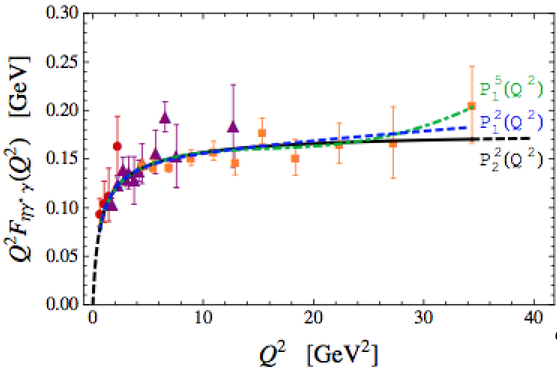
We reach P_3^3 (constraining the high-energy behaviour to $2F_\pi$)

$b_\pi = 0.0324(12)_{stat}$

$c_\pi = 1.06(9)_{stat} \cdot 10^{-3}$

Fit results

Escribano et.al. PRD89 (2014), 034014

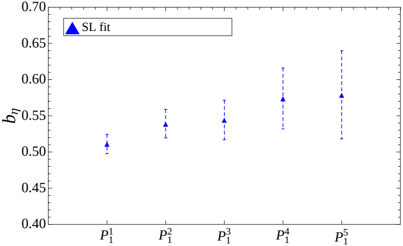


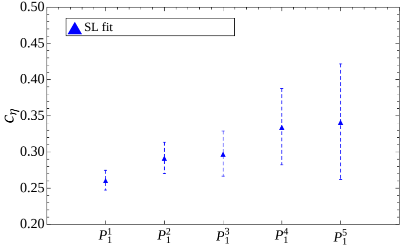
We reach P_1^5 and P_2^2

$b_\eta = 0.60(3)_{stat}(3)_{sys}$

$c_\eta = 0.37(10)_{stat}(7)_{sys}$

$\lim_{Q^2\to\infty} Q^2 F_{\eta\gamma^*\gamma}(Q^2, 0) = 0.160(24)$ GeV





Fit results

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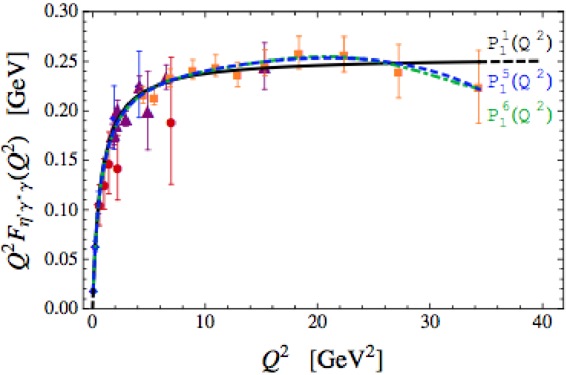
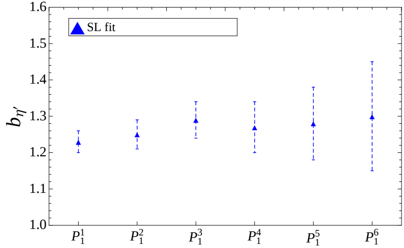


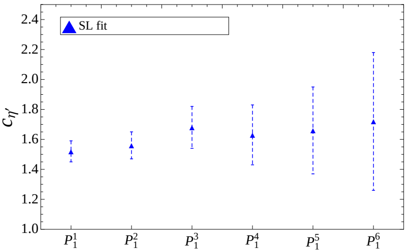
We reach P_1^6 and P_1^1

$b_{\eta'} = 1.30(15)_{stat}(7)_{sys}$

$c_{\eta'} = 1.72(47)_{stat}(34)_{sys}$

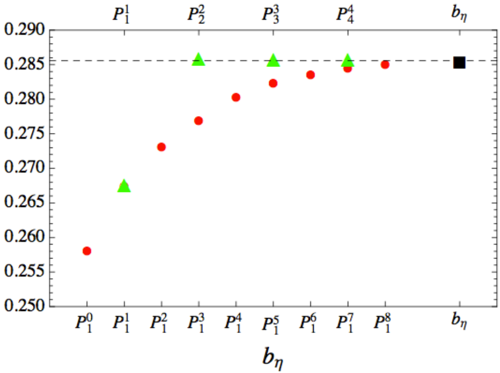
$$\lim_{Q^2\to\infty} Q^2 F_{\eta'\gamma^*\gamma}(Q^2,0) = 0.254(4) \text{ GeV}$$



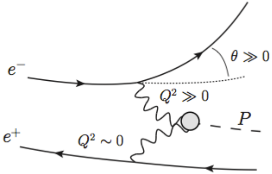


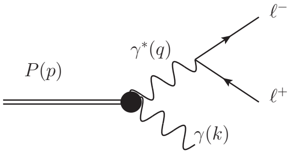
A word on the systematic error

- Consider your preferred model
- Generate a pseudodata set emulating the physical situation
- Build up your Padé Approximants sequence
- Fit and compare
- Add an extra systematic uncertainty



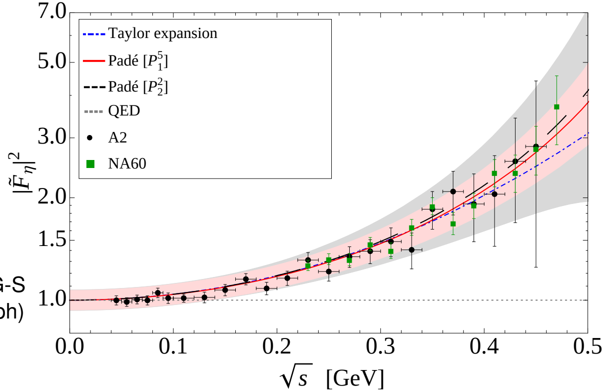
From $\gamma^*\gamma \to P$ to $P \to \gamma^*\gamma$





$e^+e^- \to e^+e^- P$

$P \to \ell\ell\gamma$



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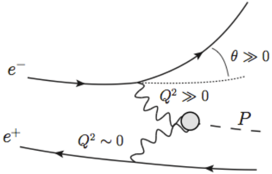
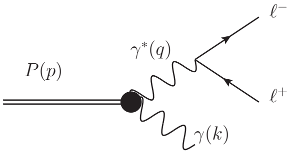
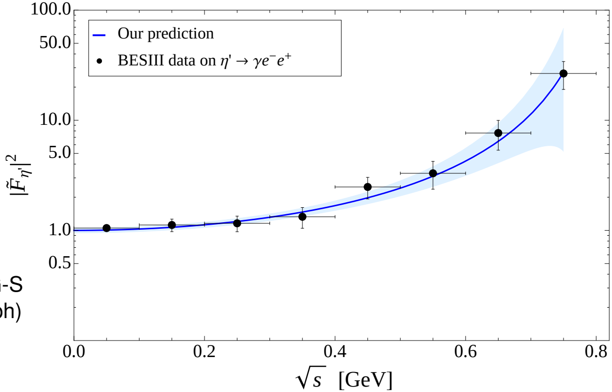
From $\gamma^*\gamma \to P$ to $P \to \gamma^*\gamma$



$e^+e^- \to e^+e^- P$



$P \to \ell\ell\gamma$



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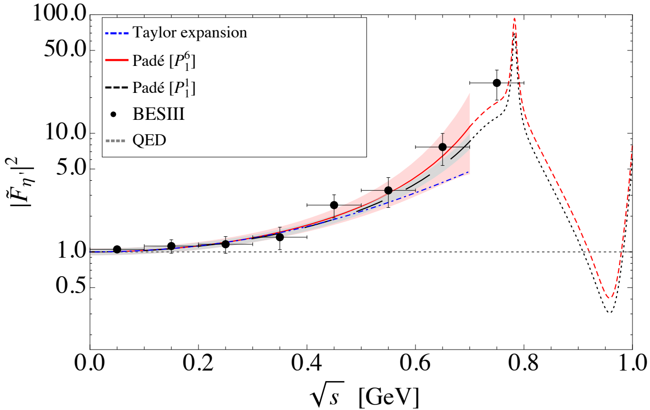
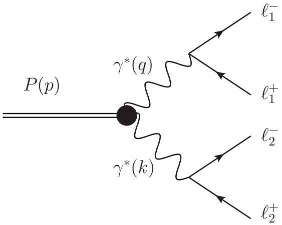
η' Transition Form Factor representation

Matching Padé description with Breit-Wigner

$$\widetilde{F}_{\mathcal{P}\gamma\gamma^*}(q^2) = \left(\sum_{V=\rho,\omega,\phi} \frac{g_{V\mathcal{P}\gamma}}{2g_{V\gamma}} \right)^{-1} \sum_{V=\rho,\omega,\phi} \frac{g_{V\mathcal{P}\gamma}}{2g_{V\gamma}} \frac{M_V^2}{M_V^2 - q^2 - iM_V\Gamma_V(q^2)}$$



Transition form factor of double virtuality



Factorization approach

$$F_{\mathcal{P}\gamma^*\gamma^*}(Q_1^2, Q_2^2) = F_{\mathcal{P}\gamma^*\gamma}(Q_1^2, 0) \times F_{\mathcal{P}\gamma\gamma^*}(0, Q_2^2)$$

However at high energies it falls as Q^{-4} instead of Q^{-2} (OPE)

Bivariate approximants (See P. Sanchez-Puertas talk)

$$P_1^0(Q_1^2, Q_2^2) = \frac{F_{\mathcal{P}\gamma\gamma}(0,0)}{1 - \frac{b_{1,0}}{M_{\mathcal{P}}^2}(Q_1^2 + Q_2^2) + \frac{b_{1,1}}{M_{\mathcal{P}}^4} Q_1^2 Q_2^2}$$

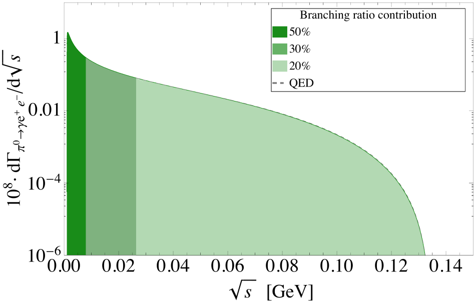
$b_{1,0} = b_{\pi^0,\eta,\eta'}$

$b_{1,1} \in \{0, 2b_{1,0}\}$ $\quad$ [$b_{1,1} = 0$ (OPE), $\quad b_{1,1} = b_{1,0}^2$ (factorization)]

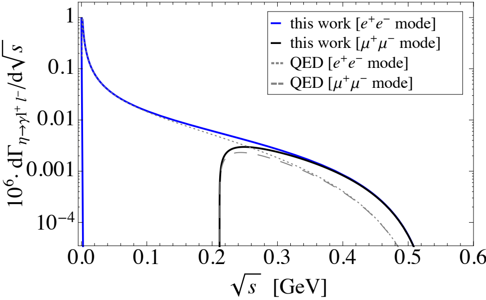
Decay distribution

$\mathcal{P} \to \ell\ell\gamma$ and $\mathcal{P} \to \ell\ell\ell\ell$ distributions are peaked at very-low energies

$\pi^0 \to ee\gamma$



$\eta \to \ell\ell\gamma$



Branching ratio predictions

Decay	This work	Experimental value	n_σ
$\pi^0 \to e^+e^-\gamma$	$1.169(1)\%$	$1.174(35)\%$	0.15
$\eta \to e^+e^-\gamma$	$6.61(59)\cdot 10^{-3}$	$6.90(40)\cdot 10^{-3}$	0.41
$\eta \to \mu^+\mu^-\gamma$	$3.27(56)\cdot 10^{-4}$	$3.1(4)\cdot 10^{-4}$	0.25
$\eta' \to e^+e^-\gamma$	$4.38(31)\cdot 10^{-4}$	$4.69(20)(23)\cdot 10^{-4}$	0.49
$\eta' \to \mu^+\mu^-\gamma$	$0.74(5)\cdot 10^{-4}$	$1.08(27)\cdot 10^{-4}$	1.24
$\pi^0 \to e^+e^-e^+e^-$	$3.36689(5)\cdot 10^{-5}$	$3.34(16)\cdot 10^{-5}$	0.17
$\eta \to e^+e^-e^+e^-$	$2.71(2)\cdot 10^{-5}$	$2.4(2)(1)\cdot 10^{-5}$	0.66
$\eta \to \mu^+\mu^-\mu^+\mu^-$	$3.98(15)\cdot 10^{-9}$	$< 3.6\cdot 10^{-4}$	
$\eta \to e^+e^-\mu^+\mu^-$	$2.39(7)\cdot 10^{-6}$	$< 1.6\cdot 10^{-4}$	
$\eta' \to e^+e^-e^+e^-$	$2.14(45)\cdot 10^{-6}$	not seen	
$\eta' \to \mu^+\mu^-\mu^+\mu^-$	$1.69(35)\cdot 10^{-8}$	not seen	
$\eta' \to e^+e^-\mu^+\mu^-$	$6.39(87)\cdot 10^{-7}$	not seen	

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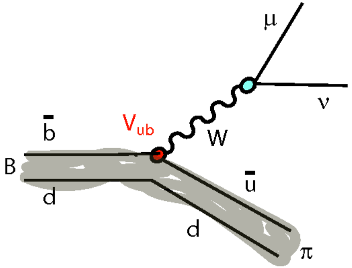
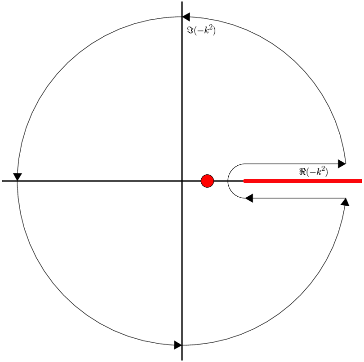
Semileptonic $B \to \pi \ell \nu_\ell$ decay



$$\frac{d\Gamma(B \to \pi \ell \nu_\ell)}{dq^2} = \frac{G_F^2 |V_{ub}|^2}{192\pi^3 m_B^3} \left((m_B^2 + m_\pi^2 - q^2)^2 - 4 m_B^2 m_\pi^2 \right)^{3/2} |F_+(q^2)|^2$$

- Determination of V_{ub}
- Main source of uncertainty within the form factor $F_+(q^2)$

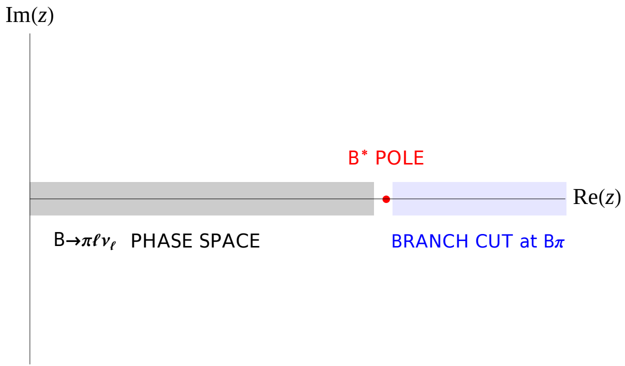
Dispersion relation for Form Factors



Form Factor analytic everywhere except for poles and cuts on the real axis

$$F(q^2) = \frac{\mathrm{Res}\,F(q^2 = s_p)}{q^2 - s_p} + \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\mathrm{Im}\,F(s')}{s' - q^2 - i\varepsilon}$$

$B \to \pi$ semileptonic Form Factor



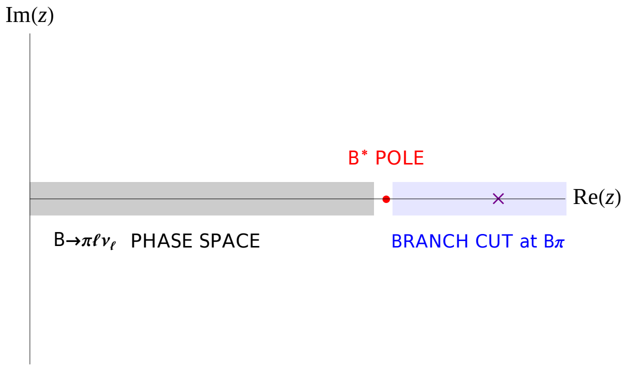
$m_{B*} = 5.325$ GeV

$$F(q^2) = \frac{\mathrm{Res}\,F(q^2{=}s_p)}{q^2 - s_p} + \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\mathrm{Im}\,F(s')}{s' - q^2 - i\varepsilon}$$

Vector Meson Dominance

$$F(q^2) = \frac{F_+(0)}{1 - q^2/m_{B*}^2}$$

$B \to \pi$ semileptonic Form Factor



$m_{B*} = 5.325$ GeV

$$F(q^2) = \frac{\mathrm{Res}\,F(q^2{=}s_p)}{q^2 - s_p} + \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\mathrm{Im}F(s')}{s' - q^2 - i\varepsilon} \quad ; \quad ImF(s') \propto \delta(s' - \Lambda)$$

Becirovic-Kaidalov (2 par.)
(PLB 478, 417 (2000))

$$F_+(q^2) = \frac{F_+(0)}{(1-q^2/m^2_{B*})(1-\alpha q^2/m^2_{B*})}$$

Ball-Zwicky (3 par.)
(PRD 71, 014015 (2005))

$$F_+(q^2) = \frac{F_+(0)}{1-q^2/m^2_{B*}} + \frac{rq^2/m^2_{B*}}{(1-q^2/m^2_{B*})(1-\alpha q^2/m^2_{B*})}$$

$B \to \pi$ semileptonic Form Factors

- VMD: $$F_+(q^2) = \frac{F_+(0)}{1-q^2/m_{B*}^2} \equiv P_1^0(q^2)$$
- Becirovic-Kaidalov: $$F_+(q^2) = \frac{F_+(0)}{(1-q^2/m_{B*}^2)(1-\alpha q^2/m_{B*}^2)} \equiv P_2^0(q^2)$$
- Ball-Zwicky: $$F_+(q^2) = \frac{F_+(0)}{1-q^2/m_{B*}^2} + \frac{rq^2/m_{B*}^2}{(1-q^2/m_{B*}^2)(1-\alpha q^2/m_{B*}^2)} \equiv P_2^1(q^2)$$
- Boyd, Grinstein, Lebed (PRL 74 (1995))

$$F_+(q^2) = \frac{1}{P(q^2)\phi(q^2,q_0^2)} \sum_{n=0}^{\infty} a_k(q_0^2)[z(q^2,q_0^2)]^k$$

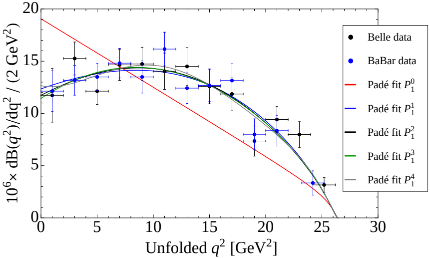
- Bourrely, Caprini, Lellouch (PRD 79 (2009) 013008)

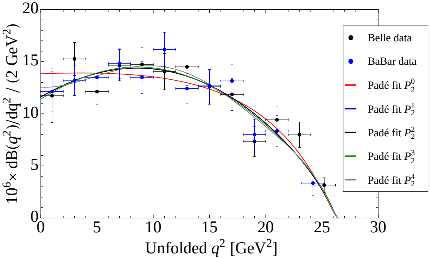
$$F_+(q^2) = \frac{1}{1-q^2/m_{B*}^2} \sum_{n=0}^{\infty} b_k(q_0^2)[z(q^2,q_0^2)]^k$$

Proposal: To analyze $B \to \pi \ell \nu_\ell$ through Padé Approximants

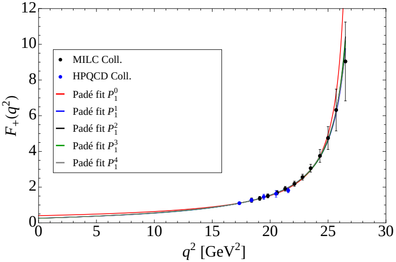
Fits to $B \to \pi \ell \nu_\ell$ spectra+Lattice data (preliminary)

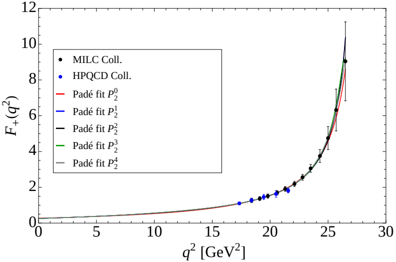
Decay spectra





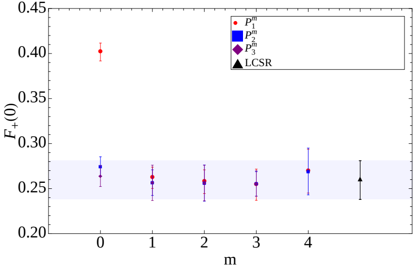
Form Factor shape

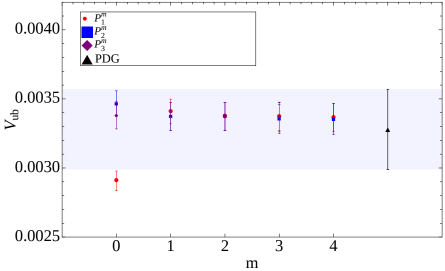




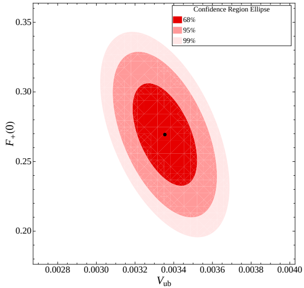
We reach P_1^4, P_2^4 and P_3^3

Fits to $B \to \pi \ell \nu_\ell$ spectra+Lattice data (preliminary)





$$F_+(0) = 0.264(16)_{stat}(7)_{sys}\, , V_{ub} = 3.37(11)_{stat}(2)_{sys} \cdot 10^{-3}$$



ex: Correlation
$F_+(0)$ vs V_{ub} $[P_1^4]$

Improvements

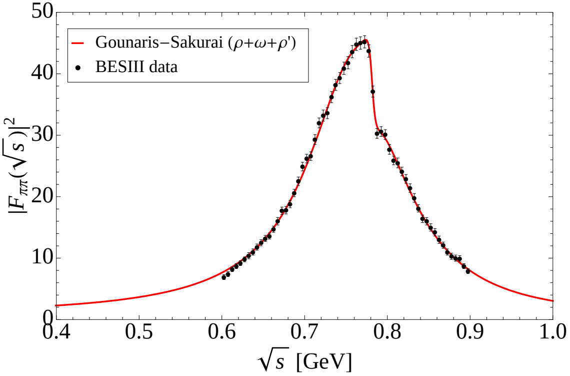
New Lattice data

Correlation of data

Fix the pole (Padé Type)

Add $F_+(0)$ as a restriction

Pion Form Factor (preliminary)



Gounaris-Sakurai (Phys.Rev.Lett. 21 (1968) , Phys.Lett. B527 (2002))

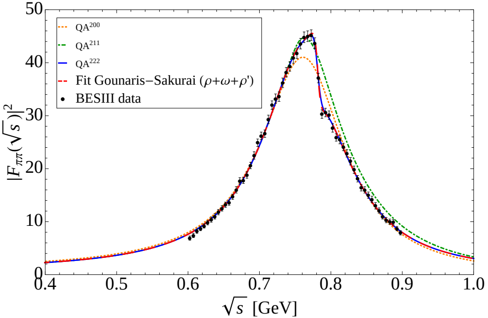
$$F_\pi(s) = \frac{BW^{GS}_{\rho(770)}(s)\cdot\left(1+\delta\,\frac{s}{M_\omega^2}\,BW_\omega(s)\right)+\beta\cdot BW^{GS}_{\rho(1450)}(s)}{1+\beta}$$

Fit results: $M_\rho = 776.81(58)$ MeV , $\Gamma_\rho = 147.73(1.42)$ MeV

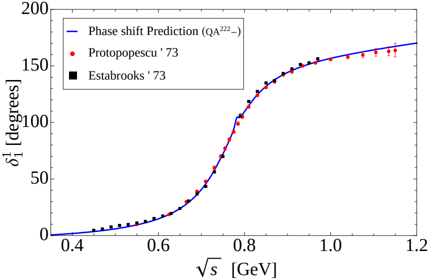
$M_\omega = 781.38(54)$ MeV , $\Gamma_\omega = 12.60(2.18)$ MeV , $\delta = 2.26(35)\cdot 10^{-3}$, $\beta = -0.0844(52)$

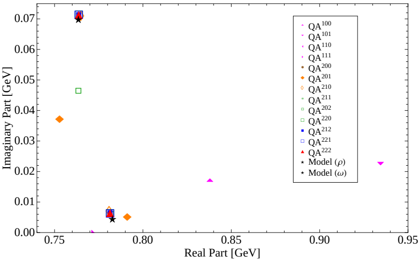
Pion Form Factor (preliminary)

Matching derivatives at the ρ pole



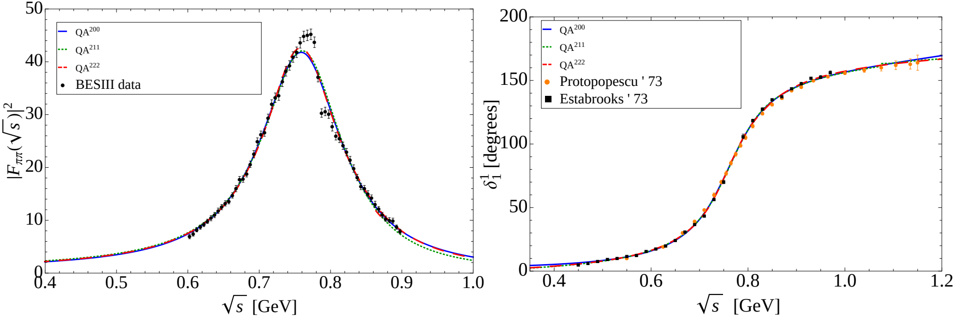
Phase shift prediction (QA^{222})





Pion Form Factor (preliminary)

Combined fit to $|F_\pi|^2$ and δ_1^1 data



Model: poles at $z_p = 0.764 - i0.070$ MeV, $z_p = 0.783 - i0.004$ MeV and $z_p = 1.423 - i0.190$ MeV

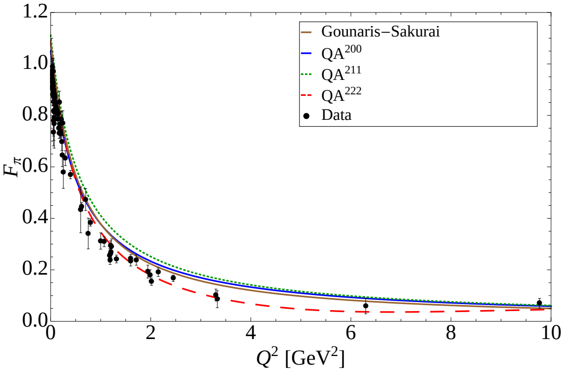
Fit: QA^{200} poles at $z_p = 0.760 - i0.060$ MeV and $z_p = 1.407 - i0.515$ MeV

QA^{211} poles at $z_p = 0.763 - i0.069$ MeV and $z_p = 0.790 - i0.068$ MeV

QA^{222} poles at $z_p = 0.759 - i0.070$ MeV and $z_p = 0.764 - i0.075$ MeV

Pion Form Factor (preliminary)

Prediction of F_π in the Space-like energy region



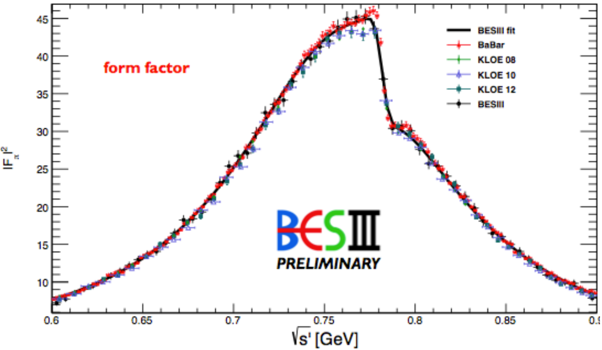
Improvements: Include Space-Like data in the fit

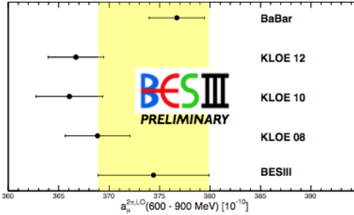
Fix the ω pole

Matching with ChPT

Include a third pole

Pion Form Factor (preliminary)





Experiment	$a_\mu^{2\pi,LO}$ (600 – 900 MeV) [10^{-10}]
BaBar	$376.7 \pm 2.0_{stat} \pm 1.9_{sys}$
KLOE 08	$368.9 \pm 0.4_{stat} \pm 2.3_{sys,exp} \pm 2.2_{sys,theo}$
KLOE 10	$366.1 \pm 0.9_{stat} \pm 2.3_{sys,exp} \pm 2.2_{sys,theo}$
KLOE 12	$366.7 \pm 1.2_{stat} \pm 2.4_{sys,exp} \pm 0.8_{sys,theo}$
BESIII (preliminary)	$374.4 \pm 2.6_{stat} \pm 4.9_{sys}$

(Figures borrowed from Benedikt Kloss, photon'15)

Tentative application: To evaluate $a_\mu^{2\pi,LO}(600-900)$ MeV and see what Space-Like data of the pion FF tell us

$$a_\mu^{2\pi,VP} \simeq \int_{4m_\pi^2}^{\infty} K(s)\sigma(e^+e^- \to 2\pi)(s)ds$$

$$|F_\pi(s)|^2 = \frac{3s}{\pi\alpha\beta_\pi^2(s)}\sigma(e^+e^- \to 2\pi)(s)\,, \quad \beta_\pi(s) = \sqrt{1-\frac{4m_\pi^2}{s}}$$

Outlook

- We have presented a method which is simple, user-friendly and predictive
- Improve systematically with new data
- Useful for describing hadronic form factors, e.g.: π^0, η, η' TFF, semileptonic $B \to \pi$, pion form factor

Backup slides

Matching η' TFF with Breit-Wigner ($g_{\rho\gamma}$ coupling as a function of the matching point)