Updated pseudoscalar contribution to the hadronic light-by-light of the muon $(g-2)$

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In collaboration with P. Masjuan

Determination of the Fundamentals Parameters in QCD
MITP, Mainz, Germany, 10^{th} March 2016

Outline

1. Current $(g_\mu - 2)$ status and hadronic contributions: HLbL
2. A transition form factor description for the HLbL
3. Updated pseudoscalar pole contribution
4. Summary & Outlook

Section 1

Current $(g_\mu - 2)$ status and hadronic contributions: HLbL

The current $(g_\mu - 2)$ status

$$a_\mu^{\rm SM} = (116\,591\,826(57)) \times 10^{-11}$$
$$a_\mu^{\rm exp} = (116\,592\,091(63)) \times 10^{-11}$$
$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = (265(85)) \times 10^{-11}$$

New Physics?
only 3σ

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Future $(g_\mu - 2)$ experiments

Fermilab & J-PARC: precision

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Future $(g_\mu - 2)$ experiments

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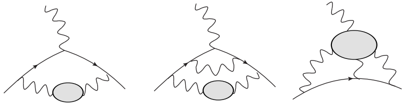
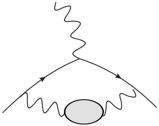
Order	Result$\times 10^{11}$
$a_\mu^{\rm HVP\cdot LO}$	6 923(42)
$a_\mu^{\rm HVP\cdot NLO}$	$-98.4(7)$
$a_\mu^{\rm HVP\cdot N^2LO}$	12.4(1)
$a_\mu^{\rm HLbL\cdot LO}$	116(39)
$a_\mu^{\rm HLbL\cdot NLO}$	3(2)
$a_\mu^{\rm QCD}$	6 956(57)

Davier et al ('12), Hagiwara et al ('11), Kurz et al ('14) Jegerlehner Nyffeler ('09, '14)

Improve on the QCD side



Hadronic contributions I: Vacuum polarization



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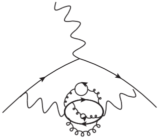

- Quark loop receives QCD corrections
- The loop integral receives contributions at Q^2 where QCD is non-perturbative
- Look for other approaches

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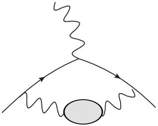

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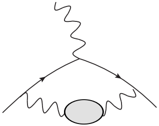

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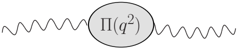

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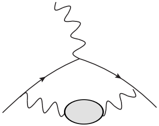

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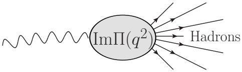

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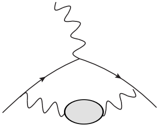

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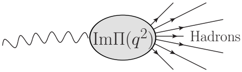

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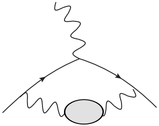

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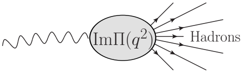
- Look for other approaches
- Ask the lattice community



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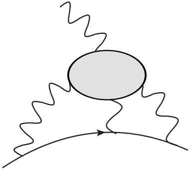
Hadronic contributions II: Hadronic Light-by-Light



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- Multi-scale problem → more difficulties
- Devise non-perturbative approach to QCD!

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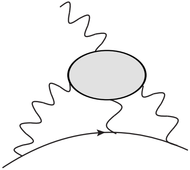

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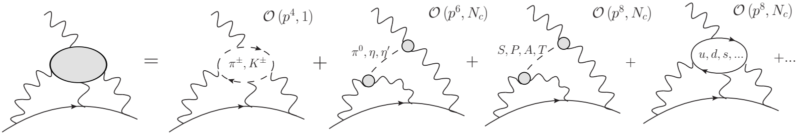

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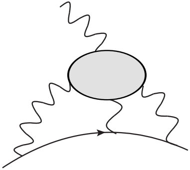

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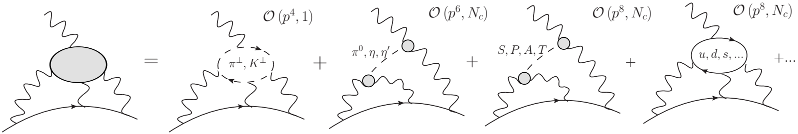

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BPP	85(13)	−19(13)	−4(3)	21(3)	83(32)
HKS	83(6)	−5(8)	2(2)	10(11)	90(15)
KN	83(12)	–	–	–	80(40)
MV	114(10)	–	22(5)	–	136(25)
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- Most results circa 15 years old
- Some advances since then

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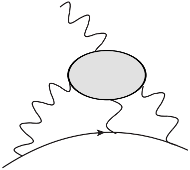

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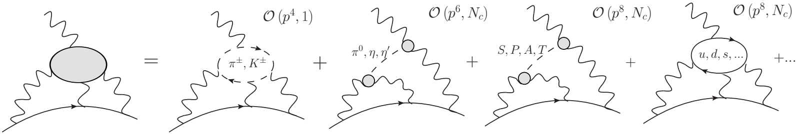

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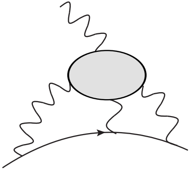

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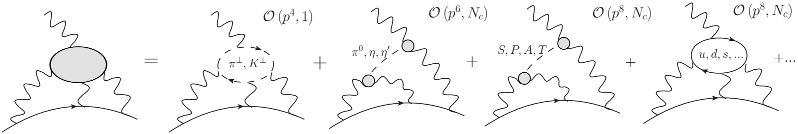

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Update
π^0, η, η'
Contributions

HLbL: the pseudoscalar-pole contribution

For the most general HLbL integral the Green's function

$$\Pi^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = \int d^4x_i e^{ip_i \cdot x_i} \langle\Omega| T\{j^\mu(x_1) j^\nu(x_2) j^\rho(x_3) j^\sigma(x_4)\} |\Omega\rangle$$

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At low energies insert lowest-lying intermediate states (close to pole):

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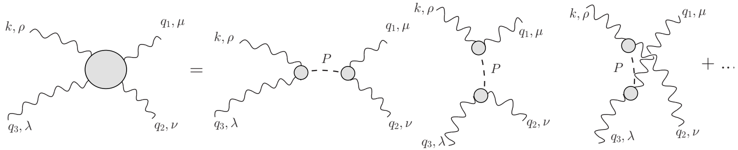
Related to physical process! Graphically, it looks like



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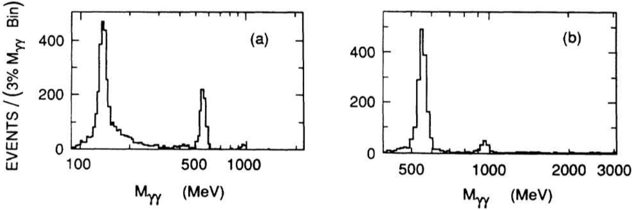
Related to physical process! Experimentally, it looks like



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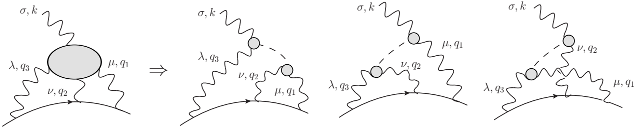
- Plug previous result into HLbL $(g-2)$ contribution



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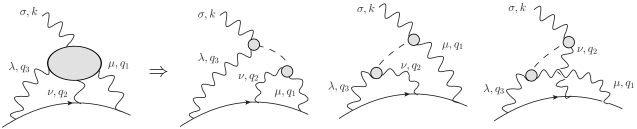

- After some fun with loops and algebra [JN Phys.Rept., 477 (2009)]

$$a_\ell^{\mathrm{HLbL};P} = \frac{-2\pi}{3}\left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 dQ_2 \int_{-1}^{+1} dt\sqrt{1-t^2}\,Q_1^3 Q_2^3$$
$$\times\left[\frac{F_{P\gamma^*\gamma^*}(Q_1^2,Q_3^2)F_{P\gamma^*\gamma}(Q_2^2,0)I_1(Q_1,Q_2,t)}{Q_2^2+m_P^2} + \frac{F_{P\gamma^*\gamma^*}(Q_1^2,Q_2^2)F_{P\gamma^*\gamma}(Q_3^2,0)I_2(Q_1,Q_2,t)}{Q_3^2+m_P^2}\right]$$

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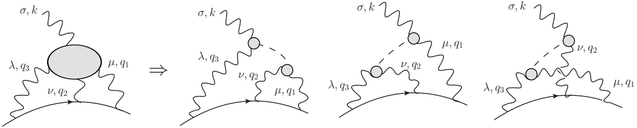

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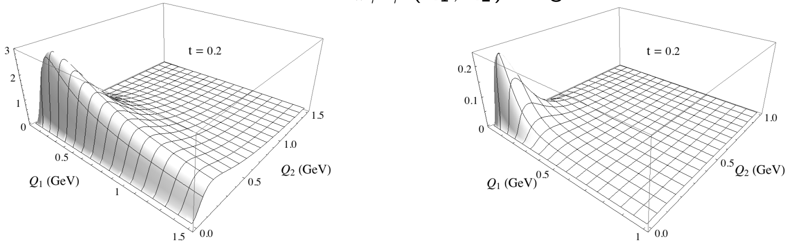

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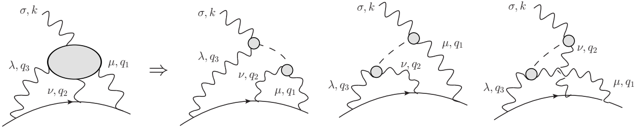

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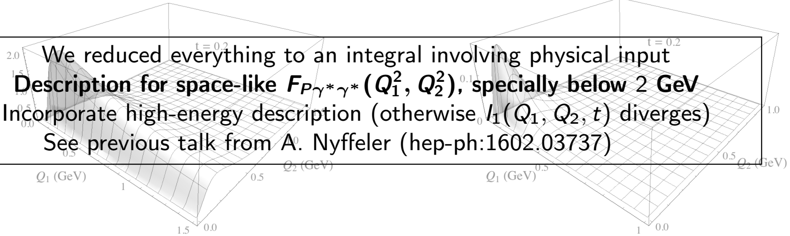


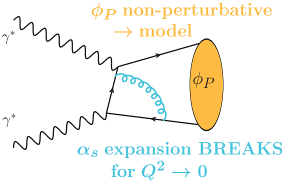
We reduced everything to an integral involving physical input
Description for space-like $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$, specially below 2 GeV
Incorporate high-energy description (otherwise $I_1(Q_1, Q_2, t)$ diverges)
See previous talk from A. Nyffeler (hep-ph:1602.03737)

Section 2

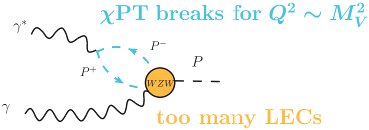
A transition form factor description for the HLbL

Describing the TFF I: First principles

—High Energies: pQCD



—Low Energies: χPT



$Q^2 \to \infty$

$$\lim_{Q^2\to\infty} F_{\pi\gamma\gamma^*}(0, Q^2) = \frac{2F_\pi}{Q^2}$$

$$\lim_{Q^2\to\infty} F_{\pi\gamma^*\gamma^*}(Q^2, Q^2) = \frac{2F_\pi}{3Q^2}$$

Guarantee convergence!

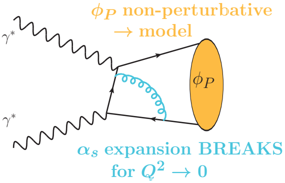
?

$Q^2 \to 0$

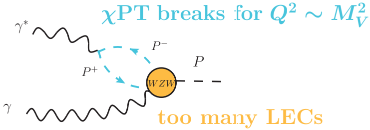
$$F_{\pi\gamma\gamma}(0,0) = (4\pi^2 F_\pi)^{-1}$$

Describing the TFF I: First principles

—High Energies: pQCD



—Low Energies: χPT



—Lattice

Ongoing progress

$Q^2 \rightarrow \infty$

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$Q^2 \rightarrow 0$

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Describing the TFF II: Model approaches

—Lagrangian-based

Nambu Jona Lasinio • Hidden Local Symmetry • Resonance chiral th. • ...

- Often provide an appropriate overall picture and ballparks
- To reach precision extremely complicated
- May be hard to consistently describe QCD properties and data
- Ok, they are models (not full QCD), problem is uncertainty estimate

Describing the TFF II: Model approaches

—Phenomenological Data-based

Large-N_c-based + Resonance saturation + Data-fitting

- Experiment is full QCD!
- Use a well motivated model to describe data

$$\text{large-}N_c \Rightarrow F_{P\gamma^*\gamma}(Q^2,0) = F_{P\gamma\gamma}(0,0)\sum_V c_V \frac{M_V^2}{M_V^2+Q^2}$$

Describing the TFF II: Model approaches

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Large-N_c-based + Resonance saturation + Data-fitting

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VMD approximation $F_{P\gamma^*\gamma}(Q^2,0) = F_{P\gamma\gamma}(0,0)\frac{M_V^2}{M_V^2+Q^2}$

Describing the TFF II: Model approaches

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Beyond VMD $F_{P\gamma^*\gamma}(Q^2,0) = F_{P\gamma\gamma}(0,0)\left(c_1 \frac{M_{V_1}^2}{M_{V_1}^2+Q^2} + c_2 \frac{M_{V_2}^2}{M_{V_2}^2+Q^2}\right)$

Describing the TFF II: Model approaches

—Phenomenological Data-based

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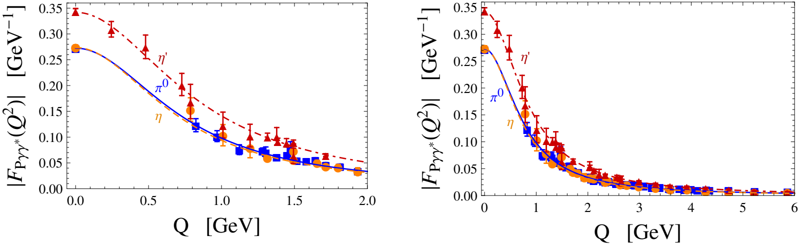

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- How reliable extrapolation is?

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- Large-N_c-corrections?

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- How reliable extrapolation is?
- Large-N_c-corrections? Systematization?

Describing the TFF II: Model approaches

—Phenomenological Data-based

Large-N_c-based + Resonance saturation + Data-fitting

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- How reliable extrapolation is?
- Large-N_c-corrections? Systematization?

—Data-based

Dispersive reconstruction

- Data based, in principle full QCD
- In practice most of QCD contributions ⇒ Not full Q^2 reconstruction

Objectives and strategies

—What do we need?

A model-independent approach for pseudoscalar transition form factors (at least in the euclidean space-like region)

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Padé Approximants: Introduction to the method

Given a function with known series expansion

$$F_{P\gamma\gamma^*}(Q^2) = F_{P\gamma\gamma^*}(0)(1 + b_P Q^2 + c_P Q^4 + ...) \quad \text{i.e. } \chi\text{PT}$$

Its Padé approximant is defined as

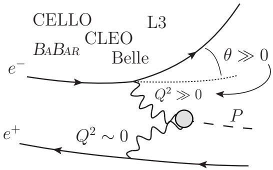
$$P_M^N(Q^2) = \frac{T_N(Q^2)}{R_M(Q^2)} = F_{P\gamma\gamma^*}(0)(1 + b_P Q^2 + c_P Q^4 + ... + \mathcal{O}(Q^2)^{N+M+1})$$

Convergence th. $\Rightarrow$ Model-independency
Increase $\{N, M\}$ $\Rightarrow$ Systematic error estimation

$$P_1^0 = \frac{F_{P\gamma\gamma^*}(0)}{1 - b_P Q^2} = F_{P\gamma\gamma^*}(0)(1+b_P Q^2+\mathcal{O}(Q^4)) \; ~~\text{VMD}~~ \rightarrow \chi\text{PT/DR} + \text{pQCD}$$

Correct low (& high) energy implementation!

Padé Approximants: Results



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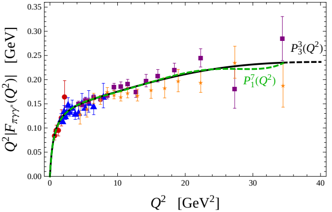

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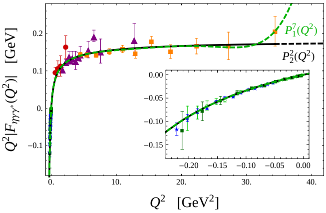

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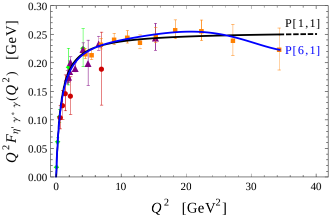

P. Masjuan, '12; R. Escribano, P. Masjuan, P. Sanchez, '14 & '15

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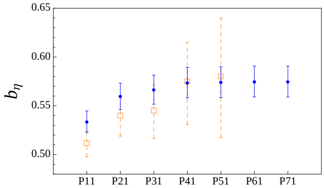

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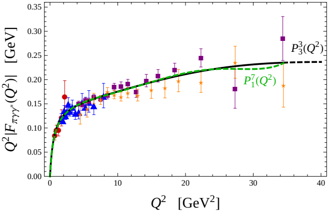

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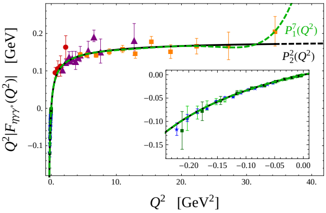

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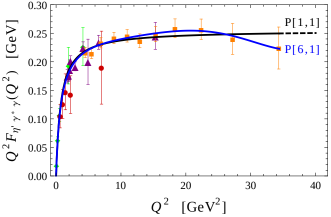

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Generalize our approach to bivariate functions: Canterbury Approximants

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

- Generalization of Padé apps. $\to$ Canterbury apps. (Chisholm 1973)

For a symmetric function with Taylor expansion

$$F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) = F_{P\gamma\gamma}(0,0)(1 + c_{1,0}(Q_1^2 + Q_2^2) + c_{2,0}(Q_1^4 + Q_2^4) + c_{1,1}Q_1^2 Q_2^2 + ...)$$

Its Canterbury appproximant is defined as

$$C_M^N(Q_1^2, Q_2^2) = \frac{T_N(Q_1^2, Q_2^2)}{Q_M(Q_1^2, Q_2^2)} = \frac{\sum_{i,j}^N a_{i,j} Q_1^{2i} Q_2^{2j}}{\sum_{k,l}^M b_{k,l} Q_1^{2k} Q_2^{2l}}$$

Fulfilling the conditions that

$$\sum_{i,j}^{M} b_{i,j} Q_1^{2i} Q_2^{2j} \sum_{\alpha,\beta}^{\infty} c_{\alpha,\beta} Q_1^{2\alpha} Q_2^{2\beta} - \sum_{k,l}^{N} a_{k,l} Q_1^{2k} Q_2^{2l} = \sum_{\gamma,\delta}^{\infty} d_{\gamma,\delta} Q_1^{2\gamma} Q_2^{2\delta},$$

$$\begin{aligned}
d_{\gamma,\delta} = \quad & 0 \quad 0 \leq \gamma+\delta \leq M+N \\
d_{\gamma,\delta} = \quad & 0 \quad 0 \leq \gamma \leq \max(M, N), \\
& \qquad 0 \leq \delta \leq \max(M, N) \\
d_{\gamma,\delta} = \quad & 0 \quad 1 \leq \gamma \leq \min(M, N), \\
& \qquad \delta = M+N+1-\gamma.
\end{aligned}$$

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}.$$

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—Reconstruction

1.Reproduce original series expansion ⇒ low energies

$$C_1^0(Q_1^2, Q_2^2) = F_{P\gamma\gamma}(0,0)(1 + b_P(Q_1^2 + Q_2^2) + a_{P;1,1} Q_1^2 Q_2^2 + \ldots)$$

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—Reconstruction

1.Reproduce original series expansion $\Rightarrow$ low energies

2.Reduce to Padé Approximants

$$C_1^0(Q^2, 0) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P Q^2} = P_1^0(Q^2) \Rightarrow F_{P\gamma\gamma}(0,0) \ \ \& \ \ b_P \ \ \text{determined}$$

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—Reconstruction

1.Reproduce original series expansion $\Rightarrow$ low energies

2.Reduce to Padé Approximants (already determined)

3.Systematically implement double virtuality: $a_{P;1,1}$ (Exp. unknown)

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3a. χPT leading logs suggest factorization at low energies

$$C_1^0(Q_1^2, Q_2^2)|_{OPE} = \frac{F_{P\gamma\gamma}(0,0)}{(1 + b_P Q_1^2)(1 + b_P Q_2^2)}; \quad (a_{P;1,1} \equiv b_P^2) \ \textit{Factorization}$$

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

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3a. χPT leading logs suggest factorization at low energies

3b. Can incorporate QCD constraints from OPE

$$C_1^0(Q_1^2, Q_2^2)|_{OPE} = \frac{F_{P\gamma\gamma}(0,0)}{1 + b_P(Q_1^2 + Q_2^2)}; \quad (a_{P;1,1} \equiv 2b_P^2)\ OPE$$

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1.Reproduce original series expansion $\Rightarrow$ low energies

2.Reduce to Padé Approximants (already determined)

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3a. χPT leading logs suggest factorization at low energies

3b. Can incorporate QCD constraints from OPE

Theoretically, we expect $a_{P;1,1} \in \{b_P^2 \div 2b_P^2\}$
Precise value ultimately from experiment (implements low energies)

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1+\alpha_1(Q_1^2+Q_2^2)+\alpha_{1,1}Q_1^2Q_2^2)}{1+\beta_1(Q_1^2+Q_2^2)+\beta_2(Q_1^4+Q_2^4)+\beta_{1,1}Q_1^2Q_2^2+\beta_{2,1}Q_1^2Q_2^2(Q_1^2+Q_2^2)+\beta_{2,2}Q_1^4Q_2^4}$$

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—Reconstruction

1.Reduce to Padé Approximants

$F_{P\gamma\gamma}(0,0), \alpha_1, \beta_1, \beta_2 \rightarrow$ from PAs

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—Reconstruction

1.Reduce to Padé Approximants

2.Reproduce the OPE behavior (high energies)

$$F_{\pi\gamma^*\gamma^*} = \frac{1}{3Q^2}(2F_\pi)\left(1-\frac{8}{9}\frac{\delta^2}{Q^2}+\mathcal{O}(\alpha_s(Q^2))\right) \Rightarrow \beta_{2,2}=0, \alpha_{1,1}, \beta_{2,1}$$

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Next element: $C_2^1(Q_1^2, Q_2^2)$

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—Reconstruction

1.Reduce to Padé Approximants

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What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

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—Reconstruction

1.Reduce to Padé Approximants

2.Reproduce the OPE behavior (high energies)

3.Reproduce the low energies ($a_{P;1,1}$)
Previous estimate $b_P^2 \leq a_{P;1,1} \leq 2b_P^2 \Rightarrow$ limited if avoiding poles
Be generous: all configurations with no poles $\Rightarrow a_{P;1,1}^{\min} < a_{P;1,1} < a_{P;1,1}^{\max}$

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What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1}Q_1^2 Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1}Q_1^2 Q_2^2 + \beta_{2,1}Q_1^2 Q_2^2(Q_1^2 + Q_2^2)}$$

—Reconstruction

1.Reduce to Padé Approximants

2.Reproduce the OPE behavior (high energies)

3.Reproduce the low energies ($a_{P;1,1}^{\min} < a_{P;1,1} < a_{P;1,1}^{\max}$)

Low- and high energies implemented
Full use of data and theory constraints
Double-virtual data for $a_{P;1,1}$ (and δ^2) desirable
Systematization up to required precision ($C_3^2(Q_1^2, Q_2^2) \to C_{N+1}^N(Q_1^2, Q_2^2)$)

Seeing is believing: toy models and systematics

—Regge Model—

$$F^{\mathrm{Regge}}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2) = \frac{aF_{P\gamma\gamma}}{Q_1^2-Q_2^2}\frac{\left[\psi^{(0)}\left(\frac{M^2+Q_1^2}{a}\right)-\psi^{(0)}\left(\frac{M^2+Q_2^2}{a}\right)\right]}{\psi^{(1)}\left(\frac{M^2}{a}\right)}$$

—Logarithmic Model—

$$F^{\log}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2) = \frac{F_{P\gamma\gamma}M^2}{Q_1^2-Q_2^2}\ln\left(\frac{1+Q_1^2/M^2}{1+Q_2^2/M^2}\right)$$

Seeing is believing: toy models and systematics

—Regge Model—

$$F^{\mathrm{Regge}}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2)=\frac{aF_{P\gamma\gamma}}{Q_1^2-Q_2^2}\frac{\left[\psi^{(0)}\left(\frac{M^2+Q_1^2}{a}\right)-\psi^{(0)}\left(\frac{M^2+Q_2^2}{a}\right)\right]}{\psi^{(1)}\left(\frac{M^2}{a}\right)}$$

$$\lim_{Q^2\to\infty}F^{\mathrm{Regge}}_{\pi^0\gamma^*\gamma}(Q^2,0)=\frac{F_{P\gamma\gamma}a\ln(Q^2)}{\psi^{(1)}(M^2/a)Q^2}$$

$$\lim_{Q^2\to\infty}F^{\mathrm{Regge}}_{\pi^0\gamma^*\gamma}(Q^2,Q^2)=\frac{F_{P\gamma\gamma}a}{\psi^{(1)}(M^2/a)Q^2}$$

—Logarithmic Model—

$$F^{\log}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2)=\frac{F_{P\gamma\gamma}M^2}{Q_1^2-Q_2^2}\ln\left(\frac{1+Q_1^2/M^2}{1+Q_2^2/M^2}\right)$$

$$\lim_{Q^2\to\infty}F^{\log}_{\pi^0\gamma^*\gamma}(Q^2,Q^2)=\frac{F_{P\gamma\gamma}M^2\ln(Q^2)}{Q^2}$$

$$\lim_{Q^2\to\infty}F^{\log}_{\pi^0\gamma^*\gamma}(Q^2,Q^2)=\frac{F_{P\gamma\gamma}M^2}{Q^2}$$

Seeing is believing: toy models and systematics

—Regge Model—

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	C_1^0	C_2^1	C_3^2	C_4^3
LE	66.0	71.9	72.8	73.1
OPE	77.4	73.4	73.3	73.3
Exact		73.3		

—Logarithmic Model—

$$F^{\log}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2) = \frac{F_{P\gamma\gamma}M^2}{Q_1^2-Q_2^2}\ln\left(\frac{1+Q_1^2/M^2}{1+Q_2^2/M^2}\right)$$

	C_1^0	C_2^1	C_3^2	C_4^3
LE	87.9	97.6	99.7	100.5
OPE	99.5	101.2	101.4	101.5
Exact		101.5		

Seeing is believing: toy models and systematics

—Regge Model—

$$F^{\mathrm{Regge}}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2)=\frac{aF_{P\gamma\gamma}}{Q_1^2-Q_2^2}\frac{\left[\psi^{(0)}\left(\frac{M^2+Q_1^2}{a}\right)-\psi^{(0)}\left(\frac{M^2+Q_2^2}{a}\right)\right]}{\psi^{(1)}\left(\frac{M^2}{a}\right)}$$

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—Logarithmic Model—

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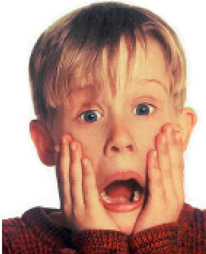
	C_1^0	C_2^1	C_3^2	C_4^3
LE	87.9	97.6	99.7	100.5
OPE	99.5	101.2	101.4	101.5
Exact		101.5		

- The convergence result is excellent!
- The OPE choice seems the best $\rightarrow$ high energy matters
- Still, low energies provide a good performance
- Error $\sim$ difference among elements $\rightarrow$ Systematics!



Section 3

Updated pseudoscalar pole contribution

Pseudoscalar-pole contribution: Final results

—$C_1^0(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	Fact ($a_{P;1,1} = b_P^2$)	OPE ($a_{P;1,1} = 2b_P^2$)
π^0	$54.0(1.1)_F(2.5)_{b_\pi}[2.7]_t$	$64.9(1.4)_F(2.8)_{b_\pi}[3.1]_t$
η	$13.0(0.4)_F(0.4)_{b_\eta}[0.6]_t$	$17.0(0.6)_F(0.4)_{b_\eta}[7]_t$
η'	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$	$16.0(0.5)_F(0.3)_{b_{\eta'}}[6]_t$
Total	$79.0[2.8]_t$	$97.9[3.2]_t$

Pseudoscalar-pole contribution: Final results

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η'	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$	$16.0(0.5)_F(0.3)_{b_{\eta'}}[6]_t$
Total	$79.0[2.8]_t$	$97.9[3.2]_t$

—$C_2^1(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\mathrm{min}}$	$a_{P;1,1}^{\mathrm{max}}$
π^0	$63.9(1.3)_L(0)_\delta[1.3]_t$	$62.5(1.2)_L(1.0)_\delta[1.6]_t$
η	$16.6(0.8)_L(0)_\delta[1.0]_t$	$16.3(0.8)_L(0.6)_\delta[1.0]_t$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t$	$14.3(0.5)_L(0.5)_\delta[0.7]_t$
Total	$95.2[1.7]_t$	$93.1[2.1]_t$

Pseudoscalar-pole contribution: Final results

$$—C_1^0(Q_1^2, Q_2^2)—$$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	Fact ($a_{P;1,1} = b_P^2$)	OPE ($a_{P;1,1} = 2b_P^2$)
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$$—C_2^1(Q_1^2, Q_2^2)—$$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$63.9(1.3)_L(0)_\delta[1.3]_t\{1.0\}_{\mathrm{sys}}$	$62.5(1.2)_L(1.0)_\delta[1.6]_t\{2.4\}_{\mathrm{sys}}$
η	$16.6(0.8)_L(0)_\delta[1.0]_t$	$16.3(0.8)_L(0.6)_\delta[1.0]_t$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t$	$14.3(0.5)_L(0.5)_\delta[0.7]_t$
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Pseudoscalar-pole contribution: Final results

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η	$16.6(0.8)_L(0)_\delta[1.0]_t\{1.4\}_{\mathrm{sys}}$	$16.3(0.8)_L(0.6)_\delta[1.0]_t\{0.7\}_{\mathrm{sys}}$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t$	$14.3(0.5)_L(0.5)_\delta[0.7]_t$
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Pseudoscalar-pole contribution: Final results

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η	$16.6(0.8)_L(0)_\delta[1.0]_t\{1.4\}_{\mathrm{sys}}$	$16.3(0.8)_L(0.6)_\delta[1.0]_t\{0.7\}_{\mathrm{sys}}$
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η'	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\mathrm{sys}}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\mathrm{sys}}$
Total	$95.2[1.7]_t\{2.7\}_{\mathrm{sys}}$	$93.1[2.1]_t\{4.8\}_{\mathrm{sys}}$

Pseudoscalar-pole contribution: Final results

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$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$63.9(1.3)_L(0)_\delta[1.3]_t\{1.0\}_{\mathrm{sys}}$	$62.5(1.2)_L(1.0)_\delta[1.6]_t\{2.4\}_{\mathrm{sys}}$
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Total	$95.2[1.7]_t\{2.7\}_{\mathrm{sys}}$	$93.1[2.1]_t\{4.8\}_{\mathrm{sys}}$

—Final Result (preliminary)

$$a_\mu^{\pi,\eta,\eta'} = (63.2[1.5]\{2.4\} + 16.5[1.0]\{1.4\} + 14.5[0.7]\{1.7\}) \times 10^{-11} = 94.2[2.3]\{4.8\} \times 10^{-11}$$

Pseudoscalar-pole contribution: Final results

—$C_1^0(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	Fact ($a_{P;1,1} = b_P^2$)	OPE ($a_{P;1,1} = 2b_P^2$)
π^0	$54.0(1.1)_F(2.5)_{b_\pi}[2.7]_t$	$64.9(1.4)_F(2.8)_{b_\pi}[3.1]_t$
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Total	$79.0[2.8]_t$	$97.9[3.2]_t$

—$C_2^1(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$63.9(1.3)_L(0)_\delta[1.3]_t\{1.0\}_{\mathrm{sys}}$	$62.5(1.2)_L(1.0)_\delta[1.6]_t\{2.4\}_{\mathrm{sys}}$
η	$16.6(0.8)_L(0)_\delta[1.0]_t\{1.4\}_{\mathrm{sys}}$	$16.3(0.8)_L(0.6)_\delta[1.0]_t\{0.7\}_{\mathrm{sys}}$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\mathrm{sys}}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\mathrm{sys}}$
Total	$95.2[1.7]_t\{2.7\}_{\mathrm{sys}}$	$93.1[2.1]_t\{4.8\}_{\mathrm{sys}}$

—Final Result (preliminary)

$$a_\mu^{\pi,\eta,\eta'} = (63.2[1.5]\{2.4\} + 16.5[1.0]\{1.4\} + 14.5[0.7]\{1.7\}) \times 10^{-11} = 94.2[2.3]\{4.8\} \times 10^{-11}$$

What has been achieved?

Final Updated Result (preliminary) — $\delta a_\mu^{\mathrm{exp}} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.2[1.5]\{2.4\} + 16.5[1.0]\{1.4\} + 14.5[0.7]\{1.7\}) \times 10^{-11} = 94.2[2.3]\{4.8\} \times 10^{-11}$$

KN: Phys.Rev., D65, 073034 (2002); **GLCR**: Phys.Rev., D89, 073016 (2014)

Final Updated Result (preliminary) $\delta a_\mu^{\mathrm{exp}} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.2(2.8) + 16.5(1.7) + 14.5(1.8)) \times 10^{-11} = 94.2(5.3) \times 10^{-11}$$

KN: Phys.Rev., D65, 073034 (2002); **GLCR**: Phys.Rev., D89, 073016 (2014)

Final Updated Result (preliminary) $\delta a_\mu^{\text{exp}} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.2(2.8) + 16.5(1.7) + 14.5(1.8)) \times 10^{-11} = 94.2(5.3) \times 10^{-11}$$

- Updated value meeting future exp. precision (if $\delta a_\mu^{\text{HVP}}$, then 11×10^{-11})
- Full use of current data with systematics and good data description
- Full QCD constraints, also for the η and η'
- η and η' relevant, of the order of $\delta a_\mu^{\text{exp}}$

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Final Updated Result (preliminary) — $\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.2(2.8) + 16.5(1.7) + 14.5(1.8)) \times 10^{-11} = 94.2(5.3) \times 10^{-11}$$

- Updated value meeting future exp. precision (if $\delta a_\mu^{\rm HVP}$, then 11×10^{-11})
- Full use of current data with systematics and good data description
- Full QCD constraints, also for the η and η'
- η and η' relevant, of the order of $\delta a_\mu^{\rm exp}$

Previous KN Result — $\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (58(10) + 13(1) + 12(1)) \times 10^{-11} = 83(12) \times 10^{-11}$$

- Study focused in sign mistake; remember $\delta a_\mu = 63 \times 10^{-11}$
- Not fully statistical errors and no systematics included ($N_c \to 30\%?$)
- Belle π^0, Babar π^0, η, η', $\pi^0 \to \gamma\gamma$ PrimEx, $\eta \to \gamma\gamma$ KLOE-2 unreleased
- η η' factorized: roughly 6×10^{-11} shift

KN: Phys.Rev., D65, 073034 (2002); **GLCR**: Phys.Rev., D89, 073016 (2014)

Final Updated Result (preliminary) $\delta a_\mu^{\mathrm{exp}} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.2(2.8) + 16.5(1.7) + 14.5(1.8)) \times 10^{-11} = 94.2(5.3) \times 10^{-11}$$

- Updated value meeting future exp. precision (if $\delta a_\mu^{\mathrm{HVP}}$, then 11×10^{-11})
- Full use of current data with systematics and good data description
- Full QCD constraints, also for the η and η'
- η and η' relevant, of the order of $\delta a_\mu^{\mathrm{exp}}$

Recent GLCR Result $\delta a_\mu^{\mathrm{exp}} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (57.5(0.6) + 14.4(2.6) + 10.8(0.9)) \times 10^{-11} = 82.7(2.8) \times 10^{-11}$$

- There are no systematic errors included above ($N_c \to 30\%$?)
- No data used for the η, η' but $SU(3)$-symmetry

KN: Phys.Rev., D65, 073034 (2002); **GLCR**: Phys.Rev., D89, 073016 (2014)

Final Updated Result (preliminary) — $\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.2(2.8) + 16.5(1.7) + 14.5(1.8)) \times 10^{-11} = 94.2(5.3) \times 10^{-11}$$

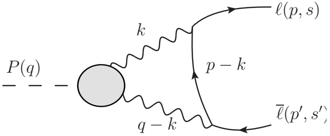
- Updated value meeting future exp. precision (if $\delta a_\mu^{\rm HVP}$, then 11×10^{-11})
- Full use of current data with systematics and good data description
- Full QCD constraints, also for the η and η'
- η and η' relevant, of the order of $\delta a_\mu^{\rm exp}$

Possible improvements

- Double virtuality measurements ($a_{P;1,1}$, δ^2): BES III
- π^0: low-energy SL at BES III & KLOE-2, TL (Dalitz) at MAMI, KLOE-2?
- η': Dalitz decay at MAMI?

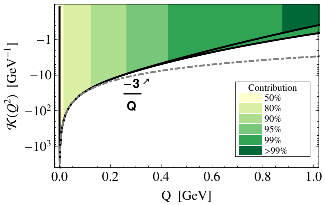
KN: Phys.Rev., D65, 073034 (2002); **GLCR**: Phys.Rev., D89, 073016 (2014)

KTeV's $\pi^0 \to e^+e^-$: a source of double-virtual information?



- The process involves a loop integral over doubly-virtual TFF (mostly $Q_1^2 = Q_2^2$)
- Loop integral saturates below 1 GeV (low-energy test)
- Indirect constraint on $a_{\pi;1,1}, \delta^2$!

KTeV's $\pi^0 \to e^+e^-$: a source of double-virtual information?



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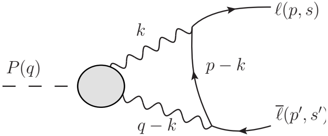

KTeV result

$$\mathrm{BR}(\pi^0 \to e^+e^-) = 7.48(38) \times 10^{-8}$$

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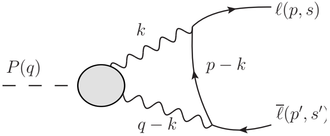

KTeV result with latest RCs

$$\mathrm{BR}(\pi^0 \to e^+e^-) = 6.87(36) \times 10^{-8}$$

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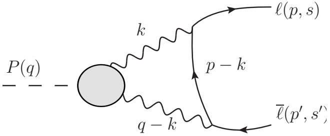

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$$\mathrm{BR}^{C^1_2}(\pi^0 \to e^+e^-) = 6.23(5) \times 10^{-8}$$

$[a_{\pi;1,1} = (1.92 - 2.07)b_\pi^2, \delta^2 = 0.2\ \mathrm{GeV}^2]$

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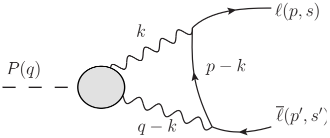

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$[a_{\pi;1,1} = (1.92-2.07)b_\pi^2, \delta^2 = 0.2\ \mathrm{GeV}^2]$

If seriously taken

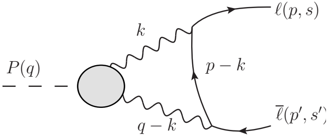
Have to damp the TFF requiring
$\delta^2 \geq 10\ \mathrm{GeV}^2, a_{P;1,1} = -(4 \div 32)b_\pi^2$

——Meanwhile, a_μ^{HLbL}——

$$a_\mu^{HLbL;\pi^0} = 36(7)\times 10^{-11}$$

[We obtained $a_\mu^{HLbL;\pi^0} = 63.2(2.8)$]

KTeV's $\pi^0 \to e^+e^-$: a source of double-virtual information?



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$$a_\mu^{HLbL;\pi^0} = 36(7) \times 10^{-11}$$
[We obtained $a_\mu^{HLbL;\pi^0} = 63.2(2.8)$]

Prospects

New π^0 measurement: NA62 possible
BES III measurements would help
New Physics?

$\eta \to \mu^+\mu^-$ some tension (larger error)
Though requires opposite solution in terms of $a_{\eta;1,1}, \delta^2$

Section 4

Summary & Outlook

Summary & Outlook

- Updated the π^0, η, η'-pole contribution to HLbL
- Full use of SL and low-energy TL data + theory constraints
- Systematic and model-independent implementation with Canterbury approximants
- New value $a_\mu^{HLbL;\pi,\eta,\eta'} = 94.2(5.4) \times 10^{-11}$ including systematics
- Error meets future experiments $\delta a_\mu \sim 16 \times 10^{-11}$ requirements
- Improvement: double-virtual measurements $\gamma^*\gamma^* \to P$ BESIII
- User friendly and potential tool for experimentalists/lattice