Test of Quark-Hadron Duality in Tau-Decay

Tau decay may be the ideal laboratory to test subtle effects of QCD in the intermediate energy region which is still accessible by the perturbation series and the operator product expansion (OPE).

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- We use only information that has been calculated explicitly, i.e. the known coefficients of a perturbative series and the Wilson coefficients of the OPE.

- This hypothesis is of course known to be violated in practice, for instance by the factorial growth of the coefficients. We assume that the asymptotic behavior sets in at high enough perturbative orders so that it does not affect this applications.

- No dimension 2 operator

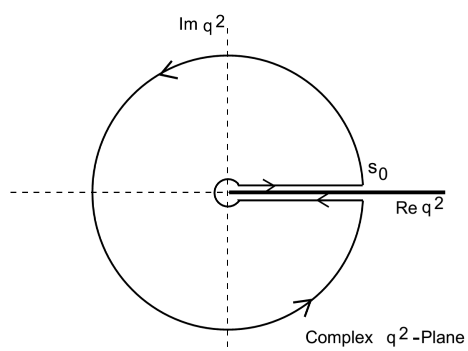
- We employ pinched FESR but no models or approximations such as large N_c, large β_0, Regge-theory, vacuum saturation,...

- We check these assumptions against measured quantities.

Operator Product Expansion

$$
\begin{aligned}
4\pi^2 \Pi^{OPE}(Q^2) &= \sum_{N=0}^{\infty} \frac{1}{Q^{2N}} C_{2N}(Q^2,\mu^2) \, \langle 0|\mathcal{O}_{2N}(\mu^2)|0\rangle \\
&\equiv \sum_{N=0}^{\infty} \frac{1}{Q^{2N}} \mathcal{O}_{2N} \, ,
\end{aligned}
$$

Finite Energy Sum Rule



$$
\begin{aligned}
0 &= \int_0^{s_0} ds\, s^N \rho(s) + \frac{1}{2\pi i} \oint_{|s|=s_0} ds s^N \Pi^{\mathrm{PQCD}}(s) \\
&\quad - \frac{1}{4\pi^2} (-)^N \, \mathcal{O}_{2N+2}^{V,A}
\end{aligned}
$$

1 Chiral Sum Rules

$$W(s_0) \equiv \int_0^{s_0} ds\; s^N \left[\rho_V(s) - \rho_A(s)\right] \;,$$

$$\begin{aligned} \int_0^{s_0\to\infty} ds\; s^N \rho_{V-A}(s) &= 2 f_\pi^2\, \delta_{N0} + (-)^N \langle \mathcal{O}_{2N+2} \rangle \\ (N &= 0, 1, 2, ...) \end{aligned}$$

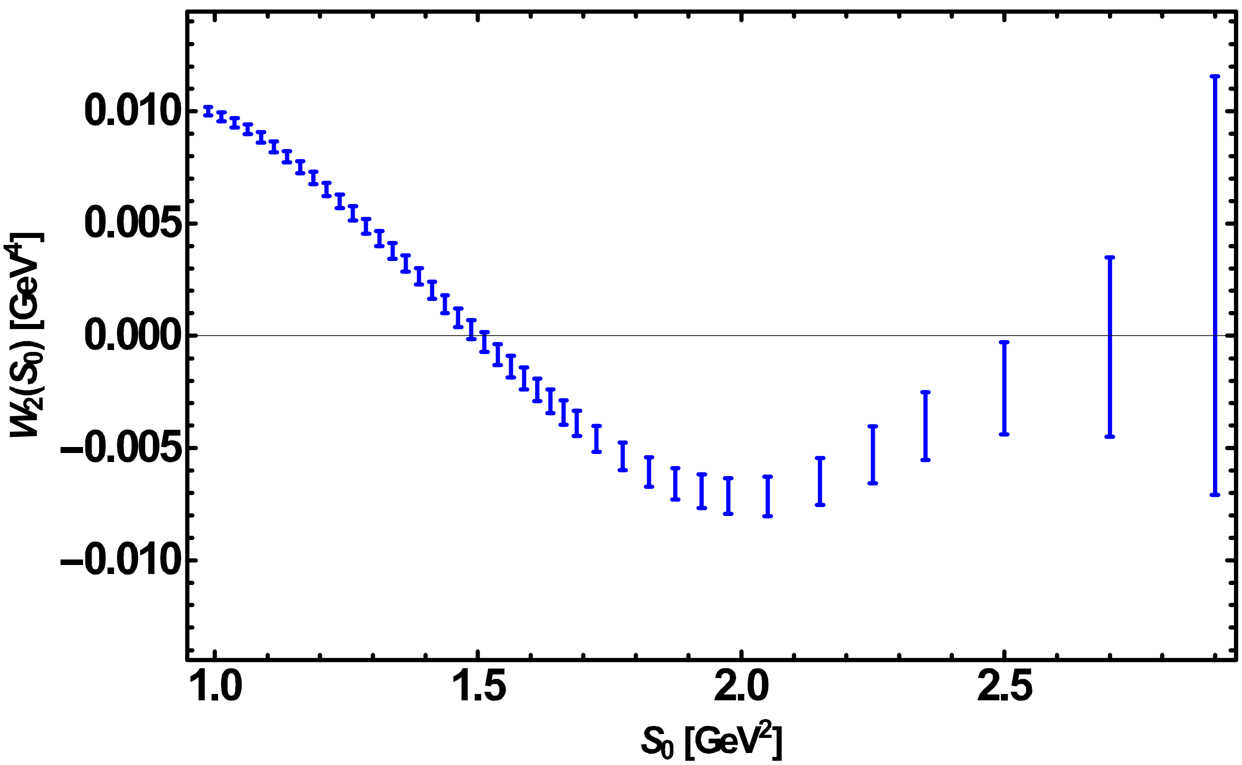
In the chiral limit $\langle \mathcal{O}_2 \rangle = \langle \mathcal{O}_4 \rangle = 0$.

1. $N = 0, 1$ Weinberg sum rules

2. $N = 2, 3$ FESR projects $d = 6, 8$ vacuum condensates,

Second Weinberg Sum Rule

$$\int_0^{s_0} ds\; s\,[\rho_V(s) - \rho_A(s)] = 0\;,$$

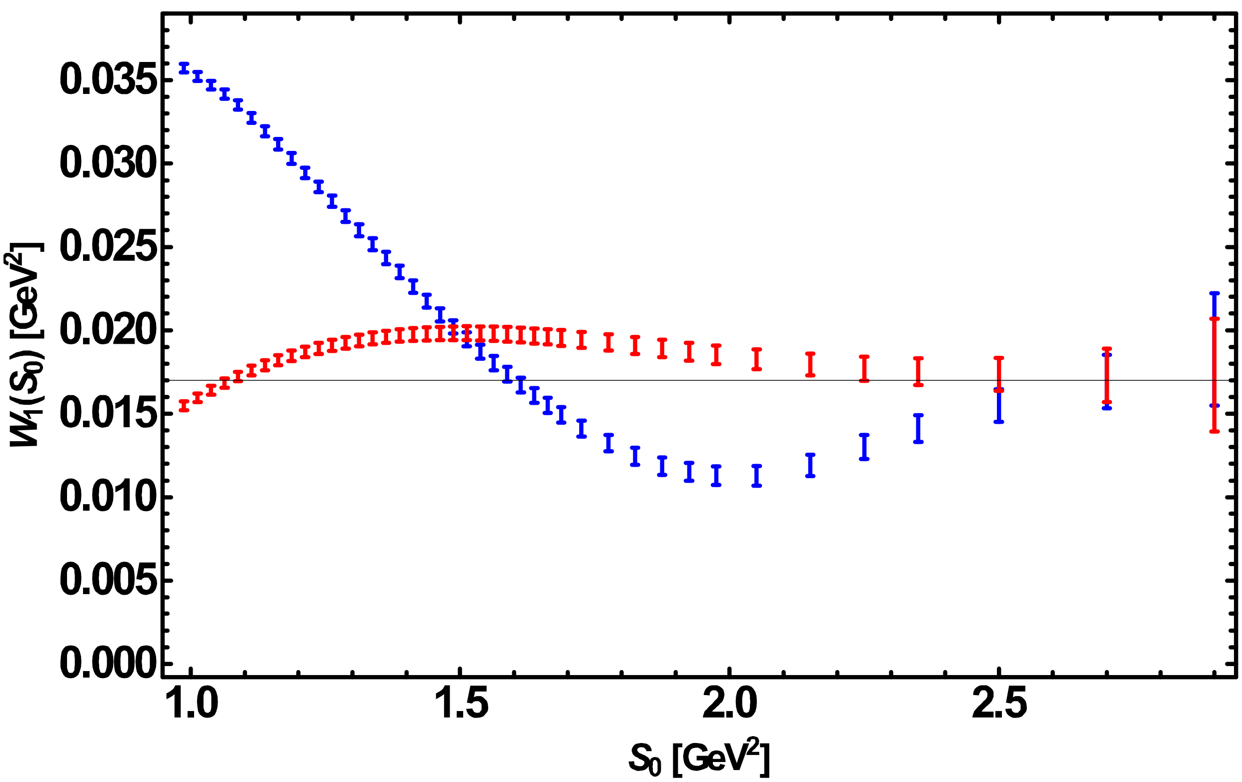


Second WSR

First Weinberg sum rule (pinched)

$$W_1(s_0) \quad : \qquad \int_{s_{thr}}^{s_0} ds \; [\rho_V(s) - \rho_A(s)] = 2f_\pi^2$$

$$W_1(s_0) : \qquad \int_{s_{thr}}^{s_0} ds \, (1 - \frac{s}{s_0}) \, [\rho_V(s) - \rho_A(s)] = 2f_\pi^2 \, .$$

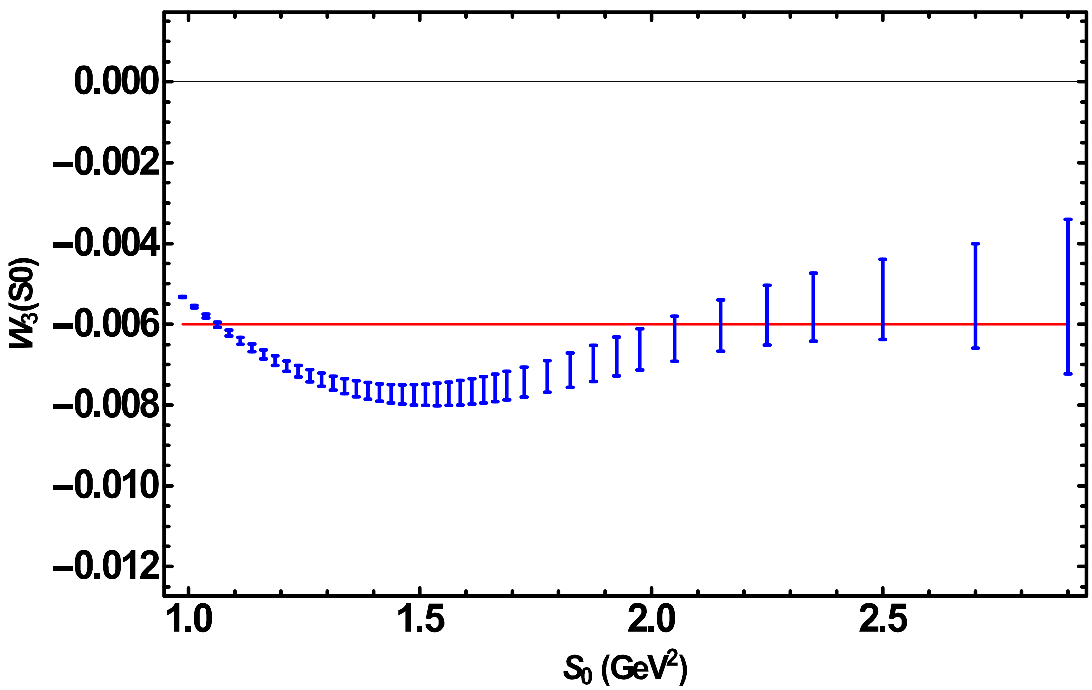


WSR 1 (blue) and pinched WSR (red)

The DGLMY sum rule:

Pinched DGLMY sum rule

$$\int_0^{s_0 \to \infty} ds s \ln \frac{s}{s_0} [\rho_V(s) - \rho_A(s)] = -\frac{8\pi f_\pi^2}{3\alpha}(m_{\pi^\pm}^2 - m_{\pi^0}^2)$$

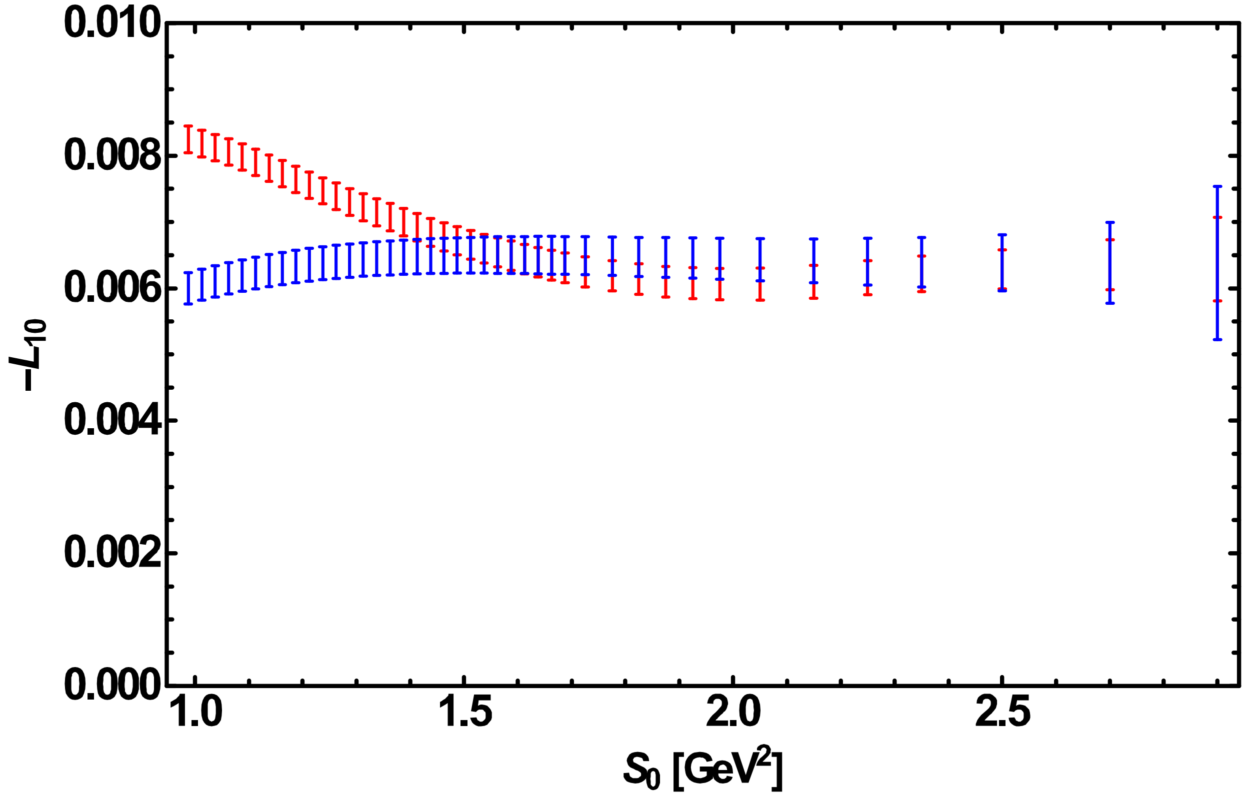


(Das, Guralnik, Low, Mathur and Young 1967). Experimentally $m_{\pi^\pm} - m_{\pi^0} = 4.59 MeV$.

The DMO sum rule:

(Pinched DMO sum rule)

$$\bar{\Pi}(0) = \int_{s_{thr}}^{s_0} \frac{ds}{s} \left(1 - \frac{s}{s_0}\right)^2 [\rho_V(s) - \rho_A(s)]$$



CHPT

$$\bar{\Pi}(0) = -8\bar{L}_{10}$$
$$= 2\left[\frac{1}{3}f_\pi^2 < r_\pi^2 > -F_A\right] = 0.052 \pm 0.002\ ,$$

$$L_{10} \underset{\text{CHPT}}{\approx} -6.33 \times 10^{-3}$$

Our result

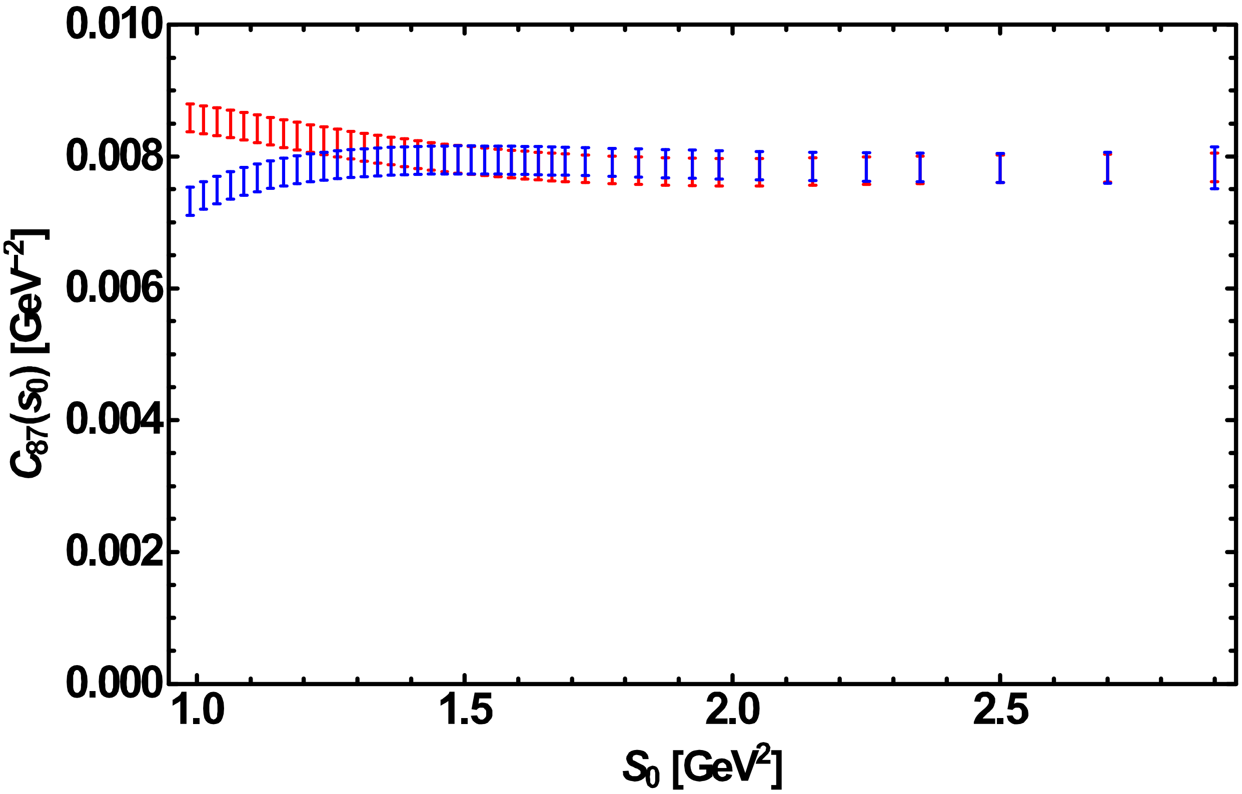
$$\bar{\Pi}(0) = -8\bar{L}_{10} = 4\frac{f_\pi^2}{s_0} + \int_{s_{thr}}^{s_0} \frac{ds}{s}\left(1 - \frac{s}{s_0}\right)^2 [\rho_V(s) - \rho_A(s)]$$

$$\bar{L}_{10} = -(6.5 \pm 0.1) \times 10^{-3}$$

$\Pi'(0)$ is related to the $\mathcal{O}(p^6)$ counter terms.

(Pinched sum rule)

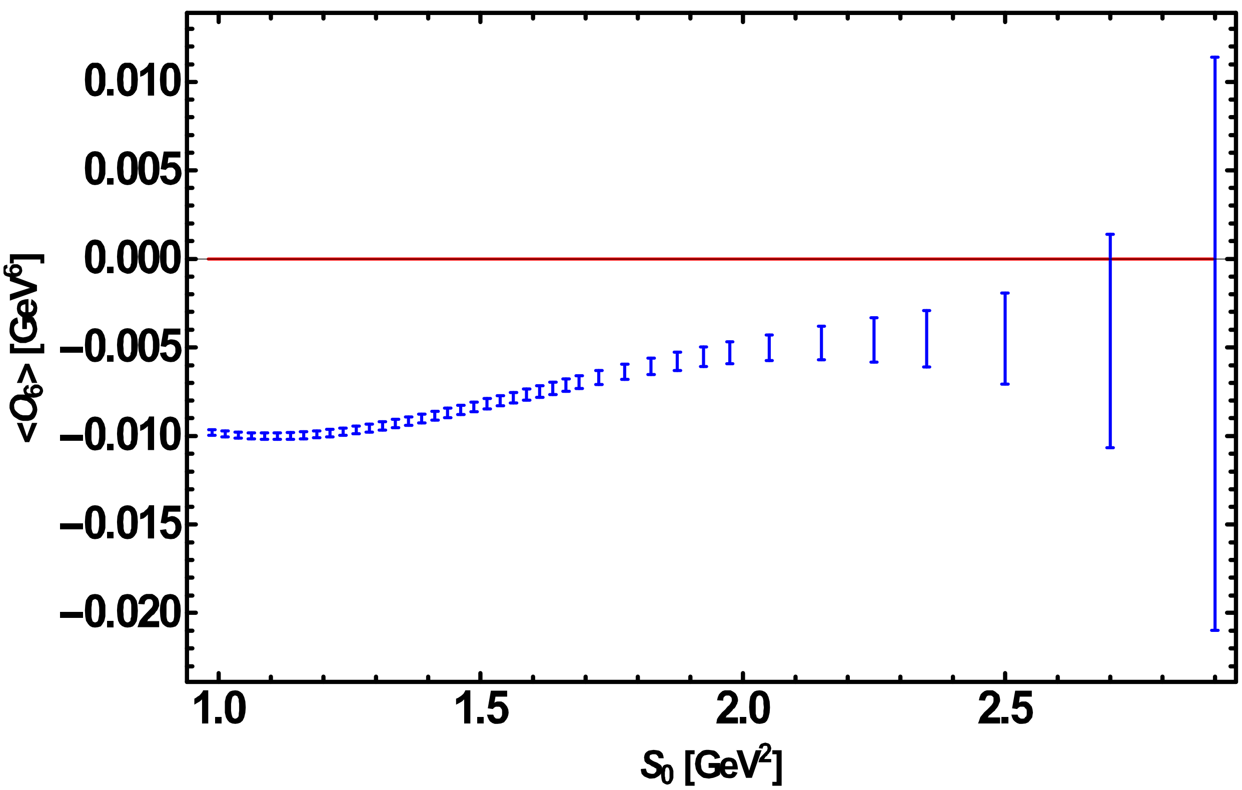
$$\bar{\Pi}'(0) = \int_{s_{thr}}^{s_0} \frac{ds}{s^2}\left(1-\left(\frac{s}{s_0}\right)^3\right)[\rho_V(s)-\rho_A(s)]$$
$$= 16\bar{C}_{87}$$



2 Chiral Condensates

Dimension 6 chiral condensate:

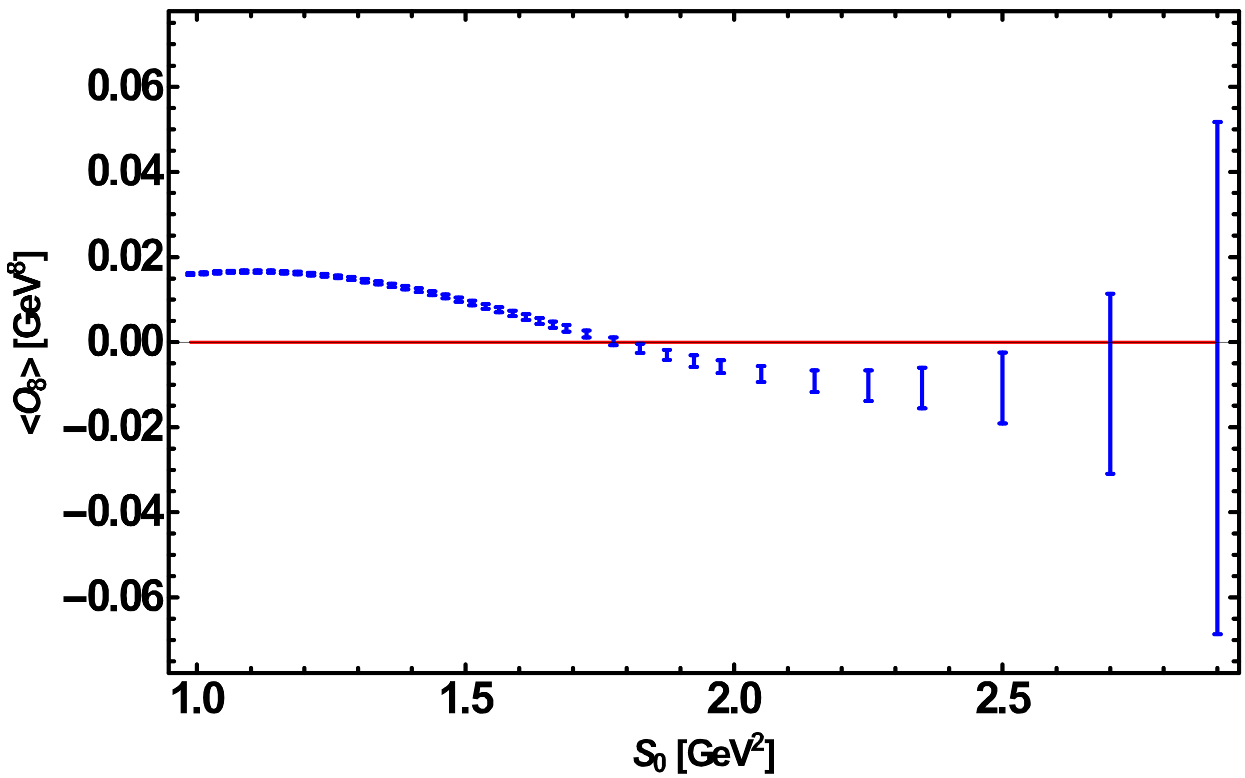
$$\int_0^{s_0} ds\, s^2 \left[\rho_V(s) - \rho_A(s)\right] = \langle \mathcal{O}_6 \rangle$$



$$\langle \mathcal{O}_6 \rangle = -(5.0 \pm 0.7) \times 10^{-3}\ \text{GeV}^6. \qquad (1)$$

- Dimension 8 chiral condensate

$$\int_0^{s_0} ds\; s^3 \left[\rho_V(s) - \rho_A(s)\right] = \langle \mathcal{O}_6 \rangle$$

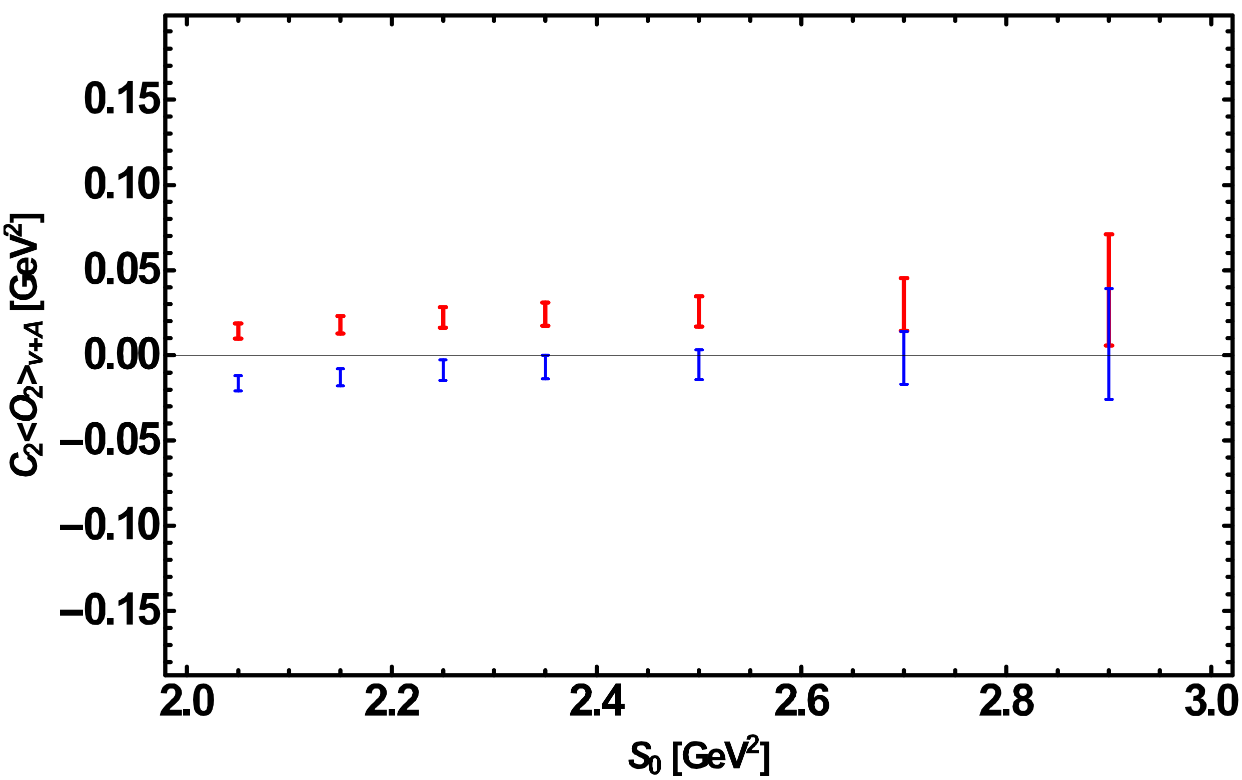


$$\langle \mathcal{O}_8 \rangle = -(9.0 \pm 5.0) \times 10^{-3}\ \text{GeV}^8\ ,$$

3 Non-Chiral Sum Rules

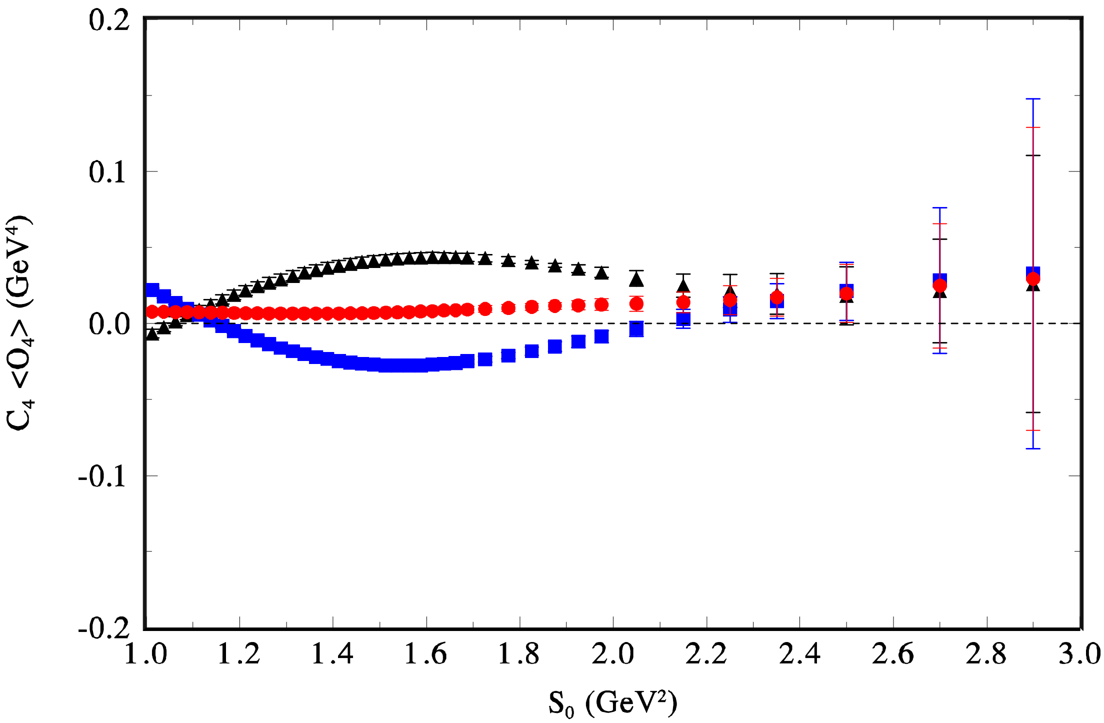
$$\frac{1}{4\pi^2}\mathcal{O}_2 = -\int_{s_{thr}}^{s_0} ds \begin{bmatrix} \rho_V(s) \\ \rho_A(s) \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} 2f_\pi^2 + s_0 \frac{1}{4\pi^2} M_0(s_0)$$

$$M_{2N+2}(s_0) = -\frac{1}{2\pi i} \oint_{|s|=s_0} \frac{ds}{s_0} \left[\frac{s}{s_0}\right]^N 4\pi^2 \Pi^{\mathrm{PQCD}}(s).$$



red dots: $\alpha_s = 0.354$ and blue dots $\alpha_s = 0.328$

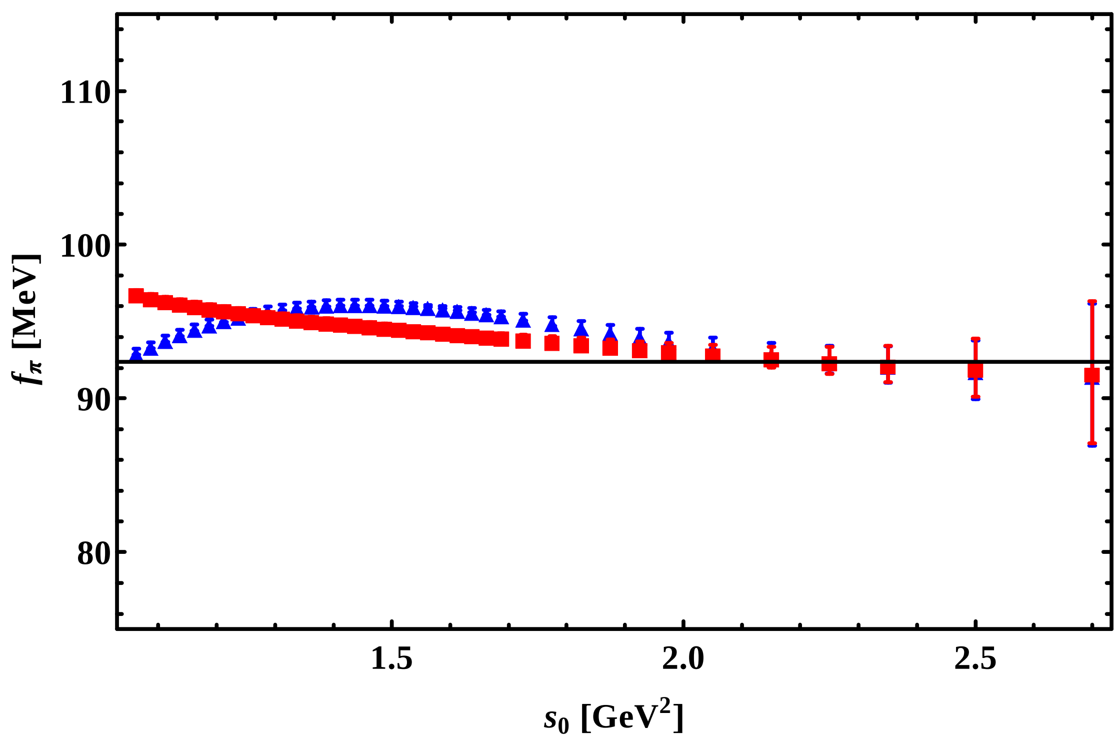
$$\frac{1}{4\pi^2}\mathcal{O}_4 = s_0\int_{s_{thr}}^{s_0} ds\left(1-\frac{s}{s_0}\right)\left[\begin{array}{c}\rho_V(s)\\ \rho_A(s)\end{array}\right] - \left[\begin{array}{c}0\\ 1\end{array}\right]2f_\pi^2$$
$$-s_0^2\frac{1}{4\pi^2}[M_0(s_0) - M_1(s_0)]$$



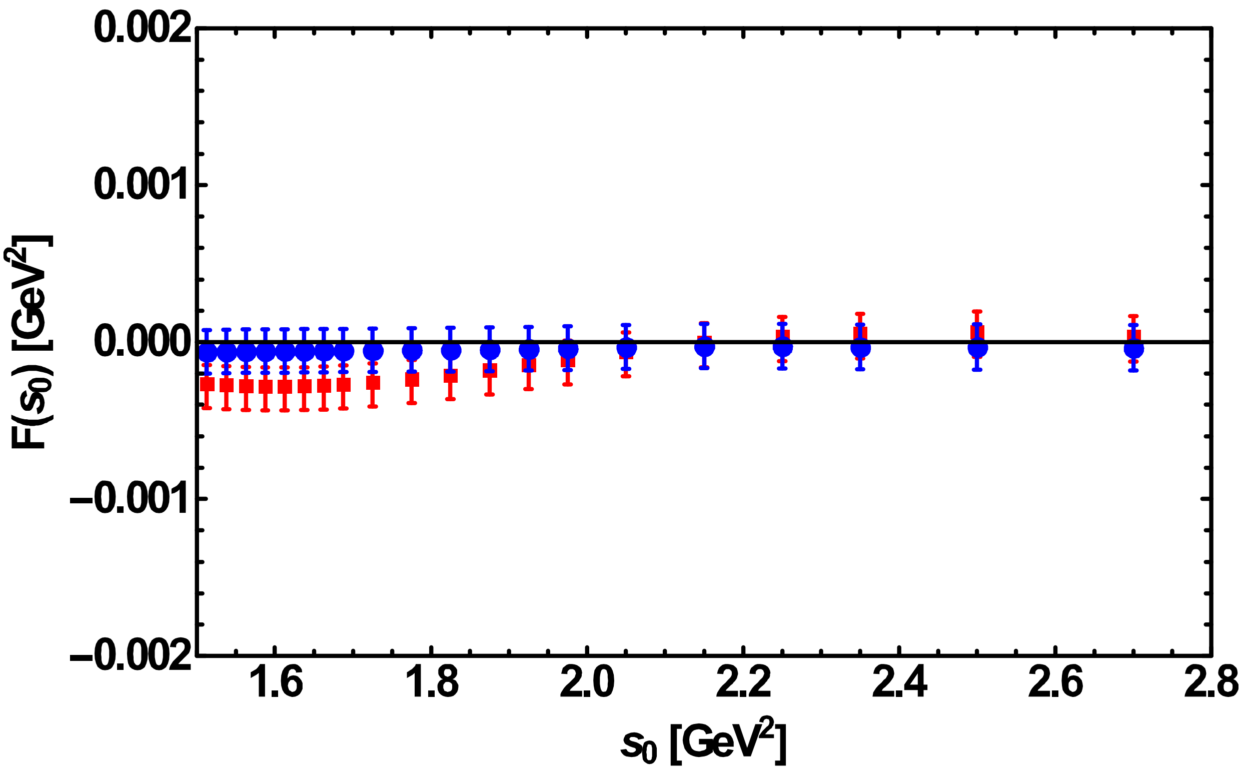
The gluon condensate is positive and equal for V and A.

$$\mathcal{O}_4|_{V,A} = \frac{\pi^2}{3}\langle\frac{\alpha_s}{\pi}G_{\mu\nu}G^{\mu\nu}\rangle = (0.017 \pm 0.012)\ GeV^2$$

$$
\begin{aligned}
2f_\pi^2 &= -\int_{s_{thr}}^{s_0} ds\ \left(1-\frac{s}{s_0}\right)\rho_A(s) \\
&\quad -\frac{1}{2\pi i}\oint_{|s|=s_0} ds\ \left(1-\frac{s}{s_0}\right) 4\pi^2\,\Pi_A^{QCD}(s) \\
&\quad +\frac{1}{4\pi^2}\left[\mathcal{O}_2-\mathcal{O}_4\right]
\end{aligned}
$$



$$
\begin{aligned}
0 \;=\; & F(s_0) = -\int_0^{s_1} ds \left(1 - \frac{s}{s_0}\right) \frac{1}{\pi} \rho_V(s) \\
& -\frac{1}{2\pi i} \oint_{C(|s_0|)} ds \, \left(1 - \frac{s}{s_0}\right) \Pi_V^{PQCD}(s)
\end{aligned}
$$



4 Generalized Pinching

$$0 = \int_0^{s_1} ds P(s)\rho_{A,V}(s) + \int_{s_1}^{s_0} ds P(s)\rho_{A,V}(s)$$
$$+\frac{1}{2\pi i} \oint_{|s|=s_0} ds\, P(s)\, \Pi^{QCD}_{A,V}(s)$$
$$\left(s_1 \simeq 2.7\,\mathrm{GeV}^2,\ s_0 \simeq 10\,\mathrm{GeV}^2\right)$$

Expectation:

$$\rho_{A,V}(s) = \frac{1}{\pi}\,\mathrm{Im}\,\Pi^{\mathrm{data}}_{A,V}(s) \approx \frac{1}{\pi}\,\mathrm{Im}\,\Pi^{\mathrm{QCD}}_{A,V}(s) \qquad (s_1 \leq s \leq s_0)$$

Choose a_n in $p(s) = \sum a_n s^n$ such that

$$\int_{s_1}^{s_0} ds\ P(s)\frac{1}{\pi}\,\mathrm{Im}\,\Pi^{PQCD}_{A}(s) = 0$$

$$\Longrightarrow \int_{s_1}^{s_0} ds\ P(s)\frac{1}{\pi}\,\mathrm{Im}\,\Pi^{data}_{A,V}(s) \approx 0$$

Sum rule for **this** $P(s)$:

$$\int_0^{s_1} ds\ P(s)\frac{1}{\pi}\,\mathrm{Im}\,\Pi^{data}_{A,V}(s) + \frac{1}{2\pi i} \oint_{|s|=s_0} ds\, P(s)\, \Pi^{QCD}_{A,V}(s) = 0$$

The powers in $P(s)$ should be not too high as otherwise higher and unknown condensates would contribute, a linear function is sufficient

$$P(s) = 1 + \gamma(s_0, s_1)s$$

$$\gamma(s_0, s_1) = -\frac{[s_0\, M_0(s_0)\ -\ s_1\, M_0(s_1)]}{\left[s_0^2\, M_1(s_0)\ -\ s_1^2\, M_1(s_1)\right]}\ .$$

If $s_0 = s_1$, then $\gamma(s_0, s_0) = -\frac{1}{s_1}$, and

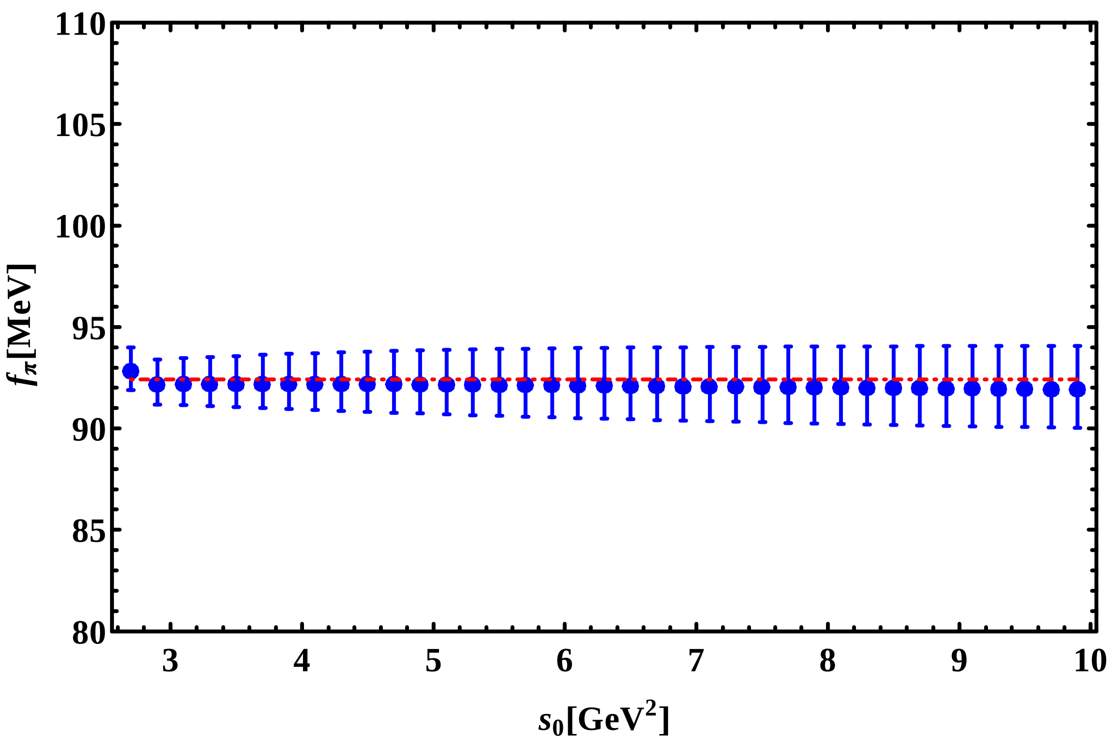
$$P(s) = 1 - \frac{s}{s_1} \quad \text{standard pinching} \qquad (2)$$

We use the average of ALEPH and OPAL data and $\alpha_s(M_\tau^2) = 0.431 \pm 0.013$ (Pich 2014)

Axial Sum Rule

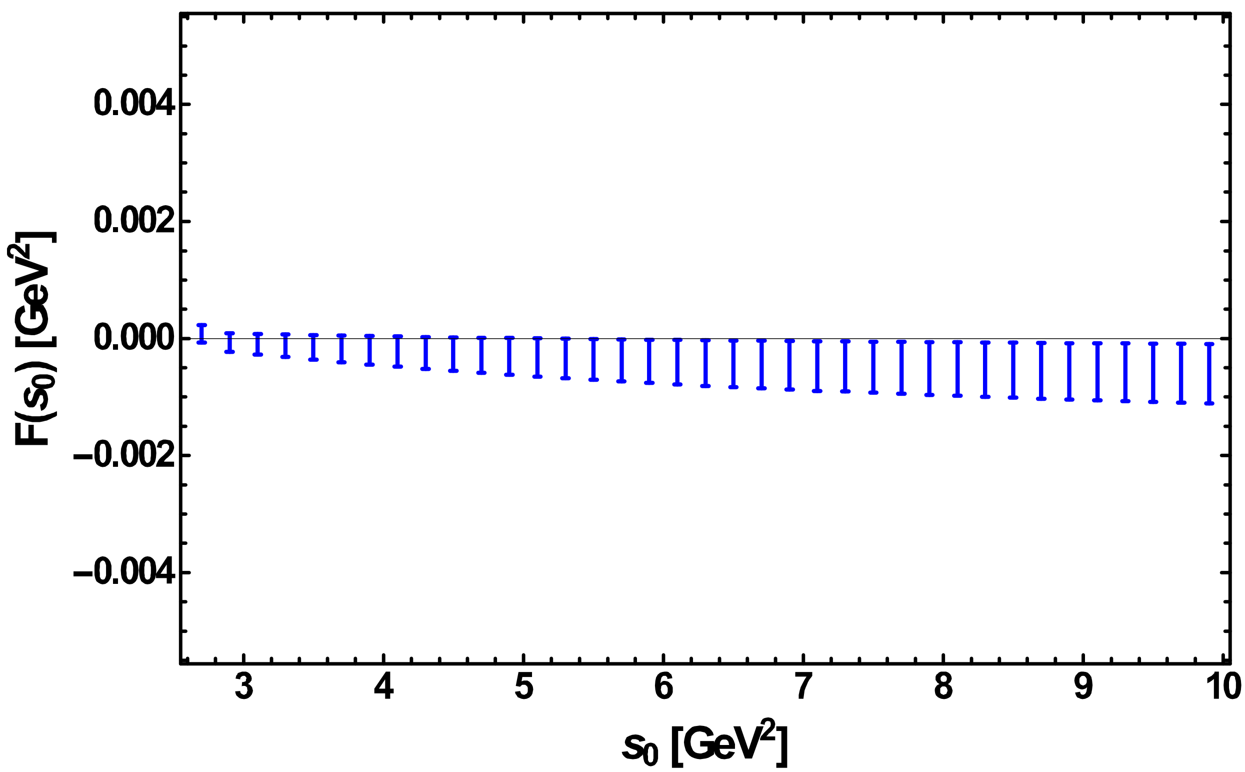
$$2f_\pi^2 = -\int_0^{s_1} ds P(s) \rho_A(s) - \frac{1}{2\pi i} \oint_{|s|=s_0} ds\, P(s)\, \Pi_A^{OPE}(s)$$

$$P(s) = 1 + \gamma(s_1, s_0)$$



Vector Sum Rule

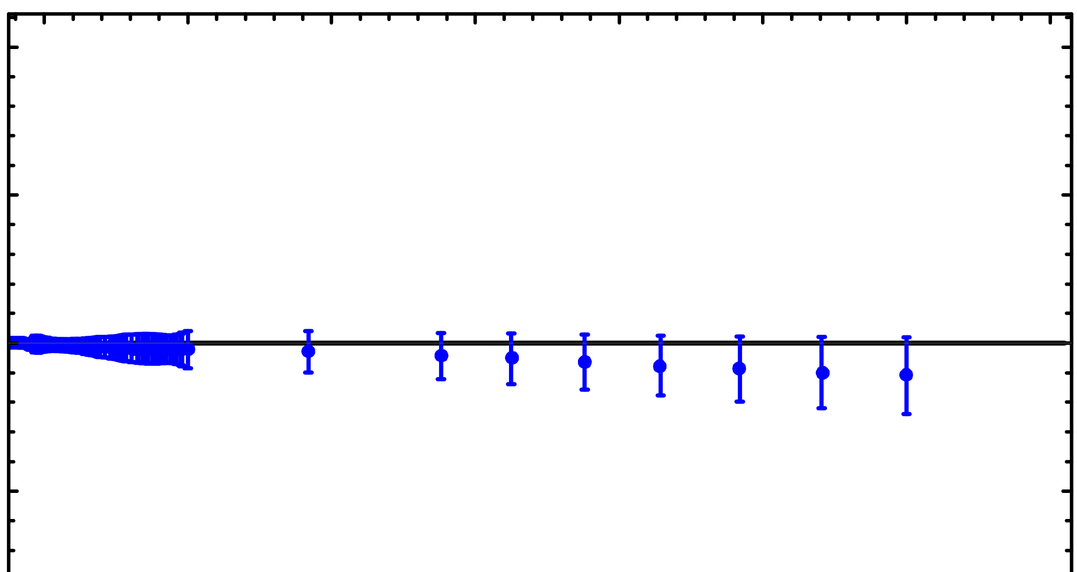
$$F(s_0) \;=\; \int_0^{s_1} ds P(s)\rho_V(s) + \int_{s_1}^{s_0} ds P(s)\rho_V(s) + \frac{1}{2\pi i} \oint_{|s|=s_0} ds\, P(s)\, \Pi_V^{OPE}(s)$$



$$P(s) = 1 + \gamma(s_1, s_0)$$

For the vector current we can use e^+e^- data beyond the end-point of the τ-spectrum

$$F(s_0) = -\int_0^{s_1} ds\, P(s)\, \rho_V(s) - \frac{1}{8\,\pi^2}\int_{s_1}^{s_0} ds\, P(s)\, R_{e^+e^-}(s) - \frac{1}{2\pi i}\oint_{|s|=s_0} ds\, P(s)\, \Pi_V^{PQCD}(s)\,,$$



Conclusions

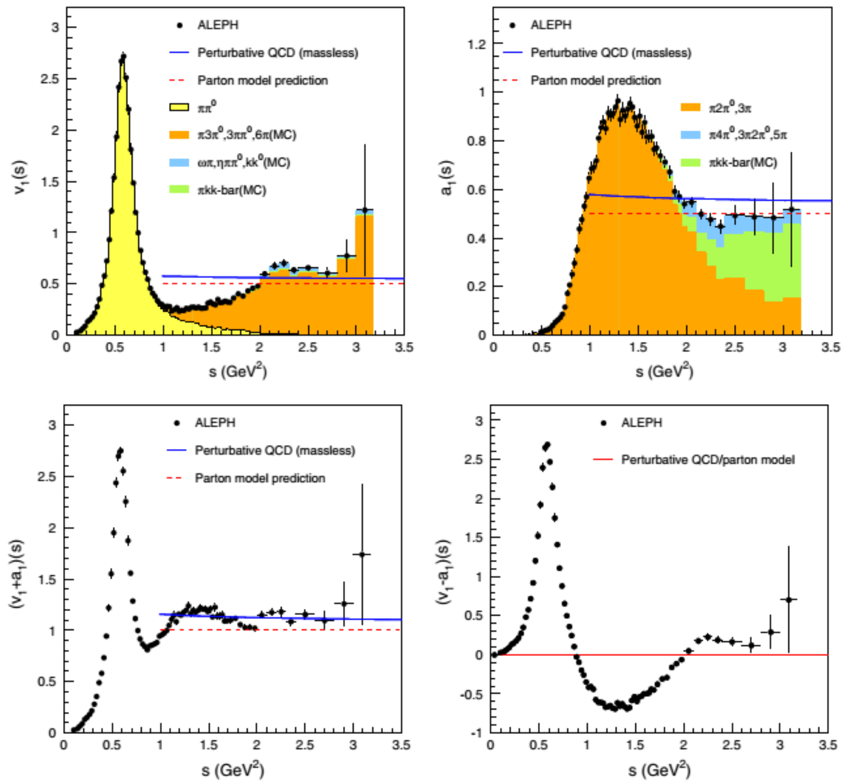
- Duality with pinching works well for $s_0 \gtrsim 2.3 - 2.6\ GeV^2$ for all observables where the answer is known.

- We use only linear fit because of unknown higher dimensional condensates.

- Errors are predominantly experimental. Little or no evidence of DV

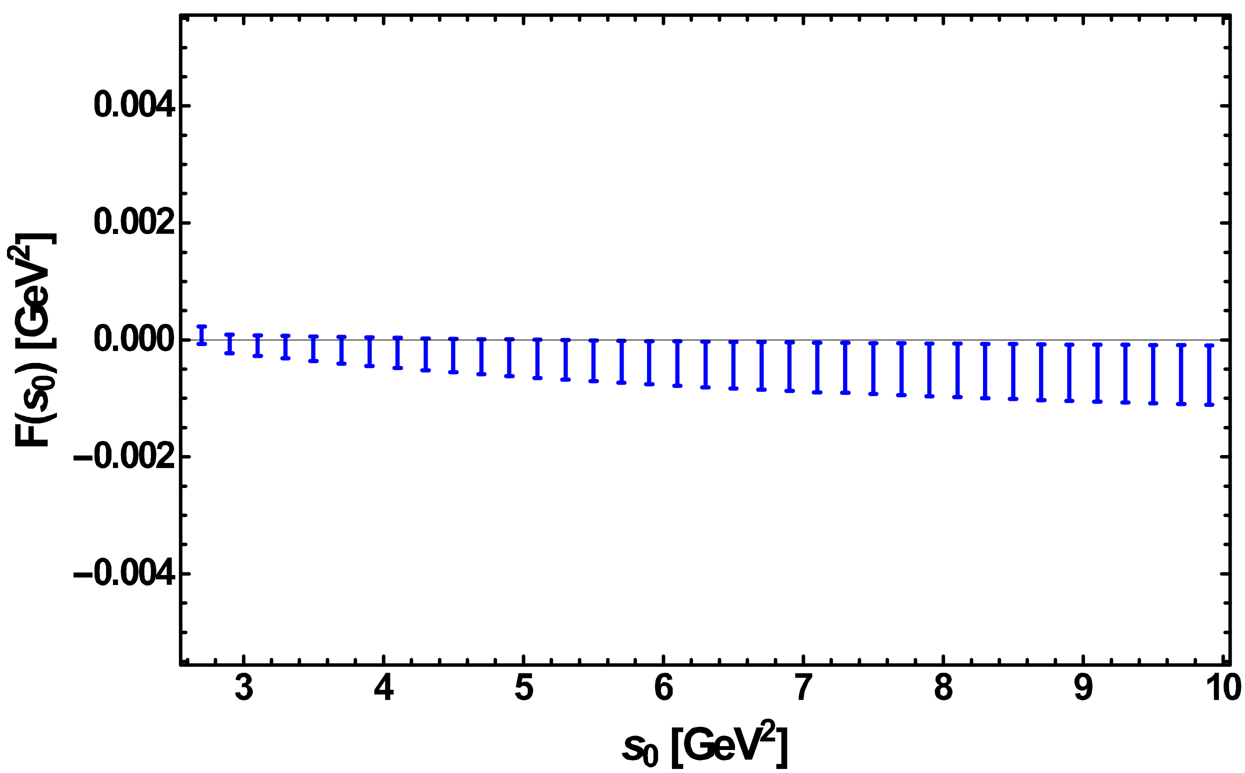
- The pinched Weinberg sum rule shows that above 2.6 GeV^2 we must have $\rho_V^{PQCD}(s) = \rho_A^{PQCD}(s)$ (within experimental errors)

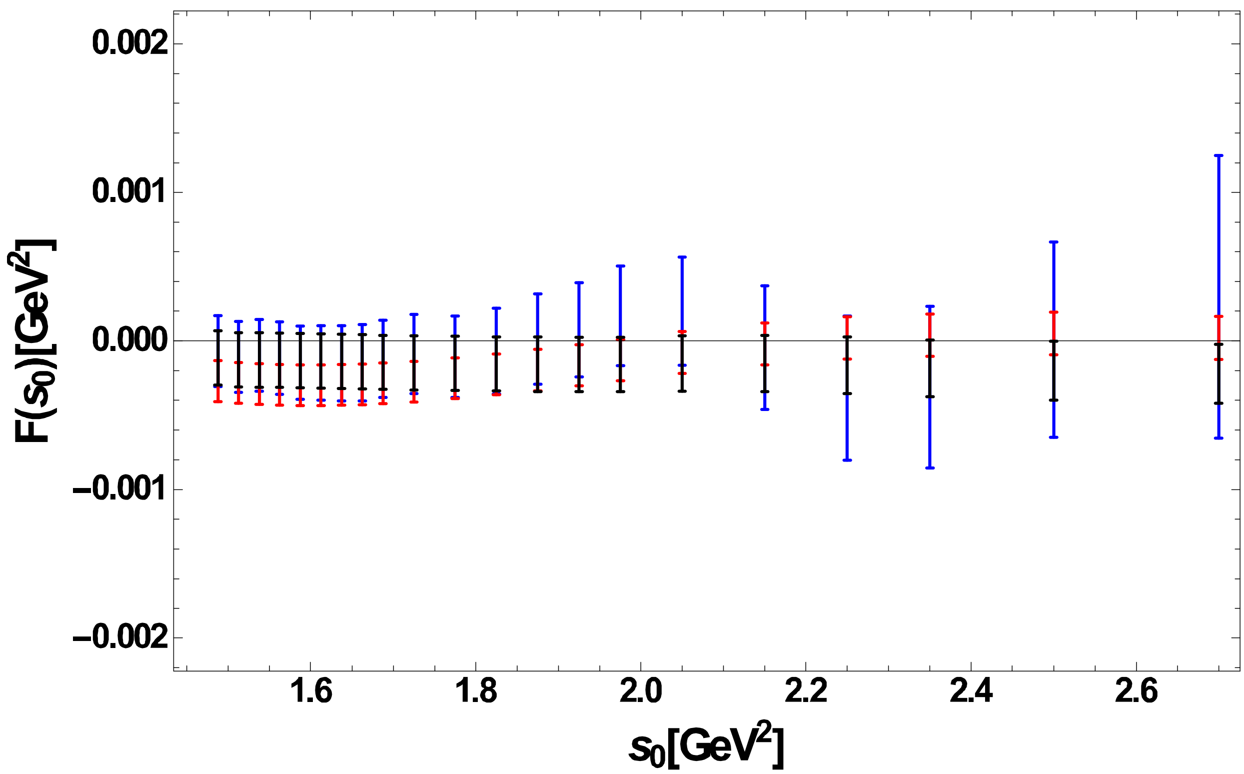
- All QCD constraints regarding condensates must be incorporated.



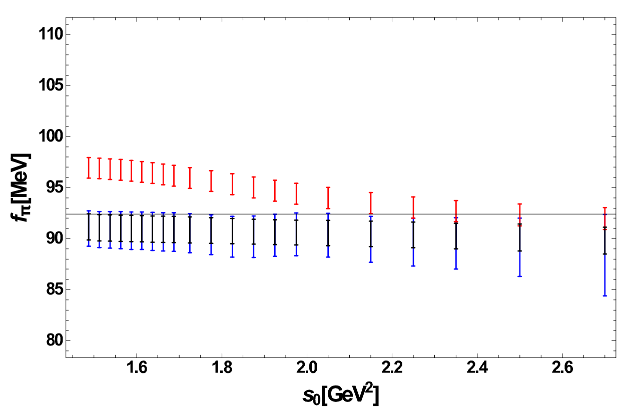
$$0 \;=\; F(s_0) = -\int_0^{s_0} ds \left(1-\frac{s}{s_0}\right)\frac{1}{\pi}\rho_V(s)$$

$$-\frac{1}{2\pi i}\oint_{|s|=s_0)} ds \left(1-\frac{s}{s_0}\right) \Pi_V^{PQCD}(s)$$


0.004
0.002
0.000
−0.002
−0.004
F(s_0) [GeV^2]
3
4
5
6
7
8
9
10
s_0 [GeV^2]



0.002
0.001
0.000
−0.001
−0.002
F(s0)[GeV2]
1.6
1.8
2.0
2.2
2.4
2.6
s0[GeV2]



110
105
100
95
90
85
80
fπ[MeV]
1.6
1.8
2.0
2.2
2.4
2.6
s0[GeV2]