Finite volume effects in the hadronic vacuum polarization

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thanks to Taku Izubuchi and Kim Maltman

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Hadronic vacuum polarization contribution to muon anomalous magnetic moment:

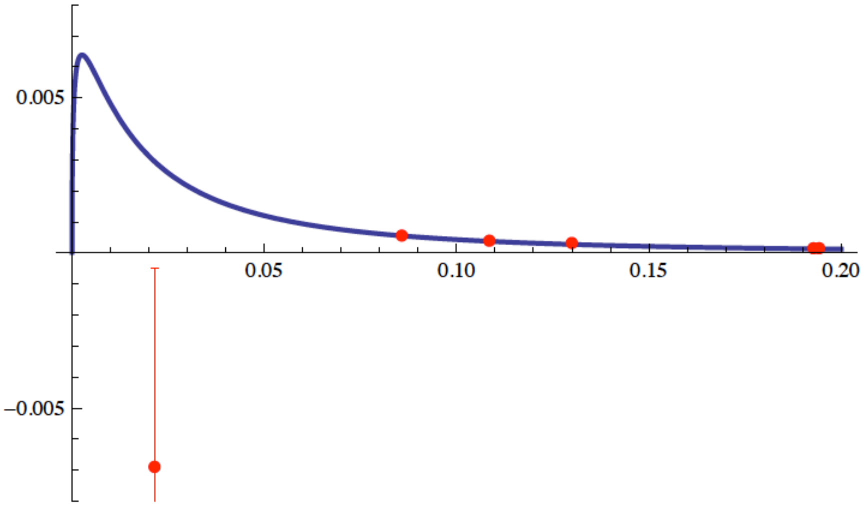
expression: $$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_o^\infty dQ^2\, f(Q^2)\, [\Pi(Q^2) - \Pi(0)]$$ (Blum, '03)

with $f(Q^2)$ a known weight function, and $\Pi(Q^2)$ the HVP obtained from

$$\Pi_{\mu\nu}(Q) = \left(\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu\right)\Pi(Q^2)$$

integrand looks like

old statistics, '12



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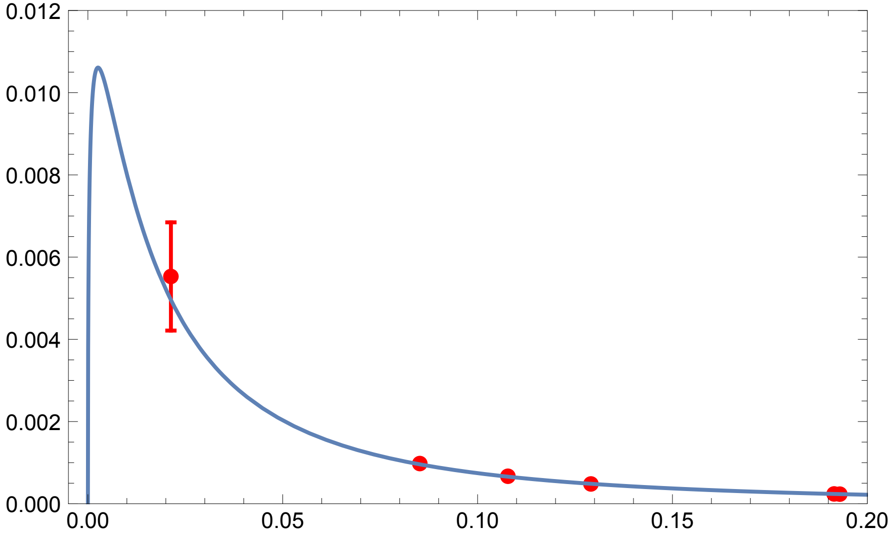
integrand looks like

new statistics '15

AMA (Blum et al., '13)



Finite volume effects (torus with periodic boundary conditions)

- First, Ward-Takahashi identity does not exclude $\Pi_{\mu\nu}(0) \neq 0$

  (see also Bernecker and Meyer, '11)

- HVP more singular for low momenta than in infinite volume

⇒ suggests considering finite-volume subtraction

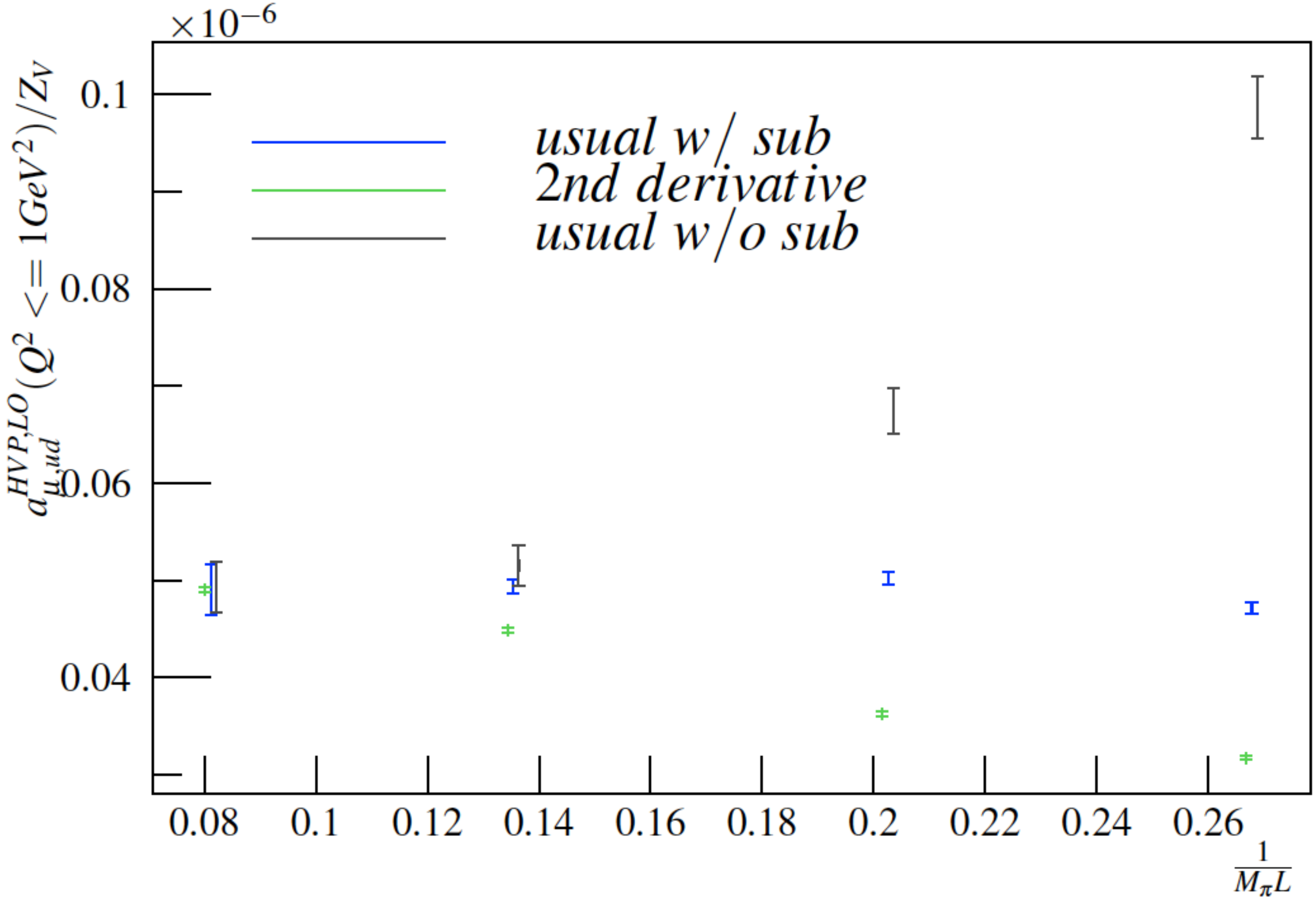
$$\bar{\Pi}_{\mu\nu}(Q) \equiv \Pi_{\mu\nu}(Q) - \Pi_{\mu\nu}(0)$$

or

$$\bar{\Pi}_{\mu\nu}(Q) \equiv P^T_{\mu\kappa}(Q)\,(\Pi_{\kappa\lambda}(Q) - \Pi_{\kappa\lambda}(0))\,P^T_{\lambda\nu}(Q)$$

with $P^T_{\mu\nu}(Q) = \delta_{\mu\nu} - \dfrac{Q_\mu Q_\nu}{Q^2}$ the transversal projector

From Malak et al., ’15



$a = 0.104$ fm , $m_\pi = 292$ MeV , $3.7 \leq m_\pi L \leq 12.3$

Compare black (unsubtracted) and blue (subtracted) points

Second, assume scaling violations small for low momenta:

$$\Pi_{\mu\nu}(Q) = \left(\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu\right)\Pi(Q^2) + \underbrace{O(a^2Q^4)}_{small}$$

then $SO(4)$ broken to cubic rotation group by finite volume $L^3 \times T$
Project onto irreps of cubic group:

$$\begin{aligned}
A_1 :&\quad \sum_i \Pi_{ii} \quad \text{and} \quad \Pi_{44} \\
T_1 :&\quad \Pi_{4i} = \Pi_{i4} \\
T_2 :&\quad \Pi_{i\neq j} = \Pi_{j\neq i} \\
E :&\quad \Pi_{11} - \sum_i \Pi_{ii}/3 \;, \Pi_{22} - \sum_i \Pi_{ii}/3
\end{aligned}$$

to obtain 5 different scalar functions Π_{A_1} , $\Pi_{A_1^{44}}$, Π_{T_1} , Π_{T_2} , Π_E
(two relations from Ward-Takahashi identities)
(see also Bernecker and Meyer, ‘11)

Chiral perturbation theory in finite volume

Assume FV effects entirely due to pions; NLO ChPT (connected part) yields

$$\Pi^{\rm ChPT}_{\mu\nu}(Q) =$$

$$\frac{10}{9}e^2\Bigg(\frac{1}{L^3T}\sum_p \frac{4\sin\left(p+Q/2\right)_\mu \sin\left(p+Q/2\right)_\nu}{\left(2\sum_\kappa(1-\cos p_\kappa)+m_\pi^2\right)\left(2\sum_\kappa(1-\cos\left(p+Q\right)_\kappa)+m_\pi^2\right)}$$

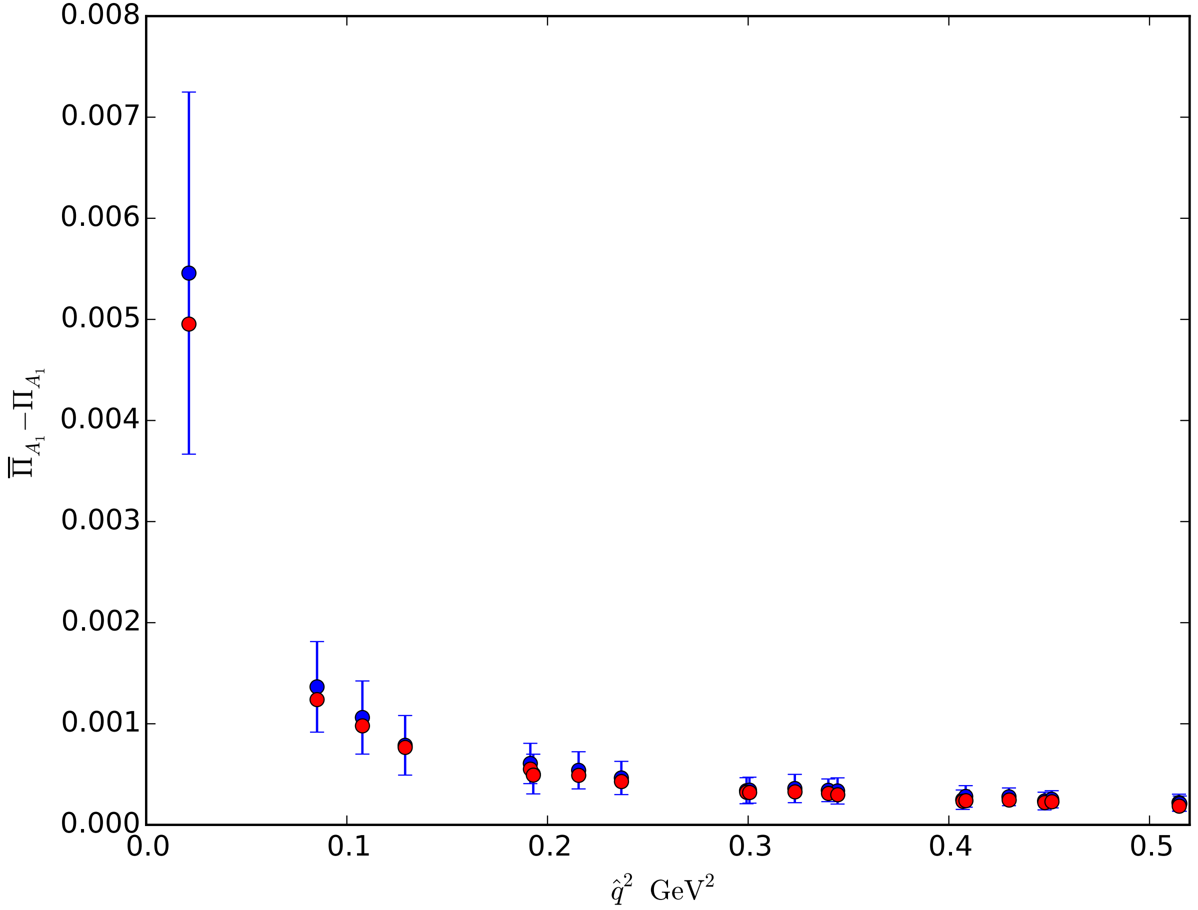
$$-\ \delta_{\mu\nu}\frac{1}{L^3T}\sum_p\left(\frac{2\cos p_\mu}{\left(2\sum_\kappa(1-\cos p_\kappa)+m_\pi^2\right)}\right)\Bigg)$$

Even NNLO ChPT gives poor description of HVP, but here interested in FV

⇒ consider only differences that vanish in infinite volume

MILC asqtad ensemble with $1/a = 3.34532\ \mathrm{GeV}\ , \quad m_\pi = 220\ \mathrm{MeV}$

$L = 64\ , \quad T = 144 \quad \Rightarrow \quad m_\pi L = 4.2$



Blue lattice data

Red NLO ChPT

Difference of $\bar{\Pi}_{A_1}(Q^2)$ (subtracted) and $\Pi_{A_1}(Q^2)$ (unsubtracted)

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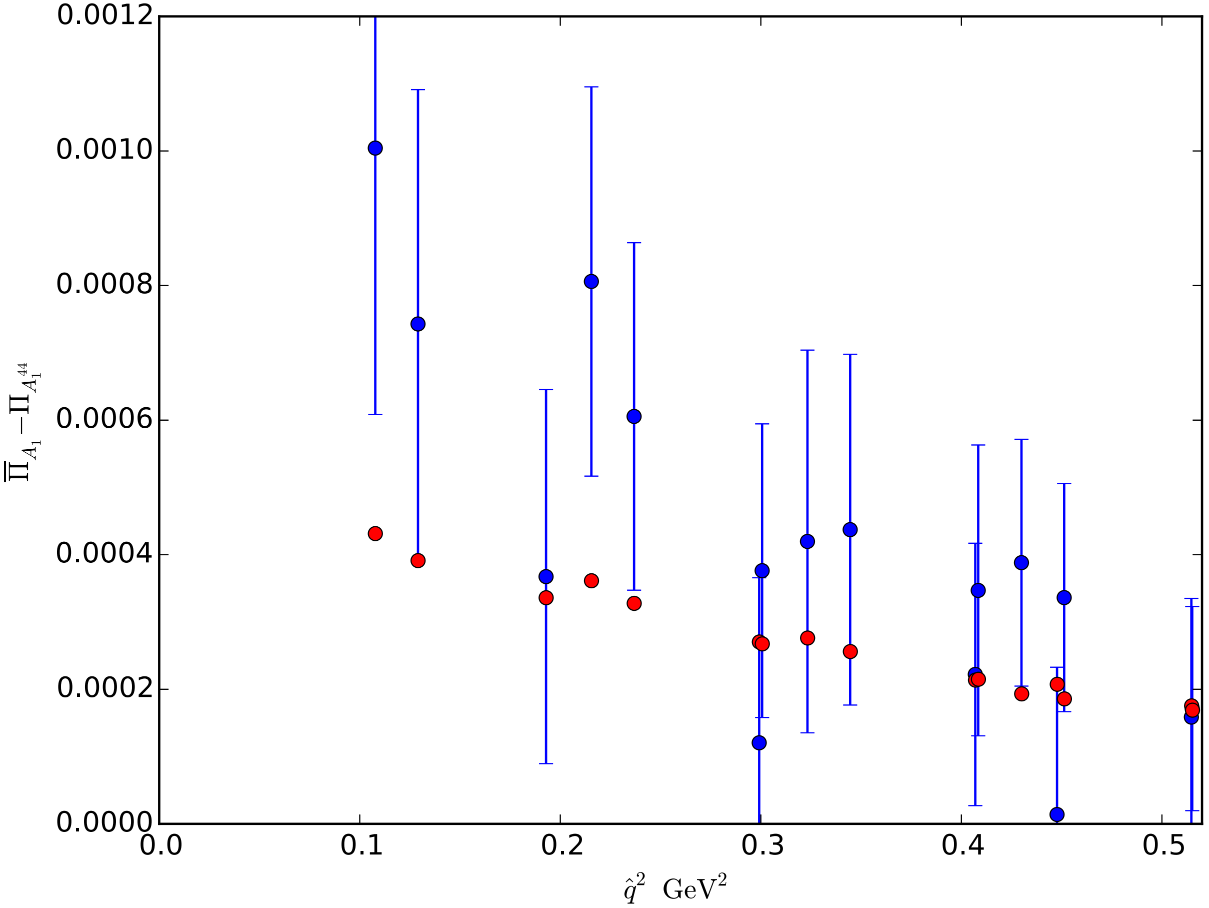

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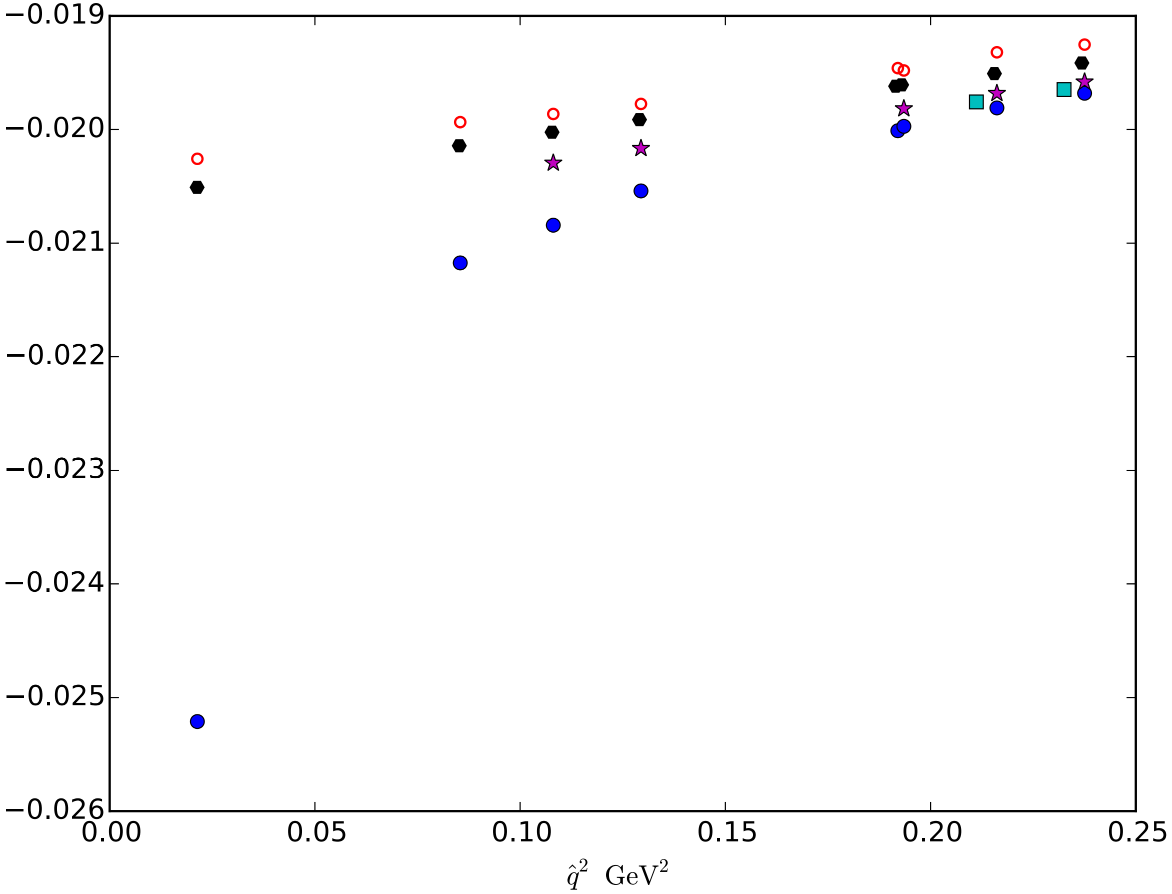
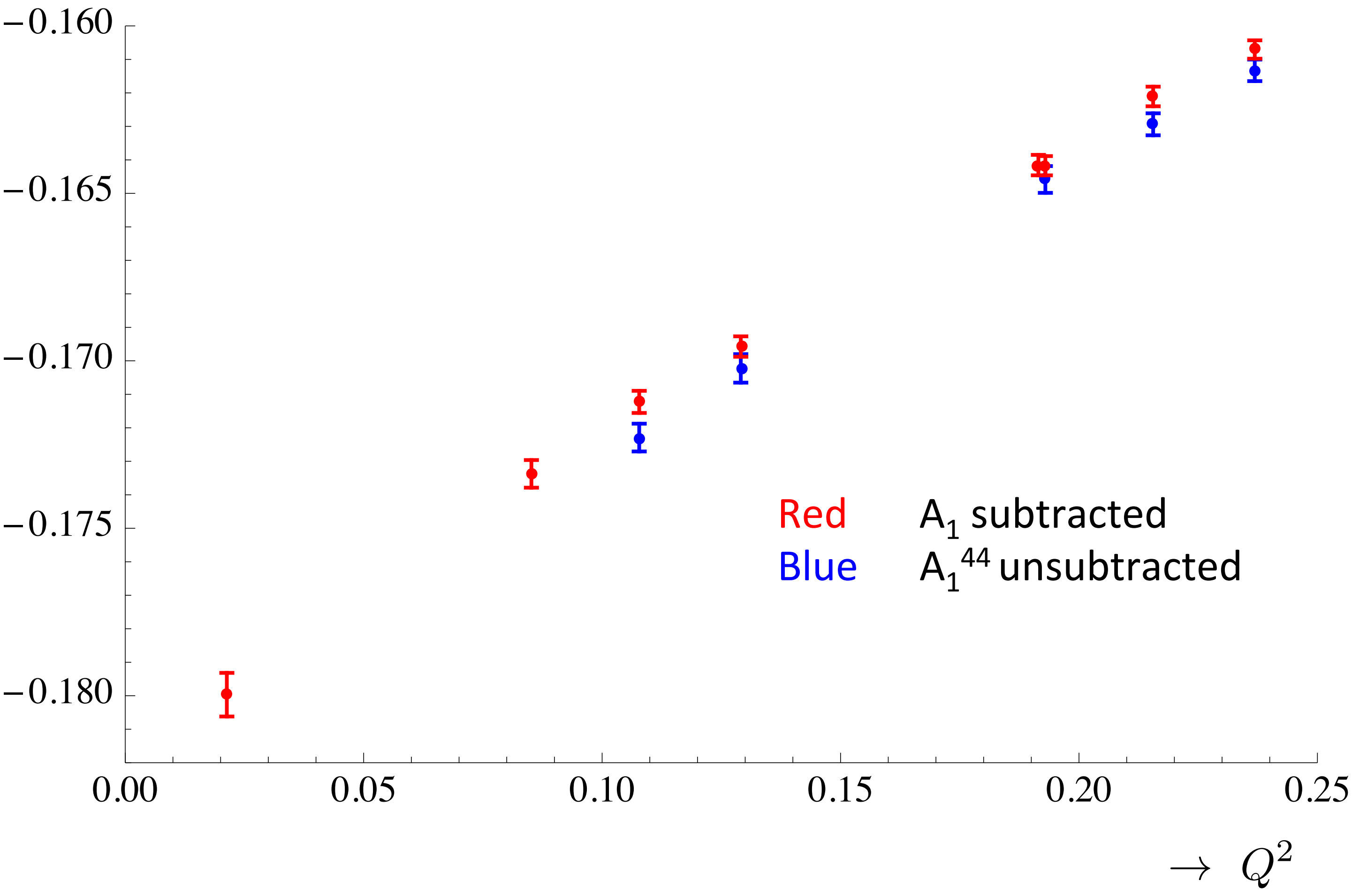


Red	A_1 subtracted
Blue	A_1 unsubtracted
Purple	A_1^{44} unsubtracted
Cyan	T_2 unsubtracted
Black	A_1 infinite volume

Comparison using NLO ChPT of different irreps – straddle infinite-volume



Comparison using AMA lattice data of different irreps (Aubin et al. ’15)

(AMA: Blum, Izubuchi and Shintani, ’13)

Effect on a_μ

Define $$a_\mu^{\mathrm{HVP}}(Q^2_{max}) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^{Q^2_{max}} dQ^2\, f(Q^2)\,[\Pi(Q^2) - \Pi(0)]$$

A$_1$:

[0,1] Padé: $a_\mu^{\mathrm{HVP}}(0.1\ \mathrm{GeV}^2) = 6.8(4) \times 10^{-8}$

quadr. conf. pol.: $a_\mu^{\mathrm{HVP}}(0.1\ \mathrm{GeV}^2) = 7.5(3) \times 10^{-8}$

A$_1^{44}$:

[0,1] Padé: $a_\mu^{\mathrm{HVP}}(0.1\ \mathrm{GeV}^2) = 6.8(4) \times 10^{-8}$

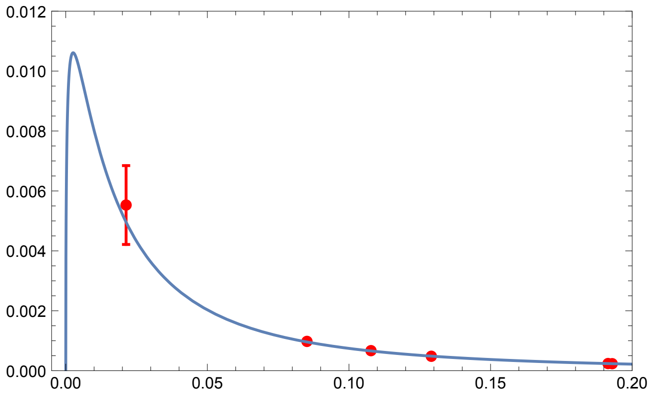
quadr. conf. pol.: $a_\mu^{\mathrm{HVP}}(0.1\ \mathrm{GeV}^2) = 7.9(4) \times 10^{-8}$

Difference of 10 – 15% as a consequence of finite volume effects

(Fits on interval between 0 and 0.3 GeV^2)

Consistent with Francis, Jaeger, Meyer and Wittig, ‘13

Conclusions



- Very low Q^2 region is important

- Need sequence of model-independent fit functions,
  approx. physical pion masses,
  and good control over finite-volume effects

- t^2 moment of current correlator is linear combination of values at all non-zero Q – moments method has similar issue (TB, Izubuchi, ‘15)

$$\Pi(0) \to \sum_{n\neq 0} 4(-1)^n \Pi\left(\frac{2\pi n}{T}\right)$$