Nuclear response theory: RPA and its extensions

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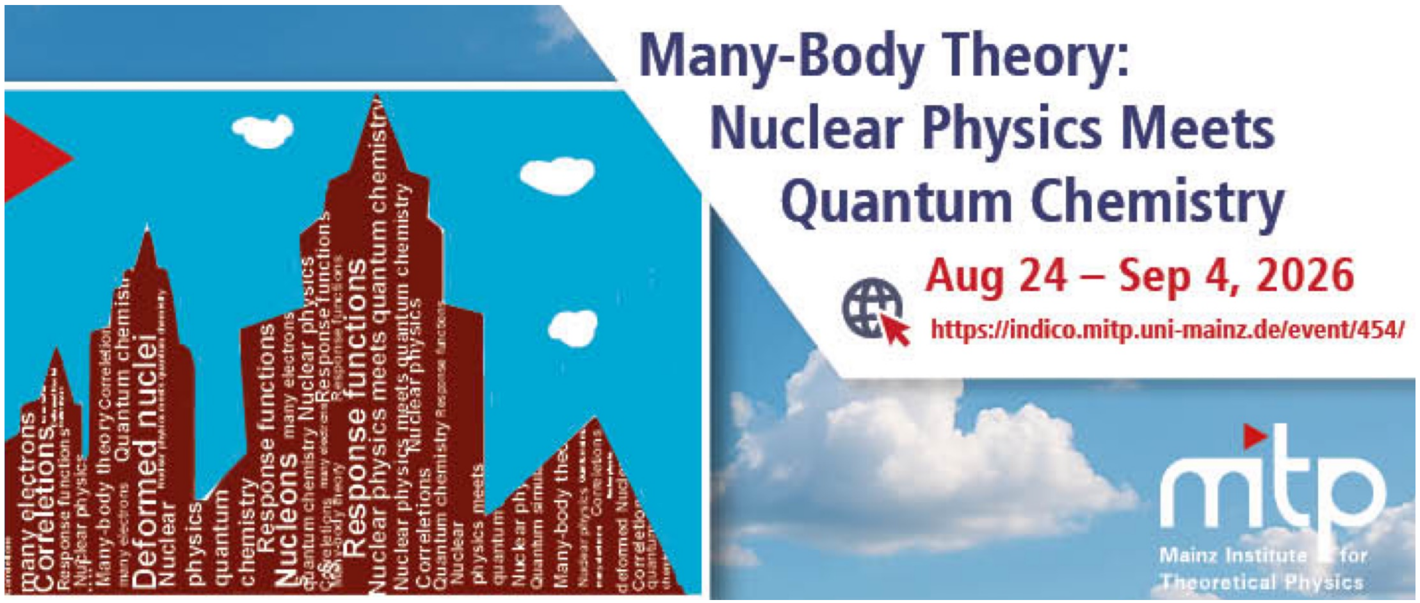
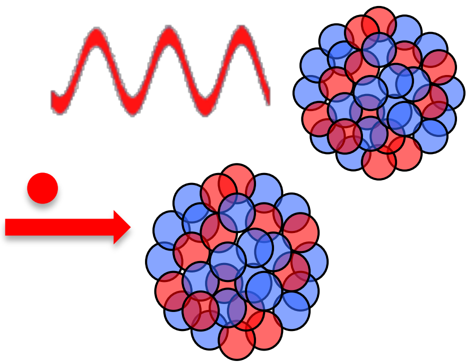
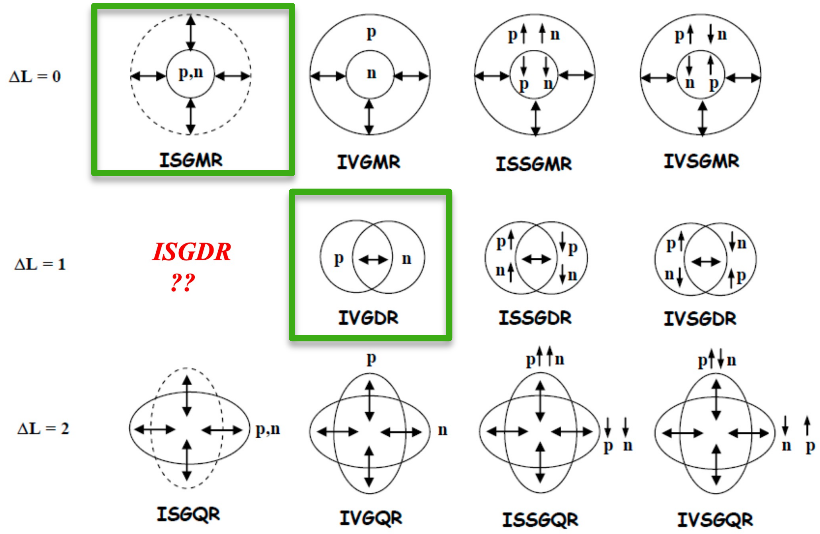
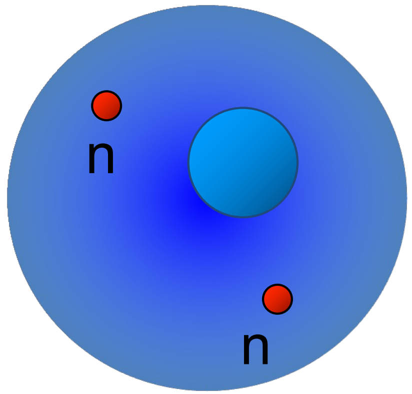


General concepts / motivation

There are many data on **collective nuclear motion,** and we hope to study in near future how it evolves when one considers exotic systems.







There is a link between collective motion and the nuclear Equation of State

There is no separation of scale (in general) between single-particle and collective motion

$\approx$ MeV

Linear response or RPA (from EOM)

We start from a Hamiltonian $$H = \sum_\alpha \varepsilon_\alpha a^\dagger_\alpha a_\alpha + \frac{1}{4}\sum_{\alpha\beta\gamma\delta} v_{\alpha\beta\gamma\delta} a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma$$

Excited states *n* ("vibrations" or "phonons") $\boxed{H\Gamma^\dagger_n|0\rangle = \hbar\omega_n \Gamma^\dagger_n|0\rangle}$

The g.s. has zero energy and is the phonon vacuum $\boxed{\Gamma_n|0\rangle = 0 \qquad \forall n}$

The ansatz for the RPA phonons is: $$\boxed{\Gamma^\dagger = \sum_{\rm ph} X_{\rm ph} a^\dagger_{\rm p} a_{\rm h} - Y_{\rm ph} a^\dagger_{\rm h} a_{\rm p}}$$

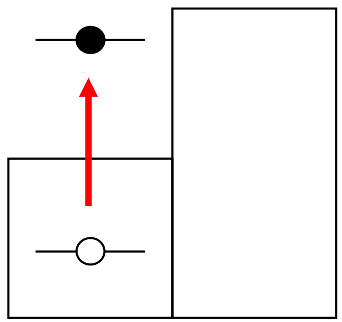
This leads to $$\langle 0|\left[\delta\Gamma_n, \left[H, \Gamma^\dagger_n\right]\right]|0\rangle = \hbar\omega_n \langle 0|\left[\delta\Gamma_n, \Gamma^\dagger_n\right]|0\rangle$$

and in RPA we also assume that the excitations are boson-like

If the equation is solved on a basis:
[cf. T. Nakatsukasa for other method(s)]

$$\begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \hbar\omega \begin{pmatrix} X \\ Y \end{pmatrix}$$

G.C. *et al.*, Computer Physics Commun. 184, 142 (2013).



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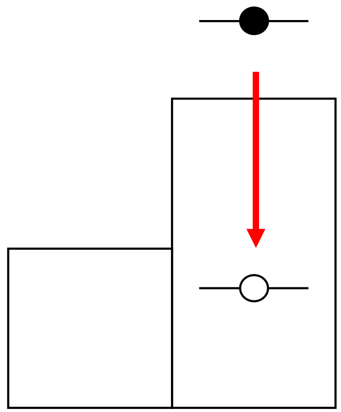

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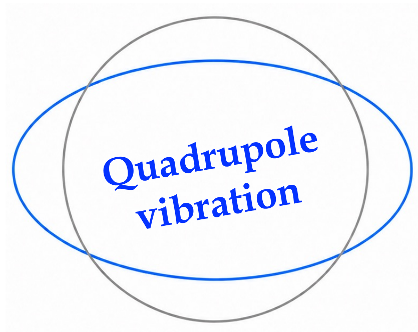
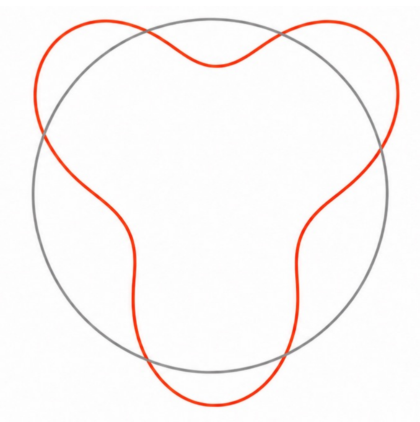
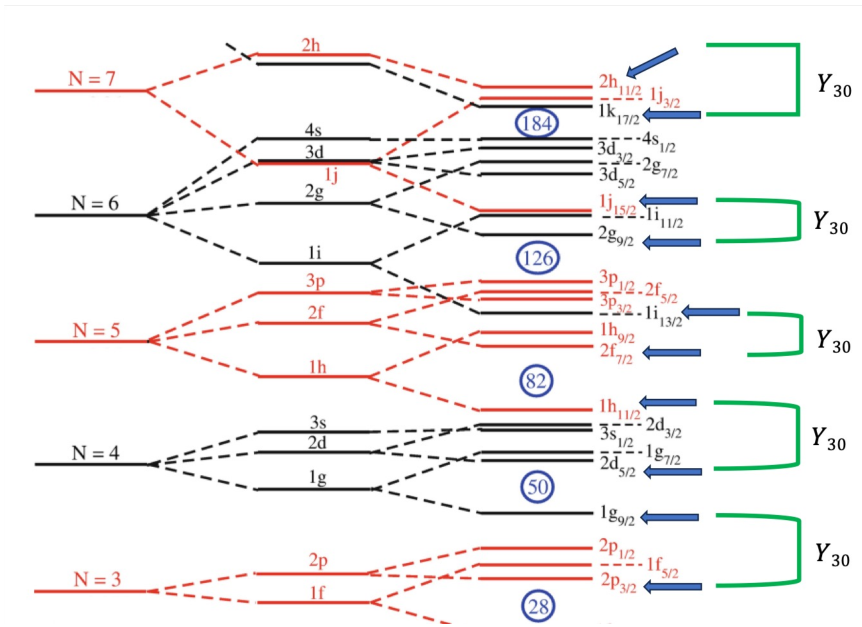


Tamm-Dancoff approximation is often poor in nuclear physics, for low-lying states (e.g. low-octupole states in ^{208}Pb, ^{132}Sn)





Second RPA (SRPA)

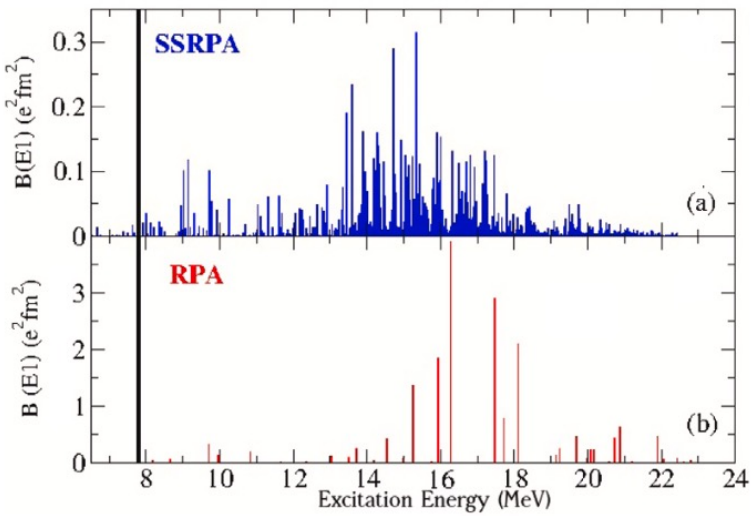
D. Gambacurta and M. Grasso, PPNP 150, 104251 (2026).

The ansatz for the SRPA phonon creation operator is:

$$\Gamma^\dagger = \sum_{\mathrm{ph}} X_{\mathrm{ph}} a_{\mathrm{p}}^\dagger a_{\mathrm{h}} - Y_{\mathrm{ph}} a_{\mathrm{h}}^\dagger a_{\mathrm{p}} + \sum_{\mathrm{p}'<\mathrm{p}\ \mathrm{h}'<\mathrm{h}} X_{\mathrm{php}'\mathrm{h}'} a_{\mathrm{p}}^\dagger a_{\mathrm{p}'}^\dagger a_{\mathrm{h}'} a_{\mathrm{h}} - Y_{\mathrm{php}'\mathrm{h}'} a_{\mathrm{h}}^\dagger a_{\mathrm{h}'}^\dagger a_{\mathrm{p}'} a_{\mathrm{p}}$$

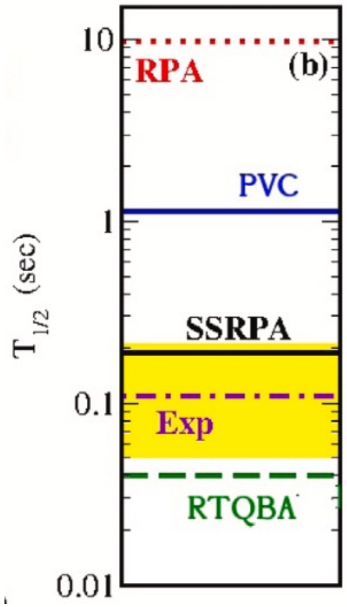
$$\begin{pmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{21} & A_{22} & B_{21} & B_{22} \\ -B_{11}^* & -B_{12}^* & -A_{11}^* & -A_{12}^* \\ -B_{21}^* & -B_{22}^* & -A_{21}^* & -A_{22}^* \end{pmatrix} \begin{pmatrix} X^{(1)} \\ X^{(2)} \\ Y^{(1)} \\ Y^{(2)} \end{pmatrix} = \hbar\omega \begin{pmatrix} X^{(1)} \\ X^{(2)} \\ Y^{(1)} \\ Y^{(2)} \end{pmatrix}$$

Computationally very demanding (10^6-10^8 2p-2h states)



Dipole states in ^{68}Ni

β-decay half-life of ^{78}Ni



Mainz, Sep. 2nd, 2026

Second QRPA and QPVC

We introduce the **Bogoliubov** transformation for treating open-shell systems, so we take **pairing** into account.

$$\alpha_k^\dagger = \sum_l U_{lk} a_l^\dagger + V_{lk} a_l$$

The excited state in second QRPA is

$$|N\rangle = \left(\sum_{\alpha\beta} X^{(1)}_{\alpha\beta} \alpha^\dagger_\alpha \alpha^\dagger_\beta - Y^{(1)}_{\alpha\beta} \alpha_\beta \alpha_\alpha + \sum_{\alpha\beta\gamma\delta} X^{(2)}_{\alpha\beta\gamma\delta} \alpha^\dagger_\alpha \alpha^\dagger_\beta \alpha^\dagger_\gamma \alpha^\dagger_\delta - Y^{(2)}_{\alpha\beta\gamma\delta} \alpha_\delta \alpha_\gamma \alpha_\beta \alpha_\alpha \right) |\tilde{0}\rangle$$

The quasi-particle vibration (QPVC) model assumes instead that

$$|N\rangle = \left(\sum_{\alpha\beta} X^{(1)}_{\alpha\beta} \alpha^\dagger_\alpha \alpha^\dagger_\beta - Y^{(1)}_{\alpha\beta} \alpha_\beta \alpha_\alpha + \sum_{\alpha\beta n} X^{(2)}_{\alpha\beta n} \alpha^\dagger_\alpha \alpha^\dagger_\beta \Gamma^\dagger_n - Y^{(2)}_{\alpha\beta n} \alpha_\beta \alpha_\alpha \Gamma_n \right) |\tilde{0}\rangle$$

Derivation of the QPVC equations

$$\mathcal{O}_N^\dagger = \sum_{\alpha\beta} X^{(1)}_{\alpha\beta}\alpha^\dagger_\alpha\alpha^\dagger_\beta - Y^{(1)}_{\alpha\beta}\alpha_\beta\alpha_\alpha + \sum_{\alpha\beta n} X^{(2)}_{\alpha\beta n}\alpha^\dagger_\alpha\alpha^\dagger_\beta\Gamma^\dagger_n - Y^{(2)}_{\alpha\beta n}\alpha_\beta\alpha_\alpha\Gamma_n$$

We can use again the Equation of Motion (EoM) in the form:

$$\langle\tilde{0}|\left[\delta\mathcal{O}_N,\left[H,\mathcal{O}_N^\dagger\right]\right]|\tilde{0}\rangle = E_\nu\langle\tilde{0}|\left[\delta\mathcal{O}_N,\mathcal{O}_N^\dagger\right]|\tilde{0}\rangle$$

green: usual QRPA matrix

yellow: interaction between 2qp and 2qp+phonon

This leads to

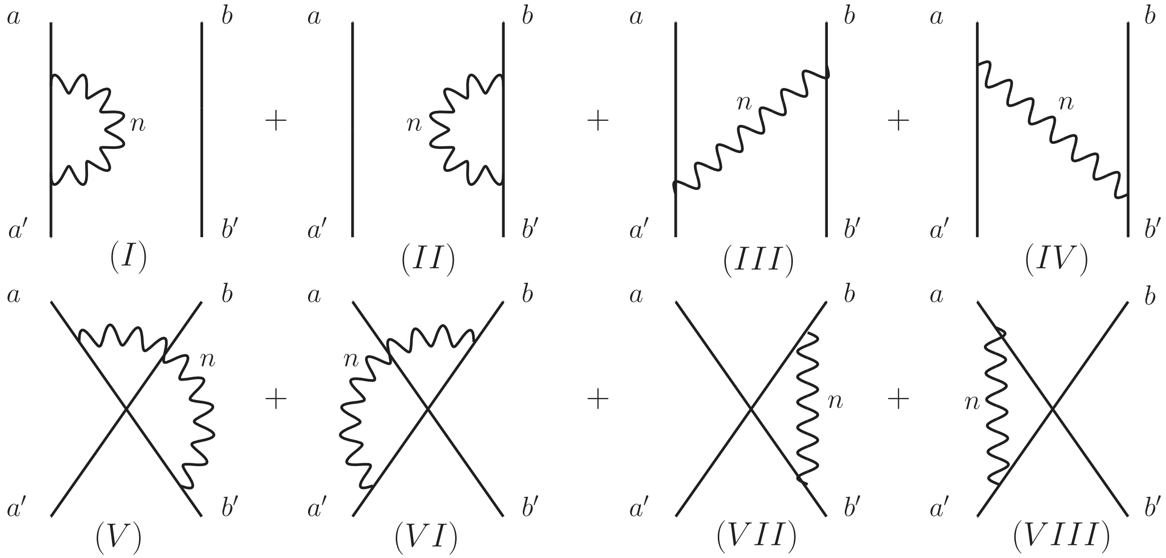
$$\begin{pmatrix} A_{\alpha\beta,\alpha'\beta'} & B_{\alpha\beta,\alpha'\beta'} & A_{\alpha\beta,\alpha'\beta'n'} & 0 \\ -B^*_{\alpha\beta,\alpha'\beta'} & -A^*_{\alpha\beta,\alpha'\beta'} & 0 & -A^*_{\alpha\beta,\alpha'\beta'n'} \\ A_{\alpha\beta,\alpha'\beta'n'} & 0 & A_{\alpha\beta n,\alpha'\beta'n'} & 0 \\ 0 & -A^*_{\alpha\beta n,\alpha'\beta'} & 0 & -A^*_{\alpha\beta n,\alpha'\beta'n'} \end{pmatrix} \begin{pmatrix} X^{(1)}_{\alpha'\beta'} \\ Y^{(1)}_{\alpha'\beta'} \\ X^{(2)}_{\alpha'\beta'n'} \\ Y^{(2)}_{\alpha'\beta'n'} \end{pmatrix} = E \begin{pmatrix} X^{(1)}_{\alpha\beta} \\ Y^{(1)}_{\alpha\beta} \\ X^{(2)}_{\alpha\beta n} \\ Y^{(2)}_{\alpha\beta n} \end{pmatrix}$$

We check the **convergence** of the results against <u>variations of the model space</u> (number of quasiparticles and of QRPA phonons).

(Q)RPA + (Q)PVC

If we project "à la Feshbach" on the space of two quasiparticle states:

$$\begin{pmatrix} A+\Sigma(E) & B \\ -B & -A-\Sigma^*(-E) \end{pmatrix} \qquad \Sigma_{ab,a'b'n} = \sum_{\alpha} \frac{\langle ab|V|\alpha\rangle\langle\alpha|V|a'b'\rangle}{E-E_\alpha+i\eta}$$



Wavy lines represent phonons and have to be seen as a sum of ring diagrams.

In this respect this is G_0W_0 including pairing.

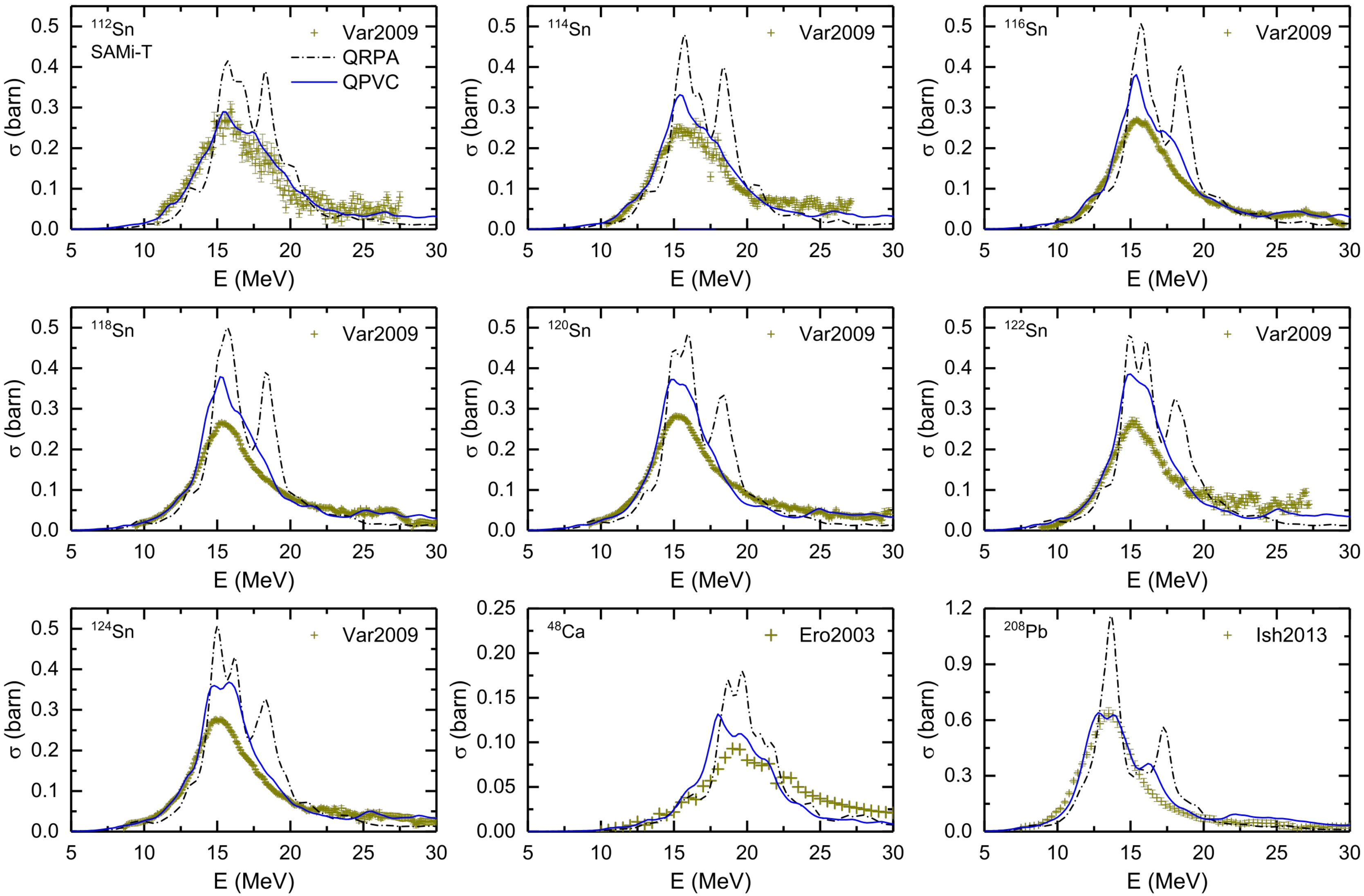
In the applications that will follow, we shall use effective Hamiltonians

$$H = H_{\rm Skyrme}$$

- They include zero-range forces with derivative terms to mimic the finite range.
- They respect basic symmetries.
- They include spin-orbit and in some cases tensor terms.
- In total: 10 free parameters to be fitted (typically).
- The idea is to fit parameters on a small set of observables (like masses or radii of a few nuclei) and use them to predict other observables.

Applications

Photoabsorbtion cross section



Z.Z. Li *et al.*, Phys. Rev. C110, 064317 (2024). SAMi-T.

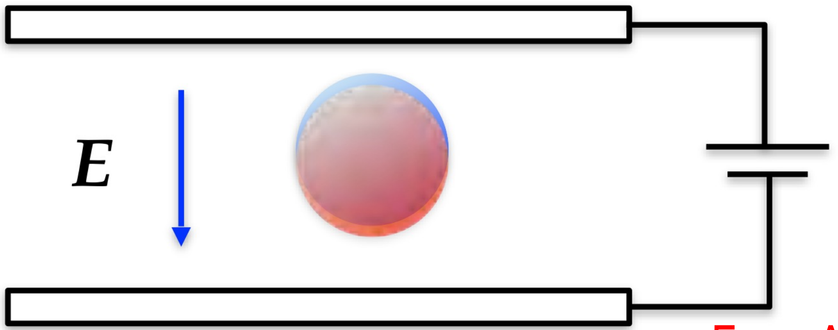
Response function and strength function

$$R(\omega) = \langle 0|F^\dagger G(\omega) F|0\rangle$$

$$S(\omega) = -\frac{1}{\pi}\Im\, R(\omega) \qquad \frac{1}{\omega - \omega_n + i\eta} \to P\frac{1}{\omega - \omega_n} - i\pi\delta(\omega - \omega_n)$$

$$S(\omega) = \sum_n |\langle n|F|0\rangle|^2 \delta(\omega - \omega_n) \qquad S(\omega) = \sum_n |\langle n|F|0\rangle|^2 \frac{\Gamma_n}{(\omega - \omega_n)^2 + \frac{\Gamma_n^2}{4}}$$

The dipole polarizability



From A. Tamii

If the nucleus were under the action of a static electric field

$$p = \alpha_{\mathrm{D}} E$$

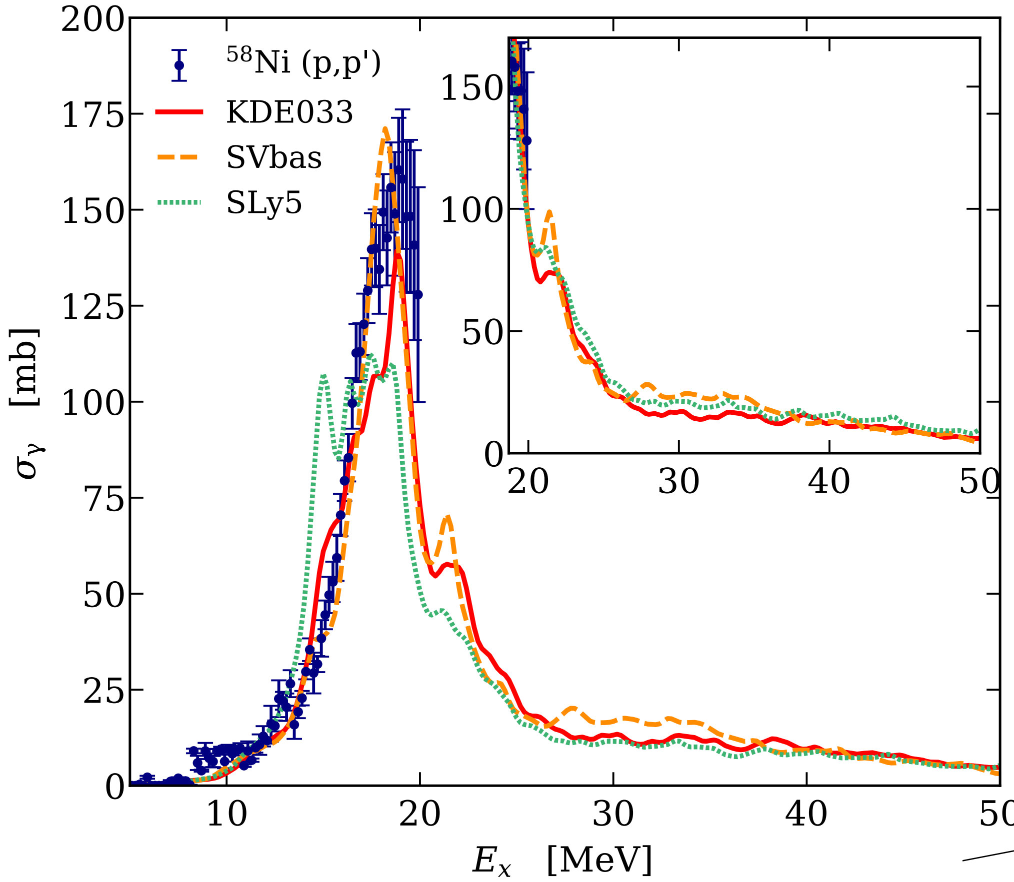
The **polarizability** can be deduced from the so-called **inverse energy-weighted sum rule** associated with dipole field

$$\alpha_{\mathrm{D}} = \frac{8\pi e^2}{9} \int dE \, \frac{dB(E1)}{dE} \equiv \frac{8\pi e^2}{9} m_{-1}$$

A. Migdal, Quadrupole and dipole γ-radiation of nuclei, J. Phys. Acad. Sci. USSR 8(1-6), 331-336 (1944)

$$\alpha_{\mathrm{D}} = \frac{\pi e^2}{54} \frac{A\langle r^2 \rangle}{J}$$

Photoabsorbtion of ^{58}Ni



Comparison with recent data from (p,p') measured at RCNP, Osaka. Exp. data from I. Brandherm *et al.*

Theory reproduces very well the dipole polarizabilty and can be used to estimate the high-energy dipole tail.

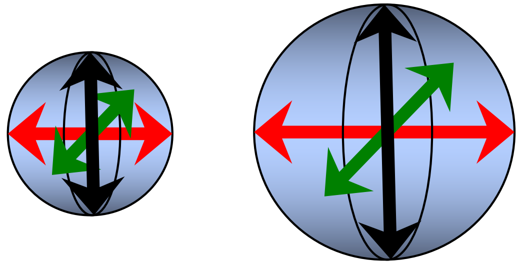
PHYSICAL REVIEW C **111**, 024312 (2025)

Electric dipole polarizability of ^{58}Ni

I. Brandherm,[1] F. Bonaiti,[2,3,4] P. von Neumann-Cosel,[1,*] S. Bacca,[2] G. Colò,[5,6] G. R. Jansen,[7,4] Z. Z. Li (李征征),[8,9,10] H. Matsubara,[11,12] Y. F. Niu (牛一斐),[9,10] P.-G. Reinhard,[13] A. Richter,[1] X. Roca-Maza,[14,15,5,6] and A. Tamii[11]

Nuclear incompressibility and the ISGMR

Isoscalar Giant Monopole Resonance or "breathing mode": its energy should be correlated with the incompressibility of nuclear matter.



$$K_\infty = 9\rho_0^2 \frac{d^2}{d\rho^2}\left(\frac{E}{A}\right)_{\rho=\rho_0}$$

$$\chi \equiv -\frac{1}{\Omega}\left(\frac{\partial P}{\partial \Omega}\right)^{-1}$$

$$\chi^{-1} = \rho^3 \frac{d^2}{d\rho^2}\left(\frac{E}{A}\right)$$

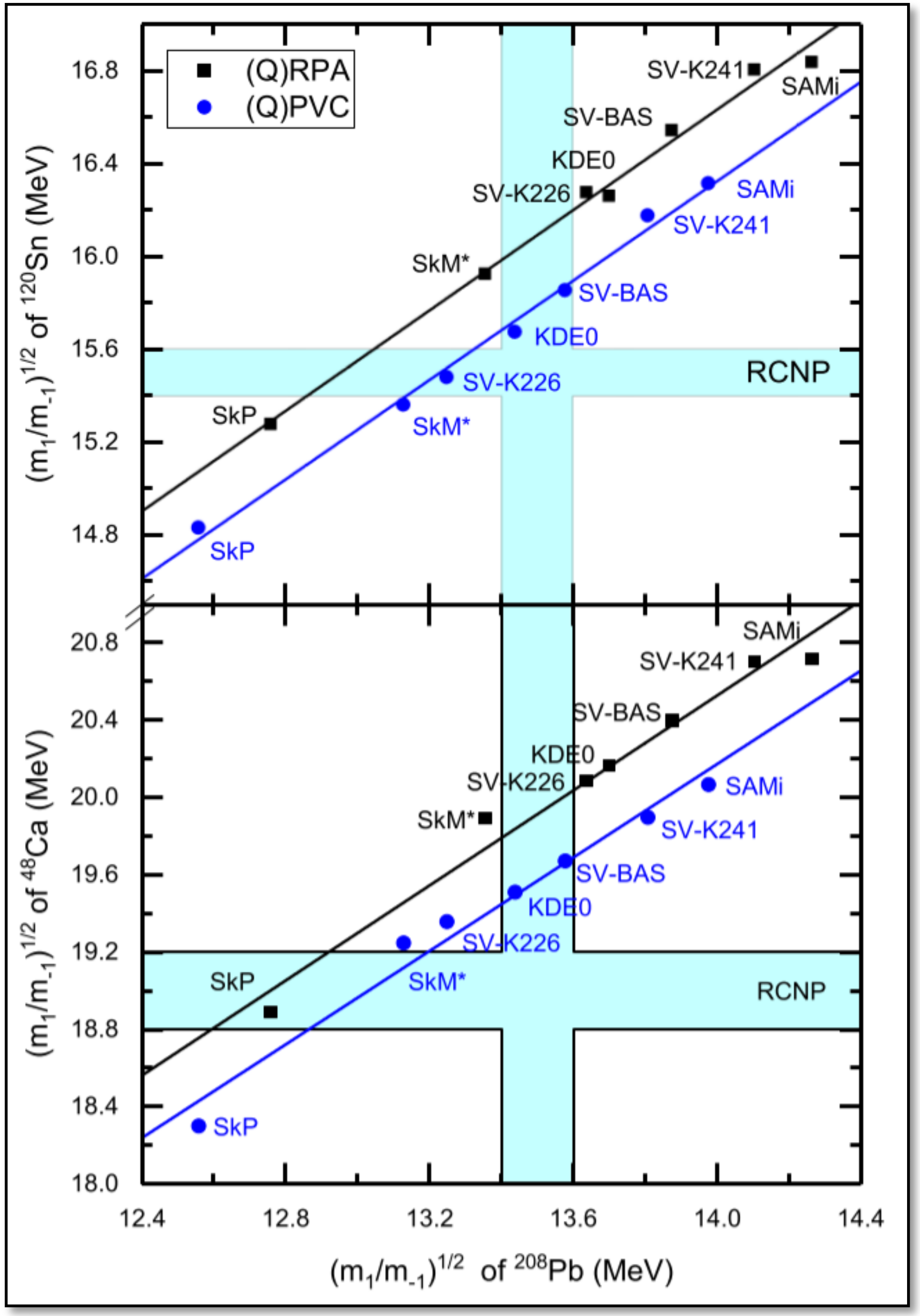
Impact on astrophysics: supernova explosion, neutron star merging

SN1987a

PHYSICAL REVIEW LETTERS **129,** 032701 (2022)

Probing the Incompressibility of Nuclear Matter at Ultrahigh Density through the Prompt Collapse of Asymmetric Neutron Star Binaries

Albino Perego,[1,2,*] Domenico Logoteta,[3,4] David Radice,[5,6,7] Sebastiano Bernuzzi,[8] Rahul Kashyap,[5,6] Abhishek Das,[5,6] Surendra Padamata,[5,6] and Aviral Prakash[5,6]



In our work, we have been able to analyse **in a systematic manner** the consistency between ISGMR energies in different nuclei.

We have used many Skyrme EDFs.

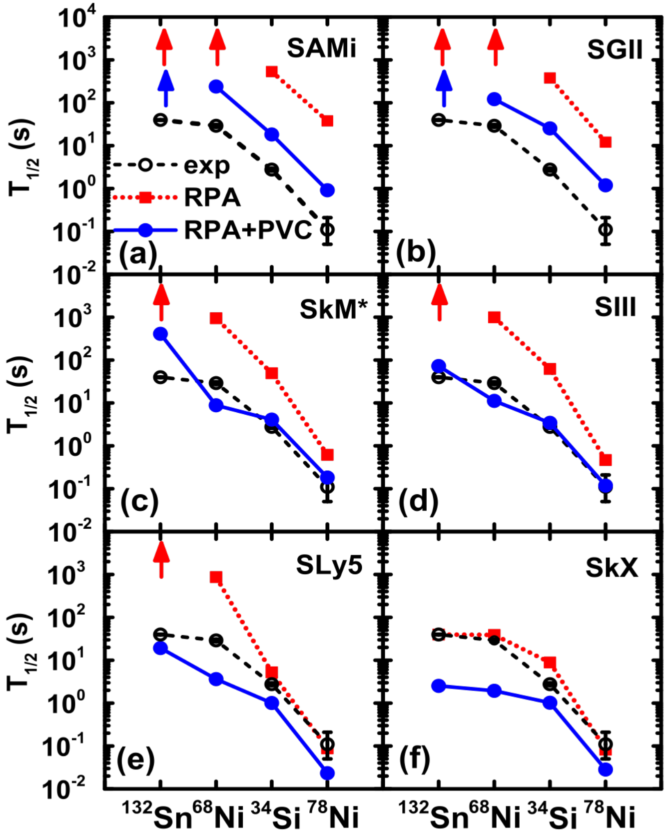
With the inclusion of QPVC effects, a big improvement is achieved.

We have, thus, a unique value for incompressibility.

Within QPVC, the ISGMR energy in ^{208}Pb is consistent with ^{120}Sn.

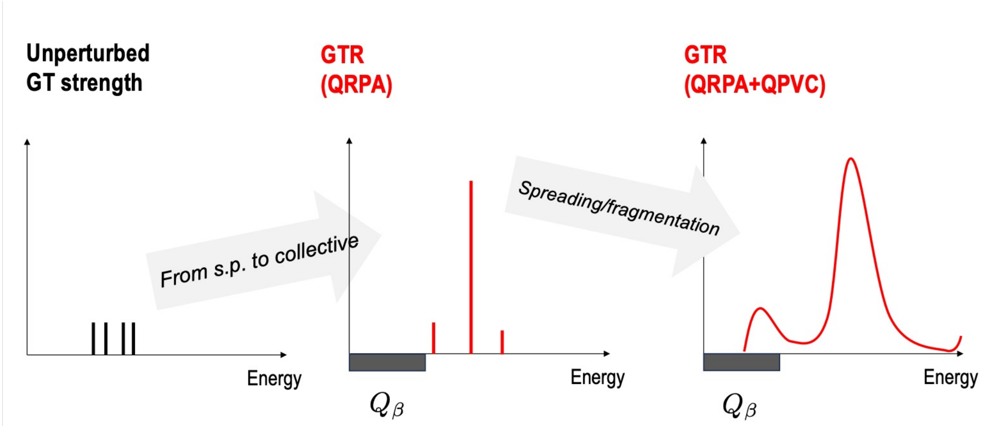
Z.Z. Li, Y.F. Niu, GC, Phys. Rev. Lett. 131, 082501 (2023)

Applying QPVC to β-decay



Y. F. Niu *et al.*, Phys. Rev. Lett. 114, 142501

Y. F. Niu *et al.*, Phys. Lett. B 780, 325



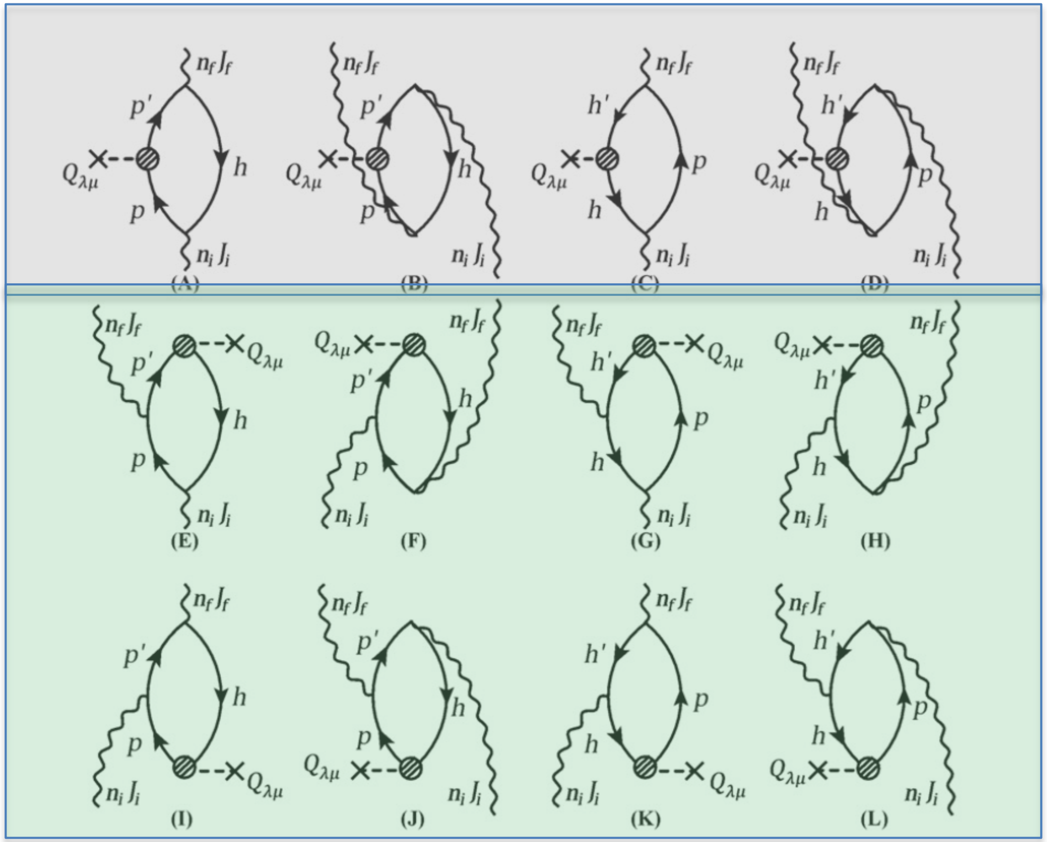
RPA may not leave strength in the Q_β window.

PVC redistributes the GT strength.

Applying QPVC to γ-decay

$$\Gamma_\gamma(E\lambda; i \to f) = \frac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]}\left(\frac{E}{\hbar c}\right)^{2\lambda+1} B(E\lambda; i \to f)$$

$$B(E\lambda; i \to f) = \frac{1}{2J_i+1}|\langle J_f||Q_\lambda||J_i\rangle|^2$$



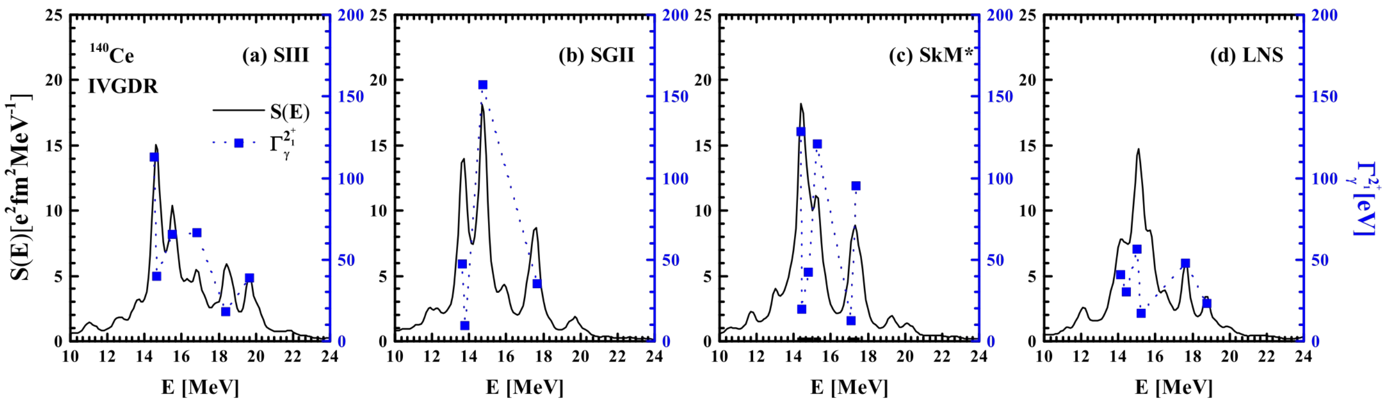
All the Feynman diagrams contribute to the matrix element:

First line: 1p-1h, RPA

Second and third line: PVC

This decay is a tiny (≈10^{-3}) part of the total width; there are cancellations between the amplitudes.

The decay from the IVGDR to 2+ in ^{140}Ce



SIII				SGII				SkM*				LNS			
E[MeV]	$\Gamma_\gamma^{2_1^+}$ [eV]	$\Gamma_\gamma^{g.s.}$[eV]	R[%]	E[MeV]	$\Gamma_\gamma^{2_1^+}$ [eV]	$\Gamma_\gamma^{g.s.}$[eV]	R[%]	E[MeV]	$\Gamma_\gamma^{2_1^+}$ [eV]	$\Gamma_\gamma^{g.s.}$[eV]	R[%]	E[MeV]	$\Gamma_\gamma^{2_1^+}$ [eV]	$\Gamma_\gamma^{g.s.}$[eV]	R[%]
14.53	112.9	2952.9	3.82	13.62	47.5	4029.6	1.18	14.38	128.3	8209.0	1.56	14.13	40.9	2790.4	1.47
14.65	39.8	9174.5	0.43	13.77	9.5	2624.9	0.36	14.45	19.6	3642.8	0.54	14.42	30.1	2282.5	1.32
15.50	65.4	7491.7	0.87	14.73	157.3	12744.4	1.23	14.78	42.4	4084.2	1.04	15.02	56.3	6966.4	0.81
16.84	66.4	4400.8	1.51	17.65	35.5	6937.5	0.51	15.26	120.8	7547.8	1.60	15.22	17.1	2682.0	0.64
18.37	18.0	6532.9	0.28					17.10	12.7	5141.9	0.25	17.63	48.0	8215.3	0.58
19.65	38.6	8592.5	0.45					17.34	95.2	6887.4	1.38	18.76	23.4	5183.9	0.45
Total	341.0	39145.2	$\boxed{0.87}$	Total	249.9	26336.5	$\boxed{0.95}$	Total	419.0	35513.1	$\boxed{1.18}$	Total	215.8	28120.5	$\boxed{0.77}$

The **partial widths** for the decay to the 2+ are or the order of **10^2 eV** but they fluctuate.

This corresponds to a ratio of 0.7-1.2% with respect to the ground-state decay.

W.L. Lv, Y.F. Niu, GC, arXiv:2603.27243

INFN Istituto Nazionale di Fisica Nucleare

Courtesy: N. Pietralla

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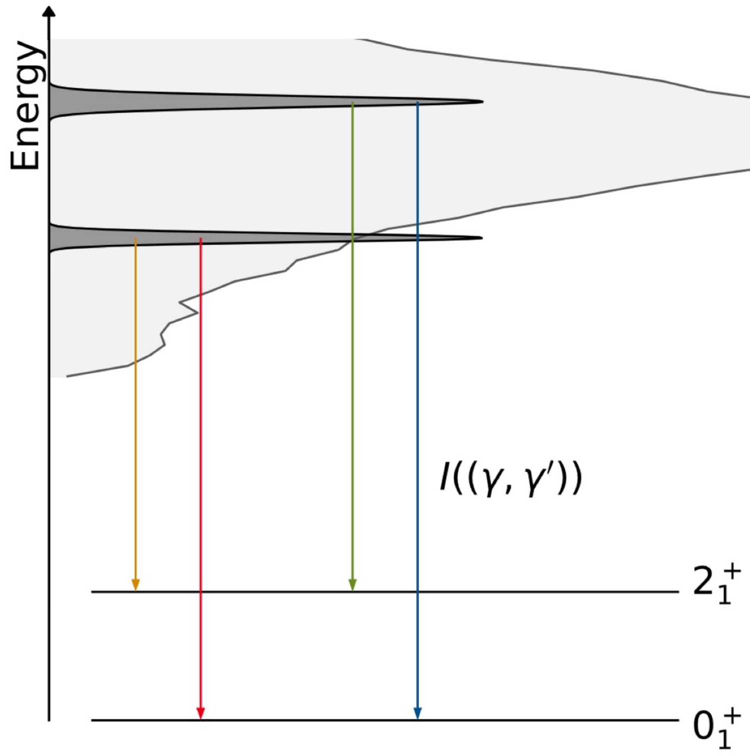


E_{beam} (MeV)	$I_{\gamma,2_1^+}/I_{\gamma,0_1^+}$
11.4	0.60(40)
12.6	0.0034(24)
14.3	0.0010(7)
15.4	0.0007(7)
16.2	0.0009(6)
17.8	0.0019(13)

Observation:

- Decays to 2^+_1 (or any other excited states) are VERY small. At all energies.

➢ The GDR decays by γ according to its collective part.

➢ The timescale for the γ–decay of the GDR must be the timescale of the collective state, i.e., 13 zs.

A similar model for the single-particle response

We can assume a **similar ansatz** for a s.p. (or single quasiparticle) state, that is a mixture with phonons:

$$\boxed{\mathcal{O}_\alpha^\dagger = X_\alpha a_\alpha^\dagger - Y_\alpha a_{\tilde{\alpha}} + \sum_{\beta n} C_{\beta n} a_\beta^\dagger \Gamma_n^\dagger - \sum_{\beta n} D_{\beta n} a_{\tilde{\beta}} \Gamma_n}$$

In this case, **we have corrected for the non-orthonormality and overcompleteness of the basis by introducing the NORM matrix.**

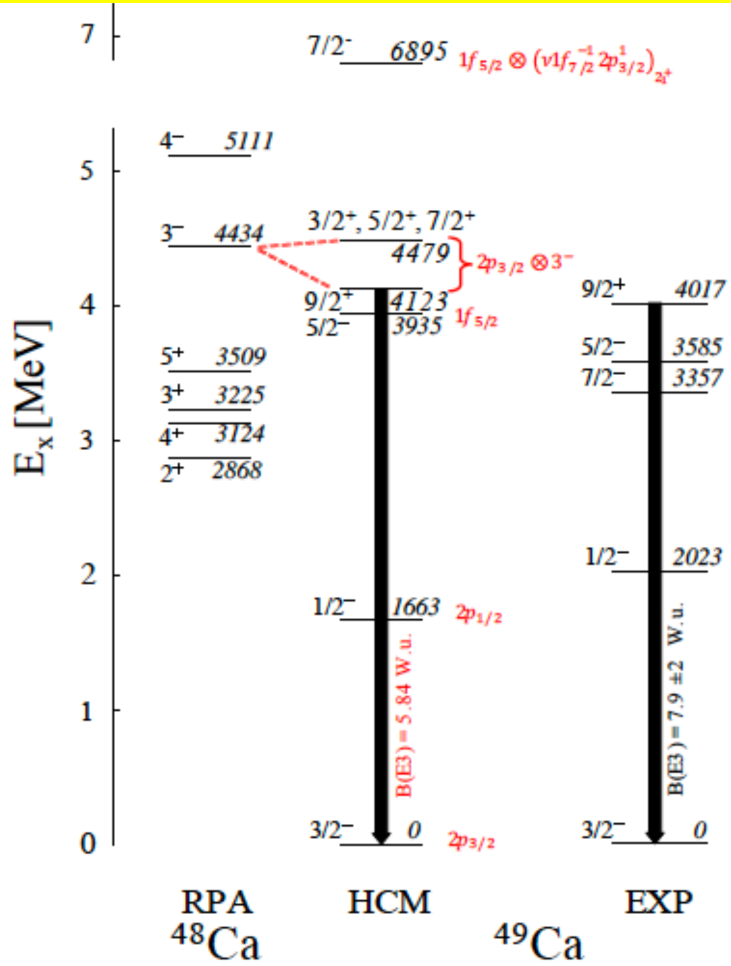
$$n(j_1' n_1 J_1, j_2' n_2 J_2) = \delta(j_1', j_2')\delta(n_1, n_2)\delta(J_1, J_2) - \sum_{h_1} (-)^{J_1+J_2+j_1'+j_2'} \hat{J}_1 \hat{J}_2 \left\{ \begin{array}{ccc} j_2' & j_{h_1} & J_1 \\ j_1' & j & J_2 \end{array} \right\} X_{j_2' h_1}^{(n_1 J_1)} X_{j_1' h_1}^{(n_2 J_2)}$$

The diagonal part reduces to 1 – $(2j+1)^{-1}$ in simple cases.

$$\boxed{(\mathcal{H} - E\mathcal{N})\Psi = 0}$$

^{49}Ca

Energies and γ transitions



Spectroscopic factors of ^{49}Ca

J^π	ℓ	C^2S (exp^a)	ξ^2(s.p.) (HCM)
$3/2^-$	1	0.84(12)	0.98
$1/2^-$	1	0.91(15)	0.99
$5/2^-$	3	0.84	0.97
$9/2^+$	4	0.14	0.11

^{a}The authors report the errors on C^2S for the $5/2^-$ and $9/2^+$ states as "a few percent".

Results obtained with SkX

PHYSICAL REVIEW C **95**, 034303 (2017)

Hybrid configuration mixing model for odd nuclei

Conclusion

- In low-energy nuclear spectra, single-particle states and collective states coexist.

- A separation of scales like the Born-Oppenheimer approximation in electronic systems is only possible for (some of the) rotational states – not for collective vibrations.

- Several models are based on the idea that the low-energy nuclear spectra arise from an interweaving of single-particle and collective degrees of freedom. We do not want to use models with *ad hoc* parameters, and we use an effective Hamiltonian consistently.

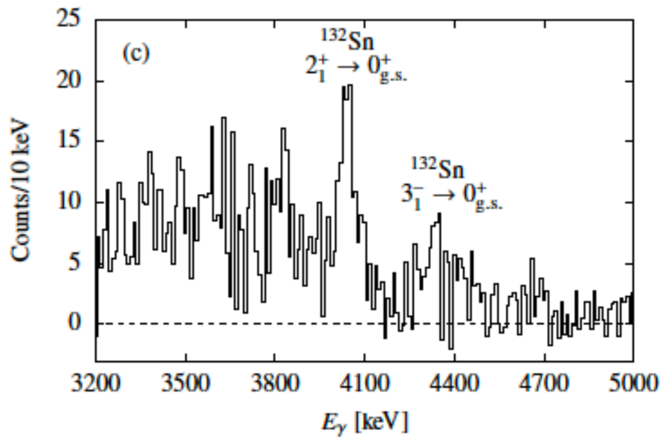
Backup slides

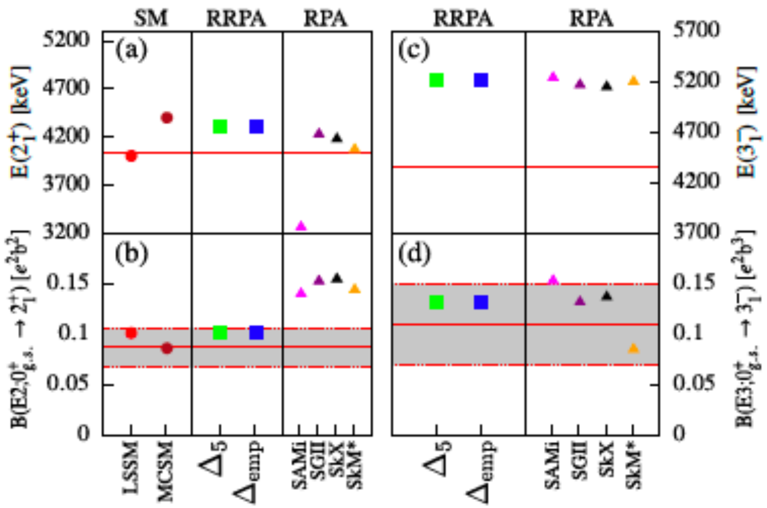
Coulomb excitation in ^{132}Sn

- **The low-lying 2^+ and 3^- states have been measured** via Coulomb excitations at HIE-ISOLDE (ions accelerated at 5.49 MeV/A and impinged on a ^{206}Pb target).

- **Energies and electromagnetic transition probabilities have been measured and compared with SM (or CI), RPA and relativistic RPA calculations. SM cannot describe the collective 3^-.**

D. Rosiak, M. Seidlitz, P. Reiter *et al.*, PRL





Skyrme force or “pseudo-potential”

attraction | **short-range repulsion**

$$v_{\mathrm{Skyrme}} = \underline{t_0\,(1+x_0P_\sigma)\,\delta(\vec r_1-\vec r_2)} + \underline{\frac{1}{2}t_1\,(1+x_1P_\sigma)\left(\vec k^{\dagger 2}\delta(\vec r_1-\vec r_2)+\delta(\vec r_1-\vec r_2)\vec k^2\right)}$$

$$+ \underline{t_2\,(1+x_2P_\sigma)\,\vec k^{\dagger}\cdot\delta(\vec r_1-\vec r_2)\vec k} + \underline{\frac{1}{6}t_3\,(1+x_3P_\sigma)\,\delta(\vec r_1-\vec r_2)\rho^\alpha\left(\frac{\vec r_1+\vec r_2}{2}\right)}$$

$$+ iW_0\,(\sigma_1+\sigma_2)\cdot\vec k^{\dagger}\times\delta(\vec r_1-\vec r_2)\vec k$$

$$\vec k = -\frac{i}{2}\left(\vec\nabla_1-\vec\nabla_2\right)$$

- There are velocity-dependent terms which mimic the finite-range.
- The last term is a zero-range spin-orbit.
- In total: 10 free parameters.

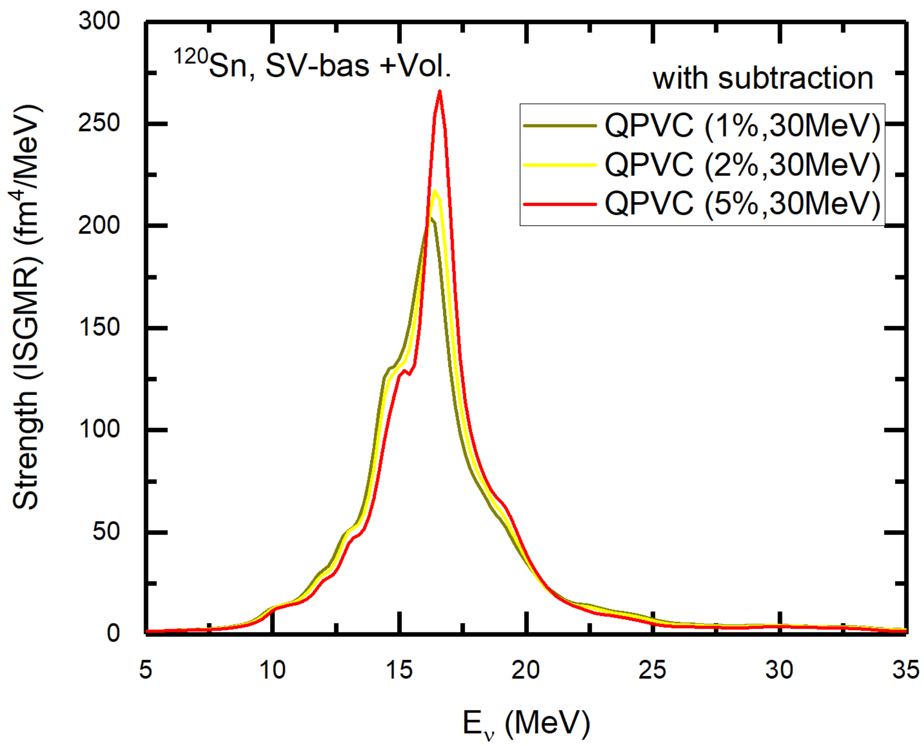
There are many existing Skyrme sets.
Cf. other talks like P. Demol

Some detail + the subtraction scheme

All QRPA calculations are performed in a model space which is large enough so that the EWSR is satisfied.

We calculate natural-parity phonons with 0^+, 1^-, 2^+ ... 5^- and select those having energy less than 30 MeV and strength larger than 2% of the total strength.

The convergence of the results with respect to the choice of the model space has been carefully assessed.



Subtraction: $$\Sigma(E) \rightarrow \Sigma(E) - \Sigma(E=0)$$

THIS PRESCRIPTION KEEPS THE VALUE OF THE m_{-1} SUM RULE AS IN QRPA

More on subtraction

The subtraction procedure has been introduced by V. Tselyaev.

At the same time, it has been shown that it emerges naturally if one:

- assumes the universality of the EDF, according to the Hohenberg-Kohn theorem
- applies this concept to $E - \lambda\hat{Q}$

This leads to the conservation of the inverse energy-weighted sum rule.

Subtraction method in the second random-phase approximation: First applications with a Skyrme energy functional

D. Gambacurta[1], M. Grasso[2], and J. Engel[3]

Phys. Rev. C **92**, 034303

The subtraction does not guarantee the conservation of the EWSR.