The Long Reach of Quantum Diffusion: Phenomenology of Universal PBH Clustering

Aurora Ireland
Stanford University

MITP | *The Phenomenology of PBHs*
July 23, 2026

Motivation

- PBH phenomenology depends on not only how many form, but also *where*
  ↪ Clustering affects mergers (and hence GWs), microlensing, isocurvature, (early) structure formation, etc

- Usual assumption: Poisson distribution
  ↪ Only correct for Gaussian statistics (generically wrong)

- Non-Gaussianity (NG) induces long-range correlations between scales
  ↪ Spatially-correlated PBH formation

- Issue: Perturbative treatments of NG (f_{NL}, g_{NL}, ...) break down in the tail
  ↪ *Precisely the regime which matters most for PBH formation!*

Motivation

- In models with flat segments of the potential (important for enhancing ζ), quantum diffusion can dominate evolution on relevant scales
- Stochastic δN formalism $\Rightarrow$ fully non-perturbative probability distribution function (PDF) for ζ
- Universal predictions for quantum-diffusion-dominated statistics
  - Exponential tail of PDF for ζ
  - Threshold-independent 2-pt statistics for structures forming in the tail

[Animali & Vennin, 2024]

This work: Phenomenological consequences

Stochastic δN formalism

δN formalism:

- Identify $\zeta \equiv \delta N$
- Curvature perturbation = variation in local duration of inflation between flat and uniform-density slices
- Non-perturbative and valid on super-horizon scales

Stochastic inflation:

- EFT for long-wavelength ($k < \sigma a H$) perturbations
- Short-wavelength modes act as stochastic noise for coarse-grained $\phi_{\mathrm{cg}} \equiv \phi$
- ϕ evolves according to Langevin eq; PDF for ϕ solves Fokker-Planck eq

Stochastic δN formalism

Stochastic δN formalism:

- Combine the two!
- Duration of inflation becomes stochastic variable $\mathcal{N}$
- Identify $\zeta \equiv \delta\mathcal{N}$
- Statistics of $\zeta \leftarrow$ statistics of $\mathcal{N} \leftarrow$ **first passage time** (FPT) problem

When does ϕ first reach ϕ_{end}?

- PDF of $\mathcal{N}$ satisfies adjoint Fokker-Planck eq

First passage time

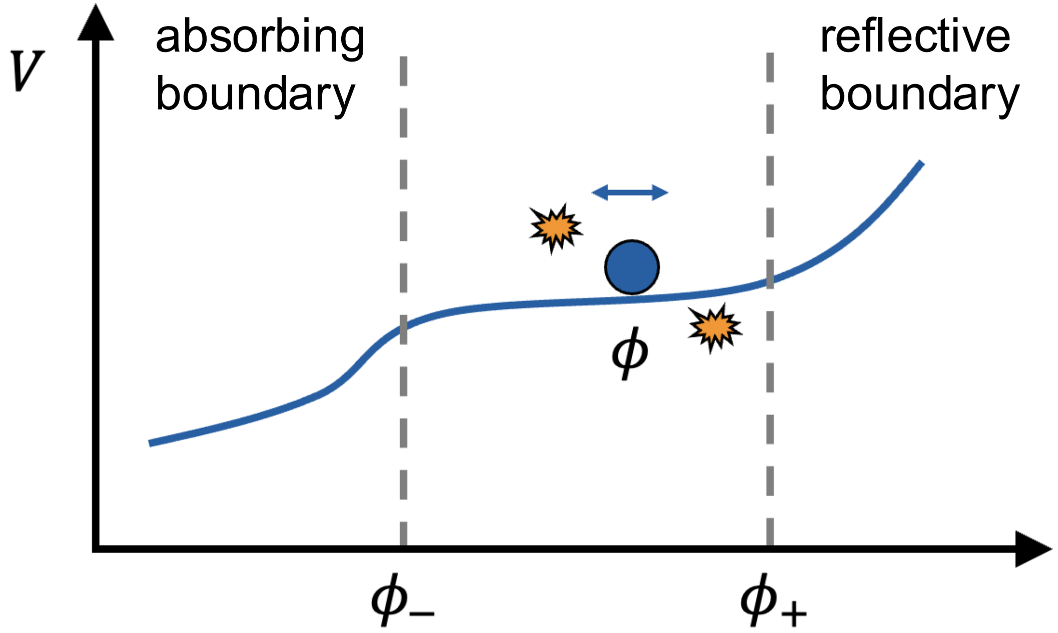
- $P_{\mathrm{FPT}}(\phi_-;\phi,\mathcal{N})$ = probability distribution for FPT from ϕ to ϕ_-
  ↪ *Starting from ϕ, what is the prob. density the trajectory reaches ϕ_- after $\mathcal{N}$ e-folds?*

- Solves adjoint Fokker-Planck:

$$\frac{dP_{\mathrm{FPT}}(\phi_-;\phi,\mathcal{N})}{d\mathcal{N}} = \mathcal{L}_{\mathrm{FP}}^{\dagger} P_{\mathrm{FPT}}(\phi_-;\phi,\mathcal{N})$$

- Boundary condition:

$$P_{\mathrm{FPT}}(\phi_-;\phi_-,\mathcal{N}) = \delta(\mathcal{N})$$



Characteristic function

- $\chi(\phi_-;\phi,t)$ = Fourier transform of PFT probability distribution

$$= \int d\mathcal{N}\, e^{it\mathcal{N}} P_{\mathrm{FPT}}(\phi_-;\phi,\mathcal{N})$$

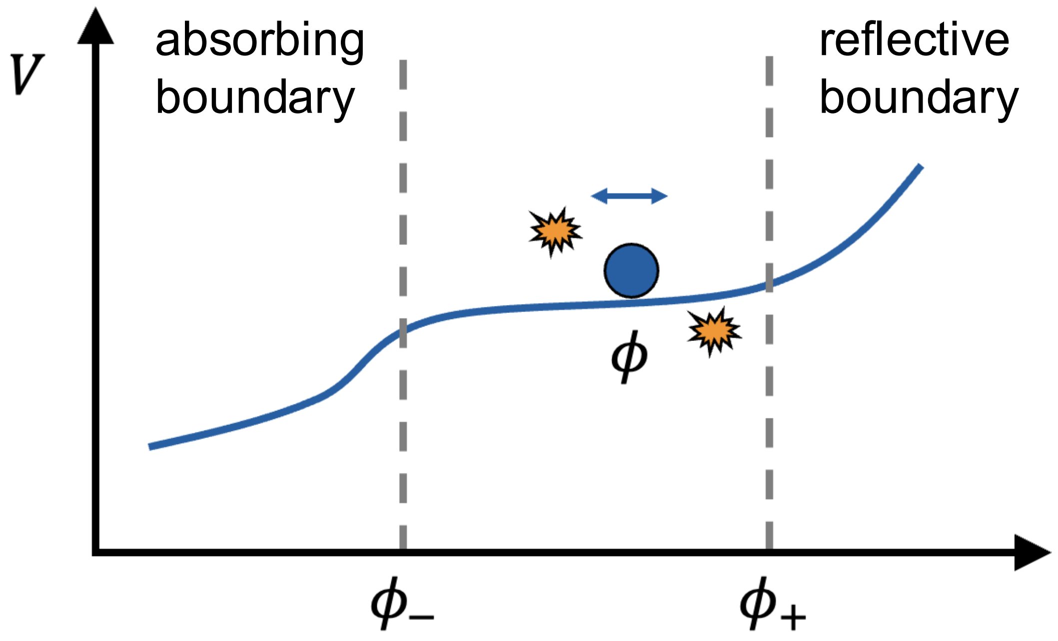
- Solves simpler equation:

$$-it\chi(\phi_-;\phi,t) = \mathcal{L}_{\mathrm{FP}}^{\dagger}\chi(\phi_-;\phi,t)$$

- Boundary conditions:

$$\chi(\phi_-;\phi_-,t) = 1$$

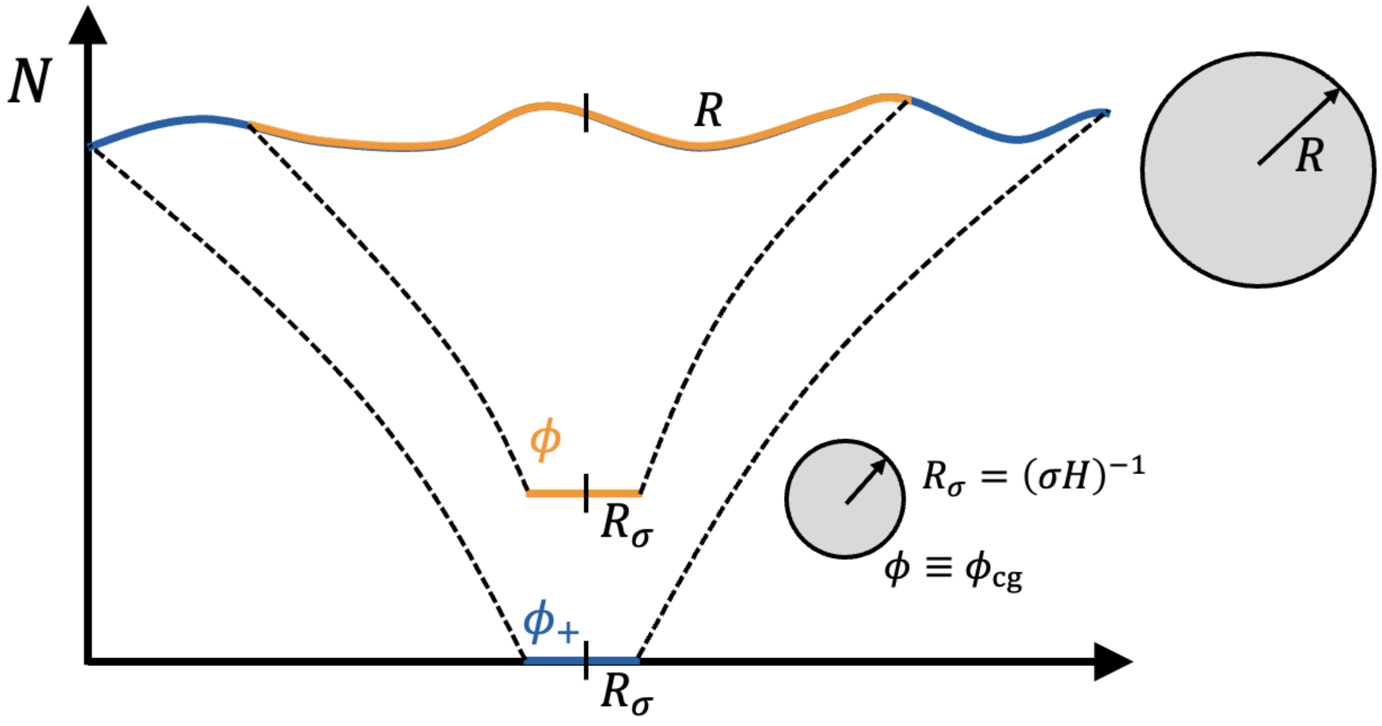
$$\partial_\phi \chi(\phi_-;\phi,t)\Big|_{\phi_+} = 0$$



Volume mapping

Stochastic δN gives statistics of $\mathcal{N}$, but observables depend on physical length scales on the end-of-inflation (EoI) hypersurface

Q. Given an initial patch of size $R_\sigma = (\sigma H)^{-1}$ associated with the coarse-grained field value $\phi_{cg} \equiv \phi$, what is the physical size R of the region it maps to at the EoI?



Volume mapping: LVA

- For a given realisation with FPT $\mathcal{N}$:

$$V_\sigma = \frac{4\pi}{3} R_\sigma^3 \quad \Rightarrow \quad V = \frac{4\pi}{3} R^3 = V_\sigma\, e^{3\mathcal{N}}$$

- *Problem:* Expansion history is stochastic, so V is stochastic
- *Solution:* **Large volume approximation** (LVA) gives a deterministic map

$$P(V|\phi) \simeq \delta\big(V - V_\sigma \langle e^{3\mathcal{N}} \rangle\big)$$

where

$$\langle e^{3\mathcal{N}} \rangle = \int d\mathcal{N}\; P_{\mathrm{FPT}}(\phi_-; \phi, \mathcal{N}) e^{3\mathcal{N}} = \chi(\phi_-; \phi, -3i)$$

- Note: LVA requires

$$(R/R_\sigma)^3 = \langle e^{3\mathcal{N}} \rangle(\phi) \gg 1$$

Volume weighting

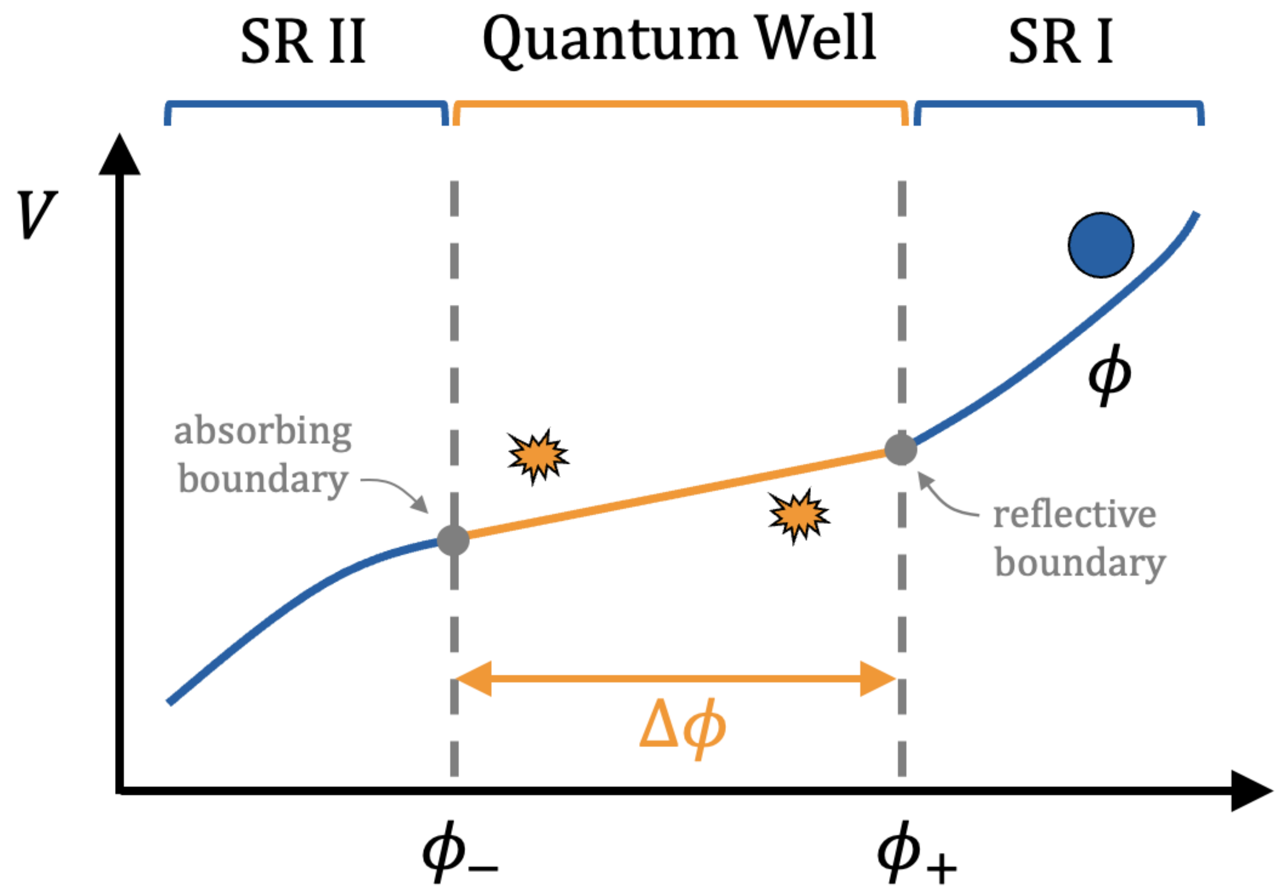
- Stochastic inflation predicts the distribution of inflationary histories for individual Hubble patches
- Physical observables are defined on the EoI hypersurface, where each trajectory is weighted by a volume factor $e^{3\mathcal{N}}$

Probability measure relevant for observables is volume-weighted version

- Define volume-weighted characteristic function & FPT distribution:

$$\chi_V(\bar{\phi};\phi,t) = \frac{\chi(\bar{\phi};\phi,t-3i)}{\chi(\bar{\phi};\phi,-3i)}, \qquad P_{\mathrm{FPT}}^{V}(\bar{\phi};\phi,\mathcal{N}) = \int \frac{dt}{2\pi}\, e^{-it\mathcal{N}} \chi_V(\bar{\phi};\phi,t)$$

Model


SR II
Quantum Well
SR I
V
ϕ
absorbing boundary
reflective boundary
$\Delta\phi$
ϕ_-
ϕ_+

Model

Two SR periods punctuated by a transient period of quantum diffusion

SR I

- Gaussian statistics, contribute to ~scale-invariant spectrum on CMB scales
- Evolution is deterministic, with well-defined mapping $\phi \leftrightarrow N \leftrightarrow k$

Quantum well

- Number of *e*-folds spent traversing the well is a stochastic variable $\mathcal{N}$
- Modes exiting during this stage have non-Gaussian statistics

SR II

- Deterministic; ratios between physical regions on EoI hypersurface established in the well are preserved

Quantum well

[Ezquiaga, García-Bellido, & Vennin, 2019]
[Animali & Vennin, 2022]

$\alpha \ll 1, \quad \alpha\Delta\phi/m_{\rm pl} \ll 1$

- Quantum well spans $\phi \in [\phi_-, \phi_+]$, with width $\Delta\phi = \phi_+ - \phi_-$
- Quantum diffusion dominates potential-driven classical drift
- Linear well

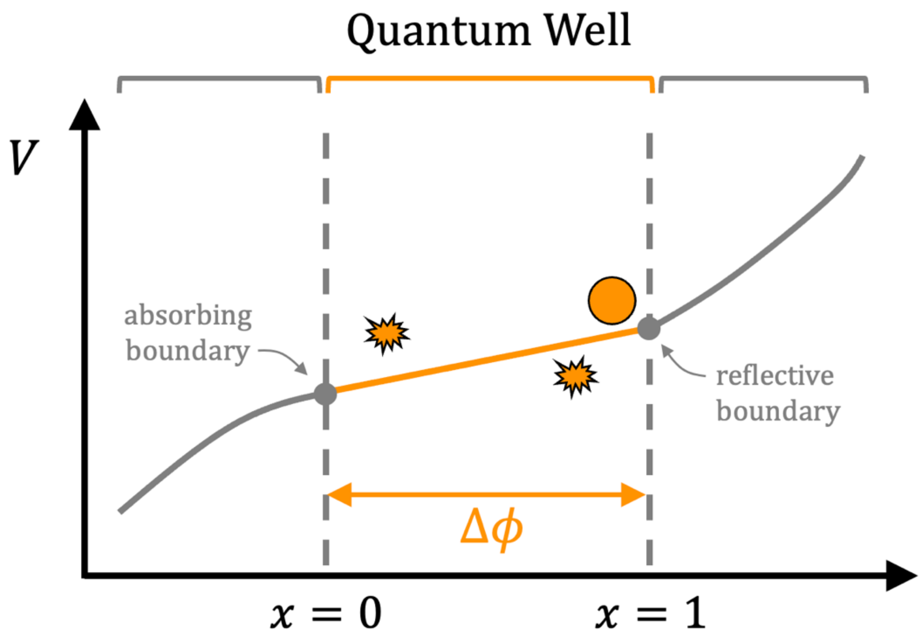
$$v(\phi) = v_0\left(1 + \alpha\frac{\phi}{m_{\rm pl}}\right), \ \mathcal{L}_{FP}^\dagger = v\partial_\phi^2 - m_{\rm pl}^2\frac{v'}{v}\partial_\phi$$

- For convenience, work with rescaled:

$$x = \frac{\phi - \phi_-}{\Delta\phi}$$

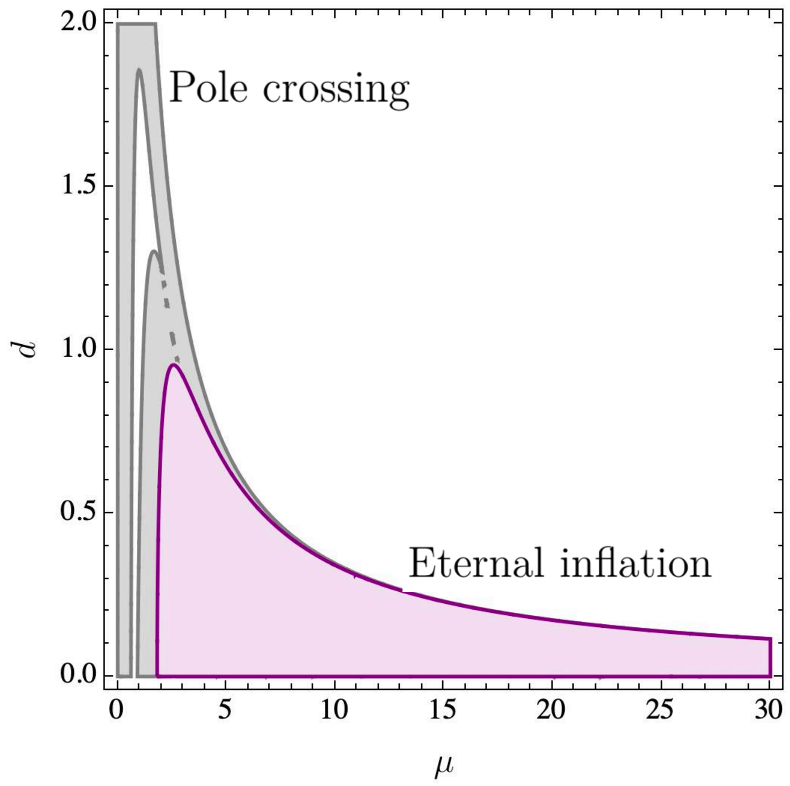
- Useful combinations:

$$\mu^2 = \frac{\Delta\phi^2}{m_{\rm pl}^2 v_0}, \ d = \frac{\alpha m_{\rm pl}}{\Delta\phi}$$



Quantum well: Parameter space

- $\mu \sim$ amplitude of quantum diffusion effects
- $d \sim$ inverse duration of inflation due to classical drift
- Limits:
  - Narrow well: $\mu^2 d \ll 1$
    $\hookrightarrow$ Recover flat well for $\mu^2 d \to 0$
  - Wide well: $\mu^2 d \gg 1$
    $\hookrightarrow$ Recover classical result for $\mu^2 d \to \infty$
- Constraints:
  - Avoid eternal inflation
  - Don't cross first pole of char. function



Statistics in the quantum well

- *Characteristic function:*

$$\chi(\bar{x};x,t)=e^{\frac{\mu^2 d(x-\bar{x})}{2}}\frac{\sqrt{4it-\mu^2d^2}\cos\left(\frac{\mu(x-1)}{2}\sqrt{4it-\mu^2d^2}\right)-d\mu\sin\left(\frac{\mu(x-1)}{2}\sqrt{4it-\mu^2d^2}\right)}{\sqrt{4it-\mu^2d^2}\cos\left(\frac{\mu(1-\bar{x})}{2}\sqrt{4it-\mu^2d^2}\right)+d\mu\sin\left(\frac{\mu(1-\bar{x})}{2}\sqrt{4it-\mu^2d^2}\right)}$$

- *Volume-weighted char. function:* $\chi_V(\bar{x};x,t)=\frac{\chi_V(\bar{x};x,t-3i)}{\chi_V(\bar{x};x,-3i)}$

- *FPT distribution:* $P_{\mathrm{FPT}}^{V}(\bar{x};x,\mathcal{N})=\int\frac{dt}{2\pi}\,e^{-it\mathcal{N}}\chi_V(\bar{x};x,t)$

- *Mean (volume-weighted) FPT to EoI:* $\langle\mathcal{N}\rangle_V(x)=-\frac{i}{\chi(0;x,-3i)}\partial_t\chi(0;x,t)|_{t=-3i}$

- *Mean expansion volume to EoI:* $\left\langle e^{3\mathcal{N}}\right\rangle(x)=\chi(0;x,-3i)$

Statistics of ζ: 1-pt function

- In LVA, choosing R on the EoI hypersurface uniquely fixes the field value x
- All worldlines ending in R share a common history up until x
- Stochastic quantity of interest: $\mathcal{N}_{1\to x}$
  ↪ Distributed according to $P^{V}_{\mathrm{FPT}}(x; 1, \mathcal{N}_{1\to x})$
- Meanwhile stochastic $\delta\mathcal{N}$ says,

$$\zeta_R = \mathcal{N}_{1\to x} - \langle \mathcal{N}_{1\to x} \rangle$$

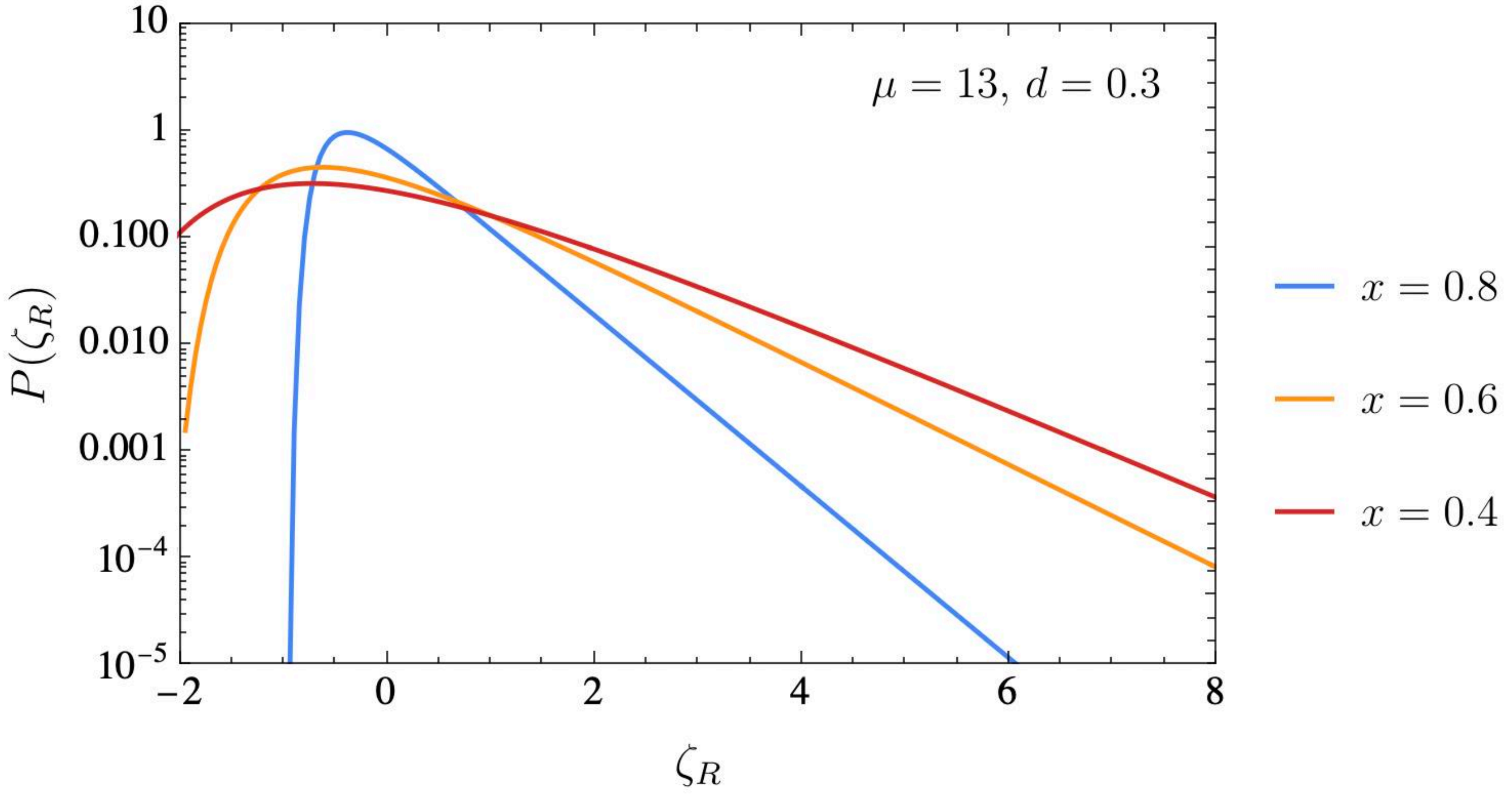
- Since $\mathcal{N}_{1\to 0} = \mathcal{N}_{1\to x} + \mathcal{N}_{x\to 0}$,

$$\mathcal{N}_{1\to x} = \zeta_R - \langle \mathcal{N} \rangle_V(x) + \langle \mathcal{N} \rangle_V(1)$$

- Thus:

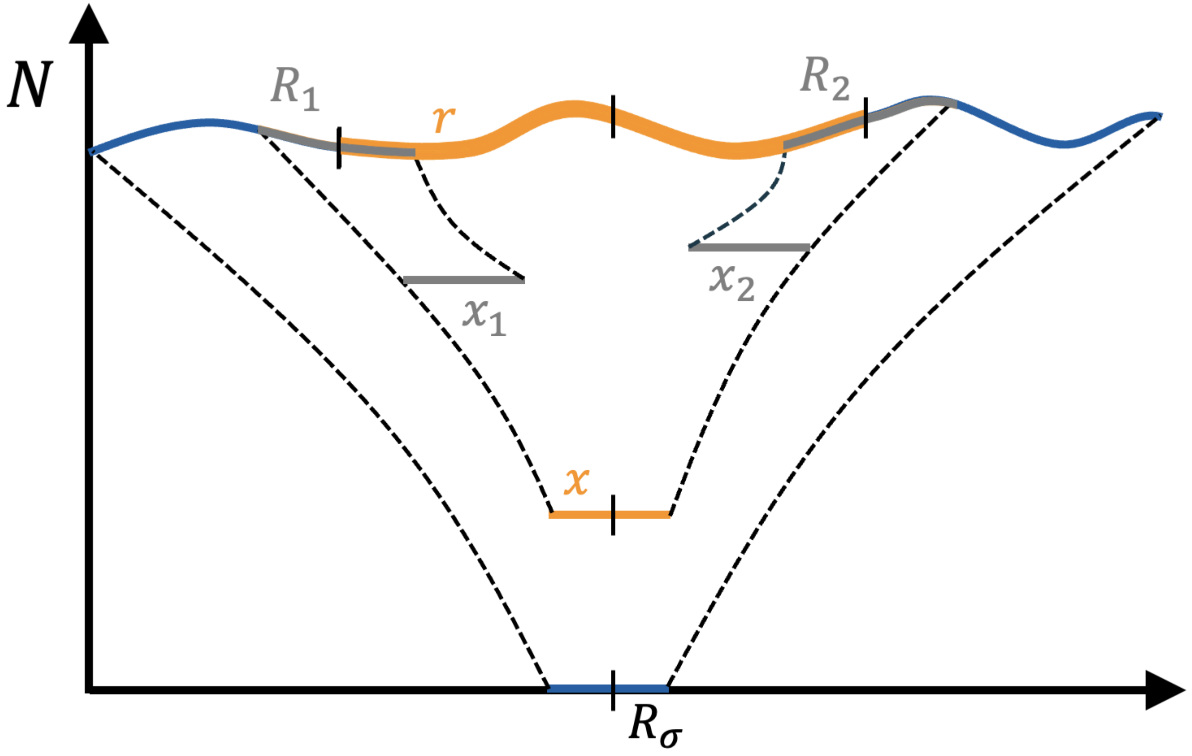
$$\boxed{P(\zeta_R) = P^{V}_{\mathrm{FPT}}\left(x; 1, 1\zeta_R - \langle \mathcal{N} \rangle_V(x) + \langle \mathcal{N} \rangle_V(1) \right)}$$

Statistics of ζ: 1-pt function

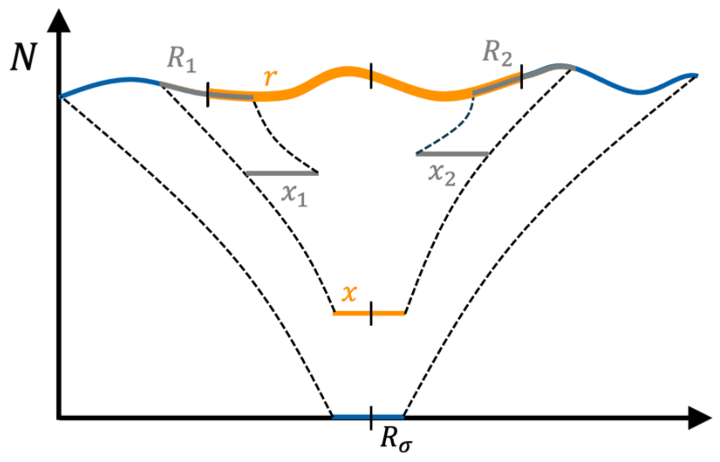


Statistics of ζ: 2-pt function

- Under LVA, each R_1, R_2 uniquely fixes the x_1, x_2 at which the corr. patch crossed R_σ
- Separation r determines a third field value x corr. to the smallest ancestor patch containing both smoothing regions
- Prior to x, regions share same stochastic history; afterwards, evolve independently



Statistics of ζ: 2-pt function



- Shared contribution $\mathcal{N}_{1\to x}$
  - Origin of correlation between ζ_{R_1}, ζ_{R_2}
  - Distributed according to $P^V_{\mathrm{FPT}}(x; 1, \mathcal{N}_{1\to x})$
- Statistically independent contributions $\mathcal{N}_{x\to x_i}$
  - Distributed according to $P^V_{\mathrm{FPT}}\left(x_i; x, \mathcal{N}_{x\to x_i}\right)$
  - From stochastic δN, $\mathcal{N}_{x\to x_i} = \zeta_{R_i} - \mathcal{N}_{1\to x} - \langle\mathcal{N}\rangle_V(x_i) + \langle\mathcal{N}\rangle_V(1)$
- Integrate product over shared common histories to obtain:

$$\boxed{\begin{aligned} P\left(\zeta_{R_1}, \zeta_{R_2}\right) = \int d\mathcal{N}_{1\to x}\, & P^V_{\mathrm{FPT}}(x; 1, \mathcal{N}_{1\to x}) \\ & \times P^V_{\mathrm{FPT}}\left(x_1; x, \zeta_{R_1} - \mathcal{N}_{1\to x} - \langle\mathcal{N}\rangle_V(x_1) + \langle\mathcal{N}\rangle_V(1)\right) \\ & \times P^V_{\mathrm{FPT}}\left(x_2; x, \zeta_{R_2} - \mathcal{N}_{1\to x} - \langle\mathcal{N}\rangle_V(x_2) + \langle\mathcal{N}\rangle_V(1)\right) \end{aligned}}$$

Statistics of PBHs

- Prob. to form PBH of mass M_i: $p_{M_i} = \int_{\zeta_c}^{\infty} d\zeta_{R_i}\, P(\zeta_{R_i})$
- Prob. to form M_1, M_2 separated by r: $p_{M_1 M_2}(r) = \int_{\zeta_c}^{\infty} d\zeta_{R_1} \int_{\zeta_c}^{\infty} d\zeta_{R_2}\, P(\zeta_{R_1}, \zeta_{R_2})$
- Reduced correlation function: $\boxed{\xi_{\mathrm{PBH}}(r) = \frac{p_{M_1 M_2}(r)}{p_{M_1} p_{M_2}} - 1}$
- In the tail, $P(\zeta_{R_1}, \zeta_{R_2}) \simeq F(r, R_1, R_2) P(\zeta_{R_1}) P(\zeta_{R_2})$ [Animali & Vennin, 2024]

$$\Rightarrow \quad \xi_{\mathrm{PBH}} \simeq F(r, R_1, R_2) - 1$$

- Note: Threshold ζ_c independent!
  ↪ *Universal clustering profile for all structures forming in the exp. tail*

Interlude: Mapping to observables

- In LVA, stochastic inflation predicts physical scales on the EoI hypersurface in terms of the coarse-graining scale

$$R^3/R_\sigma^3 = \left\langle e^{3\mathcal{N}} \right\rangle(x) = \chi(0; x, -3i)$$

- When multiple physical scales exist (e.g. R_1, R_2, r), stochastic inflation predicts ratios between them
- Trivially convert to ratios b/w comoving scales on the EoI hypersurface
- To connect with observables, need to anchor the well
  ↪ *Let k_* be the last comoving scale to leave the well*

Interlude: Mapping to observables

- Wavenumber associated with PBH $i = 1,2$:

$$k_i = k_* \, \chi(0; x_i, -3i)^{-1/3}$$

- Mass of PBH $i = 1,2$:

$$M_i = 2.8\, M_\odot \left(\frac{\gamma}{0.2}\right)\left(\frac{g_\star(T_i)}{106.75}\right)^{1/2}\left(\frac{106.75}{g_{\star,s}(T_i)}\right)^{2/3}\left(\frac{10^6\ \mathrm{Mpc}^{-1}}{k_*}\right)^{2}\chi(0; x_i, -3i)^{2/3}$$

- Comoving separation:

$$r_{\mathrm{co}} = \frac{1}{k_*}\left(2\chi(0; x, -3i)^{1/3} - \chi(0; x_1, -3i)^{1/3} - \chi(0; x_2, -3i)^{1/3}\right)$$

Reduced correlation function

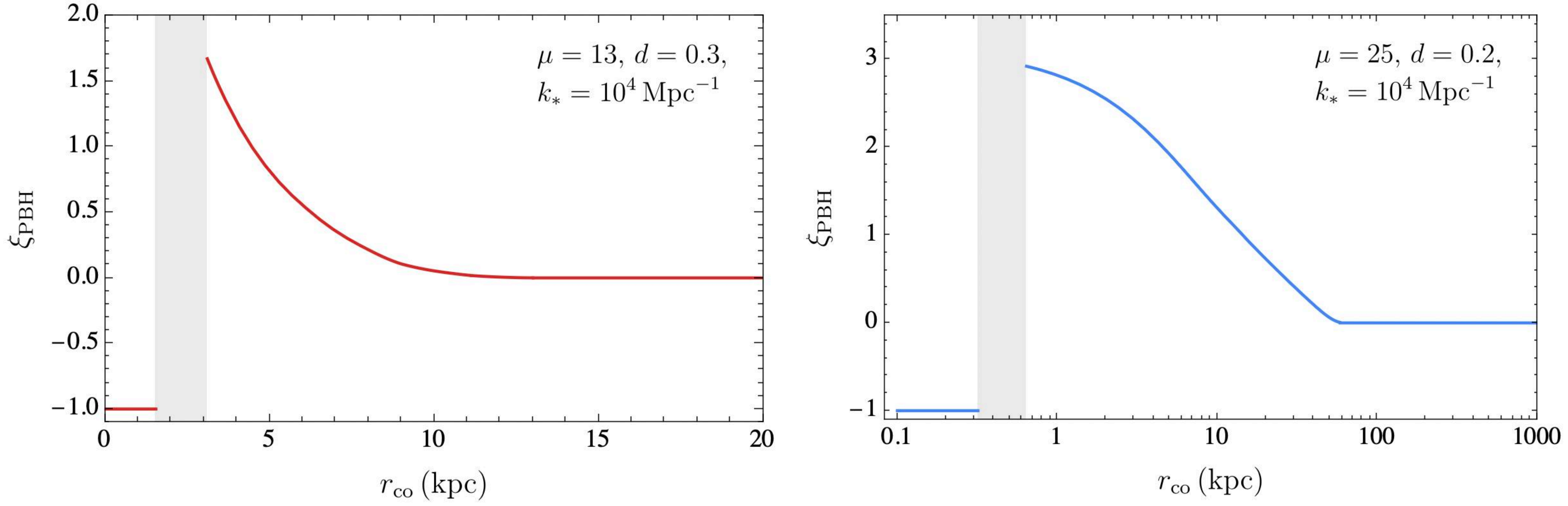
- For $\delta_{\mathrm{PBH}}(\vec{x}) = \frac{\delta\rho_{\mathrm{PBH}}(\vec{x})}{\bar{\rho}_{\mathrm{PBH}}}$: $\langle \delta_{\mathrm{PBH}}(\vec{0})\delta_{\mathrm{PBH}}(\vec{x})\rangle = \frac{1}{\bar{n}_{\mathrm{PBH}}}\delta^{(3)}(\vec{x}) + \xi_{\mathrm{PBH}}(\vec{x})$
- Recall: $\xi_{\mathrm{PBH}}(r) = \frac{p_{M_1M_2}(r)}{p_{M_1}p_{M_2}} - 1$

  ↪ In the absence of correlations, $p_{M_1M_2} = p_{M_1}p_{M_2}$ factorises and $\xi_{\mathrm{PBH}} = 0$

$\xi_{PBH}(r)$ characterises deviation from Poisson statistics

- $\xi_{\mathrm{PBH}} > 0$: Given PBH at 0, increased likelihood of finding another PBH at r
- $\xi_{\mathrm{PBH}} < 0$: Decreased likelihood
- Max correlation length set by largest common ancestor patch

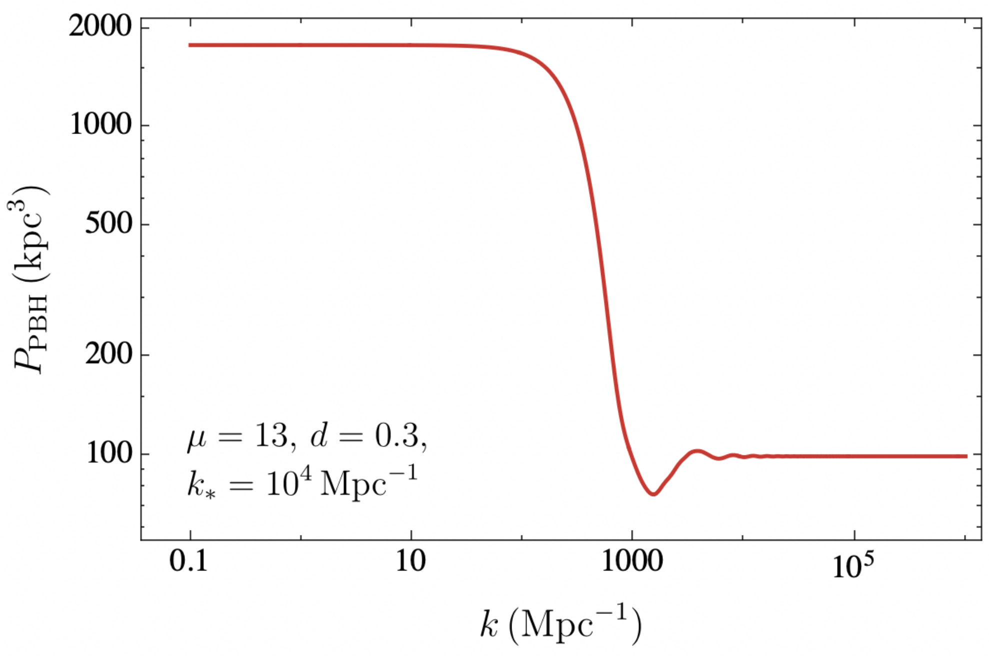
Reduced correlation function

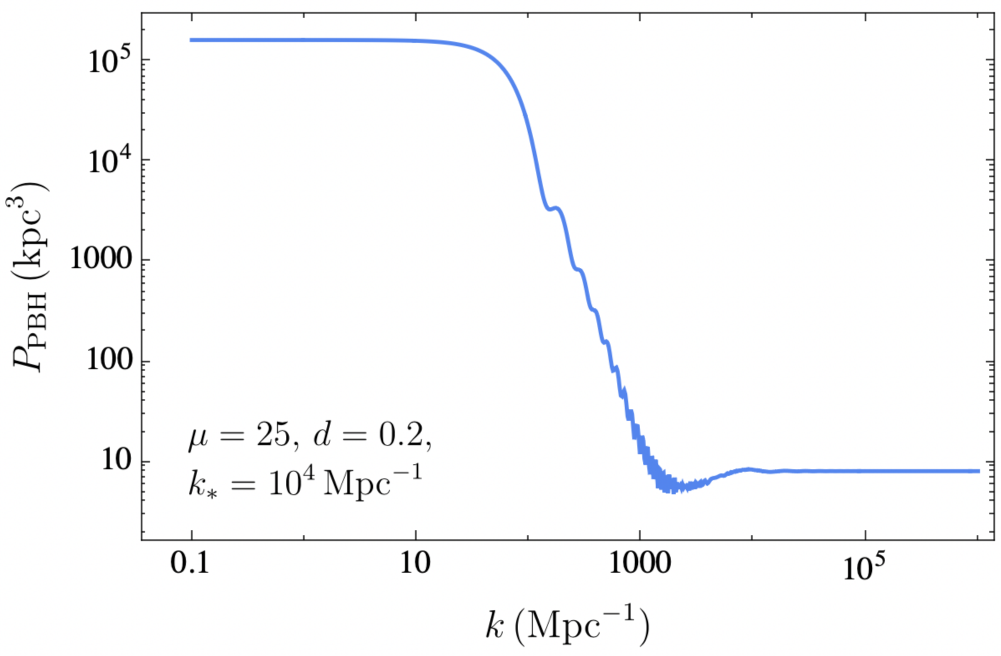


PBH power spectrum

- Define: $P_{\rm PBH}(\mathrm{k}) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} \langle \delta_{\rm PBH}(\vec{0})\delta_{\rm PBH}(\vec{x})\rangle \equiv P_{\rm Poisson} + P_\xi(\mathrm{k})$

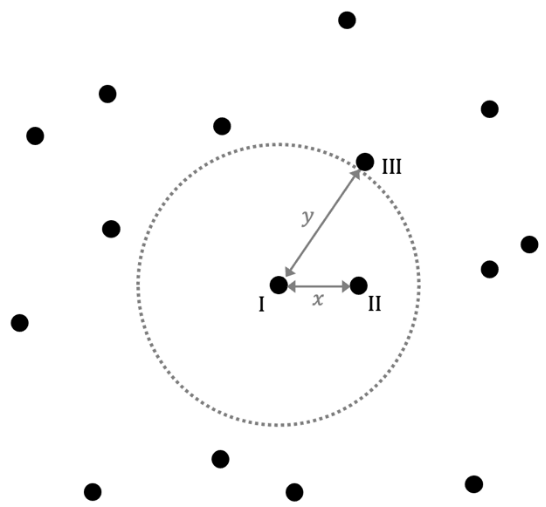
$$P_{\rm Poisson} = \frac{1}{\bar{n}_{\rm PBH}}, \quad P_\xi(k) = 4\pi \int dr\, r^2 \frac{\sin(kr)}{kr} \xi_{\rm PBH}(r)$$





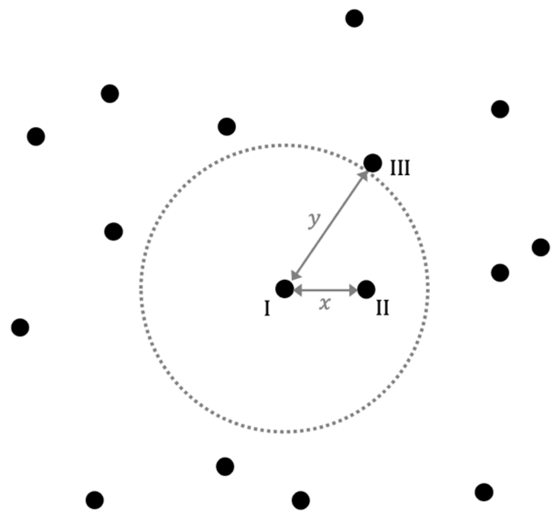
Binary formation & mergers

- Positive clustering increases likelihood of finding PBHs near one another
  ↪ Implications for binary formation & mergers?
- Consider PBHs I, II, and III
  - Step 1) Calculate prob. to produce I, II, III at comoving distances $r = 0, x,$ and y such that $0 < x \ll y$
  - Step 2) Calculate when I and II form a binary and merge (inc. angular momentum induced by III)



- Clustering modifies this calculation through increased formation rates of II and III (by factors $\xi(x)$ and $\xi(y)$)
  - More II ⇒ more binaries ⇒ larger merger rate
  - More III ⇒ larger ang. mom. ⇒ larger coalescence time (may form 3-body system)

Binary formation & mergers



- Probability that I, II are nearest neighbors (NN):

$$\Pi_{\mathrm{NN}}(x) = \bar{n}_{\mathrm{PBH}}\big(1 + \xi_{\mathrm{PBH}}(x)\big)e^{-\Gamma_{\mathrm{NN}}(x)}$$

$$\Gamma_{\mathrm{NN}}(x) = 4\pi\bar{n}_{\mathrm{PBH}} \int_0^x dr\, r^2 \left(1 + \xi_{\mathrm{PBH}}(r)\right)$$

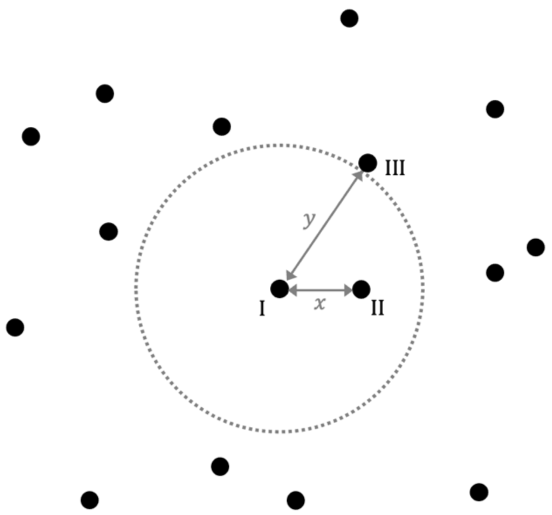
- Mean separation of NN's:

$$\langle r_{\mathrm{NN}} \rangle = \int d^3x\, x\, \Pi_{\mathrm{NN}}(x)$$

- For our benchmarks:
  - BM1: $\langle r_{\mathrm{NN}} \rangle = 2.56\ \mathrm{kpc} \rightarrow 2.22\ \mathrm{kpc}$
  - BM2: $\langle r_{\mathrm{NN}} \rangle = 0.56\ \mathrm{kpc} \rightarrow 0.44\ \mathrm{kpc}$

- (Positive) clustering shortens mean separation of NN PBHs

Binary formation & mergers



- Probability for configuration:

$$\Pi(x,y) = \Pi_{\mathrm{NN}}(x)\Pi_{\mathrm{NNN}}(x|y)$$

where cond. prob. of III at y given II at x is:

$$\Pi_{\mathrm{NNN}}(x|y) = \bar{n}_{\mathrm{PBH}}\big(1+\xi_{\mathrm{PBH}}(y)\big)e^{-\Gamma_{\mathrm{NNN}}(x,y)}\Theta(y-x)$$

$$\Gamma_{\mathrm{NNN}}(x,y) = 4\pi\bar{n}_{\mathrm{PBH}}\int_x^y dr\, r^2\left(1+\xi_{\mathrm{PBH}}(r)\right)$$

- Thus:

$$\Pi(x,y) = \bar{n}_{\mathrm{PBH}}^2\big(1+\xi_{\mathrm{PBH}}(x)\big)\big(1+\xi_{\mathrm{PBH}}(y)\big)e^{-\Gamma_{\mathrm{NN}}(y)}\Theta(y-x)$$

Binary formation & mergers

- Check whether I and II form a binary and merge
  - Condition 1) I and II must become gravitationally bound
  - Condition 2) III must not fall into the binary
- Free fall time: $\tau_{\mathrm{FF}} = \sqrt{\frac{4\pi m_{\mathrm{pl}}^2}{M_{\mathrm{PBH}}}}\,(a r_{\mathrm{co}})^{3/2}$
- During MD: $\tau_H \sim a^{3/2}$; RD: $\tau_H \sim a^2$
- Thus, $\frac{\tau_{\mathrm{FF}}}{\tau_H} \sim a^{-1/2}$ decreases only during RD $\Rightarrow$ binary formation before z_{eq}
- Decoupling time: $(1+z_{\mathrm{dec}}) = \left(\frac{r_{\max}}{r}\right)^3 (1+z_{\mathrm{eq}}),\;\; r_{\max} = \left(\frac{3}{4\pi}\frac{M_{\mathrm{PBH}}}{\rho_c^0 \Omega_m}\right)^{1/3}$
- Demand: $x < r_{\max} < y$

Binary formation & mergers

- Estimate eccentricity induced by III as: $e \simeq \sqrt{1-\left(\frac{x}{y}\right)^6}$
- Coalescence time:

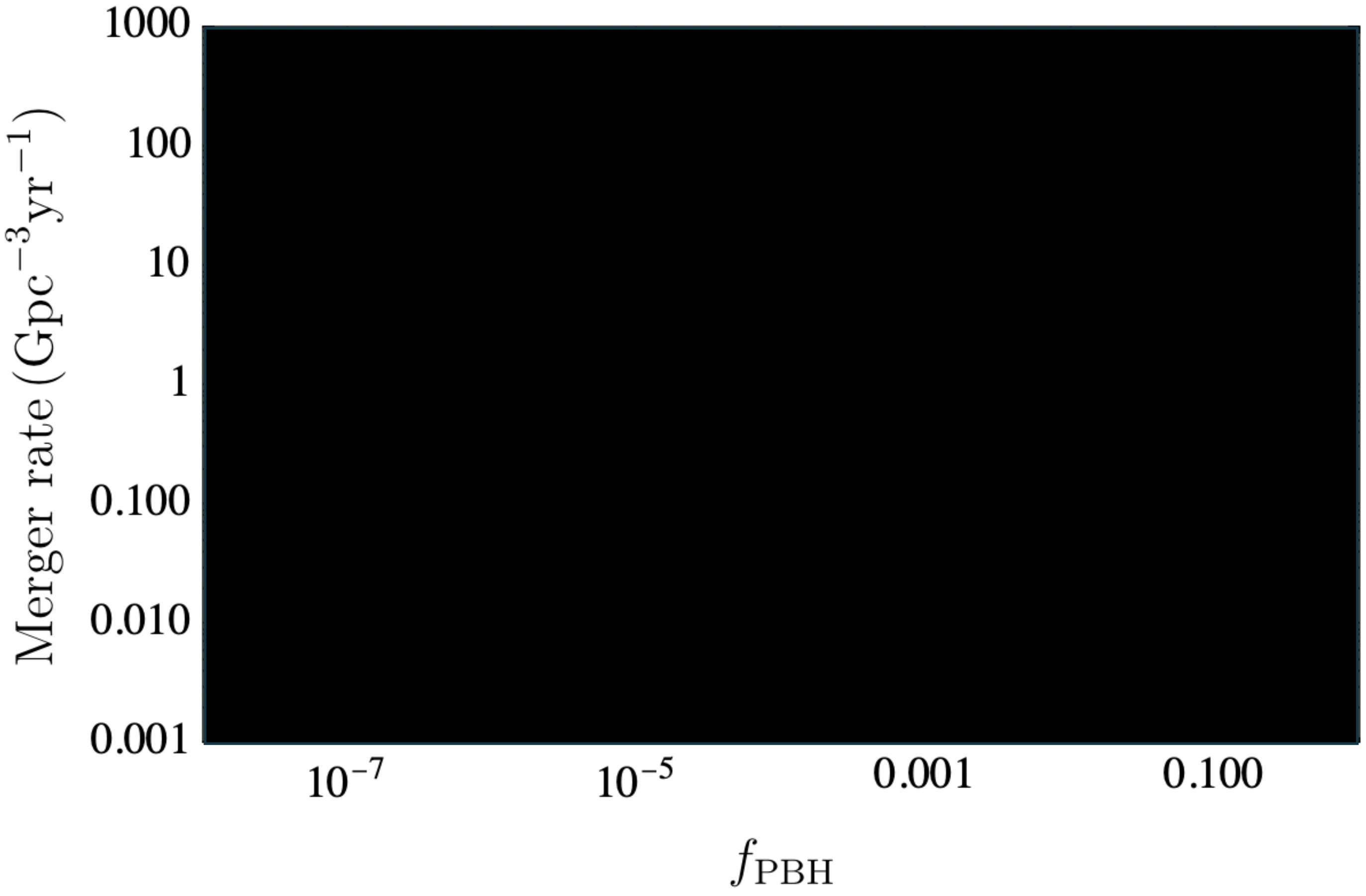
$$\tau_\mathrm{c} = \frac{x^{37}}{\left(1+z_\mathrm{eq}\right)^4 R_\mathrm{c}^3\, r_\mathrm{max}^{12}\, y^{21}} \qquad R_\mathrm{c} = \left(\frac{3}{170}\right)^{-1/3} \frac{M_\mathrm{PBH}}{8\pi m_\mathrm{pl}^2}$$

- Merger probability:

$$P_\mathrm{merge}(\mathrm{t}) = (4\pi)^2 \int_{x_t}^{r_\mathrm{max}} dx\, x^2 y_t(x)^2\, \Pi(x,y) \left(\frac{d\tau_\mathrm{c}}{dy}\right)^{-1} \Theta(\tau_\mathrm{c}(x, r_\mathrm{max}) - t)\, \Big|_{y=y_t(x)}$$

$$y_t(x) = \left(\frac{x^{37}}{\left(1+z_\mathrm{eq}\right)^4 R_\mathrm{c}^3\, r_\mathrm{max}^{12}\, t}\right)^{1/21},\ \ x_t = \left(\left(1+z_\mathrm{eq}\right)^4 R_\mathrm{c}^3\, r_\mathrm{max}^{12}\, t\right)^{1/37}$$

Binary formation & mergers



Vielen Dank!