Quantum integrable model for the quantum cohomology/K-theory of flag varieties and the double β-Grothendieck polynomials

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Gauged Linear Sigma Models 2026
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Outline

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1. Gauged linear sigma model for the flag varieties
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3. The integrable model in the Grassmannian case
4. The GL(n) asymmetric five-vertex model
5. Bethe ansatz equations and the quantum cohomology/K-theory ring of flag varieties
6. Relationship with the double β-Grothendieck polynomials

Chiral ring of GLSMs

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- For 3D theory defined on $\Sigma \times S^1$, the BPS-operators are the Wilson lines [Kapustin, Willett '13]

$$W_R = \mathrm{Tr}_R \mathrm{P} \exp\left(i \oint_{S^1} (A_\mu dx^\mu - i\sigma dl) \right),$$

  the chiral ring is the quantum K theory of X [Jockers, Mayr '18].

- If the gauge group of the GLSM is G, then the chiral ring is a quotient of the representation ring of G: $\mathcal{R}(G)/I_{\mathcal{W}}$, where $I_{\mathcal{W}}$ is determined by the effective superpotential $\mathcal{W}$ through the relations

$$\exp\left(\frac{\partial \mathcal{W}}{\partial \sigma_a}\right) = 1, \quad a = 1, \cdots, \operatorname{rank}(G).$$

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- The index and correlation functions can be computed by localization method on the Coulomb branch of the A-twisted theory [Closset, Kim '16].
- In particular, the quantum cohomology/K theory of the flag variety $\mathrm{Fl}(k_1, k_2, \cdots, k_n, N)$ is the chiral ring of the 2D/3D GLSM with gauge group $U(k_1) \times \cdots \times U(k_n)$. [Donagi, Sharpe '07],[Gu,Mihalcea,Sharpe,Xu,Zhang,Zou '23]

$$k_1 \longleftarrow k_2 \longleftarrow \cdots \longleftarrow k_n \longleftarrow \boxed{N}$$

Outline

Flag variety

Flag variety: Given a positive integer N and $\boldsymbol{k} = (k_1, k_2, \cdots, k_n)$ such that $0 < k_1 < k_2 < \cdots < k_n < N$, the flag variety $\mathrm{Fl}(\boldsymbol{k}, N)$ is defined to be the set consisting of the flags

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Tautological bundles: There is a sequence of vector bundles on $\mathrm{Fl}(\boldsymbol{k}, N)$:

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_n \subset \mathcal{O}^{\oplus N},$$

the fibre of $\mathcal{S}_i$ at a given flag $V_1 \subset V_2 \subset \cdots \subset V_n$ is V_i.

Schubert varieties: Given a fixed complete flag $F_1 \subset F_2 \subset \cdots \subset F_{N-1} \subset \mathbb{C}^N$ and $w \in S_N$ with descents in $\{k_1, k_2, \cdots, k_n\}$, the Schubert variety Ω_w is defined to be

$$\mathcal{X}_w = \{V_\bullet \in \mathrm{Fl}(\boldsymbol{k}; E) \mid \dim(V_i \cap F_p) \geq \#\{t \leq k_i : w(t) > N - p\}, \forall i, p\}.$$

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Let Ω_w denote the cohomology class dual to the Schubert variety $\mathcal{X}_w$, then the Schubert classes Ω_w form a basis of the cohomology ring $H^*(\mathrm{Fl}(\boldsymbol{k}, N), \mathbb{Z})$.

Let $\mathrm{Fl}(N) = \mathrm{Fl}(1, 2, \cdots, N-1, N)$ denote the complete flag variety. The cohomology of $\mathrm{Fl}(N)$ has the following presentation

$$H^*(\mathrm{Fl}(N)) = \mathbb{Z}[x_1, \cdots, x_N]/(e_1^N, \cdots, e_N^N),$$

where $e_i^N = e_i(x_1, \cdots, x_N)$ is the ith elementary symmetric polynomial in N variables. x_i is identified with the Chern class $c_1(\mathcal{S}_i/\mathcal{S}_{i-1})$ and the Schubert class Ω_w is represented by the **Schubert polynomial** $X_w(x_1, \cdots, x_{N-1})$.

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For general $\boldsymbol{k}$, Ω_w is represented by the Schubert polynomial $X_w(x_1, \cdots, x_{N-1})$ symmetric in $\{x_{a_{p-1}+1}, \cdots, x_{a_p}\}$ for all $1 \leq p \leq l+1$.

Classical cohomology:

$$\Omega_u \cdot \Omega_v = \sum_{w \in S_N} C_{uv}^{w} \Omega_w^{\vee},$$

where C_{uv}^{w} is the intersection number of the Schubert varieties indexed by u, v, w such that $l(u) + l(v) + l(w) = \dim \mathrm{Fl}(\boldsymbol{k}, N)$.

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The (small) quantum cohomology ring is a deformation of the classical cohomology ring

$$\Omega_u * \Omega_v = \sum_{w \in S_N} \sum_{d \in H_2(\mathrm{Fl}(\boldsymbol{k},N),\mathbb{Z})} q^d \langle \Omega_u, \Omega_v, \Omega_w \rangle_d \Omega_w^{\vee},$$

where $\langle \Omega_u, \Omega_v, \Omega_w \rangle_d$ is the 3-point, genus-zero Gromov-Witten invariant of degree d.

Quantum K-theory

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The set of the structure sheaves of the Schubert varieties $\{\mathcal{O}_{\mathcal{X}_w} \equiv \mathcal{O}_w | w \in S_N(\boldsymbol{k})\}$ constitute a basis of $K(\mathrm{Fl}(\boldsymbol{k}, N))$. These Schubert classes are represented by the **Grothendieck polynomials** $G_w(x_1, \cdots, x_n)$ symmetric in $\{x_{a_{p-1}+1}, \cdots, x_{a_p}\}$ for all $1 \leq p \leq l+1$.

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Let $\langle \mathcal{O}_{w_1},\cdots,\mathcal{O}_{w_m}\rangle_d$ denote the K-theoretic Gromov-Witten invariant. The quantum K-theory ring is a deformation of the K-theory ring:

$$(\mathcal{O}_u * \mathcal{O}_v, \mathcal{O}_w) = \sum_{d \in H_2(\mathrm{Fl}(\boldsymbol{k},N),\mathbb{Z})} q^d \langle \mathcal{O}_u, \mathcal{O}_v, \mathcal{O}_w \rangle_d,$$

where the pairing is defined by $(\mathcal{O}_u, \mathcal{O}_v) = \sum_d q^d \langle \mathcal{O}_u, \mathcal{O}_v\rangle_d$, i.e. the 2-point correlation function.

Equivariant quantum cohomology/K-theory

Given the $T = (\mathbb{C}^*)^N$ action on $E = \mathbb{C}^N$ with weights $(t_1, \cdots, t_N)$, one can define the equivariant quantum cohomology ring $QH_T^*(\mathrm{Fl}(\boldsymbol{k}, N))$ and the equivariant quantum K-theory ring $QK_T(\mathrm{Fl}(\boldsymbol{k}, N))$.

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The representatives of the Schubert classes in the equivariant cohomology and K-theory rings are given by the **double Schubert polynomials** $X_w(x_1, \cdots, x_N; t_1, \cdots, t_N)$ and the **double Grothendieck polynomials** $G_w(x_1, \cdots, x_N; t_1, \cdots, t_N)$ respectively.

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The double Schubert and double Grothendieck polynomials are special cases of a one-parameter family of polynomials called the **double β-Grothedieck polynomials** $\mathcal{G}_w^{(\beta)}, w \in S_N$ [Fomin, Kirillov '93].

Outline

Integrable model in the Grassmannian case

The quantum integrable model for the quantum cohomology and quantum K-theory ring of Grassmannians has R-matrix [Gorbounov, Korff '14]

$$R(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1+\beta(y \ominus x) & y \ominus x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x \ominus y = \frac{x-y}{1+\beta y}$, which satisfies the Yang-Baxter equation

$$R_{12}(x,y)R_{13}(x,z)R_{23}(y,z) = R_{23}(y,z)R_{13}(x,z)R_{12}(x,y).$$

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The lax operator is defined by

$$L(x,t) = \begin{pmatrix} \sigma^+\sigma^- + (x \ominus t)\sigma^-\sigma^+ & (1+\beta(x \ominus t))\sigma^+ \\ \sigma^- & \sigma^-\sigma^+ \end{pmatrix}$$

with each entry acting on $\mathbb{C}^2$.

For an auxiliary $V_a = \mathbb{C}^2$, the monodromy matrix acting on $V_a \otimes (\mathbb{C}^2)^{\otimes N}$ is

$$T_a(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} = L_{aN}(x, t_N) \cdots L_{a2}(x, t_2) L_{a1}(x, t_1),$$

which satisfies the RTT relation

$$R_{12}(x, y) T_1(x) T_2(y) = T_2(y) T_1(x) R_{12}(x, y).$$

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The transfer matrix is defined by

$$t(x) = \mathrm{Tr}_a \left(\mathrm{diag}(1, q) T_a(x) \right) = A(x) + q D(x),$$

which satisfies the commutation relation

$$[t(x), t(y)] = 0.$$

Let $v_0 = (1,0)^{\mathrm{T}}, v_1 = (0,1)^{\mathrm{T}}$ and define the pseudo vacuum $|\Omega\rangle = v_0 \otimes \cdots \otimes v_0$, then the Bethe ansatz state

$$B(x_1)B(x_2)\cdots B(x_k)|\Omega\rangle$$

is a common eigenstate of $t(x)$ if $x_1, \cdots, x_k$ satisfy the Bethe ansatz equations

$$\prod_{b=1}^{k}(1+\beta x_b)\prod_{i=1}^{N}(x_a \ominus t_i) = (-1)^{k-1}q(1+\beta x_a)^k, \quad a = 1, \cdots, k.$$

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$\beta = 0$: $e_i(x_1, \cdots, x_k) \sim c_i(\mathcal{S}) \Rightarrow$ ring relations of the equivariant quantum cohomology of the Grassmannian $Gr(k,N)$.
$\beta = -1$: $e_i(x_1, \cdots, x_k) \sim \wedge^i \mathcal{S} \Rightarrow$ ring relations of the equivariant quantum K-theory ring of the Grassmannian $Gr(k,N)$.

- The expansion coefficients of the Bethe state in the spin basis are the factorial β-Grothendieck polynomials (double β-Grothendieck polynomials corresponding to permutations with one descent), which can also be computed by the partition functions of the corresponding 2D lattice model [Motegi, Sakai '13].

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- Integrable systems for the quantum cohomology/K-theory ring of the cotangent bundles of flag varieties had been constructed, and the fibre decoupls in certain limits [Givental, Kim '93][Givental, Lee '01][Koroteev, Pushkar, Smirnov, Zeitlin '17].

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- Double β-Grothendieck polynomials can also be obtained from 2D lattice models [Brubaker, Frechette, Hardt, Tibor, Weber, '20].
- Question: What is the quantum integrable model for general flag varieties and general double β-Grothendieck polynomials?

Outline

The R-matrix

Let $V = \mathbb{C}^n$. Under the decomposition $V = \mathbb{C} \oplus \mathbb{C}^{n-1}$, the R-matrix of the $\mathrm{GL}(n)$ asymmetric five-vertex model $R^{(n)}(x, y) \in \mathrm{End}(V \otimes V)$ can be defined recursively as

$$R^{(2)}(x, y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 + \beta(x \ominus y) & x \ominus y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

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and

$$R^{(n)}(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & I_{n-1} & 0 \\ 0 & (1+\beta(x \ominus y))I_{n-1} & (x \ominus y)I_{n-1} & 0 \\ 0 & 0 & 0 & R^{(n-1)}(x,y) \end{pmatrix},$$

for $n > 2$.

Monodromy matrix

Take the auxiliary space to be $V_a = \mathbb{C} \oplus \mathbb{C}^{n-1}$, and define the monodromy matrix as

$$\begin{aligned} T_a^{(n)}(u) &= R_{aN}^{(n)}(u, t_N) \cdots R_{a2}^{(n)}(u, t_2) R_{a1}^{(n)}(u, t_1) \\ &= \begin{pmatrix} A(u) & B_1(u) & \cdots & B_{n-1}(u) \\ C_1(u) & D_{11}(u) & \cdots & D_{1,n-1}(u) \\ \vdots & \vdots & \ddots & \vdots \\ C_{n-1}(u) & D_{n-1,1}(u) & \cdots & D_{n-1,n-1}(u) \end{pmatrix} \end{aligned}$$

with $A(u), B_i(u), C_i(u), D_{ij}(u)$ acting on $\mathcal{H} = V_1 \otimes V_2 \otimes \cdots \otimes V_N$ ($V_l \cong \mathbb{C}^n$).

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with $A(u), B_i(u), C_i(u), D_{ij}(u)$ acting on $\mathcal{H} = V_1 \otimes V_2 \otimes \cdots \otimes V_N$ ($V_l \cong \mathbb{C}^n$).

Pseudo vacuum: $|\Omega^{(0)}\rangle = v_0 \otimes \cdots \otimes v_0$, $v_0 = (1, 0, \cdots, 0)^{\mathrm{T}}$.

$$A(u)|\Omega^{(0)}\rangle = |\Omega^{(0)}\rangle, \quad D_{ij}(u)|\Omega^{(0)}\rangle = \delta_{ij} \prod_{k=1}^{N} (u \ominus t_k)|\Omega^{(0)}\rangle.$$

Commutation relations and transfer matrix

The RTT relation yields the following commutation relations:

$$A(x)B_i(y) = \frac{1}{y \ominus x}\left[B_i(y)A(x) - B_i(x)A(y)\right],$$

$$D_{ij}(x)B_k(y) = \sum_{s,l}\left[\frac{1}{x \ominus y}B_s(y)D_{il}(x)R^{(n-1)}_{ls,jk}(x,y)\right] + \frac{1}{y \ominus x}B_j(x)D_{ik}(y),$$

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\end{aligned}
$$

Let $M = \mathrm{diag}(b_0, b_1, \cdots, b_{n-1})$, and define the transfer matrix

$$
t^{(n)}(u) = \mathrm{Tr}_a(MT^{(n)}_a(u)).
$$

We have $[t^{(n)}(u), t^{(n)}(v)] = 0$.

Outline

Bethe ansatz equations

The eigenvalues and common eigenstates of $t^{(n)}(u)$ can be solved by the algebraic Bethe ansatz method for the GL(n) systems [Kulish, Reshetikhin '83].

Take the Bethe ansatz state to be of the form

$$|\psi^{(0)}\rangle = \sum_{i_1,\cdots,i_{k_1}} \psi^{(1)}_{i_1\cdots i_{k_1}}(\sigma^{(2)}_{\alpha_2},\cdots,\sigma^{(n-1)}_{\alpha_{n-1}})B_{i_1}(\sigma^{(1)}_1)\cdots B_{i_k}(\sigma^{(1)}_{k_1})|\Omega^{(0)}\rangle,$$

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where $\sigma^{(s)}_{\alpha_s}$ are parameters to be determined, $\alpha_s = 1,2,\cdots,k_s$.
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$\psi^{(1)}_{i_1\cdots i_{k_1}}$ are the expansion coefficients of the Bethe state of a GL$(n-1)$ system with k_1 sites:

$$\psi^{(1)}_{i_1\cdots i_{k_1}} = \langle i_1 i_2 \cdots i_{k_1}|\psi^{(1)}\rangle\,, \quad i_s = 1,\cdots,n-1.$$

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$$\prod_{a=1}^{k_1}(1+\beta(\sigma_a^{(1)}\ominus\sigma_{\alpha_1}^{(1)}))\prod_{i=1}^{N}(\sigma_{\alpha_1}^{(1)}\ominus t_i)=(-1)^{k_1-1}q_1\prod_{b=1}^{k_2}(\sigma_b^{(2)}\ominus\sigma_{\alpha_1}^{(1)}),$$

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where $m=2,\cdots,n-2,\quad \alpha_s=1,\cdots,k_s,\quad q_s:=b_{s-1}/b_s$, $s=1,\cdots,n-1$.

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When $\beta=0$ ($\beta=-1$), $0<k_{n-1}<\cdots<k_2<k_1<N$, these equations reduce to the equations for the Coulomb vacua of the 2D (3D) GLSMs.

When $\beta = 0$, The BAE reduces to

$$\prod_{\alpha=1}^{k_{m-1}} (\sigma_a^{(m)} - \sigma_\alpha^{(m-1)}) = (-1)^{k_m - 1} q_m \prod_{\gamma=1}^{k_{m+1}} (\sigma_\gamma^{(m+1)} - \sigma_a^{(m)})$$

for $a = 1, \cdots, k_m, m = 1, 2, \cdots, n-1$, where $\sigma_i^{(0)} = t_i, k_0 = N, k_n = 0$.

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$\sigma_a^{(m)}, a = 1, \cdots, k_m$, are roots of the polynomial

$$\sum_{i=0}^{k_{m-1}} (-1)^i x^{k_{m-1}-i} e_i(\sigma^{(m-1)}) + (-1)^{k_m - k_{m+1}} q_m \sum_{j=0}^{k_{m+1}} (-1)^j x^{k_{m+1}-j} e_j(\sigma^{(m+1)}),$$

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where $e_i(\sigma^{(s)})$ is the ith elementary symmetric polynomial in the variables $\sigma_a^{(s)}$.
Assume the other $k_{m-1} - k_m$ roots are $\eta_b^{(m)}, b = 1, \cdots, k_{m-1} - k_m$, then Vieta's formula yields

$$\sum_{i=0}^{l} e_i(\sigma^{(m)}) e_{l-i}(\eta^{(m)}) = e_l(\sigma^{(m-1)}) + (-1)^{k_{m-1}-k_m} q_m e_{k_{m+1}-k_{m-1}+l}(\sigma^{(m+1)}).$$

If $e_i(\sigma^{(m)}) = c_i(\mathcal{S}_{n-m})$, $e_i(\eta^{(m)}) = c_i(\mathcal{S}_{n-m+1}/\mathcal{S}_{n-m})$, then

$$c(\mathcal{S}_i) \cdot c(\mathcal{S}_{i+1}/\mathcal{S}_i) = c(\mathcal{S}_{i+1}) + (-1)^{k_{n-i-1}-k_{n-i}} q_{n-i} c(\mathcal{S}_{i-1}),$$

for $i = 1, 2, \cdots, n-1$, which are the quantum Whitney relations of the equivariant quantum cohomology ring $QH_T^*(\mathrm{Fl}(k_{n-1}, \cdots, k_2, k_1, N))$.

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Similarly, for $\beta = -1$, if $e_i(1-\sigma_1^{(m)}, \cdots, 1-\sigma_{k_m}^{(m)}) = \wedge^i(\mathcal{S}_{n-m})$, then

$$\lambda_y(\mathcal{S}_i) * \lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i) =$$
$$\lambda_y(\mathcal{S}_{i+1}) + \frac{q_{n-i}}{1-q_{n-i}} y^{k_{n-i-1}-k_{n-i}} \left(\lambda_y(\mathcal{S}_{i-1}) - \lambda_y(\mathcal{S}_i)\right) * \det(\mathcal{S}_{i+1}/\mathcal{S}_i)$$

for $i = 1, 2, \cdots, n-1$, which are the quantum Whitney relations of the equivariant quantum K-theory ring $QK_T(\mathrm{Fl}(k_{n-1}, \cdots, k_2, k_1, N))$ proposed in [Gu, Mihalcea, Sharpe, Xu, Zhang, Zou, '23] and proved in [Huq-Kuruvilla, '24], where the λ_y-class is defined by

$$\lambda_y(\mathcal{E}) = \sum_{i=0}^{\mathrm{rank}(\mathcal{E})} y^i \wedge^i \mathcal{E}.$$

Outline

Double β-Grothendieck polynomials

Define the β**-divided difference operator** ∂_i^β acting on $\mathbb{Z}[x_1, \cdots, x_n]$ by

$$\partial_i^\beta f(x_1, \cdots, x_n) = \frac{(1 + \beta x_{i+1}) f(x_1, \cdots, x_n) - (1 + \beta x_i) f(x_1, \cdots, x_{i+1}, x_i, \cdots, x_n)}{x_i - x_{i+1}}.$$

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If $w = \omega_N$ is the permutation in S_N with maximal length, then

$$\mathcal{G}_{\omega_N}^{(\beta)}(\boldsymbol{x}; \boldsymbol{y}) = \prod_{i+j \leq N} (x_i + y_j + \beta x_i y_j),$$

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In particular, $\mathcal{G}_w^{(0)} = X_w$ (Schubert polynomial),
$\mathcal{G}_w^{(-1)} = G_w$ (Grothendieck polynomial).

In the case of complete flag variety, one can show by induction

Lemma 1

If $\pi \in S_N$ and $l(s_i\pi) > l(\pi)$, then

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\begin{aligned}
&(\langle s_i\pi| + \beta \langle\pi|)B_{N-1}(\sigma_1)B_{N-2}(\sigma_2)\cdots B_{i+1}(\sigma_{N-i-1})B_i(\sigma_{N-i})B_{i-1}(\sigma_{N-i+1}) \\
=&\langle\pi| B_{N-1}(\sigma_1)B_{N-2}(\sigma_2)\cdots B_{i+1}(\sigma_{N-i-1})B_{i-1}(\sigma_{N-i})B_i(\sigma_{N-i+1}),
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On the other hand, we have the commutation relation

$$
\begin{aligned}
B_a(x_i)B_b(x_{i+1}) &= (1+\beta x_i)\frac{B_b(x_i)B_a(x_{i+1}) - B_b(x_{i+1})B_a(x_i)}{x_i - x_{i+1}} \\
&= (\partial_i^\beta + \beta)B_b(x_i)B_a(x_{i+1})
\end{aligned}
$$

for $a < b$.

Lemma 2

For $\pi \in S_N$, if $l(s_i\pi) > l(\pi)$, then

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Proof:

$$\begin{aligned}&(\langle s_i\pi| + \beta\,\langle\pi|)B_{N-1}(\sigma_1)\cdots B_i(\sigma_{N-i})B_{i-1}(\sigma_{N-i+1})\\ =&\,\langle\pi|\, B_{N-1}(\sigma_1)\cdots B_{i-1}(\sigma_{N-i})B_i(\sigma_{N-i+1})\\ =&(\partial^{\beta}_{N-i} + \beta)\,\langle\pi|\, B_{N-1}(\sigma_1)\cdots B_i(\sigma_{N-i})B_{i-1}(\sigma_{N-i+1}).\end{aligned}$$

Theorem 1

$$B_{N-1}(\sigma_1)B_{N-2}(\sigma_2)\cdots B_1(\sigma_{N-1})\,|\Omega^{(0)}\rangle = \sum_{w\in S_N} \mathcal{G}_w^{(\beta)}(\boldsymbol{\sigma};\ominus\boldsymbol{t})\,|\omega_N w^{-1}\rangle\,,$$

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If we write $\omega_N w^{-1} = \pi\rho_k$ for $\pi\in S_{N-1}$, then one can use Lemma 2 and the recursive definition of the double β-Grothendieck polynomials to complete the proof by induction on the length of π.

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Lemma 3

For any $w \in S_{N-1}$,

$$B_{w(1)}(\sigma_1) \cdots B_{w(N-1)}(\sigma_{N-1}) = \bar{\partial}^{\beta}_{w^{-1}\omega_{N-1}} B_{N-1}(\sigma_1) \cdots B_1(\sigma_{N-1}).$$

From the generalized Cauchy identity [Brubaker, Frechette, Hardt, Tibor, Weber '20]

$$\mathcal{G}_w^{(\beta)}(\boldsymbol{x};\ominus\boldsymbol{t}) = \sum_{v\in S_N} \mathcal{G}_v^{(\beta)}(\boldsymbol{x};\ominus\boldsymbol{z})\bar{\partial}_v^{\beta}\mathcal{G}_w^{(\beta)}(\boldsymbol{z};\ominus\boldsymbol{t}),$$

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Lemma 4

For $\pi \in S_n$, if $e_l(\sigma_1,\cdots,\sigma_{n-1})$ is identified with $e_l(x_1,\cdots,x_{n-1})$ for every $l = 1,\cdots,n-1$, then

$$\sum_{w\in S_{n-1}} \mathcal{G}_w^{(\beta)}(x_1,\cdots,x_{n-2};\ominus\sigma_1,\cdots,\ominus\sigma_{n-2})\cdot\bar{\partial}_w^{\beta}\mathcal{G}_\pi^{(\beta)}(\sigma_1,\cdots,\sigma_{n-1};\ominus t_1,\cdots,\ominus t_{n-1})$$
$$= \mathcal{G}_\pi^{(\beta)}(x_1,\cdots,x_{n-1};\ominus t_1,\cdots,\ominus t_{n-1}).$$

Theorem 2

When $e_l(\sigma_1^{(i)}, \cdots, \sigma_{N-i}^{(i)})$ is identified with $e_l(x_1, \cdots, x_{N-i})$ for all $i = 1, \cdots, N-1$ and $l = 1, \cdots, N-i$, the Bethe ansatz state has the following expansion

$$\begin{aligned} |\psi^{(0)}\rangle &= \sum_{i_1, \cdots, i_{k_1}} \psi^{(1)}_{i_1 \cdots i_{k_1}}(\sigma^{(2)}_{\alpha_2}, \cdots, \sigma^{(n-1)}_{\alpha_{n-1}}) B_{i_1}(\sigma^{(1)}_1) \cdots B_{i_k}(\sigma^{(1)}_{k_1}) |\Omega^{(0)}\rangle \\ &= \sum_{w \in S_N} \mathcal{G}^{(\beta)}_w(x_1, \cdots, x_{N-1}; \ominus t_1, \cdots, \ominus t_{N-1}) \left| \omega_N w^{-1} \right\rangle . \end{aligned}$$

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\end{aligned}
$$

$\sigma_a^{(N-i)}$ ($\beta = 0$) or $1 - \sigma_a^{(N-i)}$ ($\beta = -1$), $a = 1, \cdots, i$, are interpreted as the Chern roots of $\mathcal{S}_i$. Therefore, x_i should be interpreted as the first Chern class of $\mathcal{S}_i/\mathcal{S}_{i-1}$ when $\beta = 0$ in the cohomology ring, and $1 - x_i$ should be interpreted as the K-theory class of $\mathcal{S}_i/\mathcal{S}_{i-1}$ when $\beta = -1$ in the K-theory ring.

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$$\langle \omega_m w_m^{-1}|\psi^{(N-m)}\rangle = \mathcal{G}^{(\beta)}_{w_m}(x_1, \cdots, x_{m-1}; \ominus\sigma_1^{(N-m)}, \cdots, \ominus\sigma_{m-1}^{(N-m)})$$

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$$\langle \omega_m w_m^{-1}|\psi^{(N-m)}\rangle = \mathcal{G}^{(\beta)}_{w_m}(x_1, \cdots, x_{m-1}; \ominus\sigma_1^{(N-m)}, \cdots, \ominus\sigma_{m-1}^{(N-m)})$$

yield

$$\begin{aligned}
&\langle \omega_{m+1} w_{m+1}^{-1}|\psi^{(N-m-1)}\rangle \\
=&\sum_{v_m \in S_m} \Big[\mathcal{G}^{(\beta)}_{v_m^{-1}\omega_m}(x_1, \cdots, x_{m-1}; \ominus\sigma_1^{(N-m)}, \cdots, \ominus\sigma_{m-1}^{(N-m)}) \\
&\qquad \cdot \bar{\partial}^{\beta}_{v_m^{-1}\omega_m} \mathcal{G}^{(\beta)}_{w_{m+1}}(\sigma_1^{(N-m)}, \cdots, \sigma_m^{(N-m)}; \ominus\sigma_1^{(N-m-1)}, \cdots, \ominus\sigma_m^{(N-m-1)})\Big] \\
=&\mathcal{G}^{(\beta)}_{w_{m+1}}(x_1, \cdots, x_m; \ominus\sigma_1^{(N-m-1)}, \cdots, \ominus\sigma_m^{(N-m-1)})
\end{aligned}$$

from Lemma 3 and Lemma 4.

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- Bethe/Gauge correspondence at the level of Frobenius algebras.
- q-deformation.
- Other types of boundary conditions.

Thank you for your attention!