

Grade Restriction Rule In Non-Abelian Gauged Linear Sigma Models : The Latest Update

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Based on a joint work with

Richard Eager, Johanna Knapp and Mauricio Romo

Røddland model

H-Tong 2006

$$G = U(2)$$

$$V = (\det^{-1})^{\oplus 7} \oplus (\mathbb{C}^2)^{\oplus 7} \ni (p^1, \dots, p^7, x_1, \dots, x_7)$$

$$W = \sum_{i,j,k} A_k^{ij} p^k [x_i x_j]$$

$$[x_i x_j] := x_i^1 x_j^2 - x_i^2 x_j^1,$$

$$\text{write } W = \sum_{ij} \underbrace{A^{ij}(p)}_{\text{mass matrix}} [x_i x_j] = \sum_k p^k A_k(x)$$

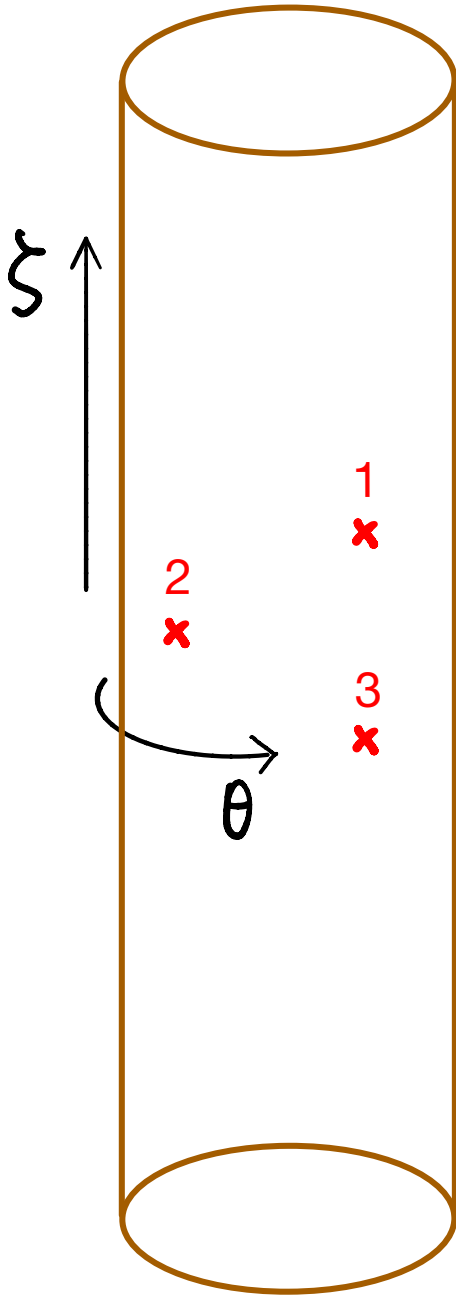
$\exists$ Coulomb branch

i.e. noncompact flat direction for $U_{\text{eff}}(\sigma)$

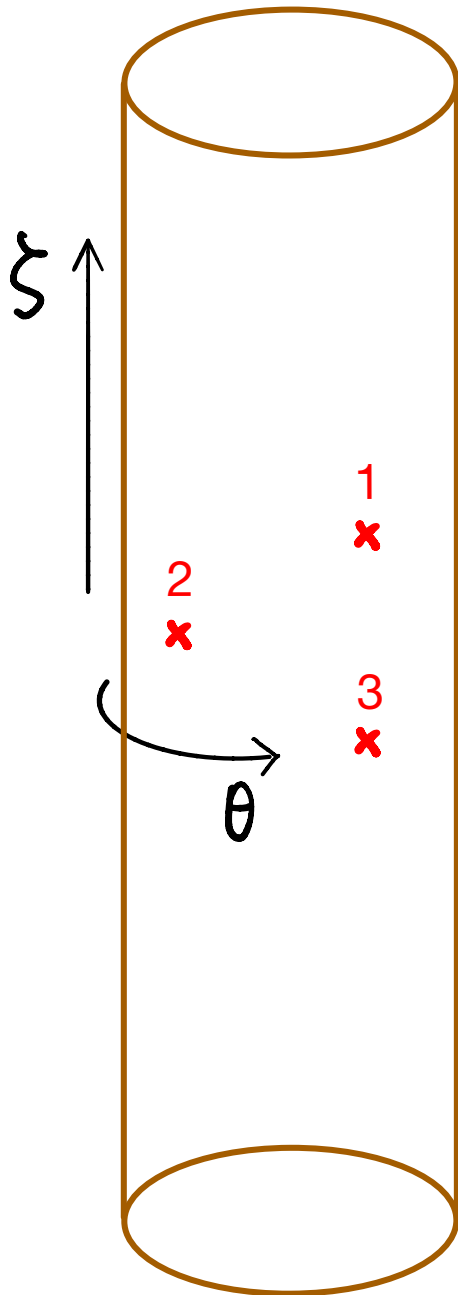
$\Leftrightarrow \exists$ solution to $d\tilde{W}_{\text{eff}}(\sigma) \equiv 0 \pmod{2\pi i \mathcal{P}}$

$$\Leftrightarrow e^t = \frac{(\sigma_1 + \sigma_2)^7}{\sigma_1^7} = \frac{(\sigma_1 + \sigma_2)^7}{\sigma_2^7} \\ = (1 + e^{-2\pi i a/7})^7 \quad a = 1, 2, 3.$$

i.e.
$$\begin{cases} \hat{S}_a = 7 \log \left(2 \cos \left(\frac{\pi a}{7} \right) \right) \\ \theta_a \equiv \pi a \end{cases}$$



Phases



$$\underline{U(2) \rightarrow \{1\}}$$

“weakly coupled phase”

$\exists$ Coulomb branch

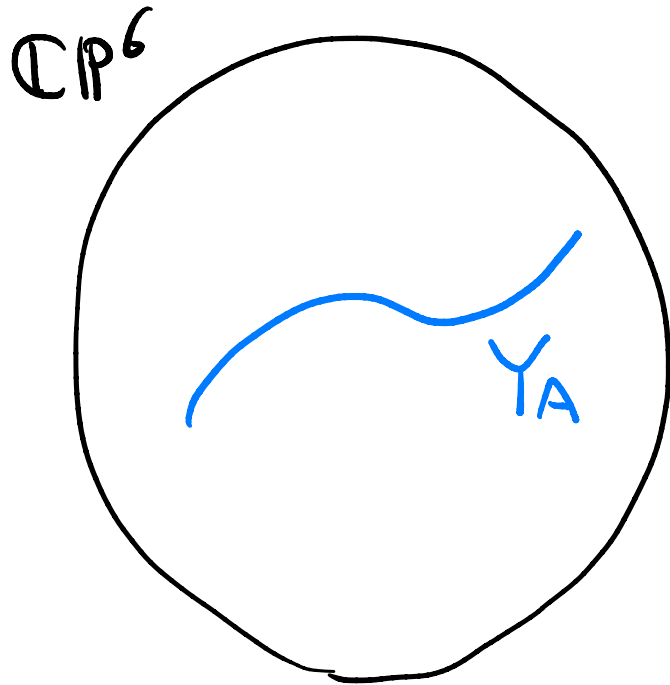
$$\text{at } \begin{cases} \vec{S}_a = 7 \log \left(2 \cos \left(\frac{\pi a}{7} \right) \right) \\ \theta_a \equiv \pi a \end{cases} \quad a = 1, 2, 3.$$

$SU(2)$ unbroken

“strongly coupled phase”

$\xi \ll 0$: $SU(2)$ gauge theory fibered over $\mathbb{CP}^6 \ni p$:

seven fundamentals with mass matrix $A^{ij}(p) \xrightarrow{\text{rank}} \{4, 6\}$



$\text{rank } A(p) = 6$: ~~SUSY~~

$\text{rank } A(p) = 4$: 3 massless $\square$

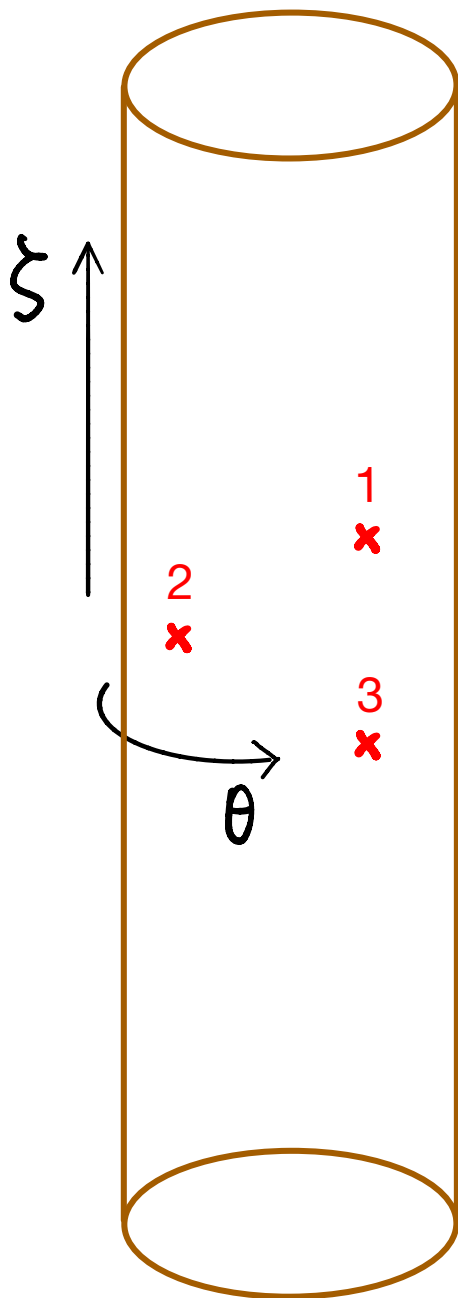
$\xrightarrow{\text{I.R.}}$ free theory of composites

$[X_1 X_2], [X_2 X_3], [X_3 X_2]$

$\xrightarrow{\text{I.R.}}$ σ -model with target Y

Pfaffian variety

Phases



$$X = \{A_1(x) = \dots = A_7(x) = 0\} \subset G(2, 7)$$

Grassmannian

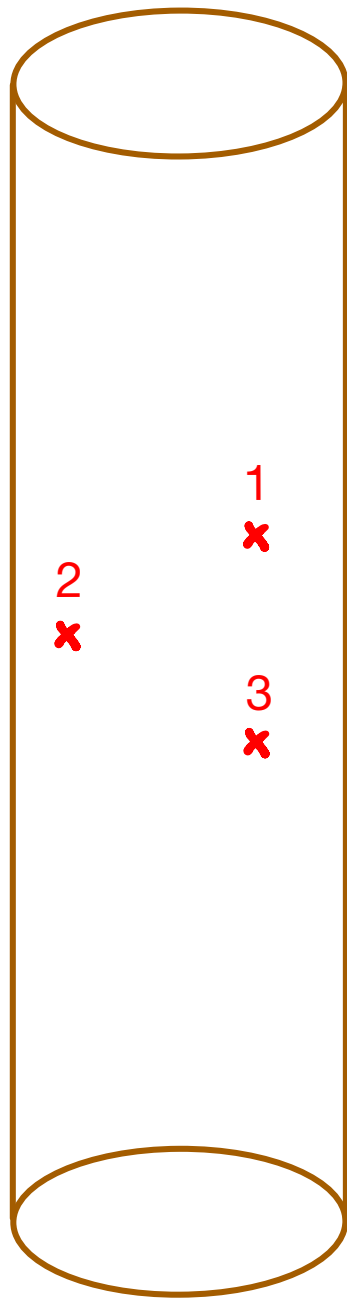
$\exists$ Coulomb branch

$$\text{at } \begin{cases} \vec{s}_a = 7 \log \left(2 \cos \left(\frac{\pi a}{7} \right) \right) \\ \theta_a \equiv \pi a \end{cases} \quad a = 1, 2, 3.$$

$$Y = \{ \text{rk } A(p) \leq 4 \} \subset \mathbb{CP}^6$$

Pfaffian

Phases



Grassmannian phase

Singular points

Pfaffian phase

This is an example of a
strong/weak Calabi-Yau/Calabi-Yau correspondence
known also as **Pfaffian/Grassmannian correspondence**

There are other examples:

Dual Rodland model

Hosono-Takagi model & its dual



H 2011

Kanazawa examples

H-Knapp 2013

Miura model

Galkin 2014, Gerhardus-Jockers 2015

2-parameter model

H-Knapp 2016

GP K^3 model

Caldararu-Knapp-Sharpe 2017

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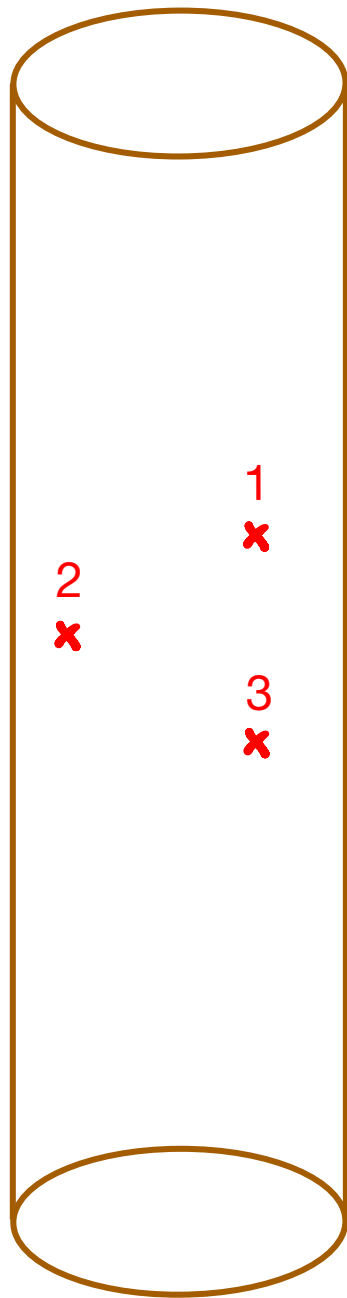
Some observables are renormalization group invariant, and have special dependence on moduli, e.g. holomorphic.

Categories of **D-branes** are examples of such observables :

for B-branes, it is locally constant on Kahler moduli.

The category of B-branes in the sigma model with target X is the derived category of coherent sheaves $D^b(X)$.

Phases

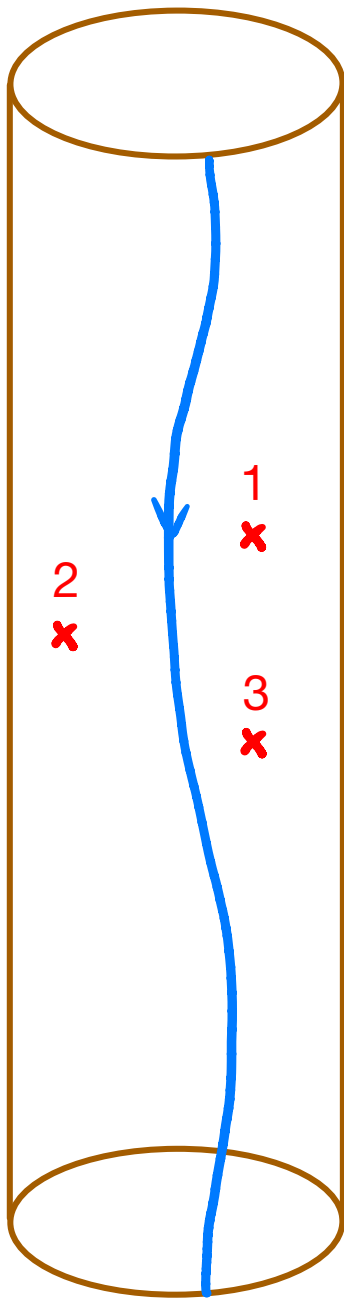


Categories of B-branes

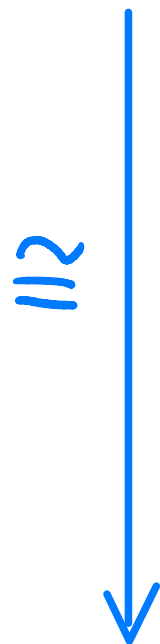
$$D^b(X)$$

$$D^b(Y)$$

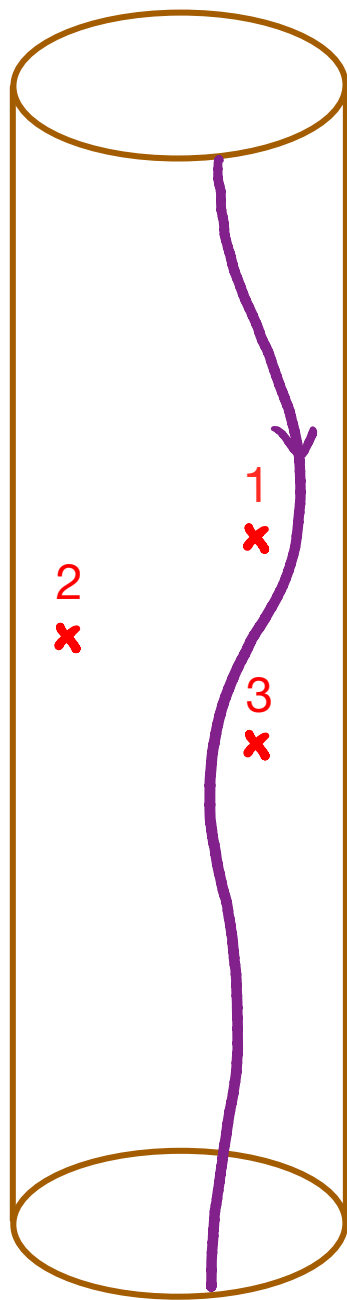
A path



An equivalence of categories

$$D^b(X)$$

$$D^b(Y)$$

Another path



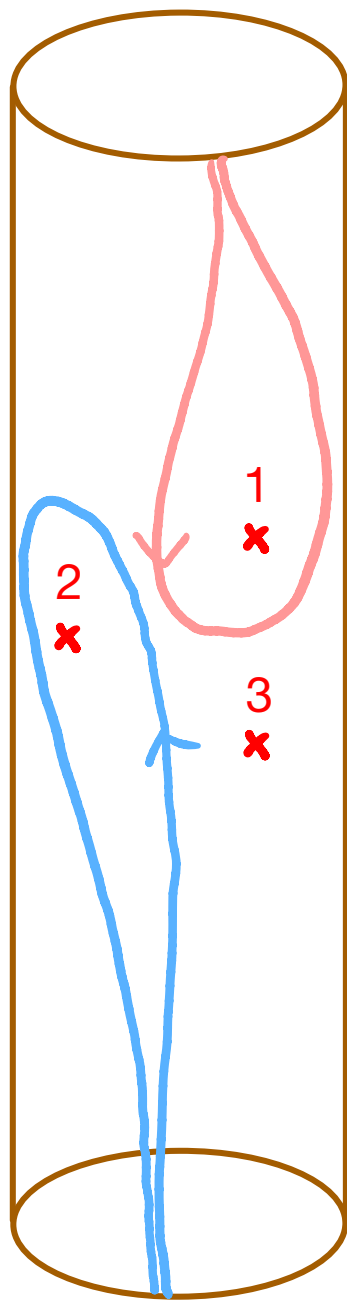
Another equivalence of categories

$D^b(X)$



$D^b(Y)$

Loops



Autoequivalences of categories

$$\begin{array}{c} D^b(X) \\ \cong \\ \text{red loop} \\ \\ \text{blue loop} \\ \cong \\ D^b(Y) \end{array}$$

The diagram illustrates the relationship between the derived categories $D^b(X)$ and $D^b(Y)$. A red loop is shown next to $D^b(X)$, and a blue loop is shown next to $D^b(Y)$. Both loops are marked with a red 'x' symbol, indicating a specific point or feature on the loops. The loops are connected by a double line, suggesting an equivalence or relationship between the two categories.

We would like to find these equivalences
using gauged linear sigma models.

In GLSM with gauge group G , matter V , superpotential W ,
a classical B-brane is specified by

$(M, Q) \dots$ a G -equivariant matrix factorization of (V, W)

$M = M^{\text{ev}} \oplus M^{\text{od}}$ a $\mathbb{Z}_2$ graded representation of G

$Q : V \rightarrow \text{End}^{\text{od}}(M)$ a G -equivariant polynomial function

s.t. $Q^2 = W \text{ id}_M$

$U(1)_V$ R-symmetry with charge integrality :

$$\begin{aligned} \lambda^{R_M} Q(\lambda^{R_V} \phi) \lambda^{-R_M} &= \lambda \cdot Q(\phi) & \left(\begin{array}{l} W(\lambda^{R_V} \phi) = \lambda^2 W(\phi) \\ e^{\pi i R_V} = J \in G \text{ on } V \end{array} \right) \\ e^{\pi i R_M} J &= \pm 1 \text{ on } M^{\text{ev/od}} \end{aligned}$$

However, a classical B-brane may induce instability in quantum theory, especially when a continuous gauge symmetry remains unbroken.

E.g. Around phase boundaries

Inside strongly coupled phases

Theories with simple gauge groups

We would like to find a condition on (M, Q) to define a well behaved B-brane in the quantum theory.

A proposal

Convergence of **hemisphere partition function**

HR, Sugishita-Terashima, Honda-Okuda 2013

$$\left\langle \text{Diagram} \right\rangle = \int_{\gamma} d^d \sigma \prod_{\alpha > 0} \langle \alpha, \sigma \rangle \sinh(\pi \langle \alpha, \sigma \rangle) \times$$

$$\prod_i \Gamma(i \langle Q_i, \sigma \rangle + \frac{R_i}{2}) \times$$

$$e^{i \langle t, \sigma \rangle} \cdot \text{tr}_M(e^{\pi i R_M} e^{2\pi \sigma})$$

$$V|_{T \times U(1)_v} = \bigoplus_i \mathbb{C}(Q_i, R_i) \quad \text{weight decomposition}$$

A proposal

(M, Q)

defines a well-behaved boundary condition when

there is a deformation of the real locus $i\mathbb{R} \rightarrow \gamma$

without hitting the poles $i\langle Q_i, \sigma \rangle + \frac{R_i}{2} = 0, -1, -2, \dots$

so that the integral $\left\langle \text{[diagram of a green circle with vertical lines inside]} \right\rangle =$

$$\int_{\gamma} d^l \sigma \prod_{\alpha \in \mathfrak{s}_0} \Gamma(\langle \alpha, \sigma \rangle) \sinh(\pi \langle \alpha, \sigma \rangle) \prod_i \Gamma\left(i\langle Q_i, \sigma \rangle + \frac{R_i}{2}\right) \times \\ e^{i\langle t, \sigma \rangle} \cdot \text{tr}_M \left(e^{\pi i R_M} e^{2\pi \sigma} \right)$$

is absolutely convergent.

This imposes a severe constraint on representations of G
that can be included in M .

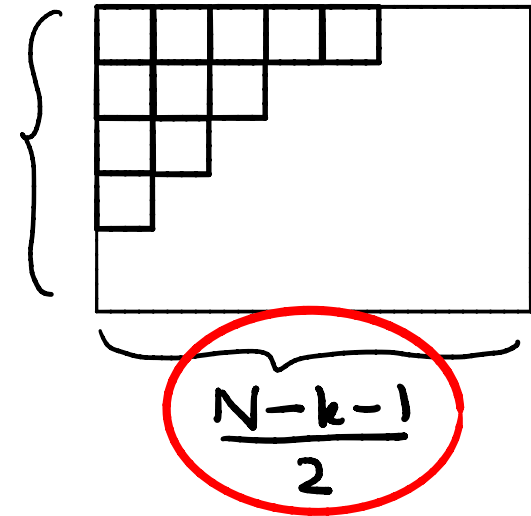
For theories with Abelian gauge groups, this reproduces
the **grade restriction rule** for D-brane transport across
phase boundaries. HHP 2008

We shall continue to call the constraint
grade restriction rule.

Example supersymmetric QCD $G = US_p(k)$, $V = (\mathbb{C}^k)^{\oplus N}$
 N odd for regularity

GRR requires

$S_Y(\mathbb{C}^k)^{\oplus |Y|}$ for Y in $\frac{k}{2}$
 Young symmetrizer J-traceless



(also Rennemo-Segal 2016)

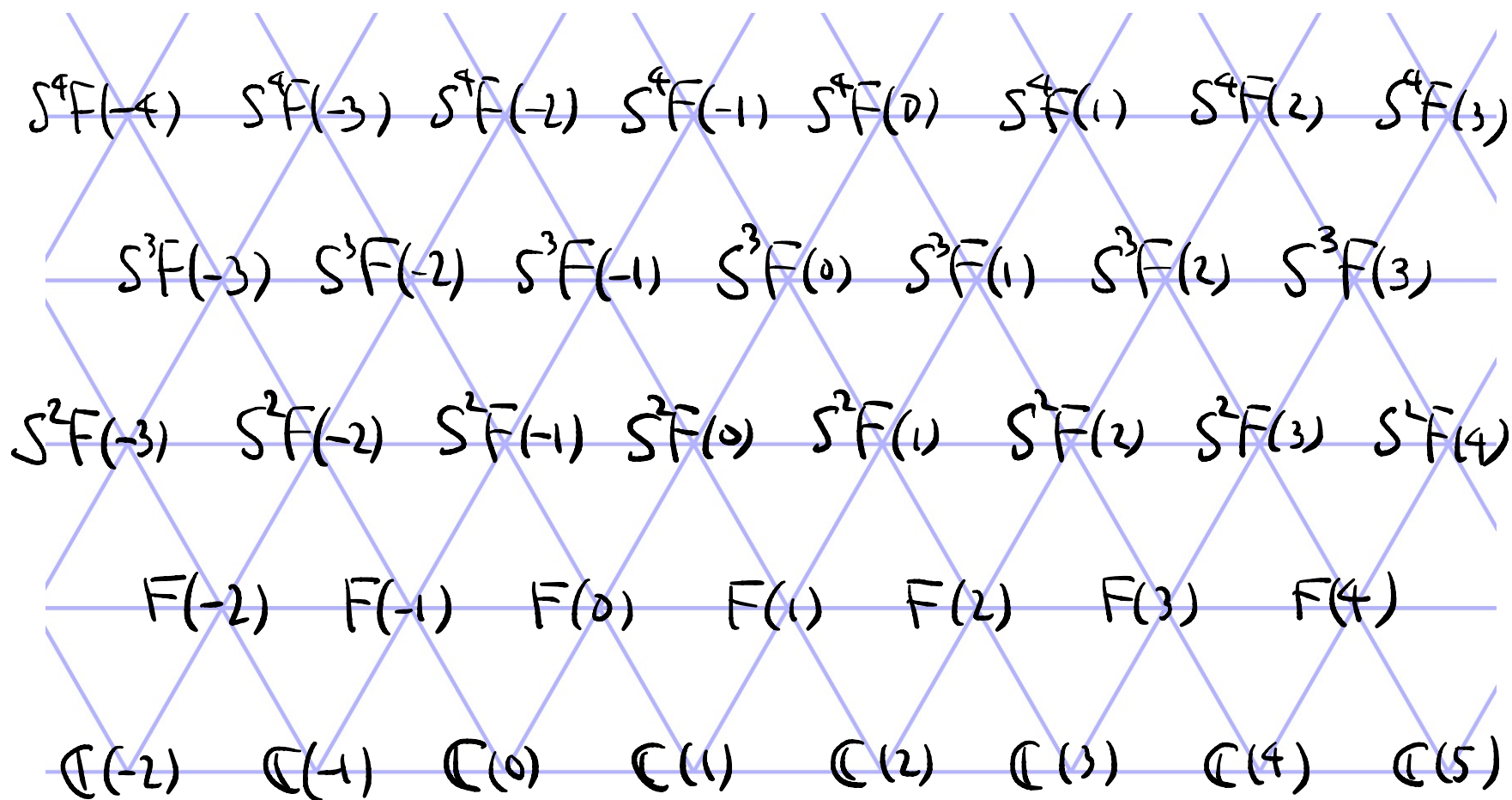
Consistent with 2d Seiberg duality H2011

$\exists$ Similar simple result for $\mathfrak{g}^G = \{0\}$, V quasi-symmetric
as long as the theory is regular.

The resulting category = NCCR of Spenko-Van den Bergh 2015

Representations of $U(2)$:

$$\text{Sym}^h \mathbb{C}^2 \otimes (\det \mathbb{C}^2)^{\otimes i} =: S^h F(i) \quad \begin{array}{l} h = 0, 1, 2, 3, \dots \\ i \in \mathbb{Z} \end{array}$$



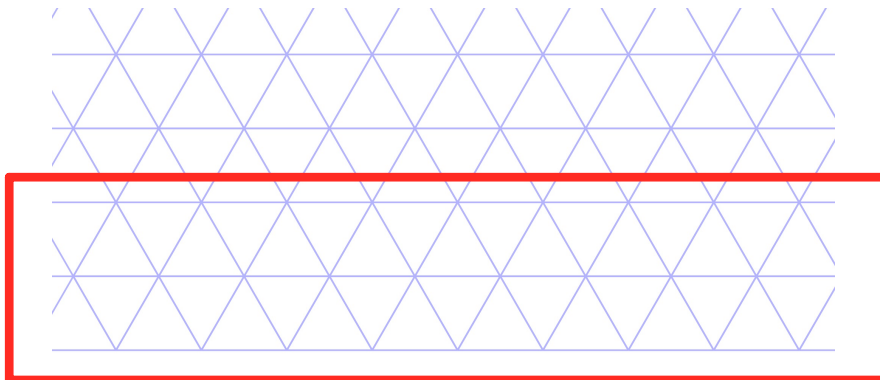
Weakly coupled phase $\zeta \gg 0$:

$$\exists \gamma_+ \text{ e.g. } \left\{ \begin{array}{l} \text{Im } \sigma_a = (\text{Re } \sigma_a)^2 \\ a = 1, 2 \end{array} \right\} \text{ that works for } \forall S^h F(i).$$

Strongly coupled phase $\zeta \ll 0$:

Only $S^h F(i)$ with $h=0,1,2$ i.e. $\mathbb{C}(i), F(i), S^2 F(i)$

admit convergent γ e.g. $\gamma_- = \left\{ \begin{array}{l} \text{Im } \sigma_1 = \text{Im } \sigma_2 \\ = -(\text{Re } \sigma_1 + \text{Re } \sigma_2)^2 \end{array} \right\}$



... **$SU(2)$ grade restriction rule**

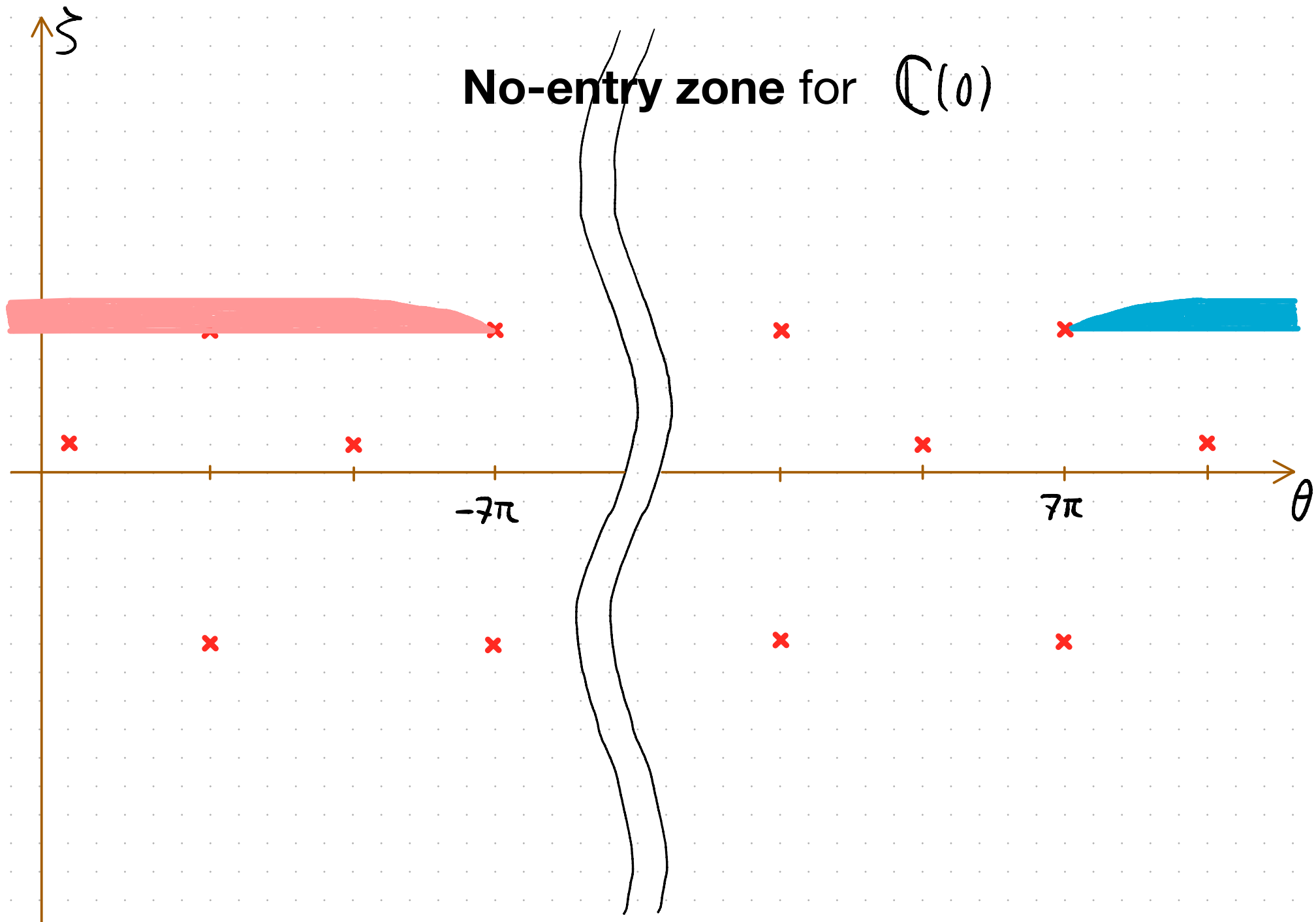
Near the phase boundary $|\xi| < O(1)$,

each $SU(2)$ grade restricted representation has

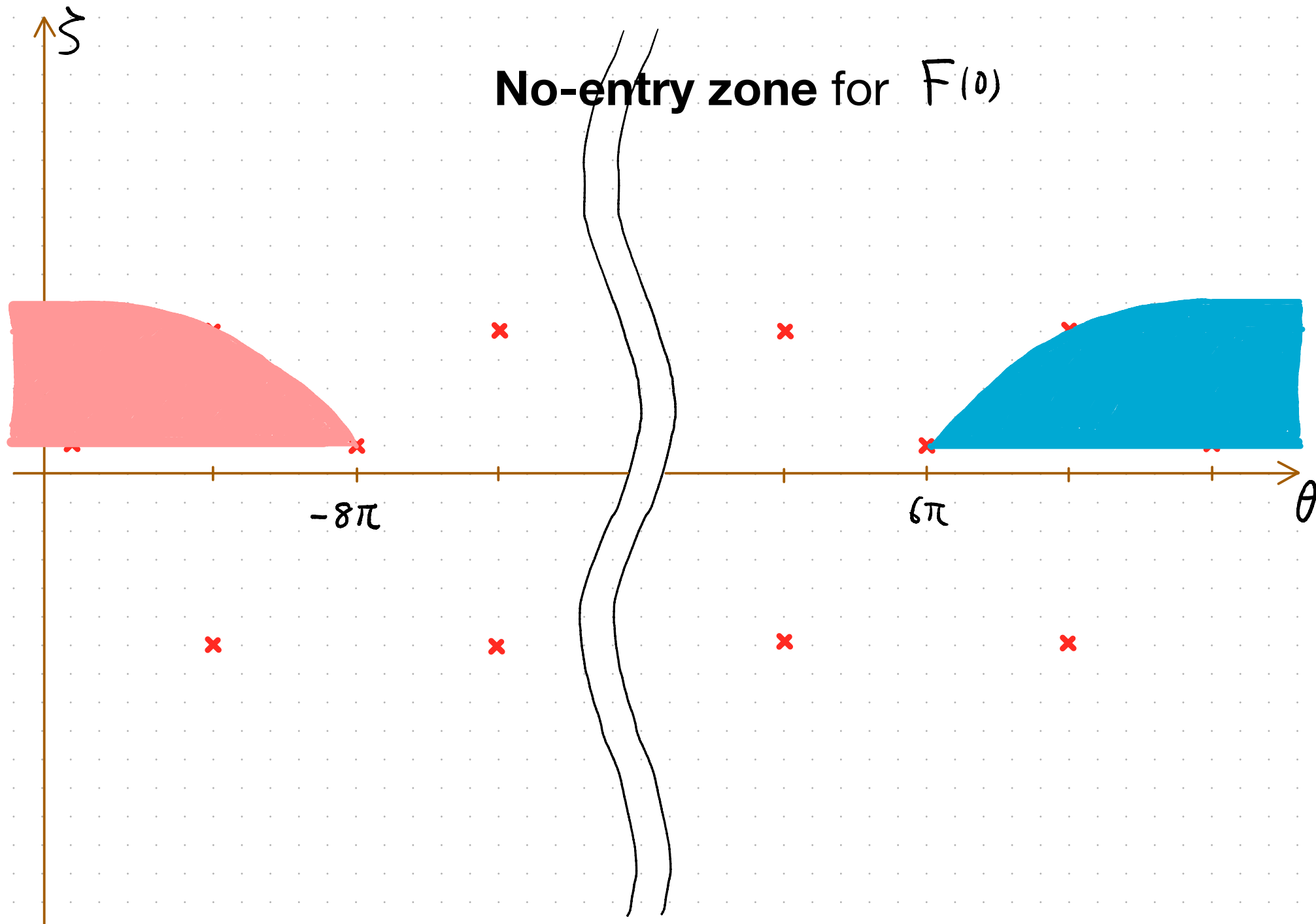
no-entry zones where convergent contour is **lost**

or is **forced to pick Gamma function poles.**

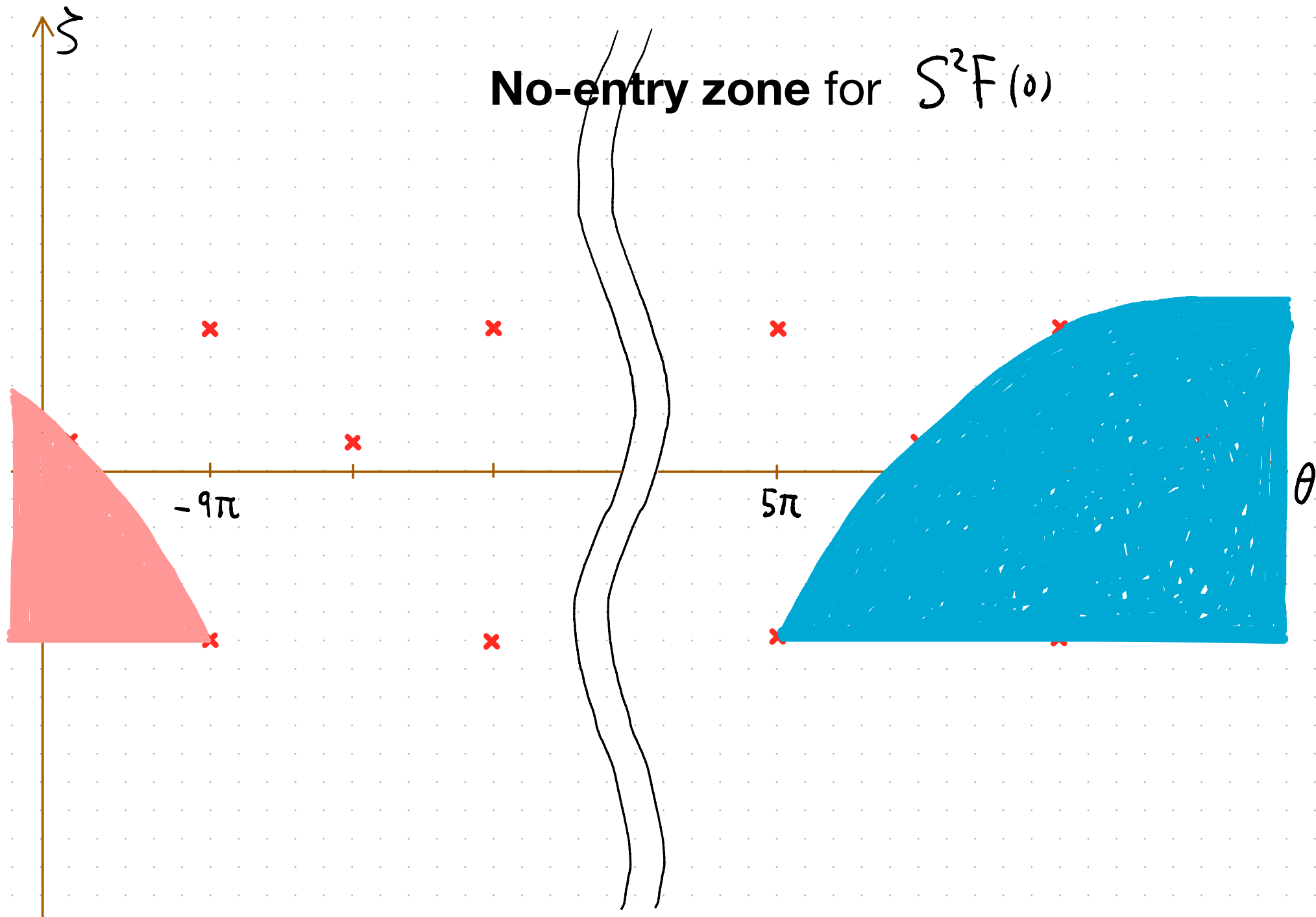
No-entry zone for $\mathbb{C}(0)$

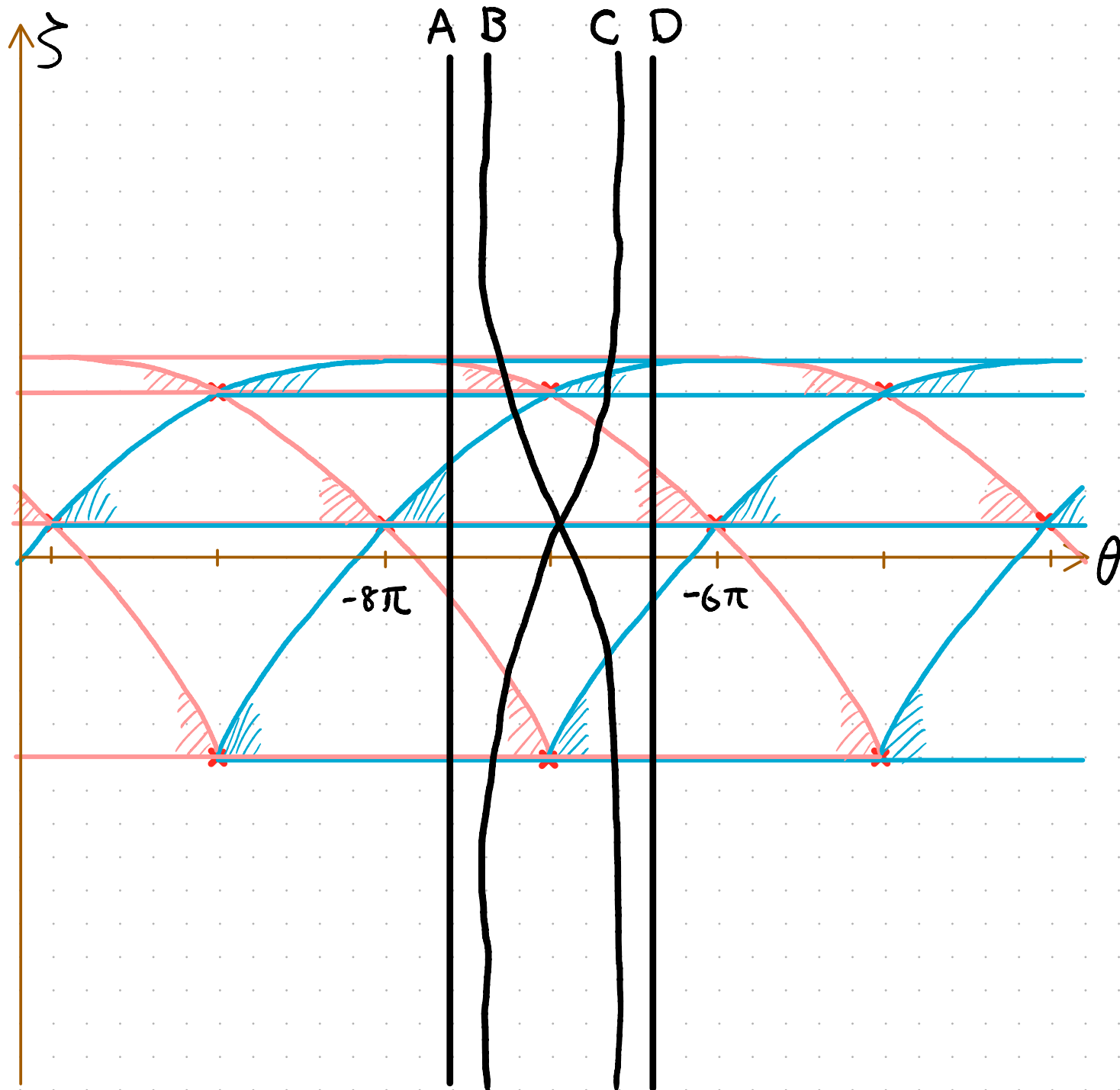


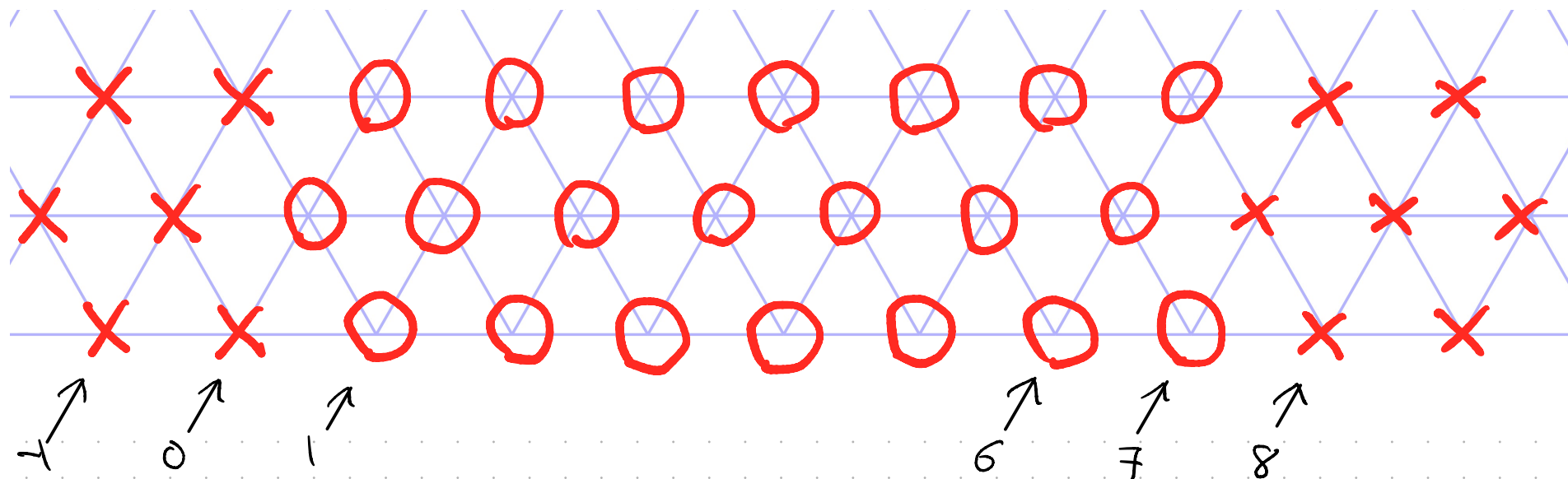
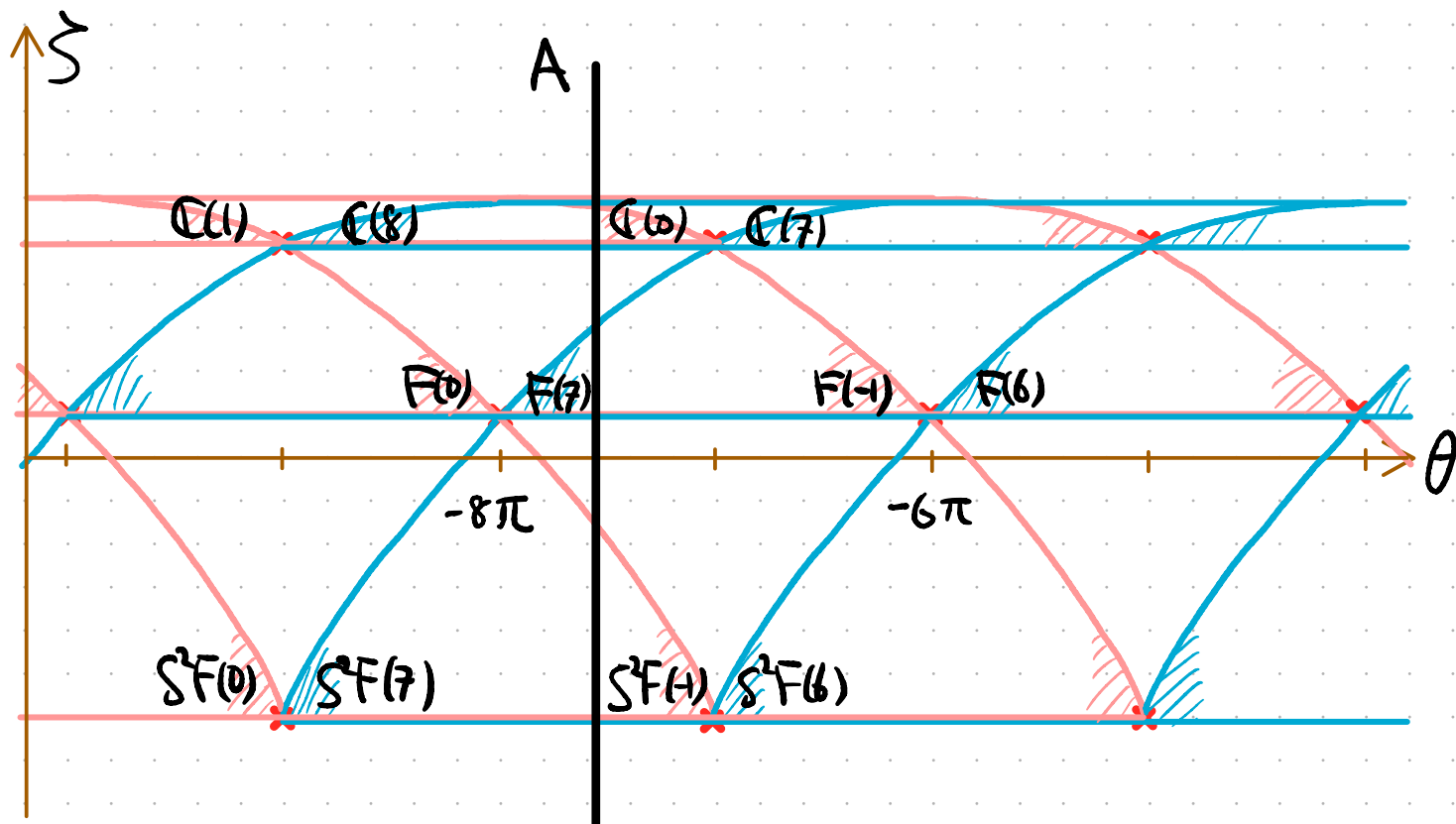
No-entry zone for $F(0)$

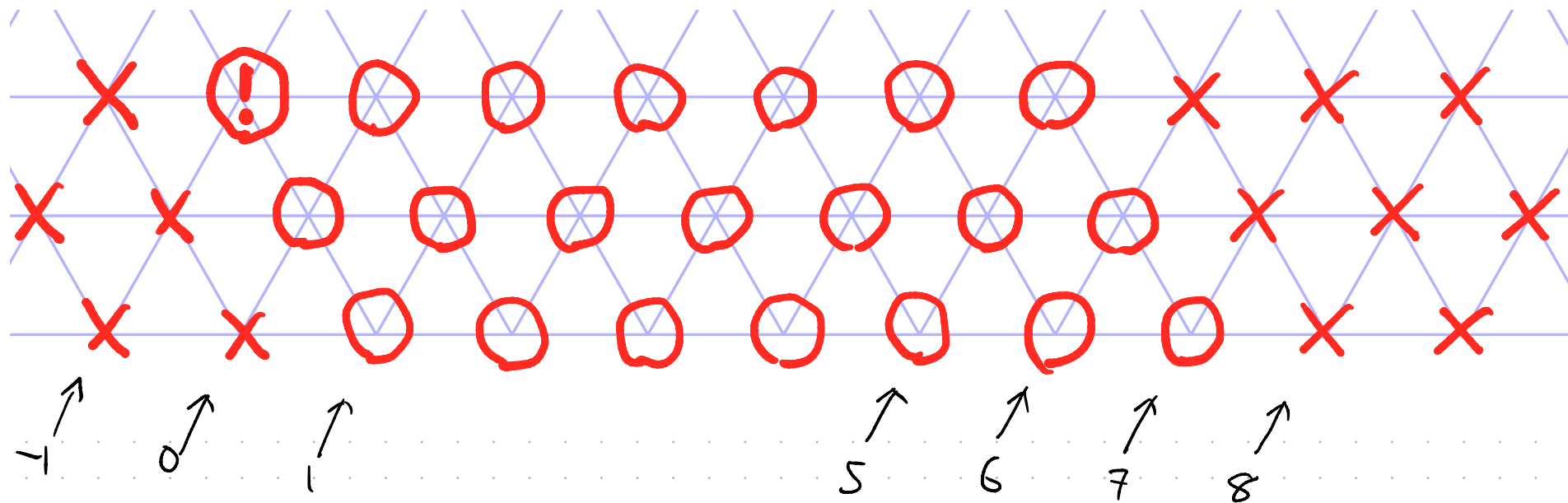
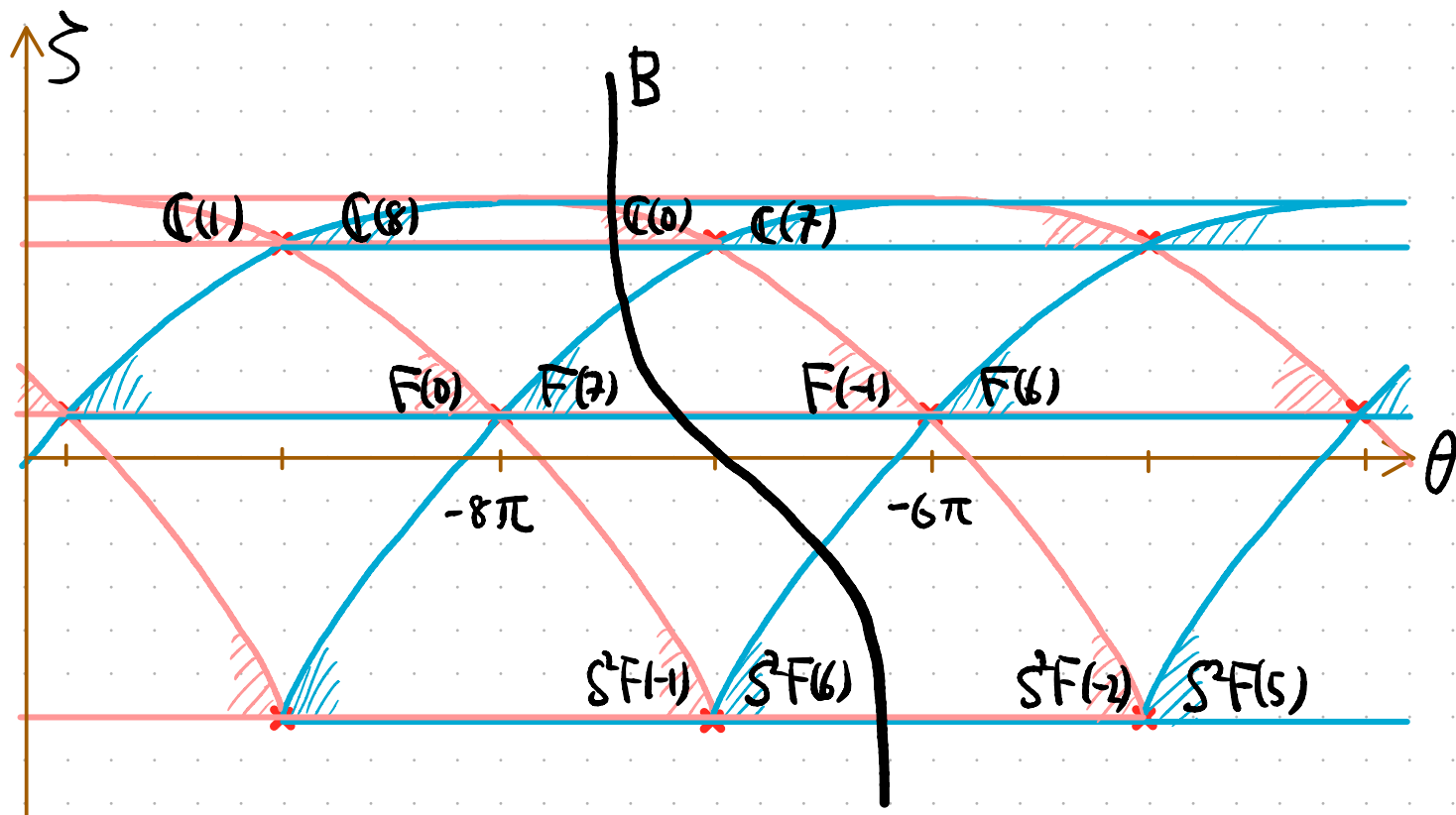


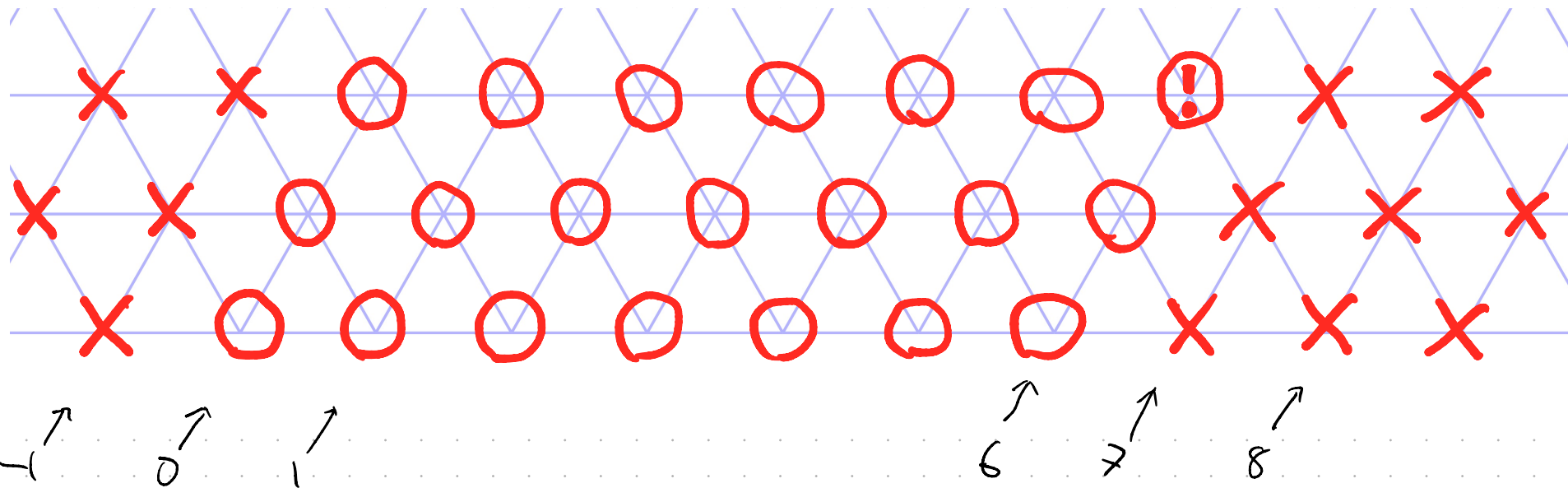
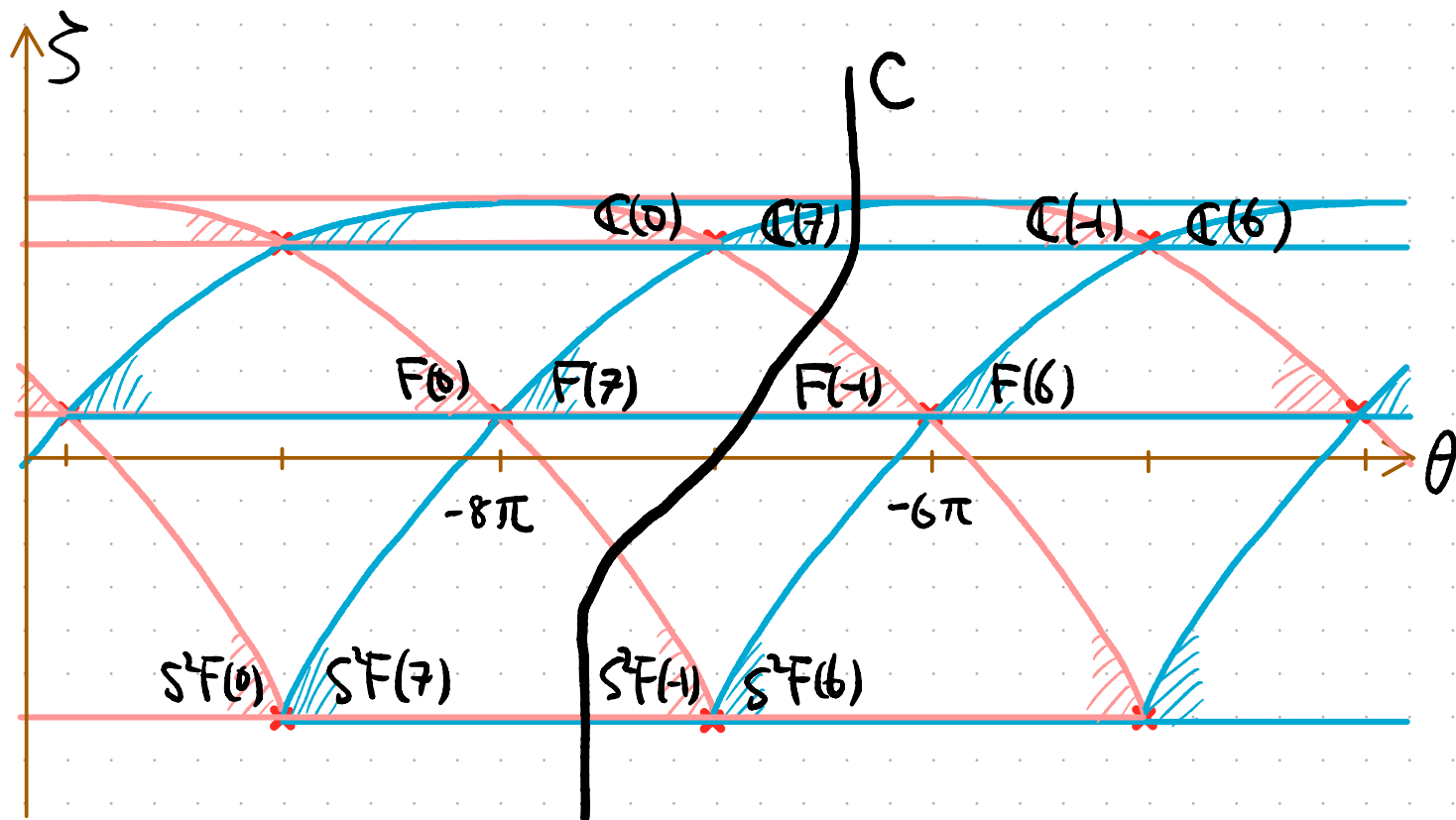
No-entry zone for $S^2F(0)$

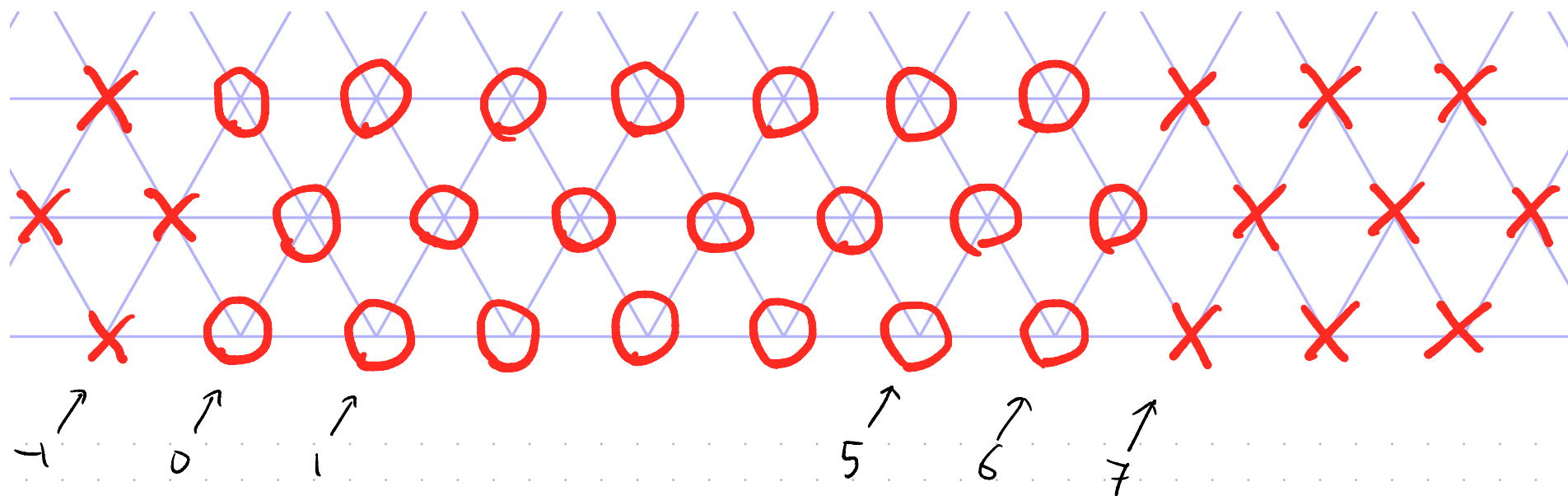
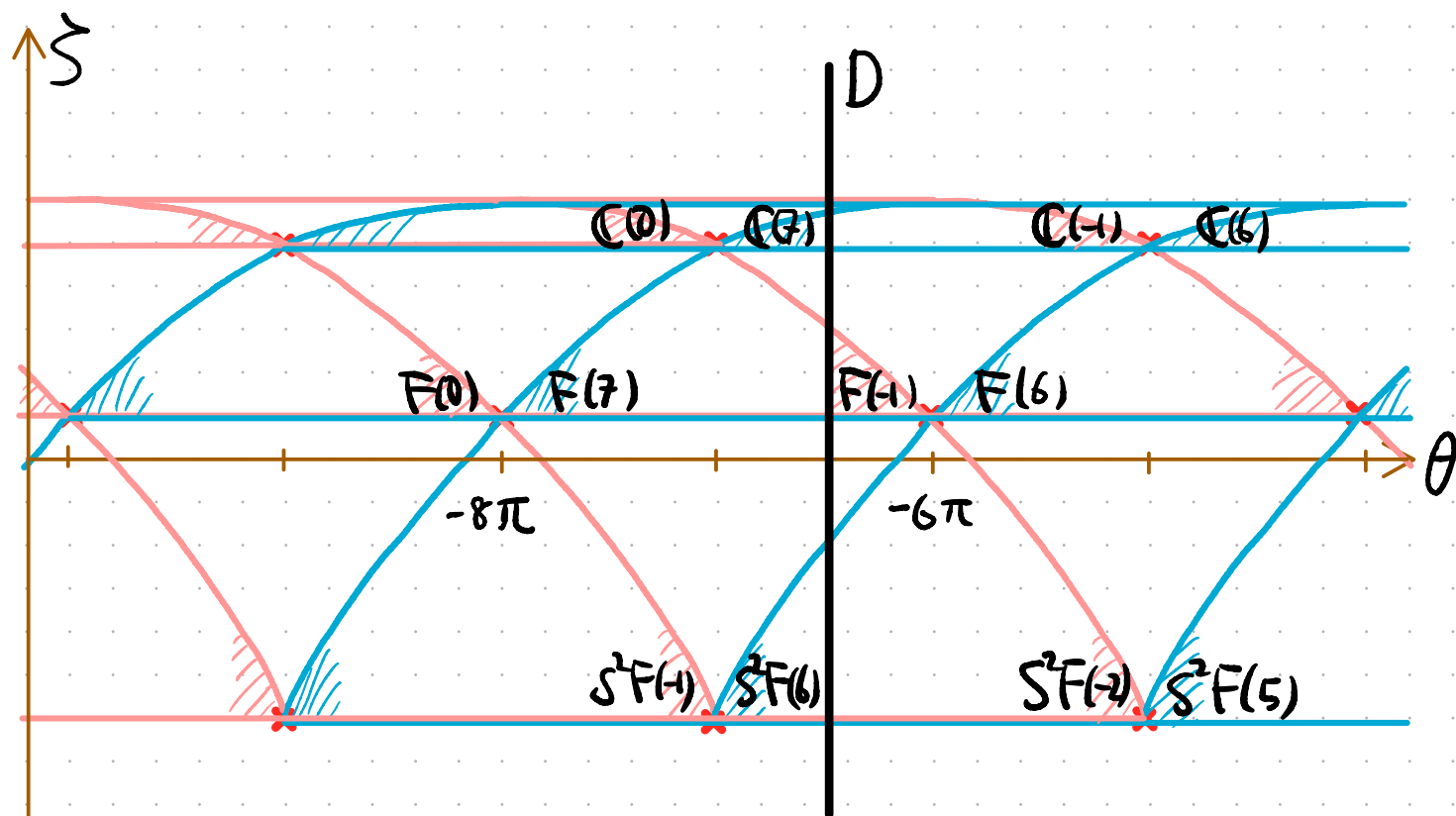


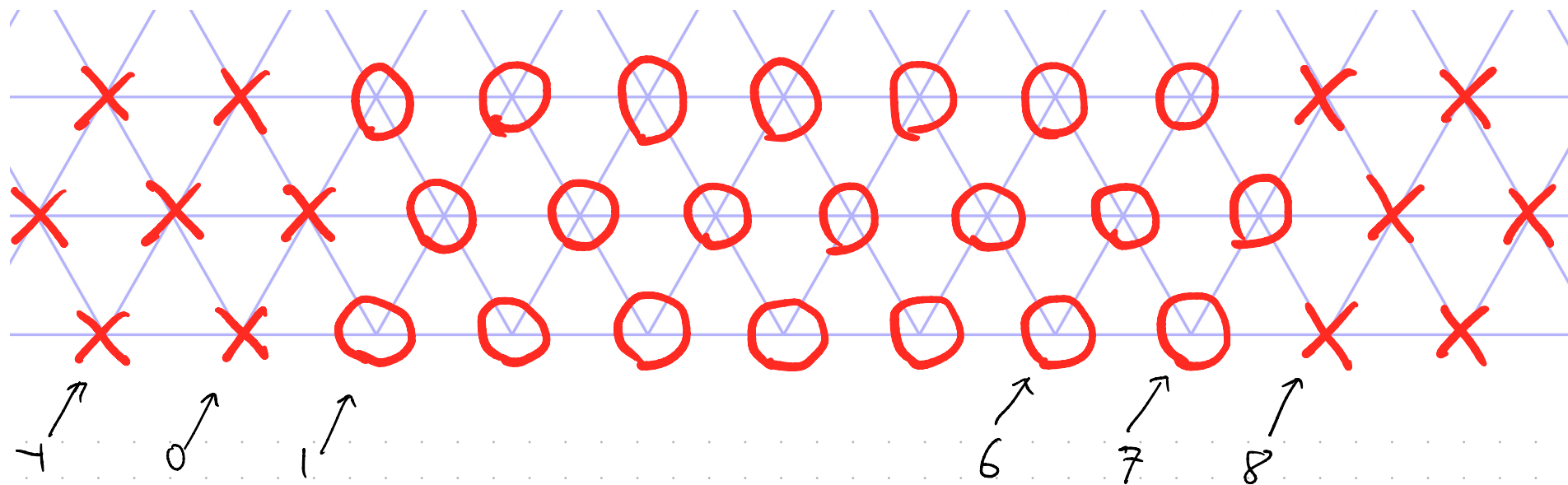
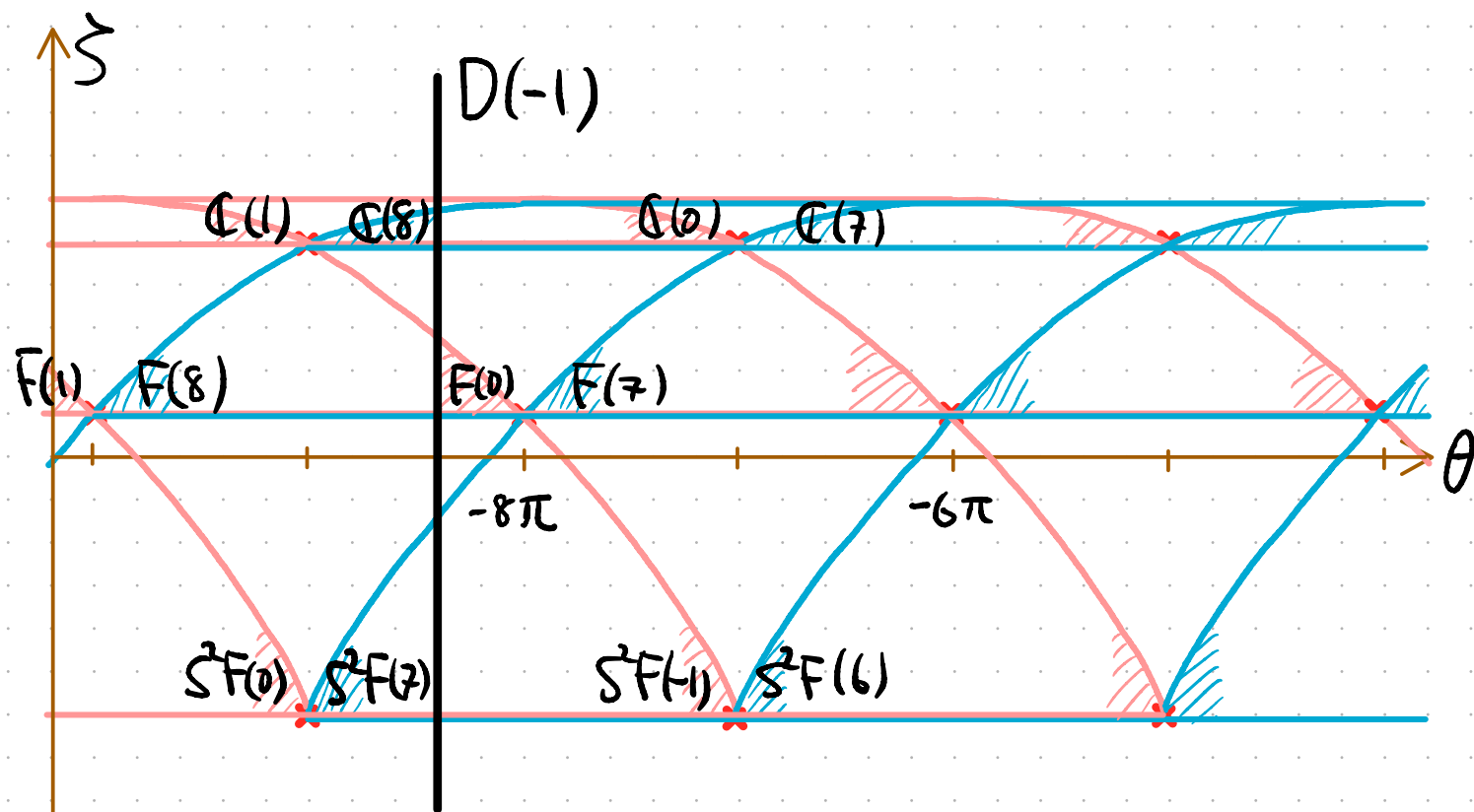




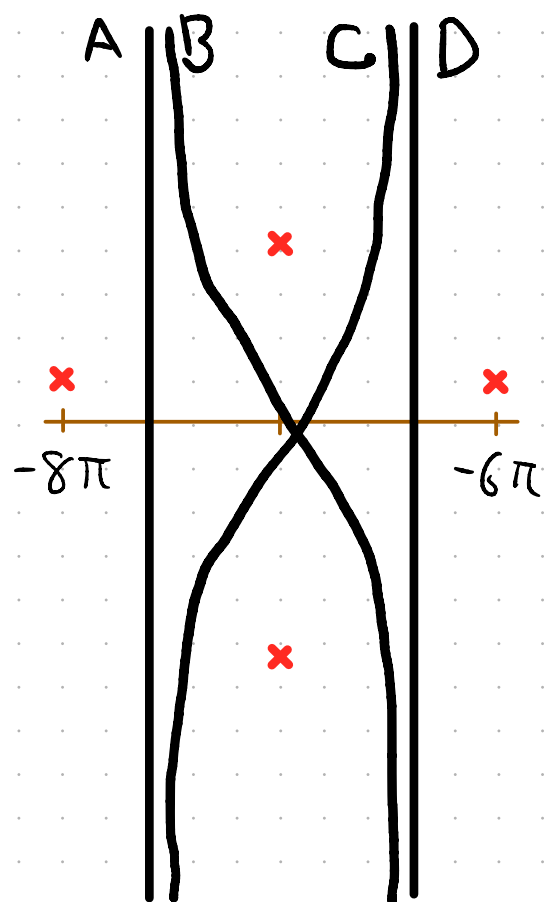






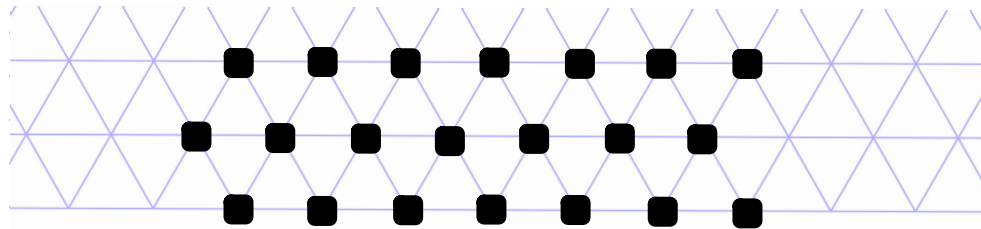


Summary

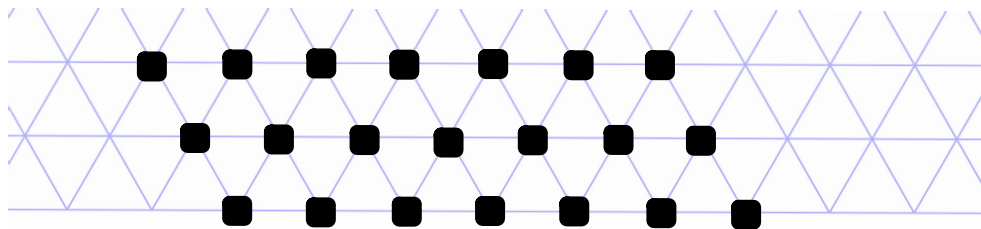


$$\text{GRR}_{w(-1)} = \text{GRR}_w(1)$$

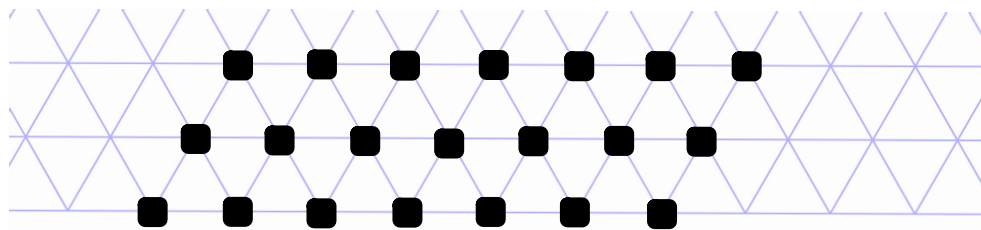
GRR_A



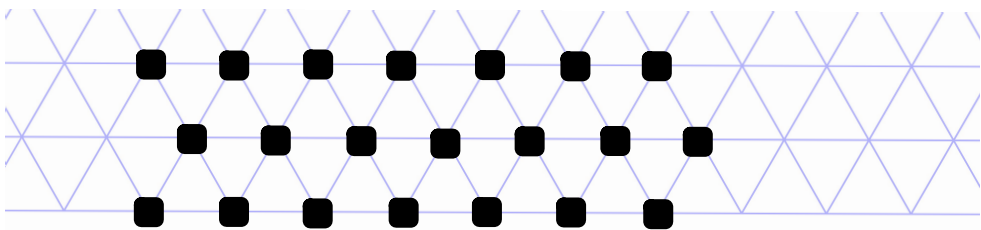
GRR_B



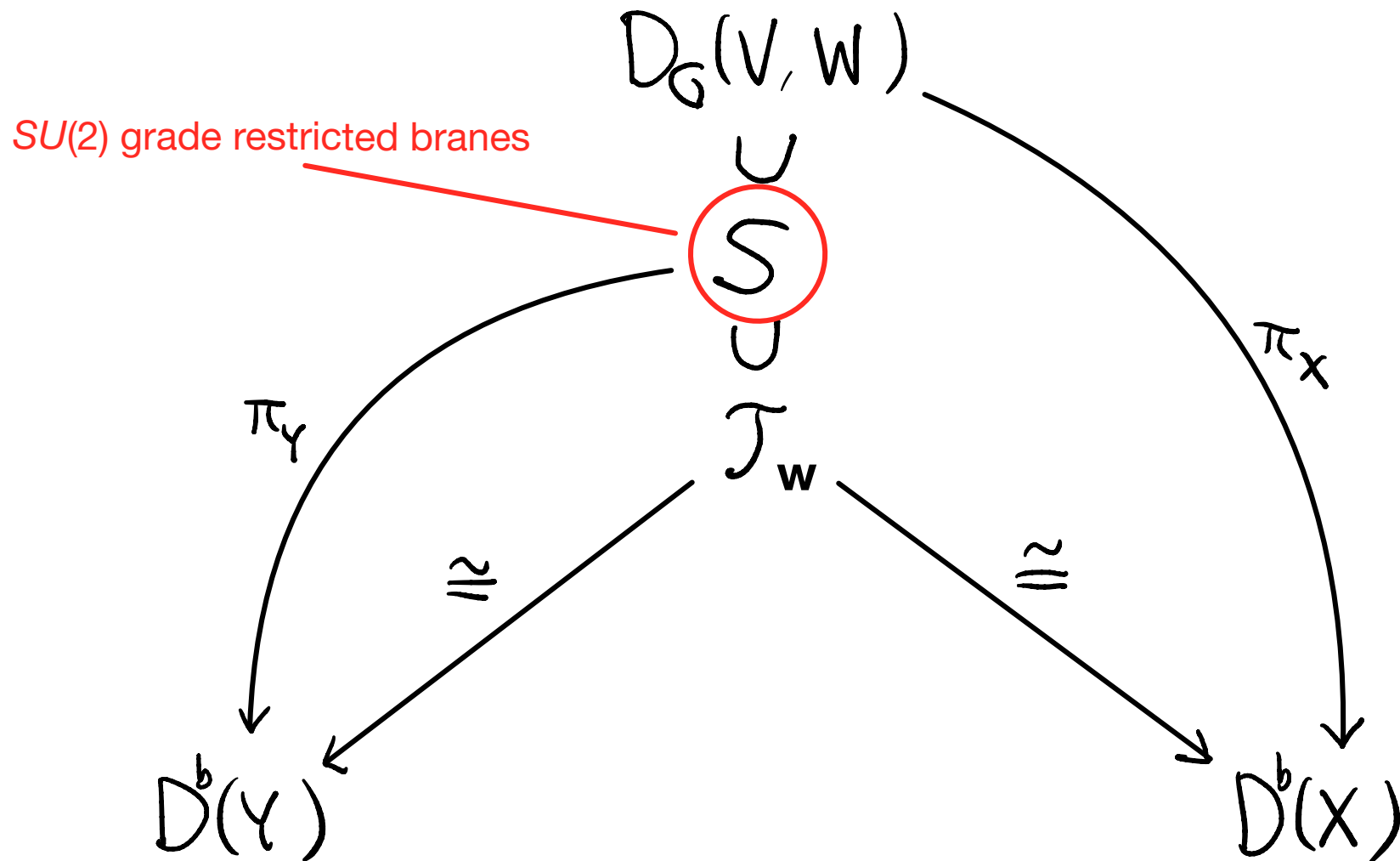
GRR_C



GRR_D

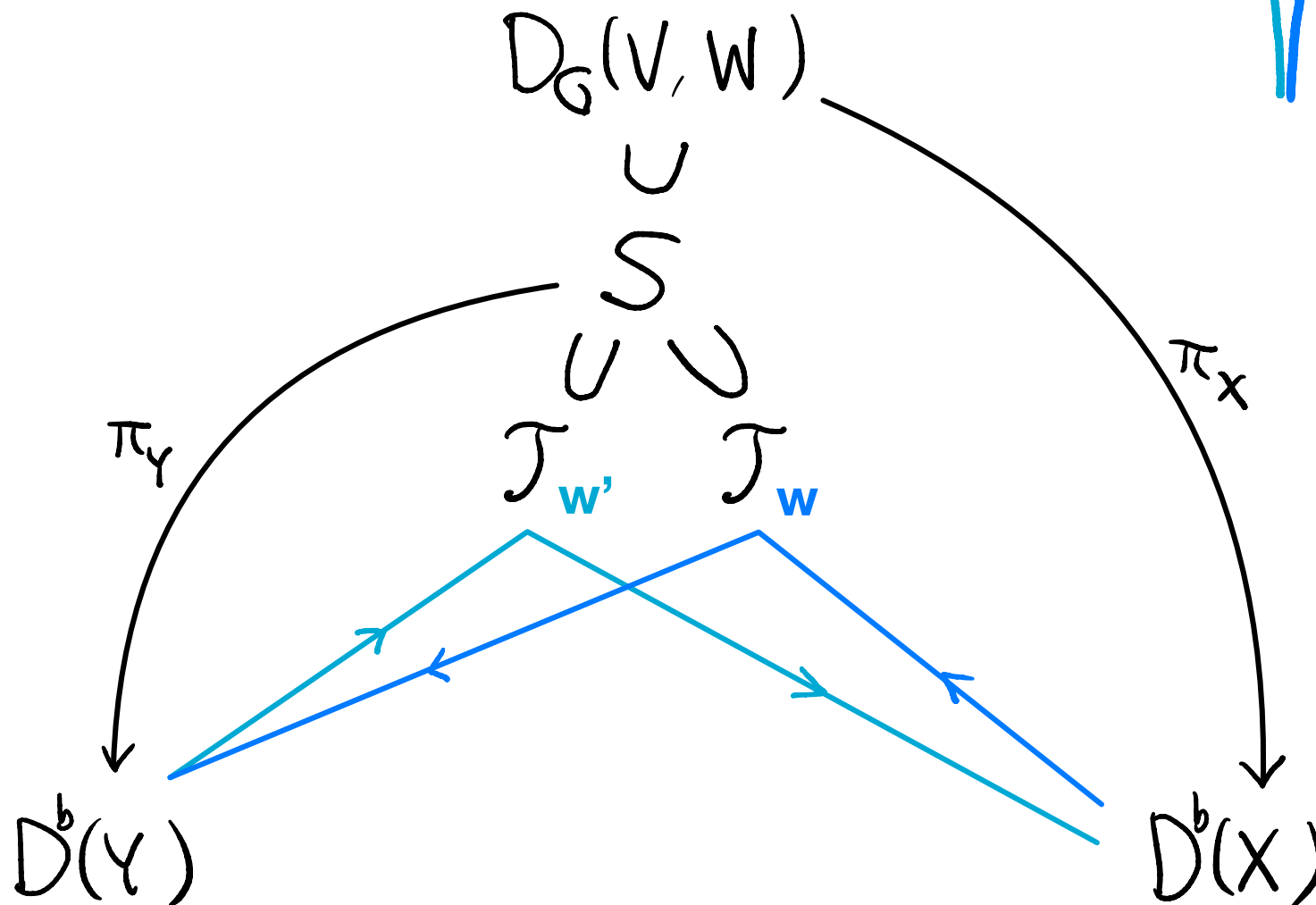
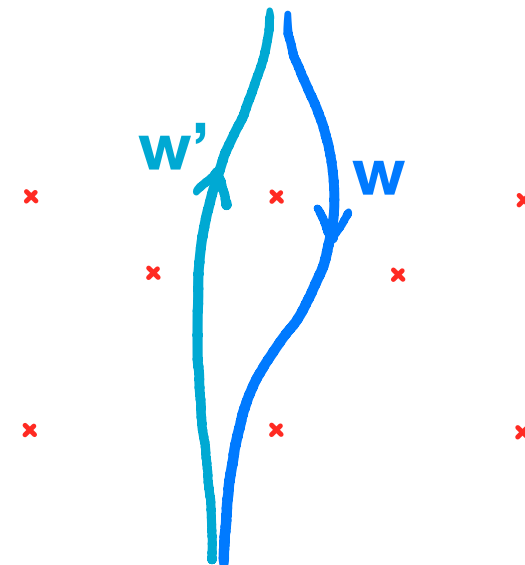


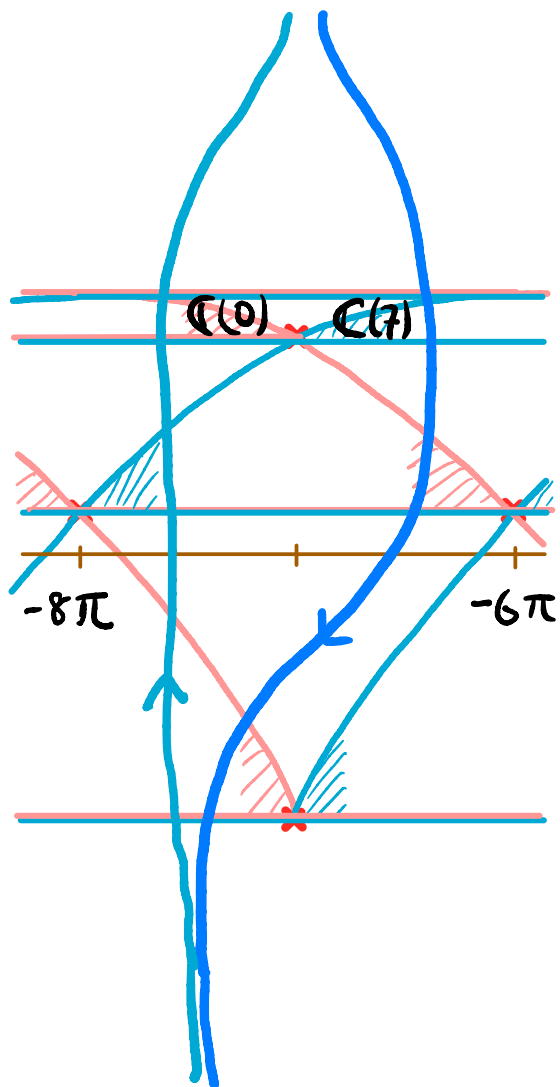
The subcategory of grade restricted branes $\mathcal{T}_w$
 produces an equivalence of categories



Change of windows : $\mathbf{w} \rightarrow \mathbf{w}'$

$\rightsquigarrow$ an autoequivalence

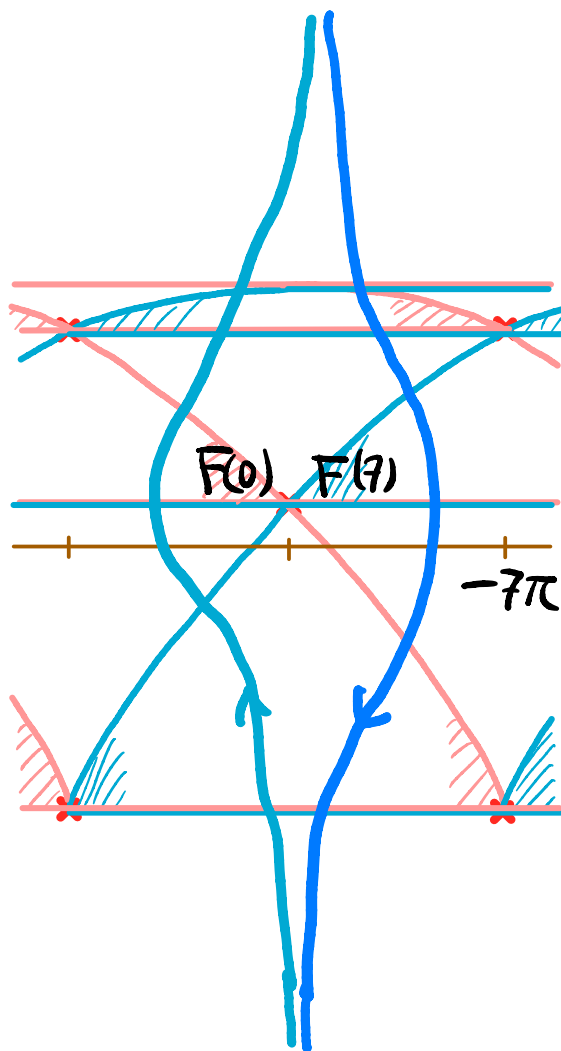




replacement

$$C(0) \rightsquigarrow C(7)$$

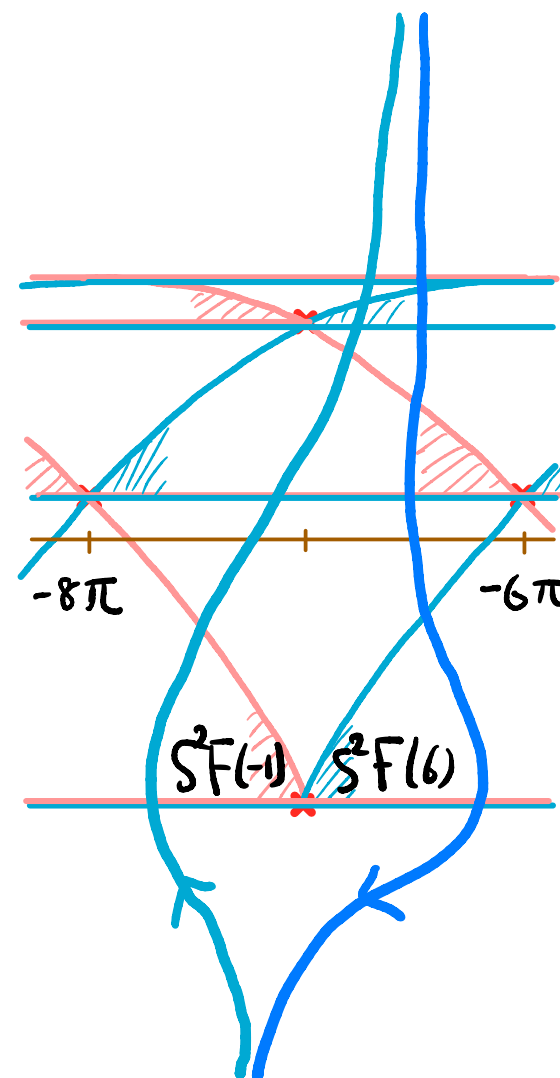
needed



replacement

$$F(0) \rightsquigarrow F(7)$$

needed



replacement

$$S^2F(-1) \rightsquigarrow S^2F(6)$$

needed

For monodromies along loops with a base point at $\zeta \gg 0$
the replacement can be done using empty branes in $\zeta \ll 0$.

Use the **Koszul branes**

$$\mathcal{K}$$

$$\mathcal{K} \otimes F(0)$$

$$\mathcal{K} \otimes S^2 F(-1)$$

where $\mathcal{K}$ is :

$$\underset{\textcolor{red}{0}}{\mathbb{C}} \xrightleftharpoons[p]{A(x)} \mathbb{C}^7(1) \xrightleftharpoons[p]{A(x)} \mathbb{C}^{\binom{7}{2}}(2) \xrightleftharpoons[p]{A(x)} \mathbb{C}^{\binom{7}{3}}(3) \xrightleftharpoons[p]{A(x)} \mathbb{C}^{\binom{7}{4}}(4) \xrightleftharpoons[p]{A(x)} \mathbb{C}^{\binom{7}{5}}(5) \xrightleftharpoons[p]{A(x)} \mathbb{C}^{\binom{7}{6}}(6) \xrightleftharpoons[p]{A(x)} \mathbb{C}^7(7) \underset{\textcolor{red}{7}}{\quad}$$

$$Q = \sum_{k=1}^7 (p^k \eta_k + A_k(x) \bar{\eta}^k) \quad \{ \eta_k, \bar{\eta}^l \} = \delta_{kl}$$

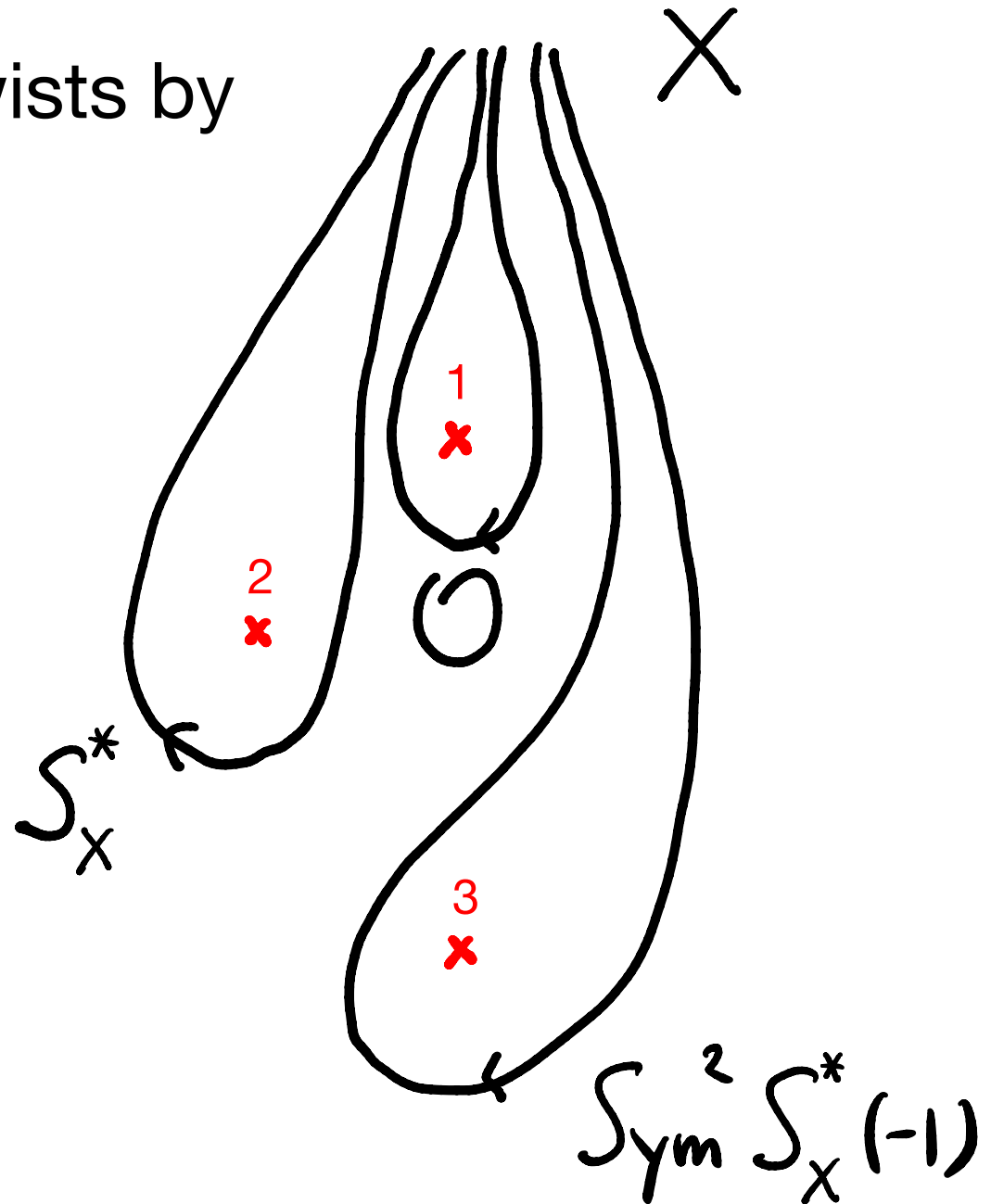
$$\{Q, Q^+\} = \|p\|^2 + \|A(x)\|^2 > 0 \quad \text{if } \zeta \ll 0 \quad \text{empty}$$

For monodromies along loops with a base point at $\zeta \gg 0$
the replacement can be done using empty branes in $\zeta \ll 0$.

Use the **Koszul branes**

$$\begin{array}{ccc}
 \mathcal{K} & \mathcal{K} \otimes F(0) & \mathcal{K} \otimes S^2 F(-1) \\
 \zeta \gg 0 \downarrow & \zeta \gg 0 \downarrow & \zeta \gg 0 \downarrow \\
 \mathcal{O}_X & S_X^* & S_{Y^m}^2 S_X^*(-1)
 \end{array}$$

Seidel-Thomas twists by



Low energy behavior of branes in the strongly coupled phase

1. Use hemisphere partition function

find candidates for D0's & D6's

2. LE behavior of branes in $SU(2)$ SQCD

3. D0's & D6's candidates are indeed D0's & D6's

4. Lift of CY branes to GLSM

Addington-Donovan-Segal to the rescue

D0 candidates

Take a point $p_* \in Y$, $A(p_*) = \begin{pmatrix} 0_{3 \times 3} & 0_{3 \times 4} \\ 0_{4 \times 3} & \boxed{\begin{matrix} * \\ 4 \times 4 \end{matrix}} \end{pmatrix}$

$\parallel$
 $(0, 0, 0, 0, 0, 0, 1)$

$$Q = \sum_{k=1}^6 (p^k \eta_k + A_k(x) \bar{\eta}^k) + \sum_{a=1,2} \left(p^7 x_4^a \tilde{\zeta}_a^4 + x_5^a \tilde{\zeta}_a^5 + p^7 x_6^a \tilde{\zeta}_a^6 + x_7^a \tilde{\zeta}_a^7 \right)$$

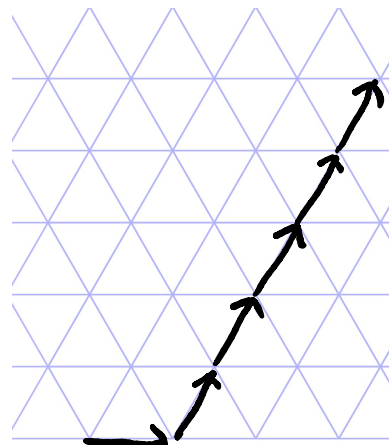
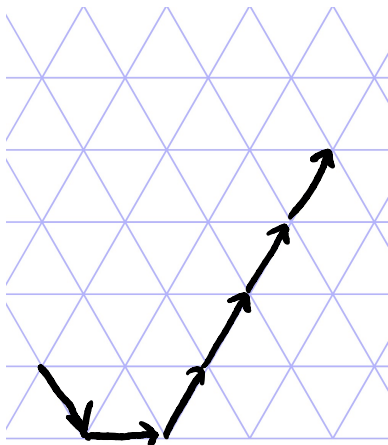
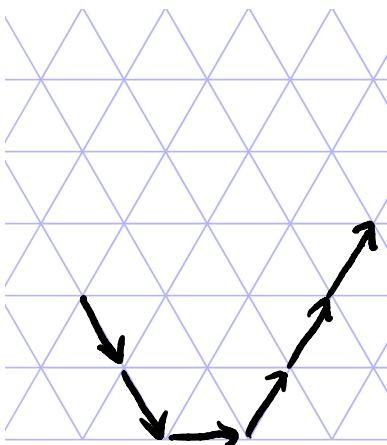
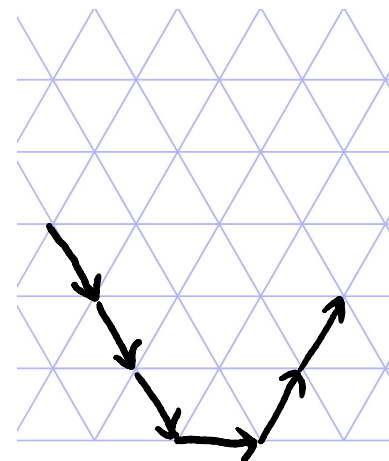
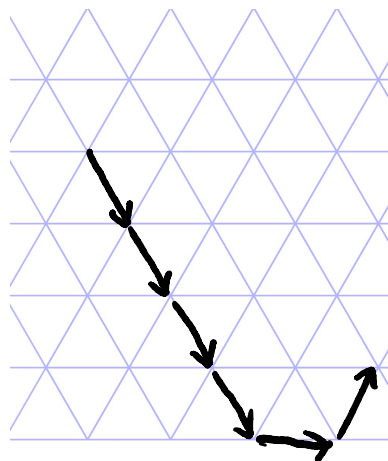
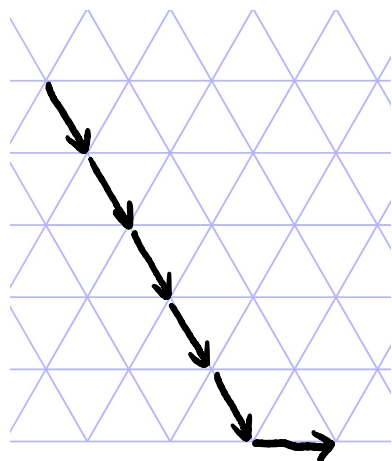
$$\left. \begin{aligned} \{ \eta_k, \bar{\eta}^l \} &= \delta_k^l \quad k, l = 1 \sim 6 \\ \{ \tilde{\zeta}_a^\mu, \tilde{\zeta}_b^\nu \} &= 2 A(p_*)^{\mu\nu} \epsilon_{ab} \quad \begin{matrix} \mu, \nu = 4 \sim 7 \\ a = 1, 2 \end{matrix} \end{aligned} \right\} \begin{array}{l} M = \text{Clifford module} \\ SU(2) \text{ grade restricted } \checkmark \end{array}$$

$$\leadsto \left\langle \text{Diagram} \right\rangle = (2\pi)^{13} e^t \cdot \underbrace{\Pi_0(e^t)}_{Z_Y(U_{pt})}$$

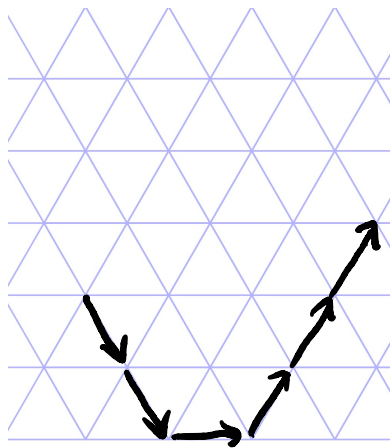
D6 candidates

Try empty branes in $\xi \gg 0$

They are obtained from **twisted Lascaux complexes**



E.g.



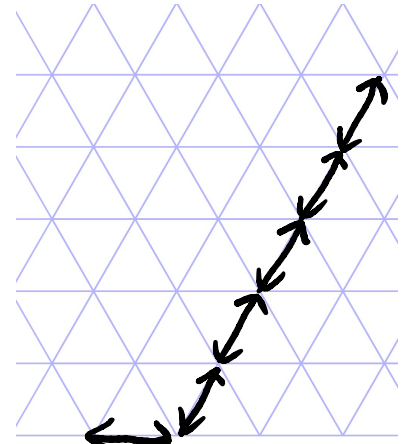
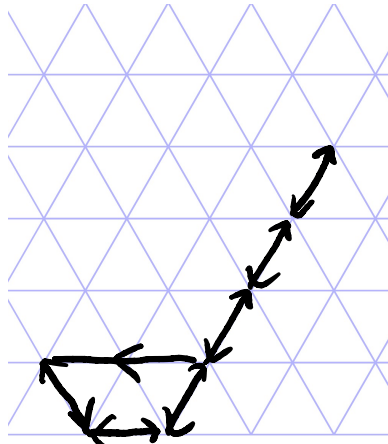
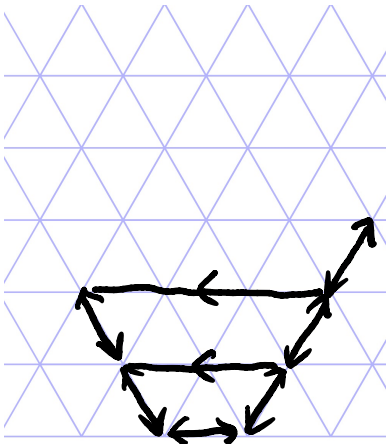
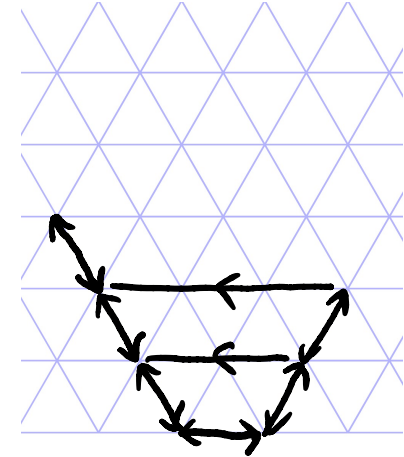
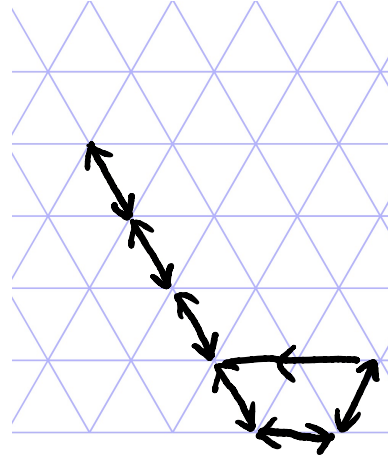
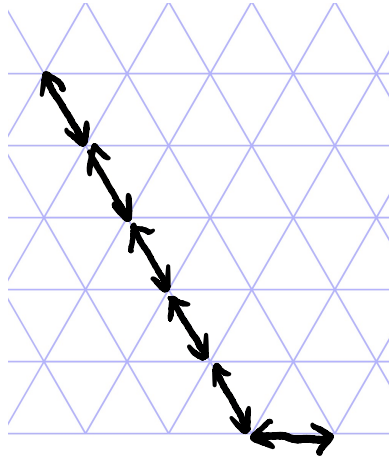
$$V = \mathbb{C}^7$$

$$F^* = \text{Hom}(F, \mathbb{C}) \cong F(-1)$$

$$S^2 F^* \otimes \wedge^7 V^* \rightarrow F^* \otimes \wedge^6 V^* \rightarrow \mathbb{C} \otimes \wedge^5 V^* \rightarrow \mathbb{C}(1) \otimes \wedge^3 V^*$$

$$\rightarrow F^*(2) \otimes \wedge^2 V^* \rightarrow S^2 F^*(3) \otimes V^* \rightarrow S^3 F^*(4)$$

They can be completed to matrix factorizations of $W = \sum A_{(p)}^{ij} [x_i x_j]$

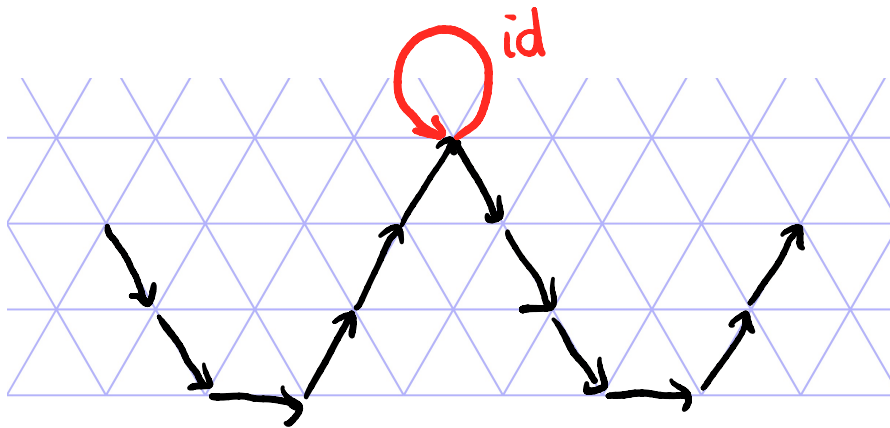


Lascaux matrix factorizations or Lascaux branes

A problem : they are **not** $SU(2)$ grade restricted

A problem : they are **not** $SU(2)$ grade restricted

Solution : bind them together



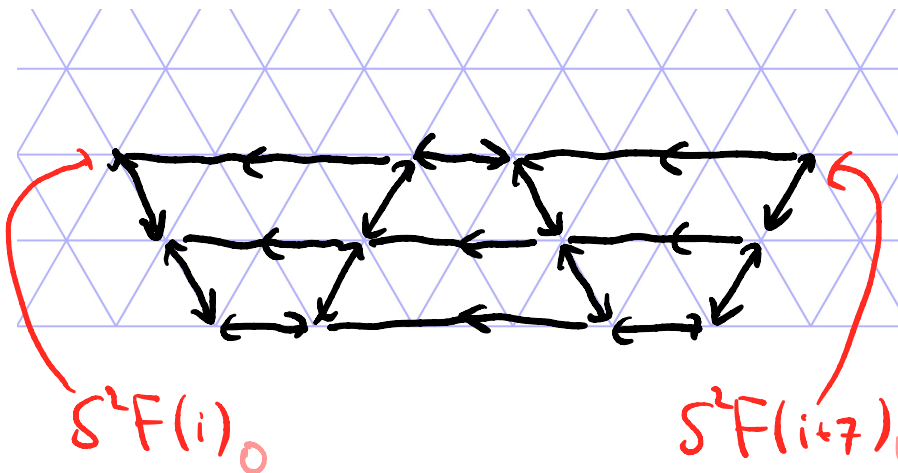
Find a homomorphism
and take its cone



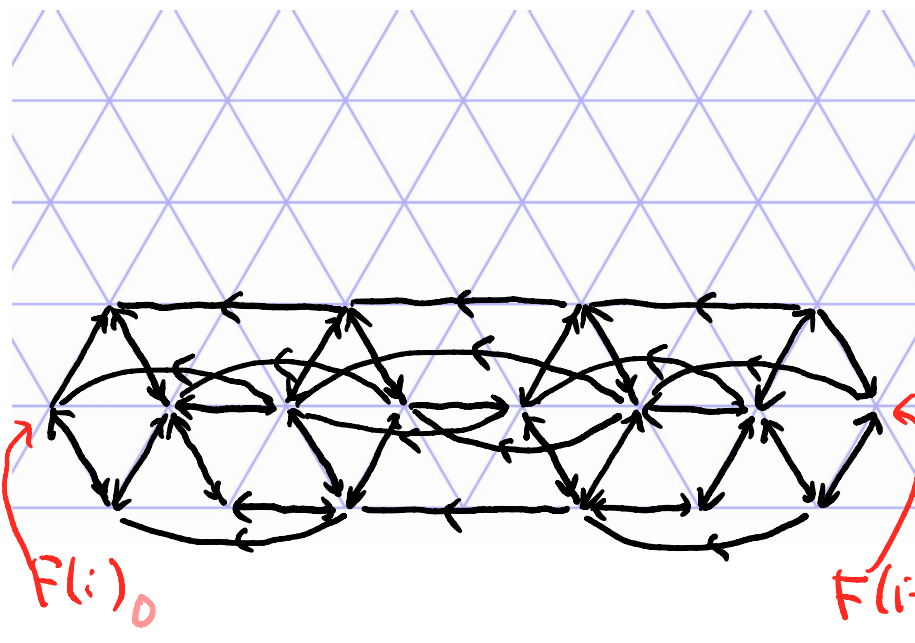
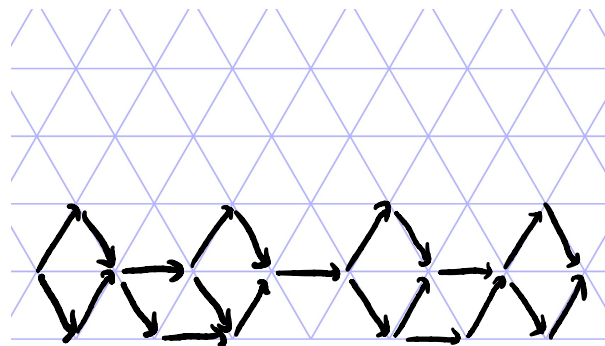
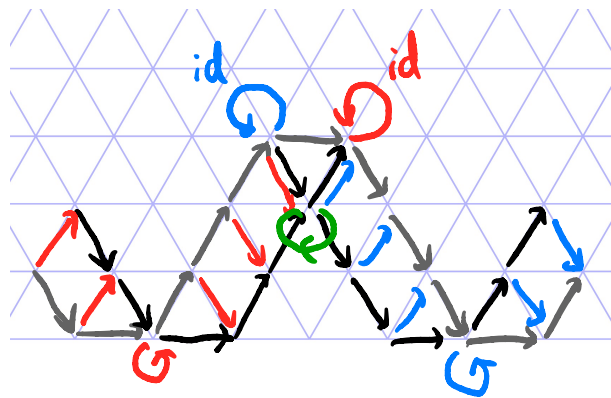
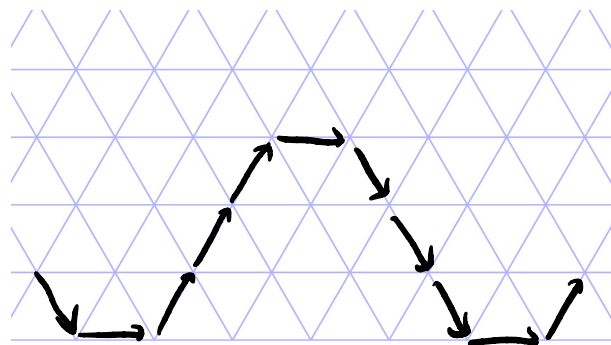
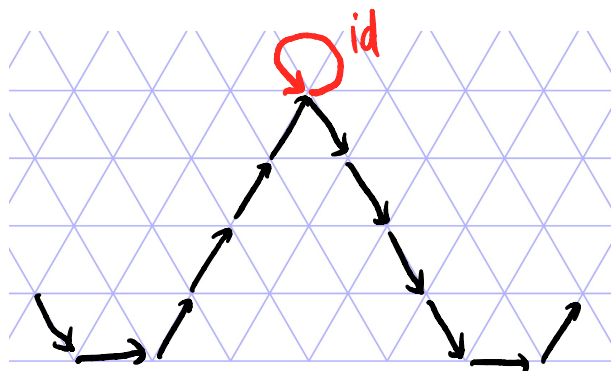
Brane-antibrane annihilation
via tachyon condensation of id



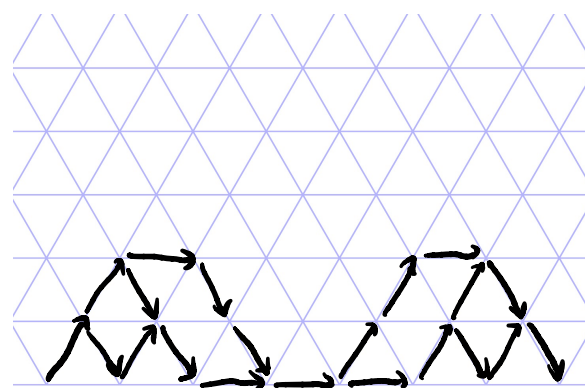
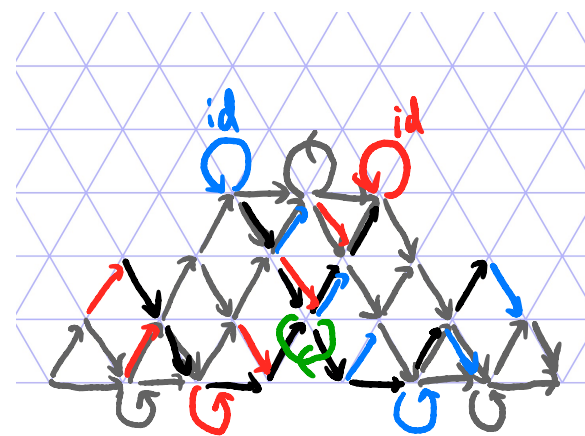
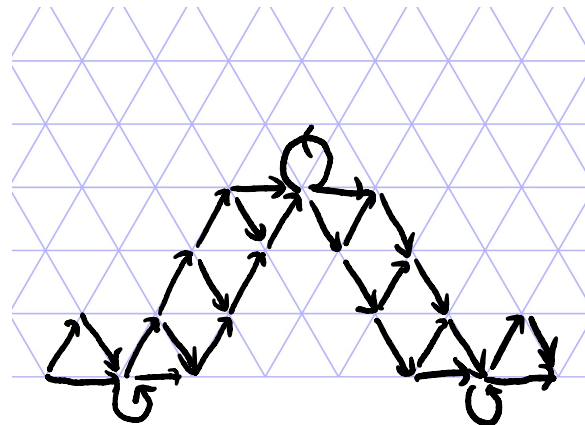
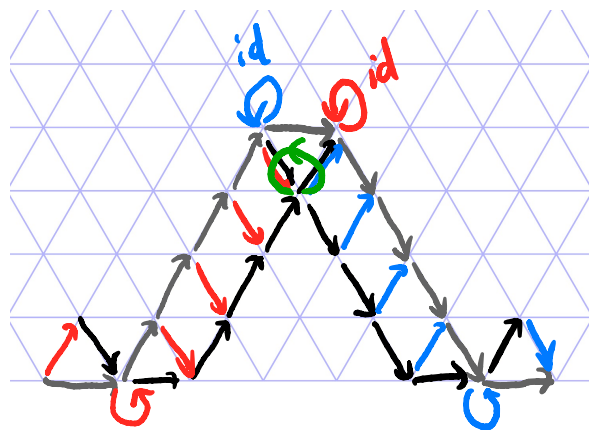
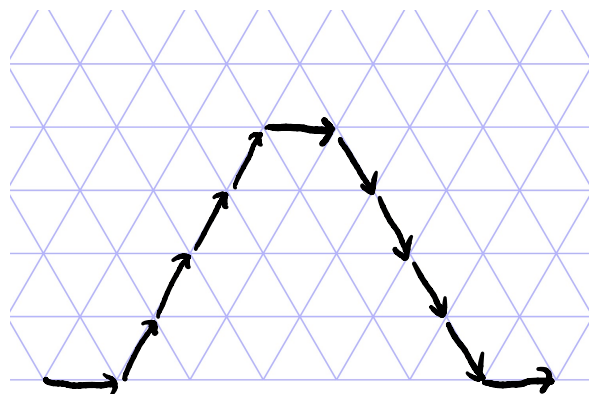
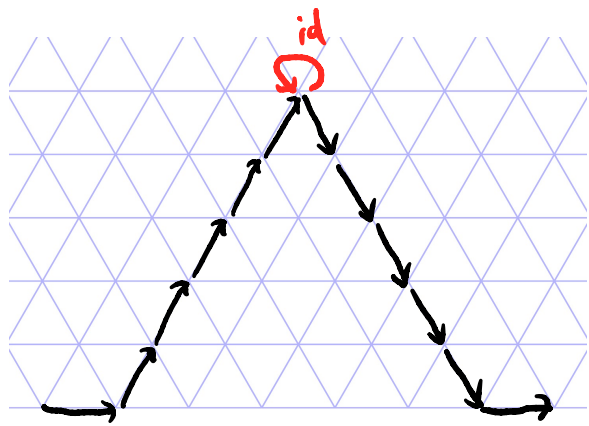
Find a matrix factorization
completion

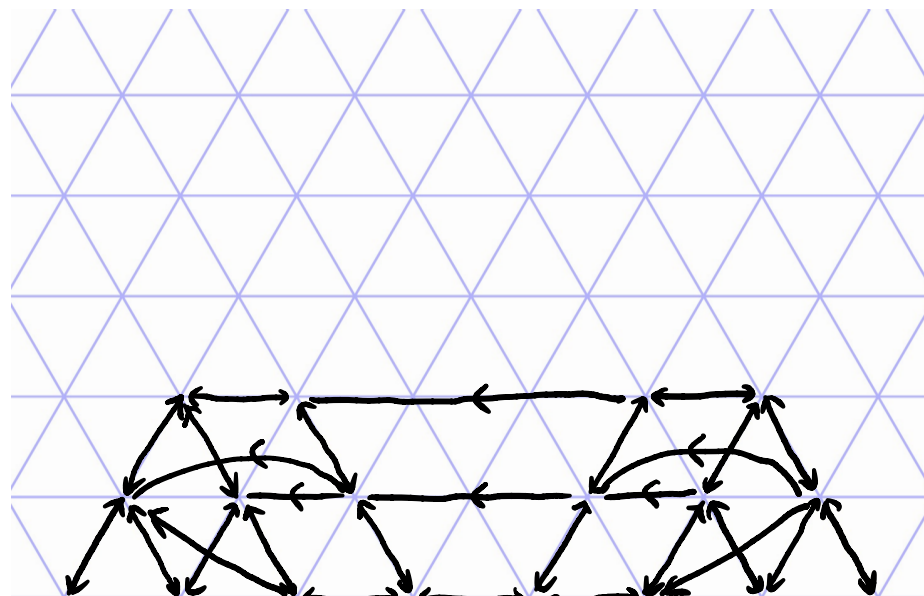


call it $\mathcal{L}^2(i)$



call it $L'(i)$





call it $\mathcal{L}^0(i)$

$\mathcal{C}(i)_0$

$\mathcal{C}(i+7)_{11}$

$\mathcal{L}^h(i)$ $h=2,1,0$ are all $SU(2)$ grade restricted

& empty in the phase $\xi \gg 0$.

$$\langle \mathcal{L}^2(-1) \rangle = (2\pi)^3 e^t \mathcal{Z}_Y(-\mathcal{O}_Y)$$

$$\text{Pic } Y = \mathbb{Z} H$$

$$H \cdot \ell = 1$$

$$\langle \mathcal{L}^1 \rangle = (2\pi)^3 e^t \mathcal{Z}_Y(4\mathcal{O}_Y - 2\mathcal{O}_H - 14\mathcal{O}_\ell + 12\mathcal{O}_{pt})$$

$$\langle \mathcal{L}^0 \rangle = (2\pi)^3 e^t \mathcal{Z}_Y(-5\mathcal{O}_Y + 14\mathcal{O}_\ell - 14\mathcal{O}_{pt})$$

$\mathcal{L}^h(i)$ are candidates for D6-branes of ranks $-1, 4, -5$.
 $h=2, 1, 0$

D-branes in $SU(2)$ supersymmetric QCD

$$Q = SU(2), \quad V = F^{\oplus N} \quad (F = \mathbb{C}^2): \text{ regular if } N \text{ is odd}$$

$$\text{GRR: only } \text{Sym}^h F \quad h = 0, 1, \dots, \frac{N-3}{2} \text{ are allowed}$$

$$\underline{N=3 \text{ with } W=0}$$

IR limit is the free theory of $b_{ij} = [X_i X_j] \quad 1 \leq i, j \leq 3 \quad (b_{ij} = -b_{ji})$

i.e. sigma model with target $\mathbb{C}^3 = \{(b_{12}, b_{23}, b_{31})\}$

Claim

$(M, Q) = (\mathbb{C}, 0)$ flows to Neumann B.C. on b_{ij} 's.

$$N=7 \text{ with } W = \sum_{\mu, \nu=4}^7 A_{*}^{\mu\nu} [X_{\mu} X_{\nu}]$$

IR limit is still the free theory of $b_{ij} = [X_i X_j] \quad 1 \leq i, j \leq 3 \quad (b_{ij} = -b_{ji})$

A matrix factorization $\mathcal{B}_{3,7}^{SU(2)}$

$$Q = \sum_{\mu=4}^7 \sum_{a=1}^2 x_{\mu}^a \xi_a^{\mu} \quad \{ \xi_a^{\mu}, \xi_b^{\nu} \} = 2 A_{*}^{\mu\nu} \epsilon_{ab}$$

$$M = \text{Clifford module} \cong (\mathbb{C} \oplus F \oplus \mathbb{C})^{\otimes 2} \quad \text{grade restricted } \checkmark$$

Claim

$\mathcal{B}_{3,7}^{SU(2)}$ flows to the Neumann B.C. on b_{ij} 's

Evidences of the claims :

Match of the hemisphere partition functions

$B_{3,7}^{SU(2)}$ has $\underbrace{Sp(F)}_{SU(2)} \times Sp(R)$ symmetry
 $\nwarrow (\mathbb{C}^4, A_*)$ symplectic vector space

A manifestly symmetric form :

$$Sym^2 F^* \otimes \Lambda^4 R^* \longleftrightarrow F^* \otimes \Lambda^3 R^* \longleftrightarrow \mathbb{C} \otimes \Lambda_0^2 R^*$$

$\uparrow$
 A_* traceless

The arrows are the same as the ones for the Lascaux branes

with inclusion/projection $\Lambda_0^2 R^* \rightleftarrows \Lambda^2 R^*$

The low energy behaviour of the D0/D6 candidates

$U(2)$ gauge coupling has two components $\begin{cases} e_1 & \tilde{u}^{(1)} \\ e_2 & \tilde{u}^{(2)} \end{cases}$
 $\{A_k^{ij}\} = \text{mass} \leadsto E_A: \text{typical energy scale}$

We work with the hierarchy of energy scales

$$\underbrace{E_A} \ll \underbrace{e_2}_{SU(2) \text{ dynamics}} \ll \underbrace{E_A \sqrt{-S}} \ll \underbrace{e_1 \sqrt{-S}}_{U(1) \text{ Higgsing}}$$

mass from $\sum A_k^{ij} p^k b_{ij}$ mass from $\sum A^{\mu\nu}(p_*) [x_\mu x_\nu]$

R-charge frame change: $x_1, \dots, x_7, p^1, \dots, p^7$
 $0 \rightarrow 1 \quad 2 \rightarrow 0$

$E \ll e_1 \sqrt{f}$ $U(1)$ is Higgsed: $SU(2)$ gauge theory / $\mathbb{P}^6$

Take $p_* \in Y$, $A(p_*) = \begin{pmatrix} O_{3 \times 3} & 0 \\ 0 & A_* \end{pmatrix}$ 4×4 full rank

$p_* \in \mathcal{U} \subset \mathbb{P}^6$ with a p -dependent change of basis for $\mathcal{X}_i$'s

$$A(p) = \begin{pmatrix} (p^{ij}) \\ A_* \end{pmatrix} \quad p^1, p^2, p^3, p^{12}, p^{23}, p^{31} \text{ coordinates of } \mathcal{U}$$

$$\mathcal{U} \cap Y = \{ p^{12} = p^{23} = p^{31} = 0 \}, \quad p_* = \{ 0, 0, 0, 0, 0, 0 \}$$

$$W = \frac{1}{2} \sum_{i,j=1}^3 p^{ij} [X_i, X_j] + \sum_{\mu,\nu=1}^3 A_*^{\mu\nu} [X_\mu, X_\nu]$$

$$W = \frac{1}{2} \sum_{i,j=1}^3 p^{ij} [x_i x_j] + \sum_{\mu,\nu=4}^7 A_{*}^{\mu\nu} [x_{\mu} x_{\nu}]$$

$E \ll E_A \sqrt{5}$ x_4, x_5, x_6, x_7 decouple

$$W = \frac{1}{2} \sum_{i,j=1}^3 p^{ij} [x_i x_j]$$

$E \ll e_2$ $SU(2)$ confines : $[x_i x_j]$ become scalars b_{ij}

$$W = \frac{1}{2} \sum_{i,j=1}^3 p^{ij} b_{ij}$$

$E \ll E_A$ $p^{12}, p^{23}, p^{31}, b_{12}, b_{23}, b_{31}$ decouple

Left with p^1, p^2, p^3 moving in $\mathcal{U} \cap Y$

A m.f. of $W = \frac{1}{2} \sum_{i,j=1}^3 p^{ij} [x_i x_j] + \sum_{\mu,\nu=4}^7 A_*^{\mu\nu} [x_\mu x_\nu]$

$$Q = \sum_{i < j} \left(p^{ij} \xi_{ij} + [x_i x_j] \bar{\xi}^{ij} \right) + \sum_{a,\mu} x_\mu^a \xi_a^\mu$$

$$\{ \xi_{ij}, \bar{\xi}^{kl} \} = \delta_i^k \delta_j^l - \delta_i^l \delta_j^k, \quad \{ \xi_a^\mu, \bar{\xi}_b^\nu \} = 2 A_*^{\mu\nu} \epsilon_{ab}$$

$\mathcal{M} =$ Clifford module

$$\underline{E_A \ll E \ll e_2} \xrightarrow{\text{Claim}} Q = \sum_{i < j} \left(p^{ij} \xi_{ij} + b_{ij} \bar{\xi}^{ij} \right)$$

all Neumann B.C.

$$\underline{E \ll E_A} \xrightarrow{\text{Knorrer}} p^1, p^2, p^3 \quad \text{Neumann B.C.}$$

This is a D6-brane. Call it $\mathcal{B}_u^{D6}$

Another m.f. of $W = \frac{1}{2} \sum_{i < j}^3 p^{ij} [x_i x_j] + \sum_{\mu, \nu}^7 A_{\mu\nu}^* [x_\mu x_\nu]$

$$Q = \sum_{k=1}^3 p^k \eta_k + \sum_{i < j} \left(p^{ij} \xi_{ij} + [x_i x_j] \bar{\xi}^{ij} \right) + \sum_{a, \mu} x_\mu^a \bar{\xi}_a^\mu$$

$$\{\eta_k, \bar{\eta}^l\} = \delta_k^l, \quad \{\xi_{ij}, \bar{\xi}^{kl}\} = \delta_i^k \delta_j^l - \delta_i^l \delta_j^k, \quad \{\bar{\xi}_a^\mu, \bar{\xi}_b^\nu\} = 2 A_{\mu\nu}^* \epsilon_{ab}$$

$M =$ Clifford module

$$\underline{E_A \ll E \ll e_2} \xrightarrow{\text{Claim}} Q = \sum_{k=1}^3 p^k \eta_k + \sum_{i < j} \left(p^{ij} \xi_{ij} + b_{ij} \bar{\xi}^{ij} \right)$$

all Neumann B.C.

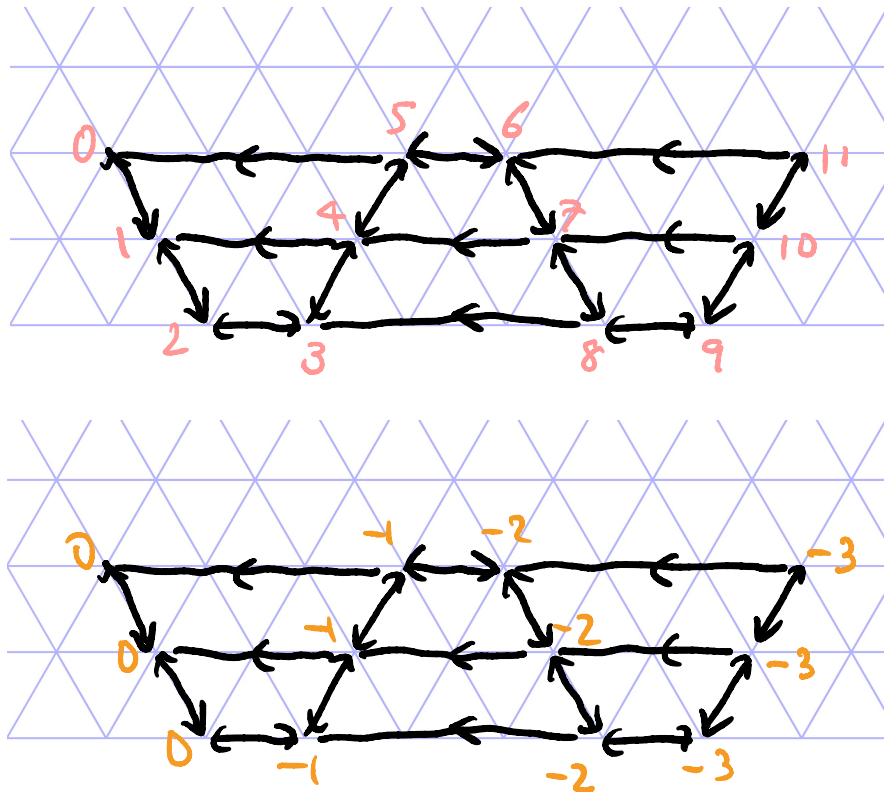
$$\underline{E \ll E_A} \xrightarrow{\text{Knorrer}} Q = \sum_{k=1}^3 p^k \eta_k \quad \text{Neumann B.C.}$$

This is a D0-brane at $p^1 = p^2 = p^3 = 0$. Call it $\mathcal{B}_{p^*, u}^{\text{D0}}$

The D0-candidate $\left| \mathcal{U} \right\rangle \cong \mathcal{B}_{p_*, \mathcal{U}}^{\text{D0}}$

So it is indeed a D0-brane at p_* , $\mathcal{O}_{p_*}$

D6-candidates. First $\mathcal{L}^2(-1)$



Change of
R-charge frame

$$\begin{array}{ccccccc}
S^2 F^* \otimes \wedge^7 V^* & \longleftarrow & S^2 F^* \otimes V^* & \longleftrightarrow & S^2 F^* \otimes V & \longleftarrow & S^2 F^* \otimes \wedge^7 V \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
F^* \otimes \wedge^6 V^* & \longleftarrow & F^* \otimes \wedge^2 V^* & \longleftarrow & F^* \otimes \wedge^2 V & \longleftarrow & F^* \otimes \wedge^6 V \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\wedge^5 V^* & \longleftrightarrow & \wedge^3 V^* & \longleftarrow & \wedge^3 V & \longleftarrow & \wedge^5 V \\
0 & & -1 & & -2 & & -3
\end{array}$$

Insert $A(p) = \begin{pmatrix} (p^{ij}) \\ A_* \end{pmatrix}$ and perform brane-antibrane annihilation at each invertible component of the arrows

$S^2 F^* \otimes \wedge^4 R^*$ $\updownarrow$ $F^* \otimes \wedge^3 R^*$ $\updownarrow$ $\wedge_0^2 R^*$	$\xleftarrow{\cong}$	$S^2 F^*$ $\updownarrow$ $F^* \otimes R^*$ $\updownarrow$ $\wedge_0^2 R^*$	$\xleftarrow{\cong}$	$S^2 F^*$ $\updownarrow$ $F^* \otimes R$ $\updownarrow$ $\wedge_0^2 R$	$\xleftarrow{\cong}$	$S^2 F^* \otimes \wedge^4 R$ $\updownarrow$ $F^* \otimes \wedge^3 R$ $\updownarrow$ $\wedge_0^2 R$
$\otimes$ $\wedge^3 T^*$ 0		$\otimes$ T^* -1		$\otimes$ T -2		$\otimes$ $\wedge^3 T$ -3

$$T \cong \mathbb{C}^3$$

The bottom line

$$\mathcal{B}_T$$

$$\cong$$

$$\begin{cases} Q = \frac{1}{2} \sum (p^i \xi_{;i} + (x_i x_j) \bar{\xi}^{ij}) \\ M = \text{Clifford module} \end{cases}$$

$$\cong \mathcal{B}_{3,7}^{SU(2)} \otimes \mathcal{B}_T \cong \mathcal{B}_u^{D6} !$$

So, it indeed flows to a single D6-brane.

To fix the global topology, use hemisphere information.

Then, we find

$$\mathcal{L}^2(-1) \xrightarrow{\pi_Y} \mathcal{O}_Y \underbrace{[-1]}_{\text{convention}}$$

Similarly, we find

$$\mathcal{L}^1 \xrightarrow{\pi_Y} \mathcal{R}^*$$

$$\mathcal{L}^0 \xrightarrow{\pi_Y} \Lambda_0^2 \mathcal{R}^*(1)[-1]$$

where $\mathcal{R} \big|_p = V / \text{Ker } A(p)$

Lifting branes in Pfaffian Calabi-Yau to gauge theory

We would like to find a lift $D(Y) \rightarrow S \subset D_G(V, W)$

Initially, we had no idea. But there is an existing work [ADS 2014](#)

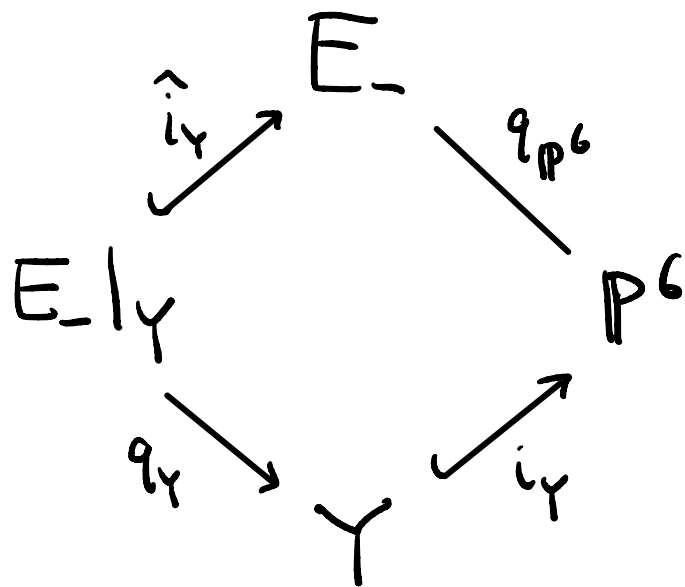
Let us examine whether this is relevant for us.

$E \ll e, \sqrt{-s}$: branes are in $D_{G_c}^b(V_-, W) \cong D^b(E_-, W_{ind})$

$$V_- = \{ (x, p) \mid p \neq 0 \}$$

$E_- = V_- / G_c$ the quotient stack

ADS has a functor $\Psi_{\text{ADS}} : D^b(Y) \rightarrow D^b(E_-, W_{ind})$



$$\Gamma \subset E_-|_Y \ni (x, p), \quad \dim \frac{\chi(F^*)}{\chi(F^*) \cap \text{Ker } A(p)} \leq 1$$

$$\mathcal{F} \in \text{Coh } Y \xrightarrow{\Psi} \left(\hat{i}_{Y*} (\mathcal{O}_\Gamma \otimes q_Y^* \mathcal{F}), 0 \right) \in \mathcal{D}^b(E_-, W_{\text{ind}})$$

“curved dg-sheaf”

$\mathcal{O}_F \in \text{Coh}(E|Y)$ has Eagon-Northcott resolution

$$\text{Sym}^2 F^*(-1) \otimes \Lambda^4(V/K)^* \rightarrow F^*(-1) \otimes \Lambda^3(V/K)^* \rightarrow \mathbb{C}(-1) \Lambda^2(V/K)^* \rightarrow \mathbb{C}$$

It has a m.f. completion and it is $\mathbb{C}(-1) \otimes$ of

$$\text{Sym}^2 F^* \otimes q_Y^* \Lambda^4 R^* \longleftrightarrow F^* \otimes q_Y^* \Lambda^3 R^* \longleftrightarrow \mathbb{C} \otimes q_Y^* \Lambda^2 R^* \longleftrightarrow \mathbb{C}(1)$$

||2

$$\text{Sym}^2 F^* \otimes q_Y^* \Lambda^4 R^* \longleftrightarrow F^* \otimes q_Y^* \Lambda^3 R^* \longleftrightarrow \mathbb{C} \otimes q_Y^* \Lambda^2_0 R^*$$

This is a globalization of $\mathcal{B}_{3,7}^{SU(2)}$!

Call it $\mathcal{B}_A$ & $\mathcal{B}_{A,i} := \mathcal{B}_A \otimes \mathbb{C}(i)$

Take $F \in \text{Wh}(Y)$. Let us first look locally.

$$p_x \in \mathcal{U} \subset \mathbb{P}^6 \text{ as before } \mathcal{U}_Y := \mathcal{U} \cap Y$$

$$\underline{E_A \ll E \ll e_2} \quad E_{\text{int}, \mathcal{U}} = \{ p^{(1,2,3)}, p^{ij}, b_{ij} \} \xrightarrow{W_{\text{int}}} \frac{1}{2} \sum p^{ij} b_{ij}$$

$$\text{Knorrer } F|_{\mathcal{U}_Y} \in D(\mathcal{U}_Y) \mapsto \kappa(F|_{\mathcal{U}_Y}) \in D(E_{\text{int}, \mathcal{U}}, W_{\text{int}})$$

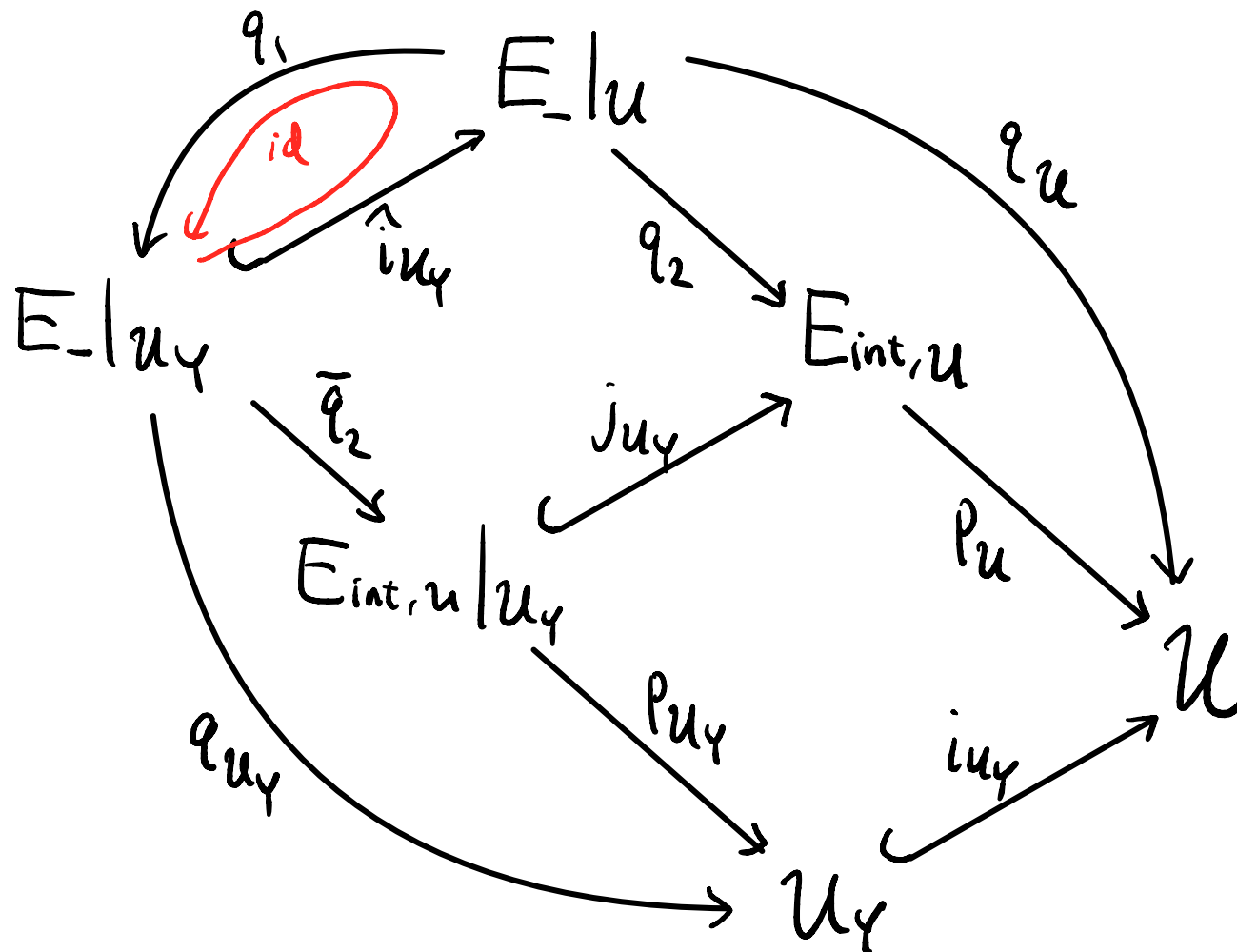
$$\underline{E_A \sqrt{-5} \ll E \ll e_1 \sqrt{-5}} \quad V_-|_{\mathcal{U}} = \{ p^{(1,2,3)}, p^{ij}, x_i^a \}$$

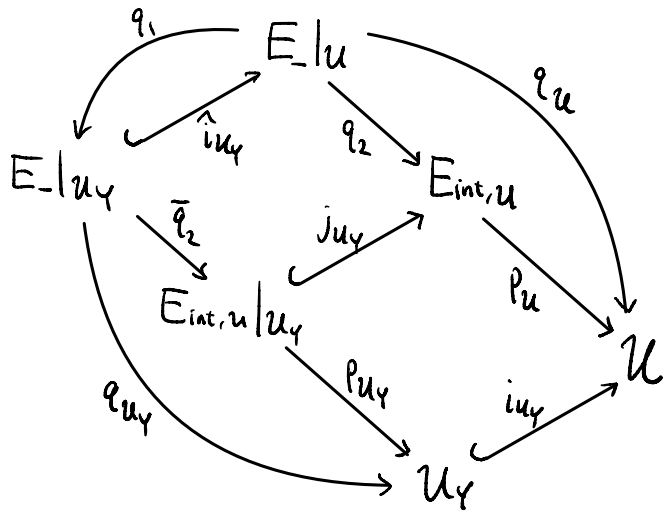
$$\underbrace{\frac{1}{2} \sum_{i,j=1}^3 p^{ij} [x_i x_j]}_{q_2^* W_{\text{int}}} + \underbrace{\sum_{\mu, \nu=4}^7 A_{\mu}^{\nu} [x_{\mu} x_{\nu}]}_{q_1^* W|_{V_-|_{\mathcal{U}_Y}}}$$

$$\begin{array}{ccc} & V_-|_{\mathcal{U}} & \\ q_1 \swarrow & & \searrow q_2 \\ V_-|_{\mathcal{U}_Y} & & E_{\text{int}, \mathcal{U}} \end{array}$$

A lift of $\mathcal{K}(F|_{u_Y})$: $q_1^* \overset{\text{say}}{\mathcal{B}_A|_{V|u_Y}} \otimes q_2^* \mathcal{K}(F|_{u_Y})$

Want to glue them together. Not obvious as things depend on choices. Need choice independent expression.





$$\begin{aligned} \kappa(\mathcal{F}|u_Y) &= j_{u_Y*} p_{u_Y}^*(\mathcal{F}|u_Y) \\ &\cong p_u^* i_{u_Y*}(\mathcal{F}|u_Y) \end{aligned}$$

$$\begin{aligned} q_1^* \mathcal{B}_A|_{E|u_Y} \otimes q_2^* \kappa(\mathcal{F}|u_Y) &\cong q_1^* \mathcal{B}_A|_{E|u_Y} \otimes \underbrace{q_2^* p_u^*}_{\hat{i}_{u_Y*} q_{u_Y}^*} i_{u_Y*}(\mathcal{F}|u_Y) \\ &\cong \hat{i}_{u_Y*} (\mathcal{B}_A|_{E|u_Y} \otimes q_{u_Y}^*(\mathcal{F}|u_Y)) \end{aligned}$$

This is choice independent and glues to

$$\hat{i}_{Y*} (\mathcal{B}_A \otimes \ell_Y^* \mathcal{F}) =: \Psi_{\mathcal{B}_A}(\mathcal{F}) \in D^b(E_-, W_{ind})$$

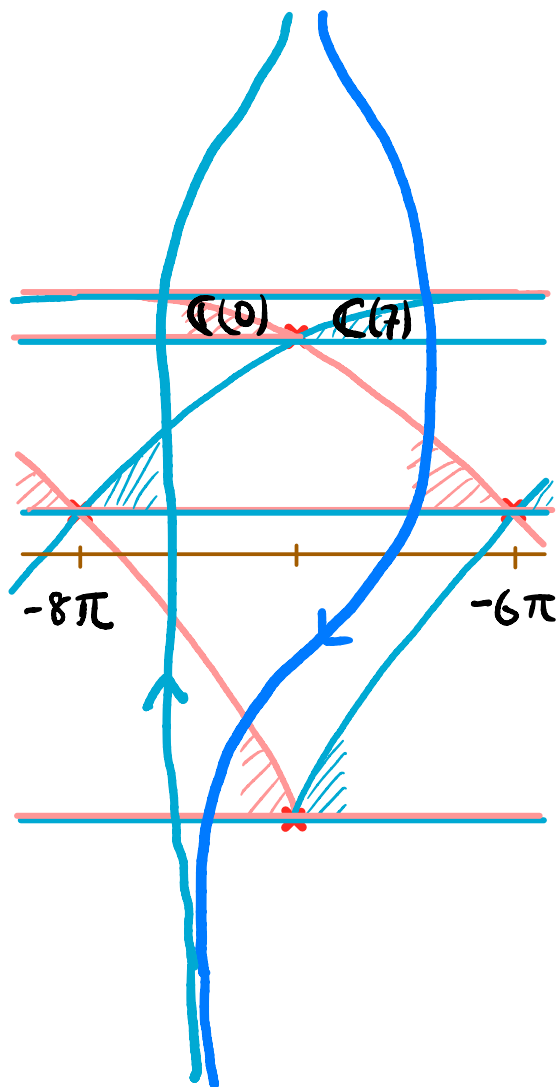
We can fix which globalization $\mathcal{B}_{A,i}$ should be taken by information from hemisphere. It is $\mathcal{B}_{A,-1}$.

The lift is $\Psi_{\mathcal{B}_{A,-1}}$.

Comparison to ADS 2014

$$\Psi_{\text{ADS}} = [-2] \circ \Psi_{\mathcal{B}_{A,-1}}$$

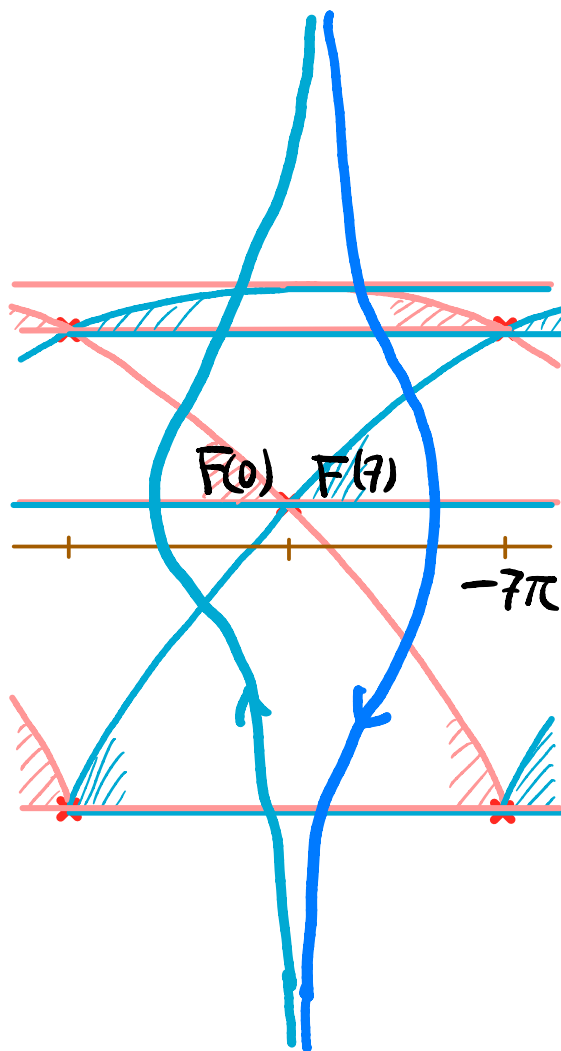
So, indeed, their lift is what we wanted up to a shift.



replacement

$$C(7) \rightsquigarrow C(0)$$

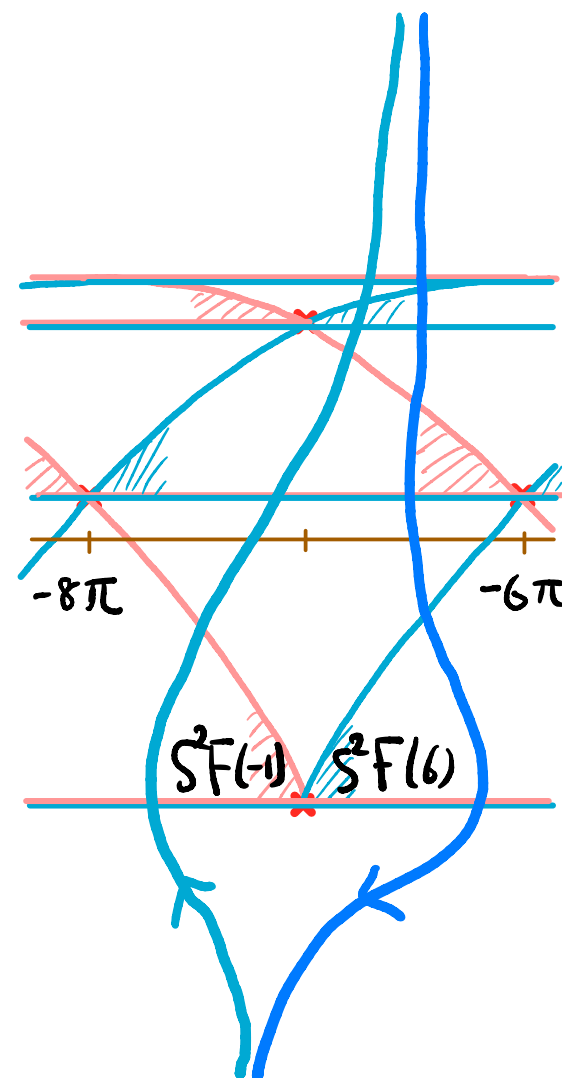
needed



replacement

$$F(7) \rightsquigarrow F(0)$$

needed



replacement

$$S^2F(6) \rightsquigarrow S^2F(-1)$$

needed

For monodromies along loops with a base point at $\zeta \ll 0$
the replacement can be done using empty branes in $\zeta \gg 0$.

Use the **Lascoux branes**

$$\mathcal{L}^0$$

$$\mathcal{L}^1$$

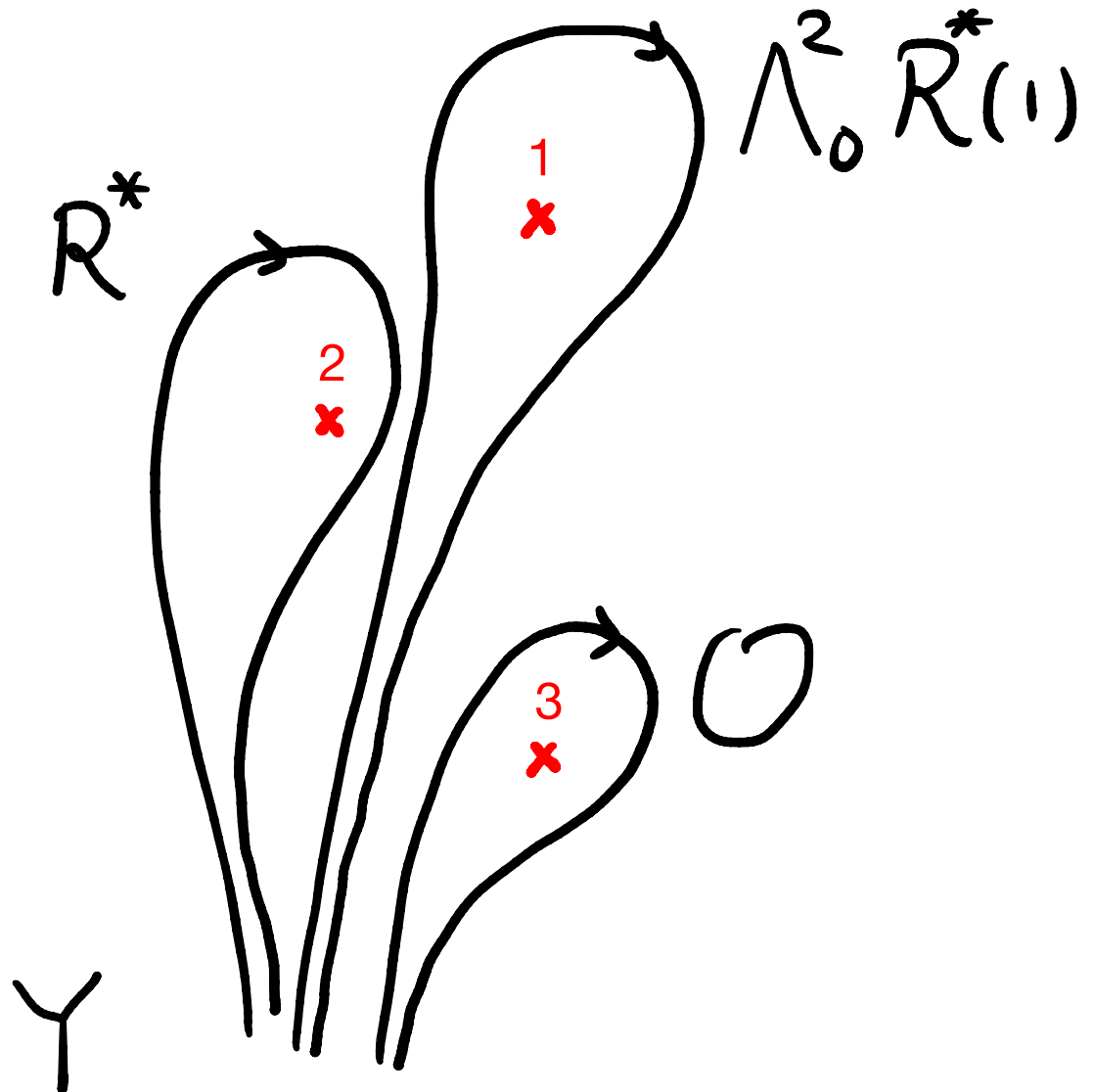
$$\mathcal{L}^2(-1)$$

For monodromies along loops with a base point at $\zeta \ll 0$
the replacement can be done using empty branes in $\zeta \gg 0$.

Use the **Lascoux branes**

$$\begin{array}{ccc}
 \mathcal{L}^0 & \mathcal{L}^1 & \mathcal{L}^2(-1) \\
 \downarrow \zeta \ll 0 & \downarrow \zeta \ll 0 & \downarrow \zeta \ll 0 \\
 \Lambda_0^2 R^*(1)[-0] & R^* & \mathcal{O}_Y[-1]
 \end{array}$$

Seidel-Thomas twists by



Other updates:

The methods applicable in other models

Dual Rodland, Hosono-Takagi & its dual, N-Rodland

Fourier-Mukai presentation of the equivalences

$\mathcal{P}_{\mathbf{w}}$	has rank	4	vector bundle	A
		1	2-term complex of v.b.'s	B
		-1	2-term complex of v.b.'s	C
		-4	vector bundle	D

Borisov-Caldadaru 2006

