Phases of torsional linear models

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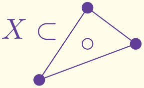
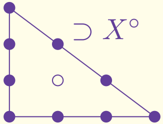
Motivations: (0,2) explorations

- (2,2) GLSMs & Calabi–Yau manifolds
  - framework for CY/LG correspondence
  - mirror symmetry for hypersurfaces & complete intersections
  - (sub)space of deformations & monomial divisor mirror map





- (0,2) linear models
  - e.g. mirror symmetry involves HYM bundle $(X, \mathcal{E}) \longleftrightarrow (X^{\circ}, \mathcal{E}^{\circ})$
  - most of the focus: (0,2) deformations of (2,2) theories

$$\mathcal{E} \simeq T_X$$

  - more general bundles? some mirror pairs from LG phase of GLSMs

- (0,2) GLSM constructions compromised by bundle stability

Motivations: (0,2) explorations

in what follows: look for new (0,2) theories with

- a large radius (LR) phase with non-Kähler geometry
- a Landau–Ginzburg (LG) phase with solvable points

Motivation 1
Target manifold
with SU(3) structure
for heterotic strings

Motivation 2
(0,2) GLSM
with non-trivial
IR dynamics

along the way:

- phases of a torsional model
- new (0,2) features
  trivial bundle degrees of freedom, field redefinitions & gauge anomalies

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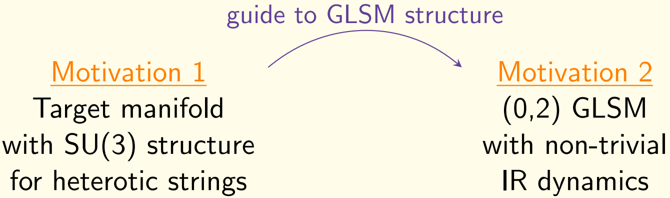

guide to GLSM structure

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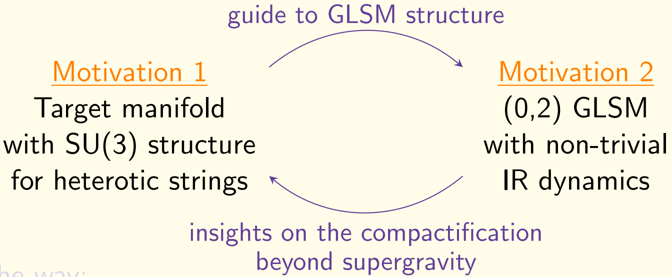

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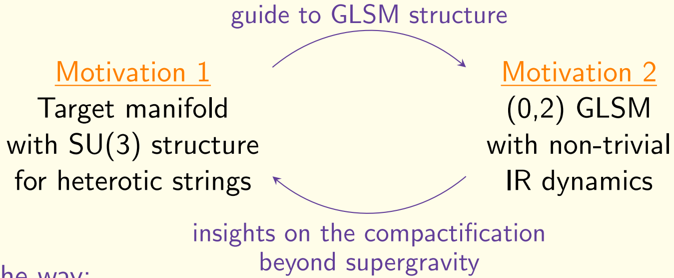

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Outline

1. heterotic flux vacua
2. the (0,2) torsional GLSM: construction
3. the (0,2) torsional GLSM: phase analysis
4. generalizations & abelian bundles

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The heterotic string on $\mathbb{R}^{1,3} \times X_6$

String worldsheet
(0,2) worldsheet susy
(& integral R-charges)

⇔

Target space
minimal susy in 4d
(four supercharges)

- in NLSM realization:
  - X complex threefold: Hermitian (1,1) form J
  - holomorphically trivial canonical bundle: (3,0) form Ω
  - holomorphic bundle $\mathcal{E}$
- differential conditions for J and Ω [Hull'86,Strominger'86]
  & Hermitian Yang–Mills (HYM) equations for $\mathcal{E}$
- Bianchi identity

$$\mathrm{d}H = \alpha' \left(\operatorname{tr} R \wedge R - \operatorname{tr} F \wedge F\right)$$

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α'= string scale
→ string-size cycles

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Compactifications with torsion

Flux $H = \mathrm{i}(\bar\partial - \partial)J$
obstructs Kähler condition for X $\quad H =$ torsion

- Calabi–Yau backgrounds with $\mathcal{E} = T_X \longrightarrow$ (2,2) worldsheet susy
  & deformations of T_X

- non-trivial H?
  - ✗ non-projective
  - ✗ no existence results for PDEs
  - ✗ no large radius supergravity (quantized cycles)

- Flux solutions with eight supercharges
  - ✓ dual to F-theory on K3×K3 [Dasgupta–Rajesh–Sethi'99]
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  - ✓ decompactification limit to 8d

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Flux vacua with eight supercharges

- $X = T^2$ principal fibration over K3 surface M [Goldstein–Prokushkin'02]
  fibration specified by two anti-self-dual (ASD) classes

$$\omega^1, \omega^2 \in H^2(M, \mathbb{Z}) \cap H^{1,1}(M)$$

flux has
"vertical" component
$H_I = -\alpha' \mathcal{G}_{IJ} \omega^J$

- HYM bundle $\mathcal{E} \to M$ pulled back to total space

K3×T^2 as a special case
$\omega^1, \omega^2 = 0$

- five-dimensional flux vacua

$$X_6 = X_5 \times S^1$$

S^1 fibration over M specified by ASD class ω

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non-trivial part
of the NLSM

necessary
for (0,2) susy

S^1 fibration over M specified by ASD class ω

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- Moduli quantization [Melnikov-Minasian-Theisen'14]

  two-form potential for H has integral periods

$$\mathrm{R}^2\omega \in H^2(M,\mathbb{Z})$$

- Bianchi identity = scalar equation on K3

  integrated over M:

$$24 = \mathrm{R}^2(-\omega.\omega) + k(\mathcal{E})$$

- "purely geometric" configurations with trivial bundle

  characterized by:

  - K3 surface with Picard rank $\varrho(M) \geq 2$
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instanton number

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Flux vacua & (non-)linear sigma models

- NLSM on X has large quantum corrections due to α'-size cycles
- IR limit of torsional linear sigma model (TLSM)

  [Adams–Ernebjerg–Lapan'08]
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Motivation 3

Construct a consistent TLSM

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the (0,2) torsional GLSM: a recipe

Ingredient 1

(0,2) superfields
for K3 hypersurface
(or complete intersection)

Ingredient 2

(0,2) superfields
for monad bundle
over K3

Ingredient 3

(0,2) shift mutiplets
“fibered” over
the GLSM

interact via U(1)$\times\cdots\times$U(1) gauge fields of the GLSM

- set-up: (0,2) superspace

left-moving fermions in
fermi multiplets Γ_A

$$Q_{\mathsf{fermi}} = \begin{pmatrix} Q^{\mathsf{f}}_1 & \cdots & Q^{\mathsf{f}}_{N_{\mathsf{fermi}}} \end{pmatrix}$$

right-moving fermions in
chiral multiplets Φ_i

$$Q_{\mathsf{chiral}} = \begin{pmatrix} Q^{\mathsf{c}}_1 & \cdots & Q^{\mathsf{c}}_{N_{\mathsf{chiral}}} \end{pmatrix}$$

- Gauge anomalies encoded in anomaly matrix

$$\mathcal{A} = Q_{\mathsf{fermi}}(Q_{\mathsf{fermi}})^{\mathrm{t}} - Q_{\mathsf{chiral}}(Q_{\mathsf{chiral}})^{\mathrm{t}}$$

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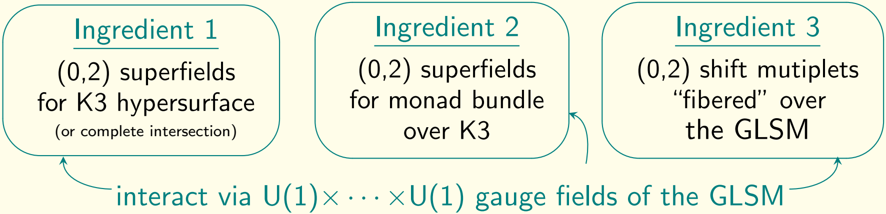

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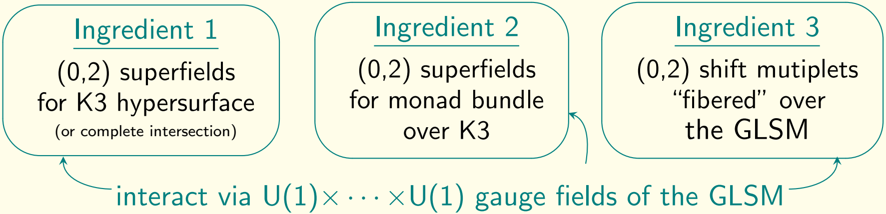

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- Toric ambient space specified by reflexive polytope $\Delta \subset \mathbb{R}^3$
  - chiral field Φ_i for each ray ρ_i
  - U(1) vector multiplet Υ_α for each linear relation between the rays
  - a single fermi multiplet Γ_0
- Superpotential

$$\mathcal{W} \supset \Gamma_0 J + (\tfrac{\vartheta^\alpha}{2\pi} + \mathrm{i}\varrho^\alpha)\Upsilon_\alpha$$

with gauge invariant hypersurface polynomial

$$J(\Phi) = \sum_{m \in \Delta \cap \mathbb{Z}^3} \alpha_m \prod_i \Phi_i^{\langle m, \rho_i \rangle + 1}$$

- Large radius phase?

remaining GLSM superfields must ensure exceptional locus

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Fayet–Iliopoulos (FI) parameter

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from F-terms & D-terms

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  - fermi multiplets $\Gamma_1, \dots, \Gamma_N$ of charge $Q_1, \dots, Q_N$
  - a single chiral multiplet Φ_0 of charge $Q_0 = -(Q_1 + \cdots + Q_N)$
  - (possibly) neutral chiral field $\mathcal{S}$

- Superpotential

$$\mathcal{W} \supset \Phi_0 \left(\Gamma_1 F^1 + \cdots + \Gamma_N F^N \right)$$

  - $F^A(\Phi)$ homogeneous polynomials in the K3 chiral fields
  - optionally: E-couplings $\bar{\mathcal{D}}\Gamma_A = E_A(\Phi)\,\mathcal{S}$

- Massless left-moving fermions in

$$0 \longrightarrow \mathcal{O} \underset{(E_1,\dots,E_N)}{\longrightarrow} \mathcal{O}(Q_1) \oplus \cdots \oplus \mathcal{O}(Q_N) \underset{(F^1,\dots,F^N)}{\longrightarrow} \mathcal{O}(-Q_0) \longrightarrow 0$$

- HYM equations? slope-stability? depends on F^A and on ϱ^α

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  - fermi multiplets $\Gamma_1, \ldots, \Gamma_N$ of charge $Q_1, \ldots, Q_N$
  - a single chiral multiplet Φ_0 of charge $Q_0 = -(Q_1 + \cdots + Q_N)$
  - (possibly) neutral chiral field $\mathcal{S}$

- Superpotential

$$\mathcal{W} \supset \Phi_0 \left(\Gamma_1 \, F^1 + \cdots + \Gamma_N \, F^N\right)$$

  - $F^A(\Phi)$ homogeneous polynomials in the K3 chiral fields
  - optionally: E-couplings $\bar{\mathcal{D}}\Gamma_A = E_A(\Phi)\, \mathcal{S}$

- Massless left-moving fermions in SU$(N-2)$ bundle $\mathcal{E}$

$$0 \longrightarrow \mathcal{O} \underset{(E_1,\ldots,E_N)}{\longrightarrow} \mathcal{O}(Q_1) \oplus \cdots \oplus \mathcal{O}(Q_N) \underset{(F^1,\ldots,F^N)}{\longrightarrow} \mathcal{O}(-Q_0) \longrightarrow 0$$

- HYM equations? slope-stability? depends on F^A and on ϱ^α

Ingredient 1
K3 surface GLSM

Ingredient 2
monad bundle GLSM

Ingredient 3
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Ingredient 2
monad bundle GLSM

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Ingredient 1
K3 surface GLSM

Ingredient 2
monad bundle GLSM

Ingredient 3
shift mutiplets

Torsion?

- in LR phase, divisor η_α for each U(1) vector field

$$c_2(T_M) - c_2(\mathcal{E}) = \tfrac{1}{2}\mathcal{A}^{\alpha\beta}\,\eta_\alpha\eta_\beta$$

- circle fibration with ASD class $\omega \to$ “charges” w^α in the GLSM s.t.

$$\omega = \mathrm{w}^\alpha\eta_\alpha$$

- topological Bianchi identity $\quad c_2(T_M) - c_2(\mathcal{E}) = \mathrm{R}^2(-\omega.\omega)$
  expanded in K3 classes: $\quad \left(\mathcal{A}^{\alpha\beta} + 2\mathrm{R}^2\mathrm{w}^\alpha\mathrm{w}^\beta\right)\eta_\alpha\eta_\beta = 0$

$\longrightarrow$ anomaly must be non-trivial & compensated by fibered circle

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Ingredient 3
shift mutiplets

Shift multiplet: chiral multiplet Ω with shift charge [Adams–Ernebjerg–Lapan'08]

$$\Omega \to \Omega + \mathrm{i}\mathrm{w}^{\alpha}\Xi_{\alpha}$$

Dynamical FI coupling (non-gauge-invariant)

$$\mathcal{W} \supset \quad -\mathrm{i}\mathrm{R}^2\mathrm{w}^{\alpha}\Omega\Upsilon_{\alpha}$$

Torsional linear model is anomaly free iff

$$\mathcal{A} = -2\mathrm{R}^2\mathrm{w}\mathrm{w}^{\mathrm{t}}$$

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for this value of the coupling:

- Ω absent from D-term
- radial field $\operatorname{re}\Omega$ decouples
- for T^2 fibration, $\operatorname{re}(\Omega^1, \Omega^2)$ = (0,2) radial multiplet

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Ingredient 1
K3 surface GLSM

Ingredient 2
monad bundle GLSM

Ingredient 3
shift mutiplets

challenge: given toric polytope for a (0,2) K3 GLSM, look for

$$\begin{array}{rcccc} & \Gamma_1 & \dots & \Gamma_N & \Phi_0 \\ Q_{\text{bundle}} = \Big(& Q_1 & \dots & Q_N & \Big| \, Q_0 \Big) \end{array} \qquad \text{with } \mathcal{A} = -2\mathrm{R}^2\mathrm{w}\mathrm{w}^{\mathrm{t}}$$

TLSM wishlist:

- multi-parameter model: at least U(1)×U(1) gauge group
  (Kähler class $J_{\text{K3}} = \varrho^1\eta_1 + \cdots + \varrho^r\eta_r$ & ASD class $\omega = \mathrm{w}^1\eta_1 + \cdots + \mathrm{w}^r\eta_r$)
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Two-parameter K3 models

K3 toric hypersurfaces $\longleftarrow$ reflexive polyhedra in 3d [Kreuzer–Skarke'98]
(4319 polytopes)

- $\#$(parameters) $= \#$(non-zero lattice points $\notin$ interior of a facet) -3
  $= \mathrm{rk}\, H^2(M,\mathbb{Z}) \cap H^{1,1}(M)$ for generic hypersurface

- twelve two-parameter models

- two of them have reflection symmetry

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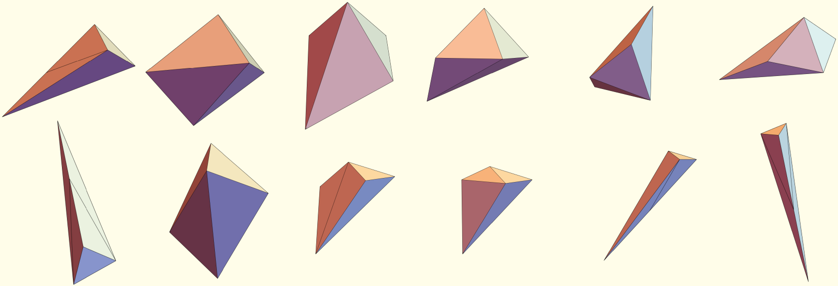

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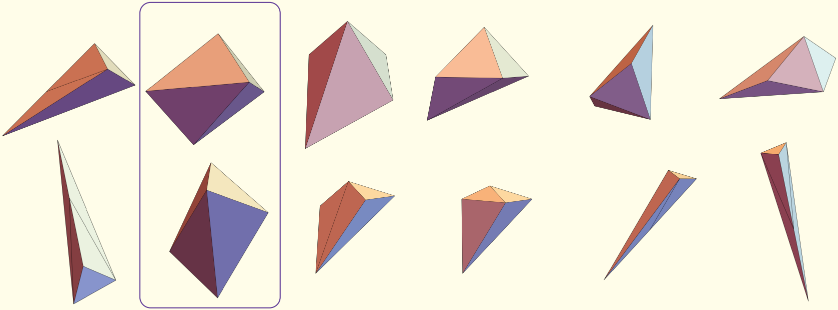

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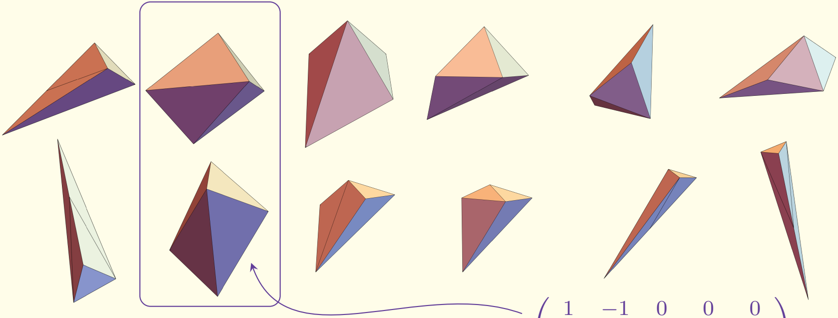

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$$P = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -2 & 0 & -2 & 1 \end{pmatrix}$$

Geometry of the K3 surface

$$Q_{\mathsf{K3}} = \begin{array}{c} \\ \end{array}\begin{array}{c|ccccc} \Gamma_0 & \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 \\ -4 & 1 & 1 & 0 & 0 & 2 \\ -4 & 0 & 0 & 1 & 1 & 2 \end{array}$$

Large radius phase of the (2,2) model?

- two LR phases with $\varrho_1, \varrho_2 > 0$
- conifold singularities when $\varrho_1 = \varrho_2$

  lines of $\mathbb{C}^2/\mathbb{Z}_2$ at $\phi_1 = \phi_2 = 0$ & $\phi_3 = \phi_4 = 0$
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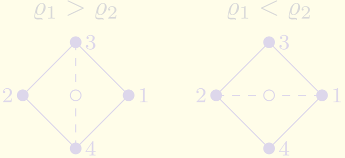

$$J = \Phi_5^2 - f(\Phi_1, \ldots, \Phi_4)$$

degree 4 in Φ_1, Φ_2
and in Φ_3, Φ_4

Branched cover description

- $\mathbb{P}^1[\Phi_1 : \Phi_2] \times \mathbb{P}^1[\Phi_3 : \Phi_4]$ (M does not intersect $\{\Phi_1 = \Phi_2 = 0\} \cup \{\Phi_3 = \Phi_4 = 0\}$)
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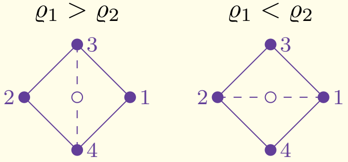

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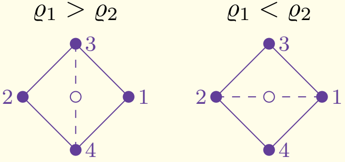

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Searching for monad bundles

$$Q_{\text{bundle}} = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc|c} \Gamma_1 & \dots & \Gamma_N & \Phi_0 \\ ? & \dots & ? & ? \\ ? & \dots & ? & ? \end{array}$$

Constraints:

- LGO? two scalars to break U(1)×U(1)
- $\mathbb{P}^1 \times \mathbb{P}^1$ structure and reflection symmetry
- trivial first Chern class
- anomaly cancelation

→ quadratic Diophantine equation

$$\sum_{A<B} \mathrm{x}_A \mathrm{x}_B = 11$$

Smallest rank ⟶ unique solution $(\mathrm{x}_1, \mathrm{x}_2) = (1, 11)$

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Searching for monad bundles

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x_A integers
$\mathrm{x}_0 > 0$

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$$\mathcal{A} = \begin{pmatrix} 10 - \mathrm{y} & 12 \\ 12 & 10 - \mathrm{y} \end{pmatrix} \text{ with } \mathrm{y} = 2 \sum_{\substack{A,B=1 \\ A<B}}^{N/2} \mathrm{x}_A \mathrm{x}_B$$

$$\det \mathcal{A} = 0 \underset{\text{ASD}}{\Rightarrow} \mathrm{y} = 22$$

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x_A integers

$\mathrm{x}_0 > 0$

$\mathrm{x}_0 = \mathrm{x}_1 + \cdots + \mathrm{x}_{N/2}$

$$\mathcal{A} = \begin{pmatrix} 10 - \mathrm{y} & 12 \\ 12 & 10 - \mathrm{y} \end{pmatrix} \text{ with } \mathrm{y} = 2 \sum_{\substack{A,B=1 \\ A<B}}^{N/2} \mathrm{x}_A \mathrm{x}_B$$

$$\det \mathcal{A} = 0 \underset{\text{ASD}}{\Rightarrow} \mathrm{y} = 22$$

→ quadratic Diophantine equation

$$\sum_{A<B} \mathrm{x}_A \mathrm{x}_B = 11$$

Smallest rank ⟶ unique solution $(\mathrm{x}_1, \mathrm{x}_2) = (1, 11)$

Searching for monad bundles

$$Q_{\mathsf{bundle}} = \left(\begin{array}{cccc|cc} \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Phi_6 & \Phi_7 \\ x_1 & x_2 & 0 & 0 & -(x_1+x_2) & 0 \\ 0 & 0 & x_1 & x_2 & 0 & -(x_1+x_2) \end{array}\right)$$

Constraints:

- LGO? two scalars to break U(1)×U(1)
- $\mathbb{P}^1 \times \mathbb{P}^1$ structure and reflection symmetry
- trivial first Chern class
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x_A integers

$$x_0 > 0$$

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$$\mathcal{A} = \begin{pmatrix} 10-y & 12 \\ 12 & 10-y \end{pmatrix} \text{ with } y = 2 \sum_{\substack{A,B=1 \\ A<B}}^{N/2} x_A x_B$$

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$$Q_{\text{bundle}} = \left(\begin{array}{cccc|cc} \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Phi_6 & \Phi_7 \\ \mathrm{x}_1 & \mathrm{x}_2 & 0 & 0 & -(\mathrm{x}_1+\mathrm{x}_2) & 0 \\ 0 & 0 & \mathrm{x}_1 & \mathrm{x}_2 & 0 & -(\mathrm{x}_1+\mathrm{x}_2) \end{array}\right)$$

Constraints:

- LGO? two scalars to break U(1)×U(1)
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$$\sum_{A<B} \mathrm{x}_A \mathrm{x}_B = 11$$

Smallest rank $\longrightarrow$ unique solution $(\mathrm{x}_1, \mathrm{x}_2) = (1, 11)$

Candidate holomorphic bundle

$$Q_{\mathsf{bundle}} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc|cc} \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Phi_6 & \Phi_7 \\ 1 & 11 & 0 & 0 & -12 & 0 \\ 0 & 0 & 1 & 11 & 0 & -12 \end{array}$$

Outline

1. heterotic flux vacua
2. the (0,2) torsional GLSM: construction
3. the (0,2) torsional GLSM: phase analysis
4. generalizations & abelian bundles

The model

$$Q_{\mathsf{K3}} = \begin{array}{c|ccccc} \Gamma_0 & \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 \\ -4 & 1 & 1 & 0 & 0 & 2 \\ -4 & 0 & 0 & 1 & 1 & 2 \end{array} \quad Q_{\mathsf{bundle}} = \begin{array}{cccc|cc} \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Phi_6 & \Phi_7 \\ 1 & 11 & 0 & 0 & -12 & 0 \\ 0 & 0 & 1 & 11 & 0 & -12 \end{array} \quad \mathrm{w} = \begin{array}{c} \Omega \\ 1 \\ -1 \\ \mathrm{R}^2 = 6 \end{array}$$

- non-anomalous $\mathrm{U}(1)_{\mathsf{L}} \times \mathrm{U}(1)_{\mathsf{R}}$ global symmetry
- generic superpotential
- from UV R-symmetry and c-extremization (presuming no accident)

$$c_{\mathsf{L}} = c_{\mathsf{R}} = 6$$

The model

$$Q = \begin{array}{ccccc|ccccccc} \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Gamma_5 & \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 & \Phi_6 & \Phi_7 \\ 1 & 11 & 0 & 0 & -4 & 1 & 1 & 0 & 0 & 2 & -12 & 0 \\ 0 & 0 & 1 & 11 & -4 & 0 & 0 & 1 & 1 & 2 & 0 & -12 \end{array} \qquad \mathrm{w} = \begin{array}{c} \Omega \\ 1 \\ -1 \end{array}$$

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$$\begin{array}{cccccc|ccccccc}
 & \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Gamma_5 & \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 & \Phi_6 & \Phi_7 \\
Q = & 1 & 11 & 0 & 0 & -4 & 1 & 1 & 0 & 0 & 2 & -12 & 0 \\
 & 0 & 0 & 1 & 11 & -4 & 0 & 0 & 1 & 1 & 2 & 0 & -12 \\
\mathrm{U}(1)_{\mathrm{L}} & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & +1 \\
\mathrm{U}(1)_{\mathrm{R}} & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & +1 & +1
\end{array}
\qquad
\begin{array}{cc}
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\mathrm{w} = & 1 \\
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- non-anomalous $\mathrm{U}(1)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}$ global symmetry

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$$\mathcal{W} = \Phi_6 \left(\Gamma_1 F^1 + \Gamma_2 F^2\right) + \Phi_7 \left(\Gamma_3 F^3 + \Gamma_4 F^4\right) + \Gamma_5 J \\ + (\tau_1 - 6\mathrm{i}\,\Omega)\,\Upsilon_1 + (\tau_2 + 6\mathrm{i}\,\Omega)\,\Upsilon_2$$

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The model

$$\begin{array}{c|ccccc|ccccccc|}
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- non-anomalous $\mathrm{U}(1)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{R}$ global symmetry

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$$\mathcal{W} = \Phi_6 \left(\Gamma_1 \Phi_1^{11} + \Gamma_2 \Phi_2 \right) + \Phi_7 \left(\Gamma_3 \Phi_3^{11} + \Gamma_4 \Phi_4 \right) + \Gamma_5 \left(\Phi_5^2 - f \right)$$
$$+ \left(\tau_1 - 6\mathrm{i}\,\Omega \right) \Upsilon_1 + \left(\tau_2 + 6\mathrm{i}\,\Omega \right) \Upsilon_2$$

polynomial in $\Phi_1, \ldots, \Phi_4$

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\hline
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\hline
\mathsf{U(1)}^1_{\mathsf{L}} & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\
\mathsf{U(1)}^2_{\mathsf{L}} & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \\
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second $\mathsf{U(1)}_{\mathsf{L}}$ symmetry!

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$$c_{\mathsf{L}} = c_{\mathsf{R}} = 6$$

extra $(c_{\mathsf{L}}, c_{\mathsf{R}}) = (2, 3)$
from torus d.o.f.

Large radius phase

- K3 geometry

  double cover of $\mathbb{P}^1[\Phi_1 : \Phi_2] \times \mathbb{P}^1[\Phi_3 : \Phi_4]$ branched over $f = 0$

Picard group $\mathrm{Pic}(M) = \langle \eta_1, \eta_2 \rangle$ (for generic f)

$\eta_1.\eta_1 = 0$ $\eta_2.\eta_2 = 0$ $\eta_1.\eta_2 = 2$

realized by toric divisors in GLSM

- Circle fibration $\omega = \eta_1 - \eta_2$

  ✓ quantized radius $\mathrm{R}^2 = 6$
  ✓ ASD condition for $\varrho^1 = \varrho^2$

- Holomorphic bundle $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$

  ◇ topologically trivial (line bundles with $c_1 = 0$)
  ◇ nowhere vanishing holomorphic section

→ "purely geometric" configuration with $\mathrm{R}^2(-\omega.\omega) = 24$

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Consequences of $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}$

- IR theory: (0,2) NLSM with

$$(c_{\mathsf{L}}, c_{\mathsf{R}}) = \overset{M}{(4,4)} + \overset{T_M}{(0,2)} + \overset{\mathcal{O}\oplus\mathcal{O}}{(2,0)}$$

two decoupled
Weyl fermions

- spacetime gauge group

$$G = \mathrm{E}_8 \times \mathrm{E}_8$$

consistent with supergravity analysis of Hull–Strominger equations*

- enhancement of the linear model currents?

$$\mathfrak{u}(1)^1_{\mathsf{L}} \oplus \mathfrak{u}(1)^2_{\mathsf{L}} \subset \mathfrak{so}(4) \qquad \text{and} \qquad \mathfrak{so}(4) \oplus \mathfrak{so}(12) \underset{\text{GSO}}{\subset} \mathfrak{e}_8$$

- hints in other phases of the model?

* omitting string-scale subtleties

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Field redefinitions and gauge anomalies

- flow in LR to ASD locus $\varrho^1 = \varrho^2$ in FI parameter space (ϱ^1, ϱ^2)
  GLSM mechanism?

- Holomorphic field redefinitions in (0,2) linear models

$$\Phi_i \to U_i \Phi_i \qquad\qquad \Gamma_A \to V_A \Gamma_A$$

act on parameter space of J^A... but also on FI couplings $q^\alpha = \mathrm{e}^{2\pi \mathrm{i} \tau^\alpha}$

$$q^\alpha \to \frac{\prod_{i=1}^{N_{\text{chiral}}} U_i^{Q_i^\alpha}}{\prod_{A=1}^{N_{\text{fermi}}} V_A^{Q_A^\alpha}} \times q^\alpha$$

- Rescaling weighted by gauge charges leads to

$$\tau^\alpha \to \tau^\alpha + \mathcal{A}^{\alpha\beta} \lambda_\beta$$

spacetime perspective: $H^2(X, \mathbb{Z}) = H^2(M, \mathbb{Z}) / \langle \omega \rangle$

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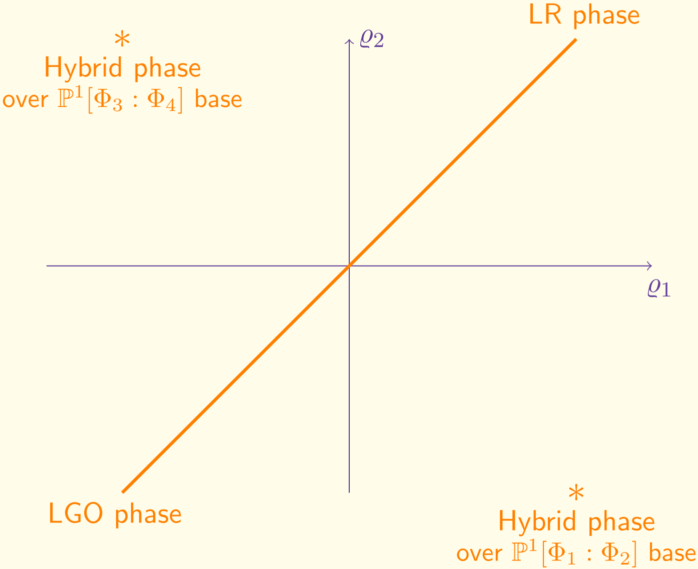
$$\tau^\alpha \to \tau^\alpha + \mathrm{w}^\alpha \xi$$

with $\xi = 2\mathrm{R}^2 \mathrm{w}\cdot\lambda$

spacetime perspective: $H^2(X, \mathbb{Z}) = H^2(M, \mathbb{Z}) / \langle \omega \rangle$

Phase diagram



LR phase

$*$
Hybrid phase
over $\mathbb{P}^1[\Phi_3 : \Phi_4]$ base

LGO phase

$*$
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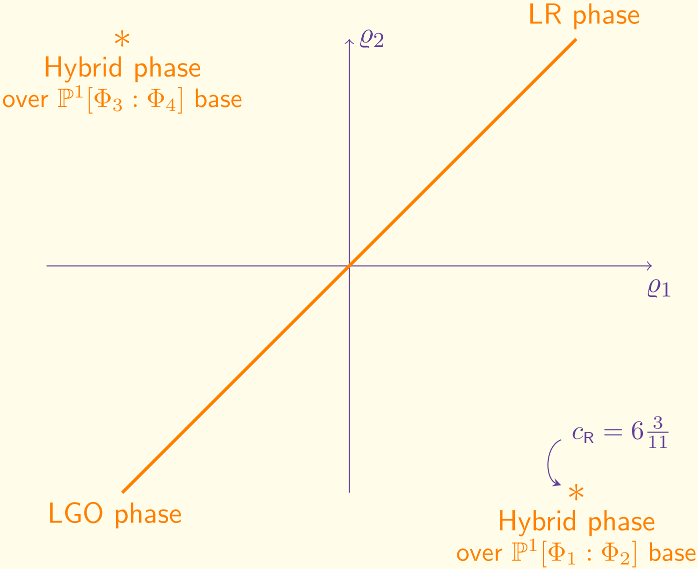
Phase diagram



LR phase

$*$

Hybrid phase

over $\mathbb{P}^1[\Phi_3 : \Phi_4]$ base

$c_\mathsf{R} = 6\,\frac{3}{11}$

$*$

LGO phase

Hybrid phase

over $\mathbb{P}^1[\Phi_1 : \Phi_2]$ base

The LGO phase

- U(1)×U(1) broken to $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$ by vev of Φ_6, Φ_7

- LG superpotential

$$\mathcal{W} = \Gamma_1\ \Phi_1^{11} + \Gamma_2\Phi_2 + \Gamma_3\Phi_3^{11} + \Gamma_4\Phi_4 + \Gamma_5\left(\Phi_5^2 - f\right)$$

(0,2) deformation of (2,2) LG model with $X_i = (\Phi_i, \Gamma_i)$

$$W = X_1^{12} + X_2^2 + X_3^{12} + X_4^2 + X_5^3$$

$$c_{\mathsf{L}} = c_{\mathsf{R}} = \frac{5}{2} + 0 + \frac{5}{2} + 0 + 1 = 6$$

- orbifold action:

$$\mathbb{Z}_{12} \times \mathbb{Z}_{12}$$

- on $f = 0$ locus, SCFT is solvable

$$(\text{K3 Gepner model} \times S^1)/\mathbb{Z}_{12}$$

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commutes with (2,2)
& projects on integer R-charge
$\longrightarrow c = 6$ Gepner model

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chiral,
asymmetric shift
action on S^1

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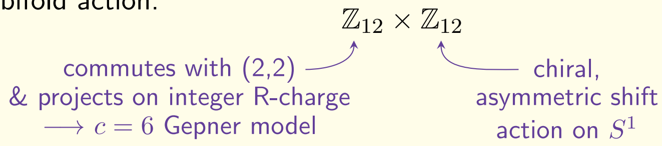

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Dressed elliptic genus

- dressed elliptic genus (extra insertion of $\bar{J}_0 = \bar{J}_0^{\mathsf{K3}} + \bar{J}_0^{\mathsf{torus}}$) [Israël–Sarkis'15]
  - a non-holomorphic "index"
  - factorization from quantized radius

$$\mathbb{E}(\tau, z, \bar{\tau}) = \sum_s \mathbb{E}_s(\tau, z)\, \overline{\chi}_s(\bar{\tau})$$ (stripping off free circle)

- encodes topological data of non-Kähler target space X

$$\mathbb{E}_s(\tau, z) = q^{\frac{r-2}{12}} y^{-\frac{r}{2}} \int_M \mathrm{td}(M)\, \mathrm{ch}(\mathscr{V}_{\tau,z})\, \mathrm{ch}(\mathscr{T}_\tau)\, \chi_s(\tau, \tfrac{1}{\pi \mathrm{i}}\omega)\ ,$$

- computed by localization in GLSM

$$\mathbb{E}_s = \frac{1}{12^2} \sum_{a,b,c,d \in \mathbb{Z}_{12}} \mathrm{e}^{\frac{2\pi\mathrm{i}}{12}(a(z+a\tau+b)+c(z+c\tau+d))} \frac{\vartheta_1(\frac{a\tau+b+z}{12} - z)}{\vartheta_1(\frac{a\tau+b+z}{12})} \frac{\vartheta_1(\frac{c\tau+d+z}{12} - z)}{\vartheta_1(\frac{c\tau+d+z}{12})}$$

$$\times \frac{\vartheta_1(-\frac{a\tau+b+c\tau+d+2z}{3})}{\vartheta_1(\frac{a\tau+b+c\tau+d+2z}{6})} \chi_s(\tau, \tfrac{a\tau+b-c\tau-d}{12})$$

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Outline

1. heterotic flux vacua
2. the (0,2) torsional GLSM: construction
3. the (0,2) torsional GLSM: phase analysis
4. generalizations & abelian bundles

A "class" of TLSMs

Two other models (branched cover of $\mathbb{P}^1 \times \mathbb{P}^1$ & trivial bundle)

K3 geometry

$\mathbb{Z}_3$ branched cover of $\mathbb{P}^1 \times \mathbb{P}^1$

Holomorphic bundle

$\mathcal{O} \oplus \mathcal{O}$

Circle fibration

ASD class with $\omega.\omega = -6$

(quantized radius $\mathrm{R}^2 = 4$)

LGO phase

$\mathbb{Z}_8 \times \mathbb{Z}_8$ orbifold group

solvable locus in LG

(2,2) Gepner model / chiral $\mathbb{Z}_8$

$$X_1^8 + X_2^2 + X_3^8 + X_4^2 + X_5^4$$

K3 geometry

$\mathbb{Z}_2 \times \mathbb{Z}_2$ branched cover of $\mathbb{P}^1 \times \mathbb{P}^1$

Holomorphic bundle

$\mathcal{O} \oplus \mathcal{O}$

Circle fibration

ASD class with $\omega.\omega = -8$

(quantized radius $\mathrm{R}^2 = 3$)

LGO phase

$\mathbb{Z}_6 \times \mathbb{Z}_6$ orbifold group

solvable locus in LG

(2,2) Gepner model / chiral $\mathbb{Z}_6$

$$X_1^6 + X_2^2 + X_3^6 + X_4^2 + X_5^3 + X_6^3$$

K3 compactifications with line bundles

Consider K3 model with trivial $\mathcal{O} \oplus \mathcal{O}$ bundle (gauge-anomalous)

- anomaly-free model with line bundles? $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$

$$c_1(\mathcal{L}_1) = \mathrm{w}(\eta_1 - \eta_2) \qquad c_1(\mathcal{L}_2) = \mathrm{w}(\eta_2 - \eta_1) \quad \begin{matrix} \mu(\mathcal{L}) = 0 \\ c_1(\mathcal{E}) = 0 \end{matrix}$$

- GLSM realization: fermi pair

$$Q_{\substack{\text{line}\\ \text{bundle}}} = \begin{array}{c} \begin{array}{cc} \Gamma_1 & \Gamma_2 \end{array} \\ \begin{pmatrix} \mathrm{w} & -\mathrm{w} \\ -\mathrm{w} & \mathrm{w} \end{pmatrix} \end{array}$$

- (0,2) GLSM with
  - LR phase: (0,4) K3 NLSM with configurations of line bundles

$$\mathcal{E} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_N$$

  - LGO phase: solvable point ($c = 6$ Gepner model $\times$ free fermions)$/\mathbb{Z}_{12}$

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gauge anomaly

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  - LR phase: (0,4) K3 NLSM with configurations of line bundles

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Calabi–Yau models

Generalization of the (0,2) GLSM for $c_{\mathsf{R}} = 9$?

$$Q_{\mathsf{K3}} = \begin{array}{c} \\ \\ \end{array}\begin{array}{c|ccccc} \Gamma_0 & \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 \\ -4 & 1 & 1 & 0 & 0 & 2 \\ -4 & 0 & 0 & 1 & 1 & 2 \end{array}$$

- double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched over $\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix}\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$
- CY threefold with $h^{1,1}(X) = 3$ and $h^{2,1}(X) = 115$
- line bundle configurations? for $\varrho^1 = \varrho^2 = \varrho^3$

$$\begin{aligned} c_1(\mathcal{L}_1) &= \mathrm{w}(\eta_1 - \eta_2) \\ c_1(\mathcal{L}_2) &= \mathrm{w}(\eta_2 - \eta_3) \\ c_1(\mathcal{L}_3) &= \mathrm{w}(\eta_3 - \eta_1) \end{aligned}$$

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Generalization of the (0,2) GLSM for $c_{\mathsf{R}} = 9$?

$$Q_{\mathsf{CY}} = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{c|ccccccc} \Gamma_0 & \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 & \Phi_6 & \Phi_7 \\ -4 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ -4 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ -4 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{array}$$

- double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched over $\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix}\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$
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- CY threefold with $h^{1,1}(X) = 3$ and $h^{2,1}(X) = 115$ $\qquad \chi(X) = -224$
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$$\mathcal{A}_{\substack{\text{line}\\\text{bundle}}} = \mathrm{w}^2 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Calabi–Yau models

- trivial bundle?

$$\begin{array}{rl} & \begin{array}{cccccc}\Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Phi_6 & \Phi_7\end{array} \\ Q_{\mathsf{bundle}} = & \left(\begin{array}{cccc|cc} 1 & 11 & 0 & 0 & -12 & 0 \\ 0 & 0 & 1 & 11 & 0 & -12 \end{array}\right)\end{array}$$

anomalous (0,2) GLSM for CY threefold with $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ bundle

- couple with configuration of fermi triples
  $\longrightarrow$ abelian bundle $\mathcal{E}$ with

$$c_2(T_X) - c_2(\mathcal{E}) = 0$$

- LGO phase with $\mathbb{Z}_{18} \times \mathbb{Z}_{18} \times \mathbb{Z}_{18}$ orbifold
  solvable point: ($c = 9$ Gepner model & free fermions)/($\mathbb{Z}_{18} \times \mathbb{Z}_{18}$)

$$W = X_1^{18} + X_2^2 + X_3^{18} + X_2^2 + X_5^{18} + X_6^2 + X_7^3$$

Calabi–Yau models

- trivial bundle?

$$Q_{\text{bundle}} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccccc|ccc} \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Gamma_5 & \Gamma_6 & \Phi_8 & \Phi_9 & \Phi_{10} \\ 1 & 17 & 0 & 0 & 0 & 0 & -18 & 0 & 0 \\ 0 & 0 & 1 & 17 & 0 & 0 & 0 & -18 & 0 \\ 0 & 0 & 0 & 0 & 1 & 17 & 0 & 0 & -18 \end{array}$$

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Calabi–Yau models

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- framework to ask (and answer) questions on heterotic flux vacua
    - massless spacetime spectrum?
    - moduli space of deformations?

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    - anomaly cancelation by fields which decouple in the IR
    - holomorphic field redefinitions with $c_2(T_X) - c_2(\mathcal{E})$ non-zero

- some future directions
    - non-trivial stable bundles? [Festi–Platt–Singhal–Tanaka'25]
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Thank you!

K3 compactifications with line bundles

How is the condition $\varrho^1 = \varrho^2$ realized?

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- general GLSM mechanism for (0,2) K3 models with abelian bundles
  - line bundle $\mathcal{L}$ with first Chern class

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  - rescalings of Γ act on Kähler class by $\tau^\alpha \to \tau^\alpha - \mathrm{w}^\alpha\lambda$
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