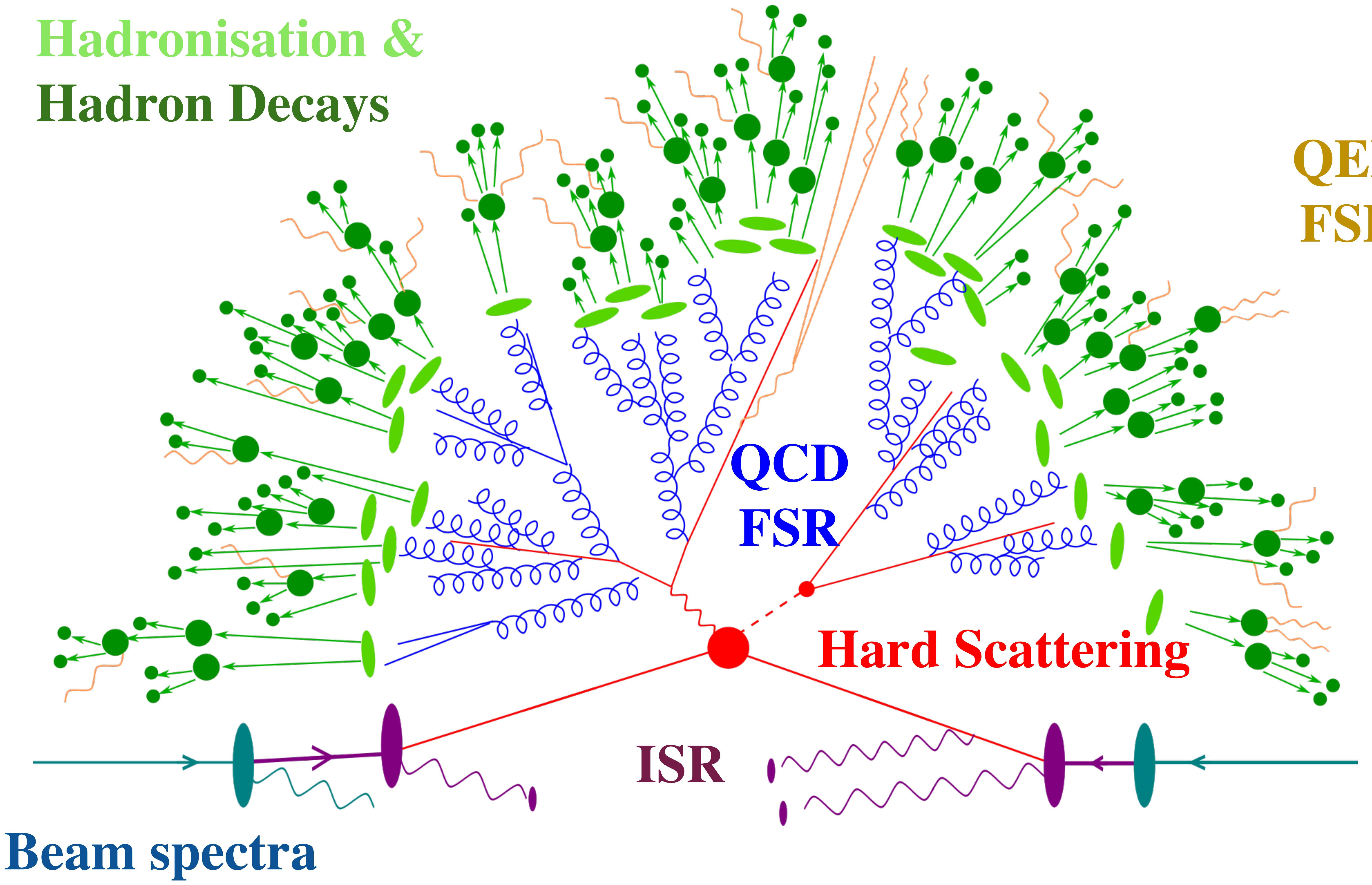
QED Factorization and Resummation/ Showering

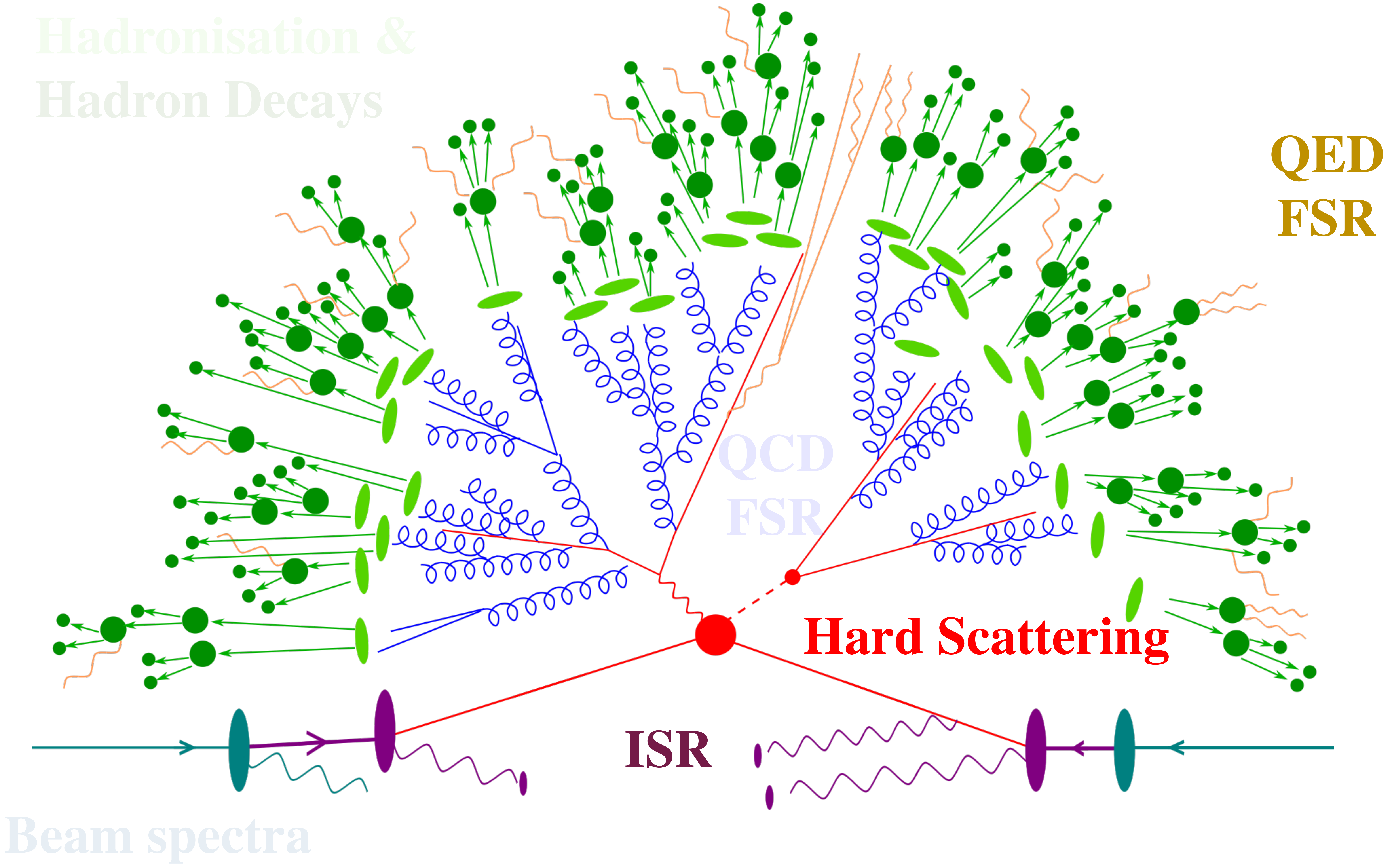
Alan Price,
5/05/26

Electroweak Corrections at Current and Future Accelerators

e^+e^- Collisions



e^+e^- Collisions



QED Cross-Section

QED cross-sections are built from towers of soft and collinear logarithms.

$$d\sigma(L,\hat{L}) = \alpha_s^k \sum_n^\infty \alpha^n \sum_{i=0}^\infty \sum_{j=0}^\infty \hat{\sigma}_{n,i,j} L^i \hat{L}^j$$

Soft Logarithms $\hat{L} = \log\left(\frac{Q^2}{E_\gamma^2}\right)$

$L = \log\left(\frac{Q^2}{m_e^2}\right)$ **Collinear Logarithms**

How to treat QED Corrections?

Soft Resummation

- All order resummation of soft/ soft-collinear logs
  - Universal approach for resummation e.g Sherpa
  - Collinear logs introduced order-by-order

Collinear Resummation

- Collinear logs are resummed with universal PDFs
  - Known to NNLO/NLL
  - Same PDFs describe all processes
  - Non trivial to include soft logs

Soft Logarithms

$$\hat{L} = \log\left(\frac{Q^2}{E_\gamma^2}\right)$$

$$L = \log\left(\frac{Q^2}{m_e^2}\right)$$

Collinear Logarithms

Collinear Approach

$$\mathrm{d}\sigma\left(p_k, p_l\right) = \sum_{ij} \int \mathrm{d}z_+\,\mathrm{d}z_-\,\Gamma_{i/k}\left(z_+, \mu^2\right)\hat{\sigma}_{ij\to X}\left(z_+ p_k, z_- p_l, \mu^2\right)\Gamma_{j/l}\left(z_-, \mu^2\right)$$

$k \in \{e^+, \gamma\}$ $\quad l \in \{e^-, \gamma\}$

$i, j \in \{e^+, e^-, \gamma\}$

Collinear Factorisation

$$\mathrm{d}\sigma\left(p_k, p_l\right) = \sum_{ij} \int \mathrm{d}z_+\, \mathrm{d}z_-\, \Gamma_{i/k}\left(z_+, \mu^2\right) \hat{\sigma}_{ij\to X}\left(z_+ p_k, z_- p_l, \mu^2\right) \Gamma_{j/l}\left(z_-, \mu^2\right)$$

- Universal Electron parton distribution (PDF)
- Can be calculated using perturbative QED to yield analytical formula
  - **LO/LL** **Z. Phys. C49**,577 – 584(1991)
    - Readily available in most generators
  - **NLO/NLL** **JHEP 03 (2020) 135**
    - Implemented in Madgraph
  - **NNLO** **JHEP 12 (2025) 167** **JHEP 11 (2025) 171**

$$k \in \left\{e^+, \gamma\right\} \quad l \in \left\{e^-, \gamma\right\}$$

$$i, j \in \left\{e^+, e^-, \gamma\right\}$$

Collinear Factorisation

$$\mathrm{d}\sigma\left(p_k, p_l\right) = \sum_{ij} \int \mathrm{d}z_+ \,\mathrm{d}z_- \,\Gamma_{i/k}\left(z_+, \mu^2\right) \hat{\sigma}_{ij\to X}\left(z_+ p_k, z_- p_l, \mu^2\right) \Gamma_{j/l}\left(z_-, \mu^2\right)$$

- Short Distance cross-section
- Incorporates fixed order corrections
- NLO from standard approaches FKS or CS JHEP 07 (2018) 185 Eur.Phys.J.C 78 (2018) 2, 119
- Some care needed when combining with PDF but well understood

$$k \in \left\{e^+, \gamma\right\} \quad l \in \left\{e^-, \gamma\right\}$$

$$i,j \in \left\{e^+, e^-, \gamma\right\}$$

QED Shower

Currently no **matched $\mathbf{NLO_{EW}}$** QED Shower like we have for QCD

Main Problem

QED is U(1) theory and there is no large-Nc equivalent. All charged dipoles contribute equally

Collinear Emissions

Determined by the charge of emitter

Soft Emissions

Sum of Eikonals with alternating signs. Leads to large negative weights

QED Shower

For the initial state we also need to preform a backwards evolution with corresponding

$$J_{SF} = \frac{f_{e^\pm}(x/z, Q^2)}{f_{e^\pm}(x, Q^2)}$$

However the electron structure function is not regular in the limit $x \to 1$

$$f_{e^\pm}(x, Q^2) = \beta \frac{\exp\left(-\gamma_E \beta + \frac{3}{4}\beta_S\right)}{\Gamma\left(1+\beta\right)}(1-x)^{\beta-1} + \beta_H \sum_{n=0}^{\infty} \beta_H^n \mathscr{H}_n(x) \ ,$$

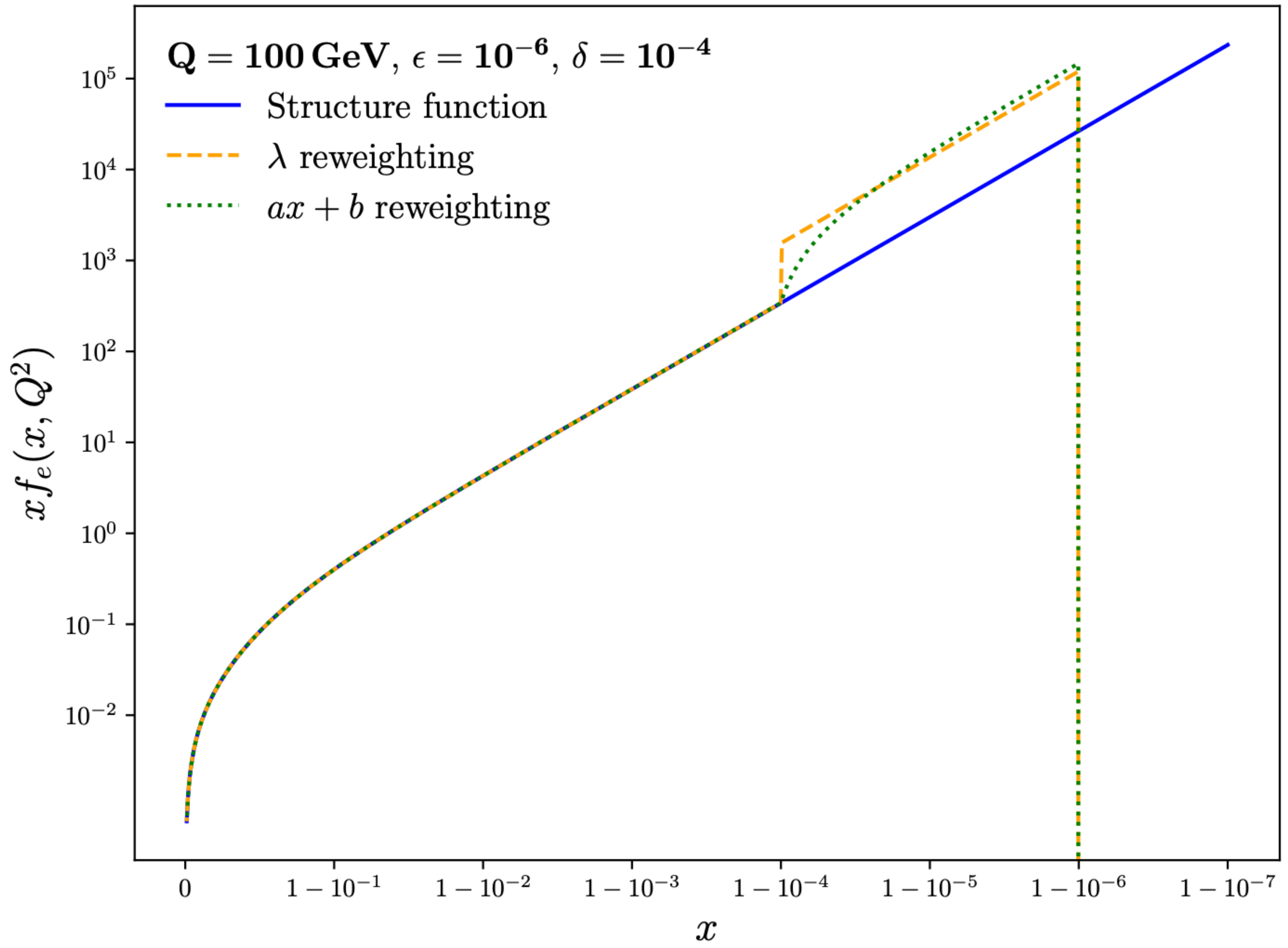
QED Shower

Problem: Integrable singularity $x \to 1$

Solution

$$W_e = \begin{cases} f_e(x) & 0 \le x \le 1-\delta \\ w(x) f_e(x) & 1-\delta < x < 1-\epsilon \\ 0 & \text{else} \end{cases}$$

where $w(x)$ solves $$\int_{1-\delta}^{1} W_e(x)dx = \int_0^1 f_e(x)dx\,.$$



$$f_{e^\pm}(x, Q^2) = \beta \frac{\exp\left(-\gamma_E \beta + \frac{3}{4}\beta_S\right)}{\Gamma(1+\beta)}(1-x)^{\beta-1} + \beta_H \sum_{n=0}^{\infty} \beta_H^n \mathscr{H}_n(x)\,,$$

[2603.05585, L.Flower, M. Schönherr]

YFS Approach

$$d\sigma = \sum_{n_\gamma=0}^{\infty} \frac{e^{Y(\Omega)}}{n_\gamma!} d\Phi_Q \left[\prod_{i=1}^{n_\gamma} d\Phi_i^\gamma \, \tilde{S}(k_i) \, \Theta(k_i, \Omega) \right] \left(\tilde{\beta}_0 + \sum_{j=1}^{n_\gamma} \frac{\tilde{\beta}_1(k_j)}{\tilde{s}(k_j)} + \sum_{\substack{j,k=1 \\ j<k}}^{n_\gamma} \frac{\tilde{\beta}_2(k_j, k_k)}{\tilde{s}(k_j)\tilde{s}(k_k)} + \cdots \right)$$

ANNALS OF PHYSICS: **13:** 379–452 (1961)

The Infrared Divergence Phenomena and High-Energy Processes*

D. R. YENNIE†

School of Physics, University of Minnesota, Minneapolis, Minnesota

S. C. FRAUTSCHI‡

Department of Physics, University of California, Berkeley, California

AND

H. SUURA

Department of Physics, Nihon University, Tokyo, Japan

A general treatment of the infrared divergence problem in quantum electrodynamics is given. The main feature of this treatment is the separation of the infrared divergences as multiplicative factors, which are treated to all orders of perturbation theory, and the conversion of the residual perturbation expansion into one which has no infrared divergence, and hence no need for an infrared cutoff. In the infrared factors, which are exponential in form, the infrared divergences arising from the real and virtual photons cancel out in the usual way. These factors can then be expressed solely in terms of the momenta of the initial and final charged particles and an integral over the region of phase space available to the undetected photons; they do not depend upon the specific details of the interaction. Electron scattering from a static potential is treated in considerable detail, and several other examples are discussed briefly. As an important byproduct of the general treatment, it is found that when the infrared contributions are separated in a particular way, they dominate the radiative corrections at high energies and together with certain "magnetic terms" and vacuum polarization corrections seem to give all the contributions proportional to ln (E/m). All of these corrections can be easily estimated (in most cases) simply from a knowledge of the external momenta of the charged particles; this then provides a very powerful and accurate way of estimating radiative corrections to high-energy processes.

* Supported in part by the U. S. Atomic Energy Commission, Contract AT(11-1)-50.

† Various refinements were added to the manuscript (particularly in Appendix C) during the academic year 1960–1961 while this author was a National Science Foundation Senior Fellow visiting the University of Paris. He is grateful to Professor M. M. Lévy for the hospitality afforded by the Laboratoire de Physique Théorique et Hautes Énergies at Orsay.

‡ Supported by National Science Foundation Grant.

379

Yennie, Frautschi, and Suura showed how to reorder the **entire perturbative series** such that all IR divergences are **resummed** leaving a finite residuals defined to **all-orders**

YFS Approach

$$d\sigma = \sum_{n_\gamma=0}^{\infty} \frac{e^{Y(\Omega)}}{n_\gamma!} d\Phi_Q \left[\prod_{i=1}^{n_\gamma} d\Phi_i^\gamma \, \tilde{S}(k_i) \, \Theta(k_i, \Omega) \right] \left(\tilde{\beta}_0 + \sum_{j=1}^{n_\gamma} \frac{\tilde{\beta}_1(k_j)}{\tilde{s}(k_j)} + \sum_{\substack{j,k=1 \\ j<k}}^{n_\gamma} \frac{\tilde{\beta}_2(k_j, k_k)}{\tilde{s}(k_j)\tilde{s}(k_k)} + \cdots \right)$$

- **Multi-Photon Phasespace**
  - Provides a Monte-Carlo algorithm for the emission of multiple resolved photons **Comput.Phys.Commun. 56 (1990) 351-384 S.Jadach, B.Ward**
- Generalised to arbitrary decays in HERWIG and Sherpa **JHEP 12 (2008) 018 M.Schonherr, F. Krauss** **JHEP 07 (2006) 010 K.Hamilton, P.Richardson**
- Generalised to arbitrary e^+e^- processes **SciPost Phys. 13 (2022) 2, 026 F.Krauss, A.P, M. Schonherr**

YFS Approach

$$d\sigma = \sum_{n_\gamma=0}^{\infty} \frac{e^{Y(\Omega)}}{n_\gamma!} d\Phi_Q \left[\prod_{i=1}^{n_\gamma} d\Phi_i^\gamma \, \tilde{S}(k_i) \, \Theta(k_i, \Omega) \right] \left(\tilde{\beta}_0 + \sum_{j=1}^{n_\gamma} \frac{\tilde{\beta}_1(k_j)}{\tilde{s}(k_j)} + \sum_{\substack{j,k=1 \\ j<k}}^{n_\gamma} \frac{\tilde{\beta}_2(k_j, k_k)}{\tilde{s}(k_j)\tilde{s}(k_k)} + \cdots \right)$$

- **Multi-Photon Phasespace**
  - Provides a Monte-Carlo algorithm for the emission of multiple resolved photons **Comput.Phys.Commun. 56 (1990) 351-384 S.Jadach, B.Ward**
- Generalised to arbitrary decays in HERWIG and Sherpa **JHEP 12 (2008) 018 M.Schonherr, F. Krauss** **JHEP 07 (2006) 010 K.Hamilton, P.Richardson**
- Generalised to arbitrary e^+e^- processes **SciPost Phys. 13 (2022) 2, 026 F.Krauss, A.P, M. Schonherr**

Gives us access to full inclusive and differential events

No need for a QED Shower

YFS Approach

$$d\sigma = \sum_{n_\gamma=0}^{\infty} \frac{e^{Y(\Omega)}}{n_\gamma!} d\Phi_Q \left[\prod_{i=1}^{n_\gamma} d\Phi_i^\gamma \tilde{S}(k_i)\,\Theta(k_i,\Omega)\right] \left(\tilde{\beta}_0 + \sum_{j=1}^{n_\gamma} \frac{\tilde{\beta}_1(k_j)}{\tilde{s}(k_j)} + \sum_{\substack{j,k=1\\ j<k}}^{n_\gamma} \frac{\tilde{\beta}_2(k_j,k_k)}{\tilde{s}(k_j)\tilde{s}(k_k)} + \cdots\right)$$

Championed by the group of Jadach & Ward who calculated it to second order for LEP

Simplified notation

$$\tilde{\beta}_{n_R} = \sum_{n_V=0}^{\infty} \tilde{\beta}_{n_R}^{n_V+n_R}$$

- Rearrangement leaves IR finite residuals, defined to all-orders
- Provides us with a **local subtraction scheme** to include higher-order corrections
- Essential at LEP: Allowed for a precision of **0.11%** in luminosity predictions

NLO Contributions

Virtual

$$\tilde{\beta}_0^1(\Phi_n) = \mathscr{V}(\Phi_n) - \sum_{ij} \mathscr{D}_{ij}(\Phi_{ij})$$

Local Subtraction Term

One-Loop amplitude

$$\mathscr{D}_{ij}\left(\Phi_{ij} \otimes \Phi_n\right) = \tilde{\beta}_0^0\left(\Phi_n\right) \mathscr{R}e\, \mathscr{B}_{ij}(\Phi_n)$$

Real

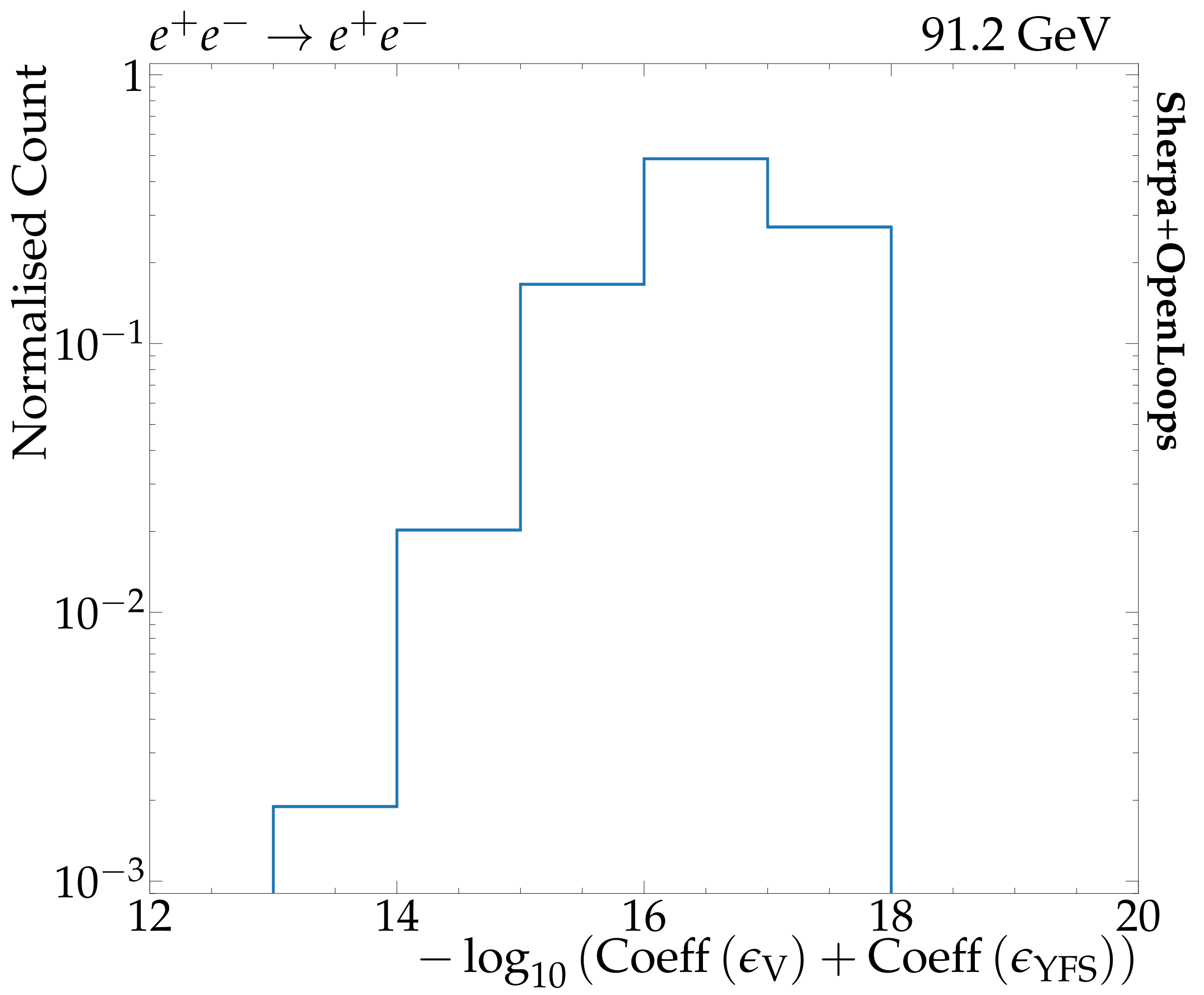
$$\tilde{\beta}_1^1\left(\Phi_{n+1}\right) = \mathscr{R}\left(\Phi_{n+1}\right) - \sum_{ij} \tilde{\mathscr{D}}_{ij}\left(\Phi_{ij+1} \otimes \Phi_n\right)$$

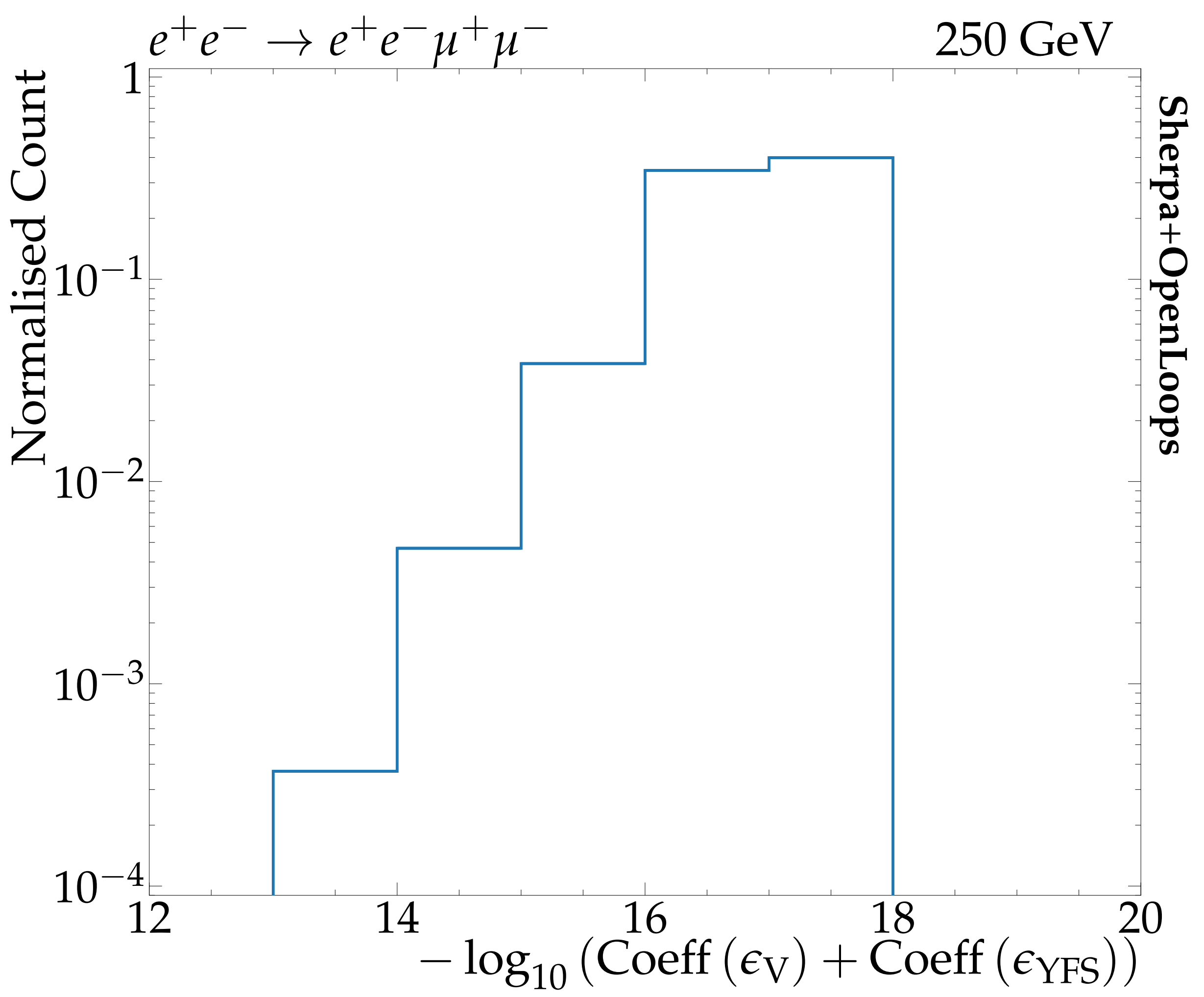
Tree level amplitude

Local Subtraction Term

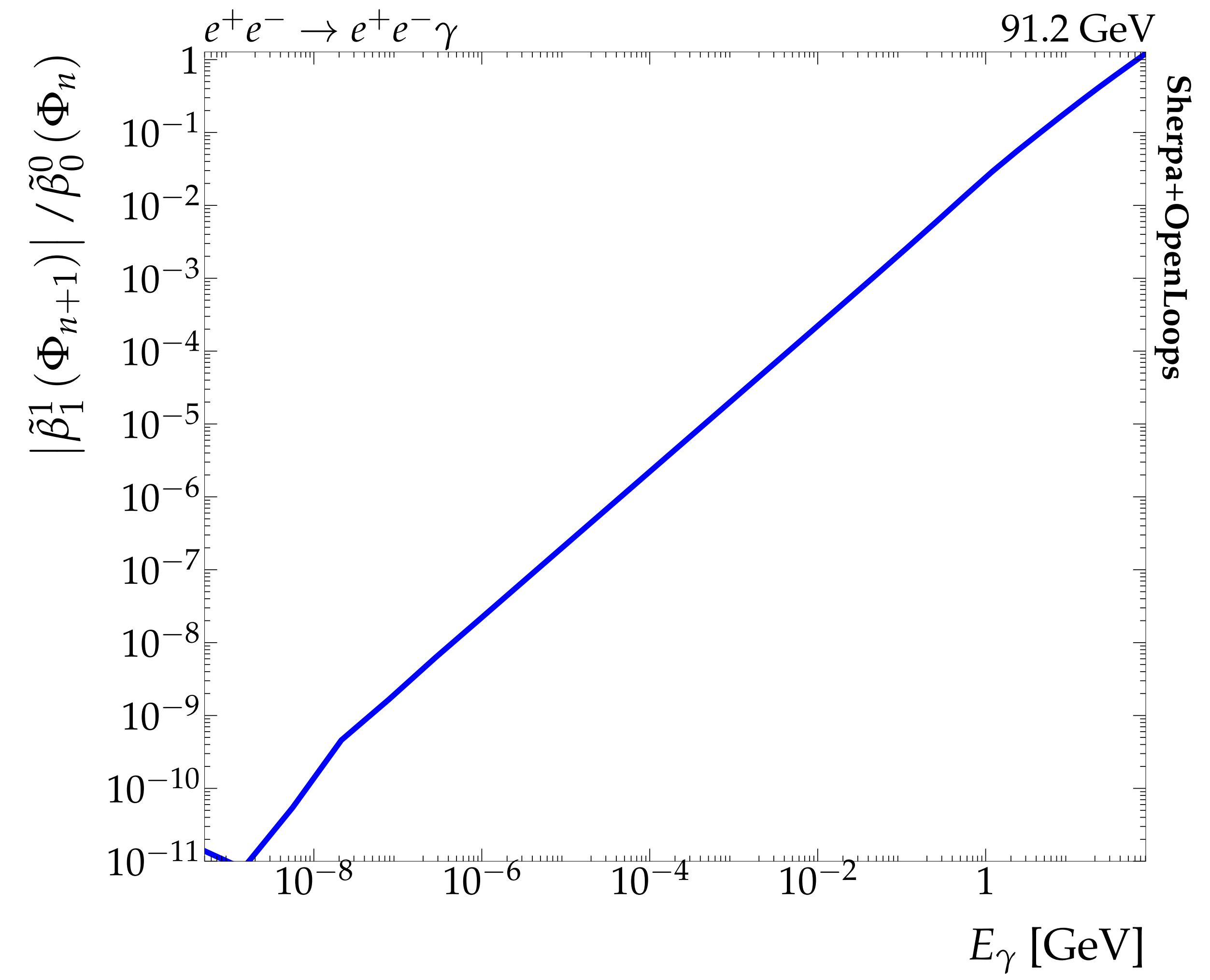
NNLO Contributions

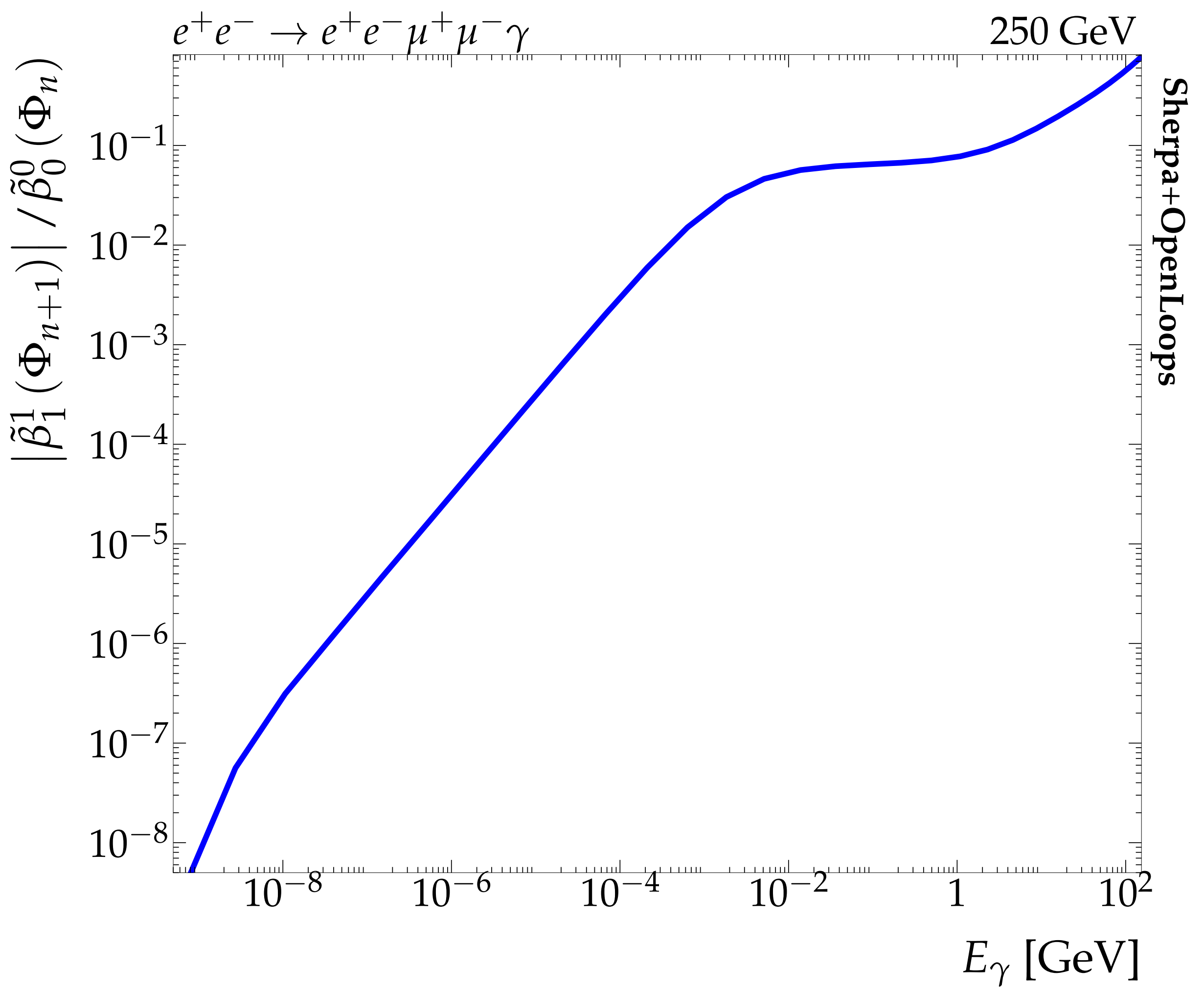
Virtual





Real





NNLO Contributions

Real-Virtual

$$\tilde{\beta}_1^2\left(\Phi_{n+1}\right) = \mathscr{RV}(\Phi_{n+1}) - \sum_{ij} \mathscr{D}_{ij}^{(1)}\left(\Phi_{ij+1} \otimes \Phi_n\right)$$

Constructed from **NLO** contribution

One-Loop amplitude

Local Subtraction Term

$$\tilde{D}_{ij}^{(1)}\left(\Phi_{ij+1} \otimes \Phi_n\right) = \tilde{S}_{ij}\left(k\right)\tilde{\beta}_0^1\left(\Phi_n\right)$$

Real-Real

$$\tilde{\beta}_2^2\left(\Phi_{n+2}\right) = \mathscr{RR}(\Phi_{n+2}) - \tilde{S}(k_1)\tilde{\beta}_1^1\left(\Phi_{n+1}; k_2\right) - \tilde{S}(k_2)\tilde{\beta}_1^1\left(\Phi_{n+1}; k_1\right) - \tilde{S}(k_1)\tilde{S}(k_2)\tilde{\beta}_0^0\left(\Phi_n\right)$$

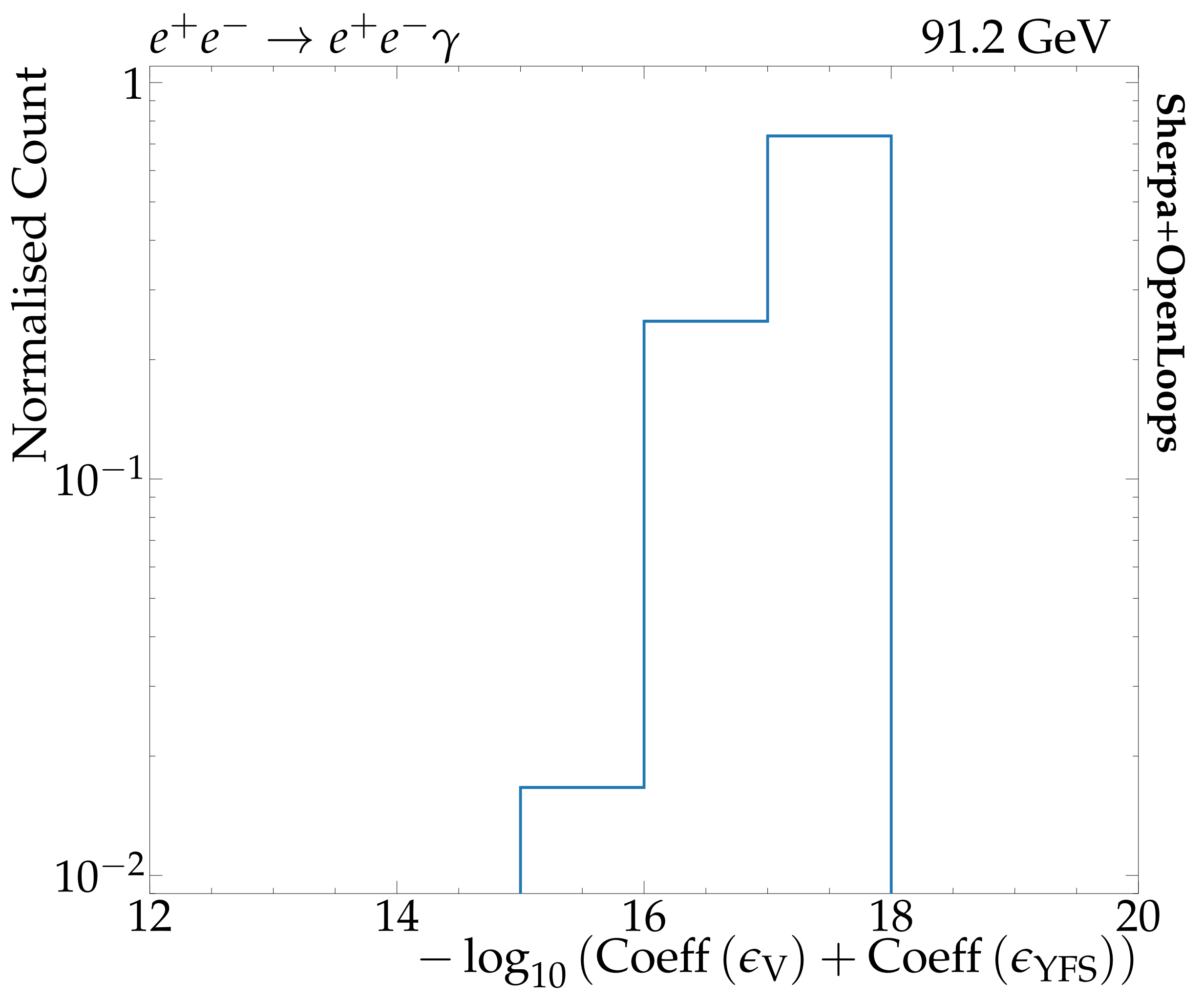
Tree level amplitude

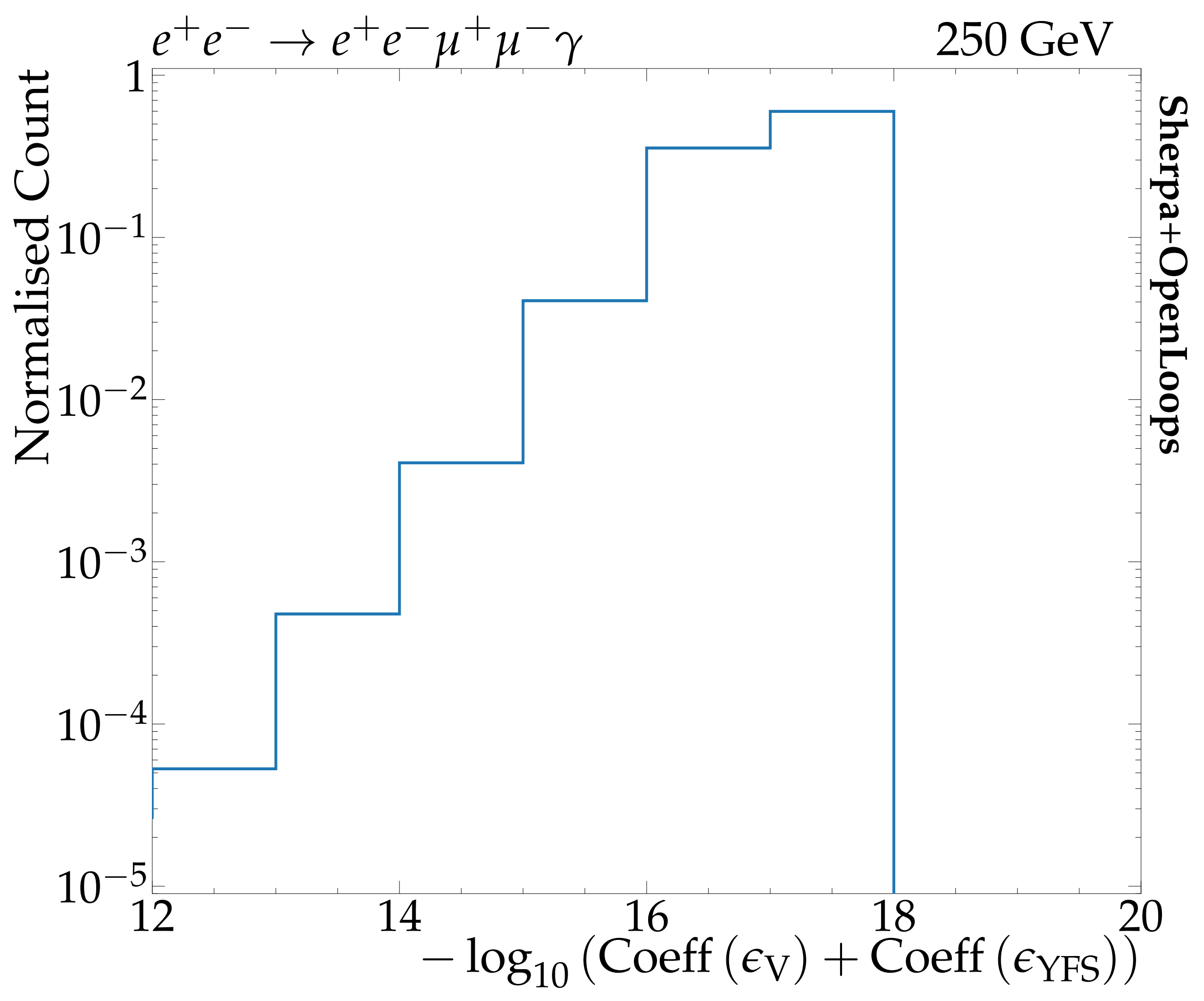
Single Soft Photon

Double Soft Photon

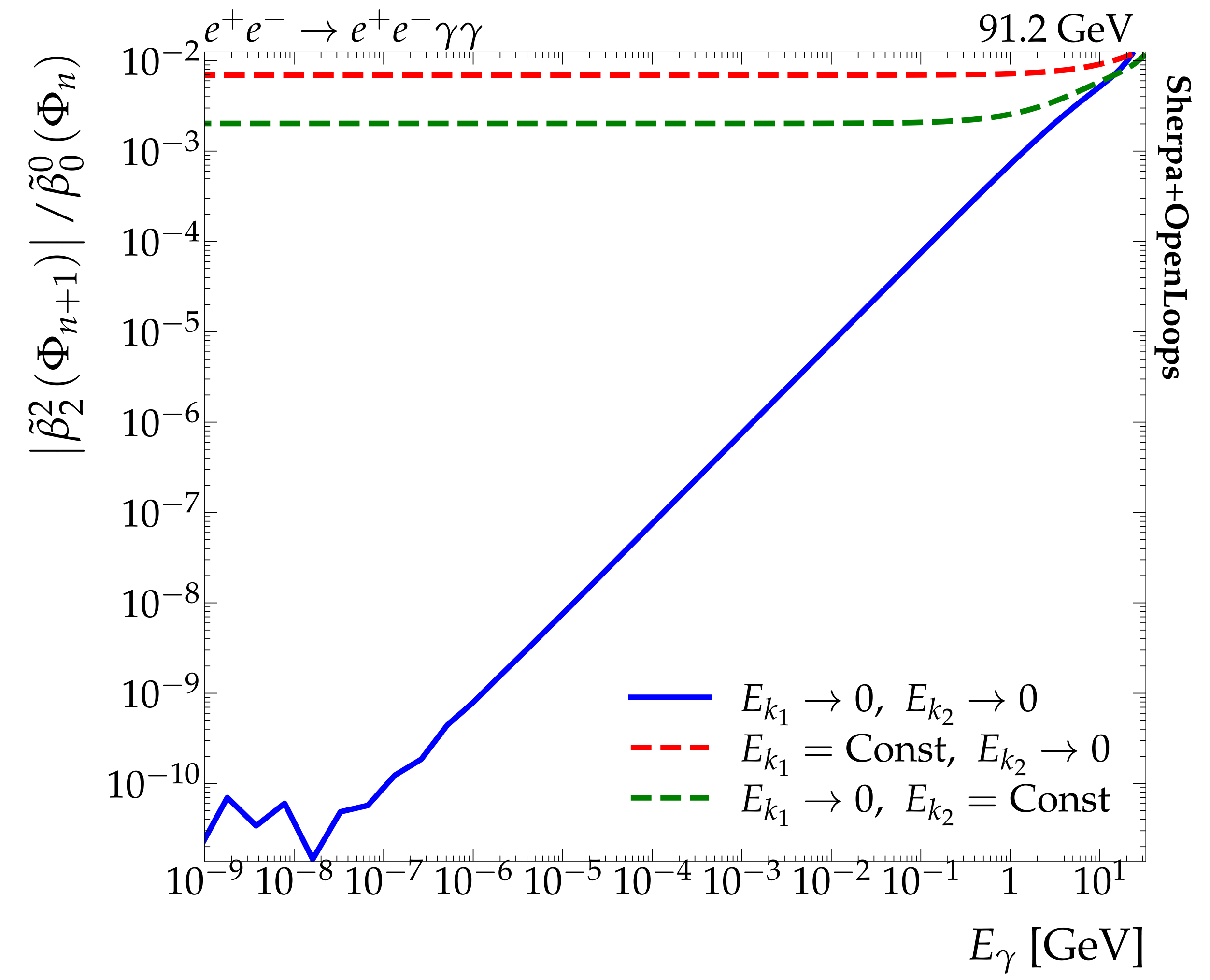
NNLO Contributions

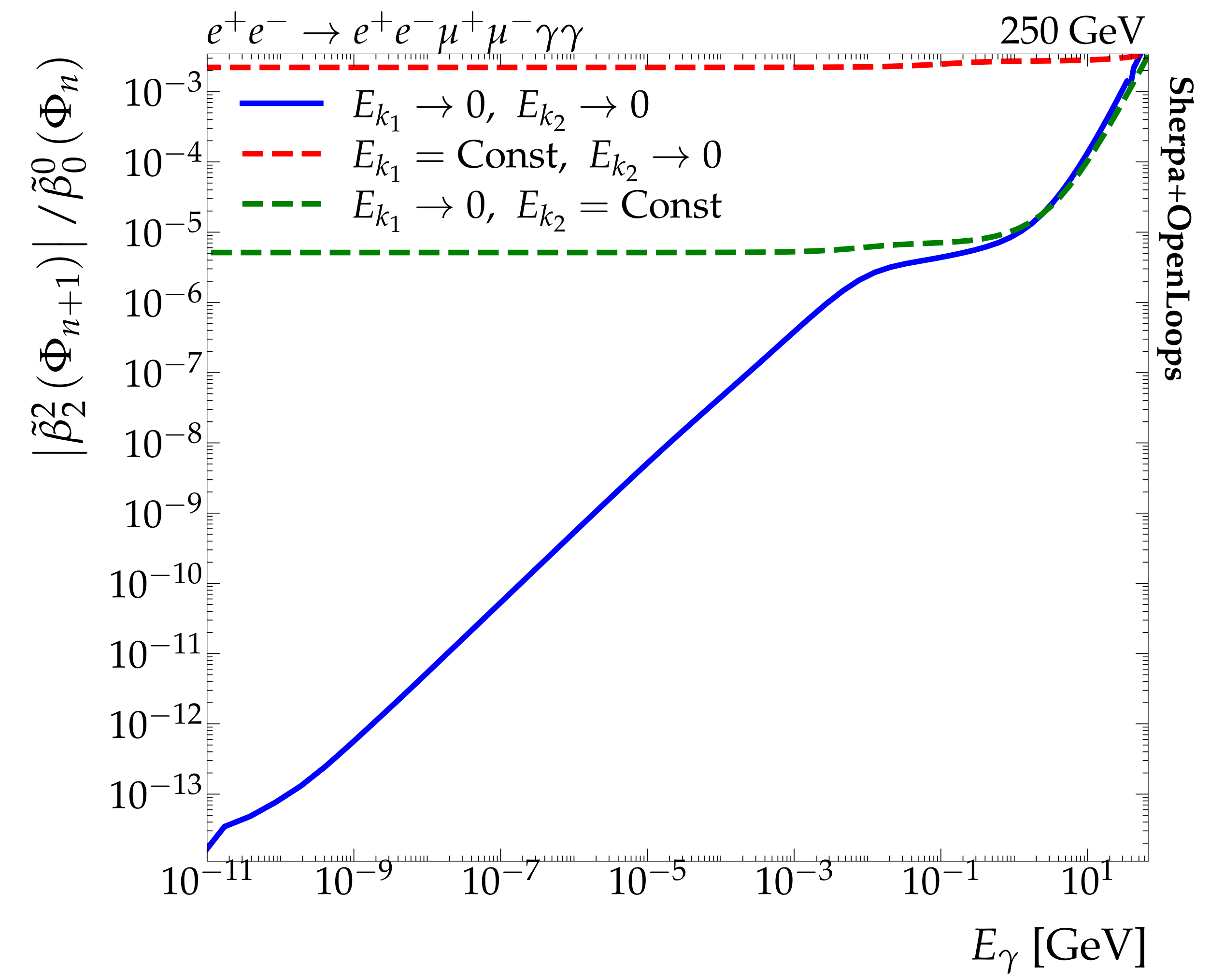
Real-Virtual





Real-Real





NNLO Contributions

Double-Virtual

$$\tilde{\beta}_0^2(\Phi_n) = \mathscr{V}\mathscr{V}(\Phi_n) - \sum_{ij} \mathscr{D}_{ij}\left(\Phi_{ij} \otimes \Phi_n\right)\tilde{\beta}_0^1(\Phi_n) - \frac{1}{2}\left(\sum_{ij} \mathscr{D}_{ij}\left(\Phi_{ij} \otimes \Phi_n\right)\right)^2 \tilde{\beta}_0^0(\Phi_n)$$

Two-Loop amplitude

- Mostly limited to $2 \rightarrow 2$
- Recently a lot of work on two loop calculations but we are far away from automation

Phys.Rev.Lett. 132 (2024) 23

JHEP 11 (2023) 148

JHEP 11 (2023) 041

Phys.Rev.Lett. 130 (2023) 3

`GRIFFIN v1.0`: $f\bar{f} \rightarrow f'\bar{f}'$ with NLO EW corrections and h.o. @ Z-pole

Future upgrades:

- Bbhabha scattering ($f = f'$)
- Higher-order off-resonance corrections, e.g.
  $\mathcal{O}(\alpha\alpha_s)$, Heller, v.Manteuffel, Schabinger, Spiesberger '20
  $\mathcal{O}(N_f\alpha^2)$ Bonciani et al. '21
- SMEFT d=6 operator effects
- W production and decay (a.k.a. charged-current DY)

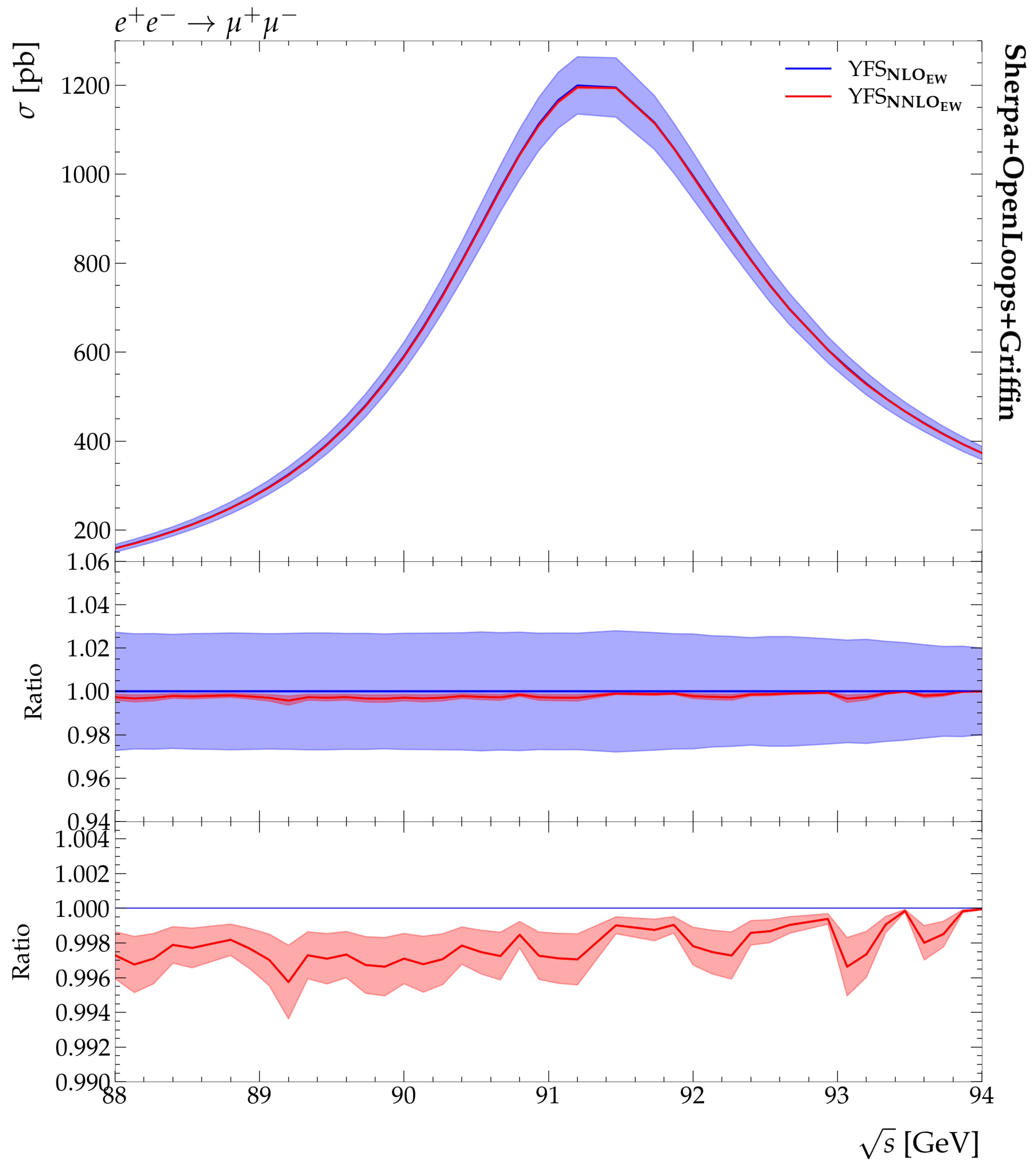
Try out the code: `github.com/lisongc/GRIFFIN/releases`

Feedback welcome!

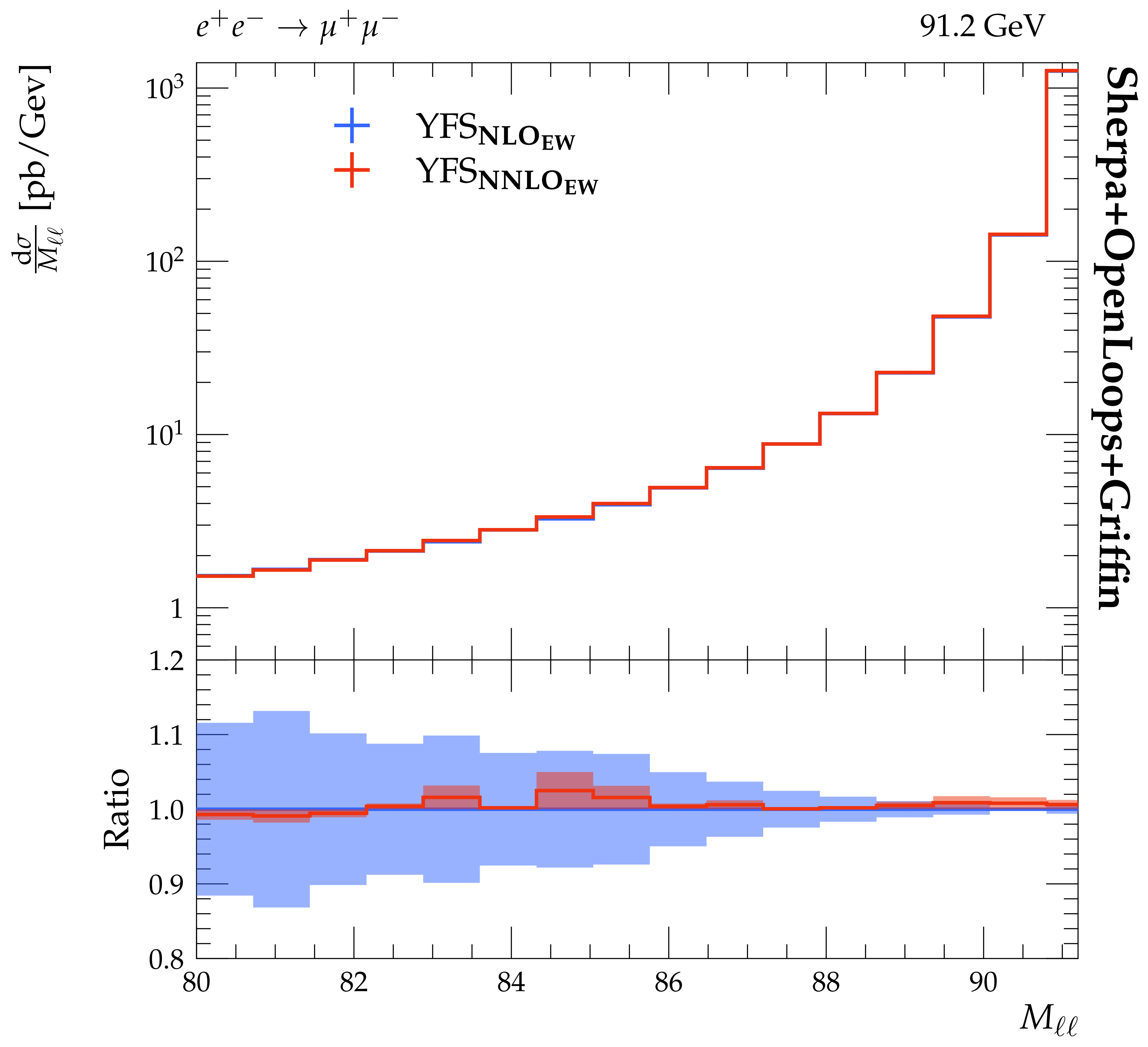
Cross-Section

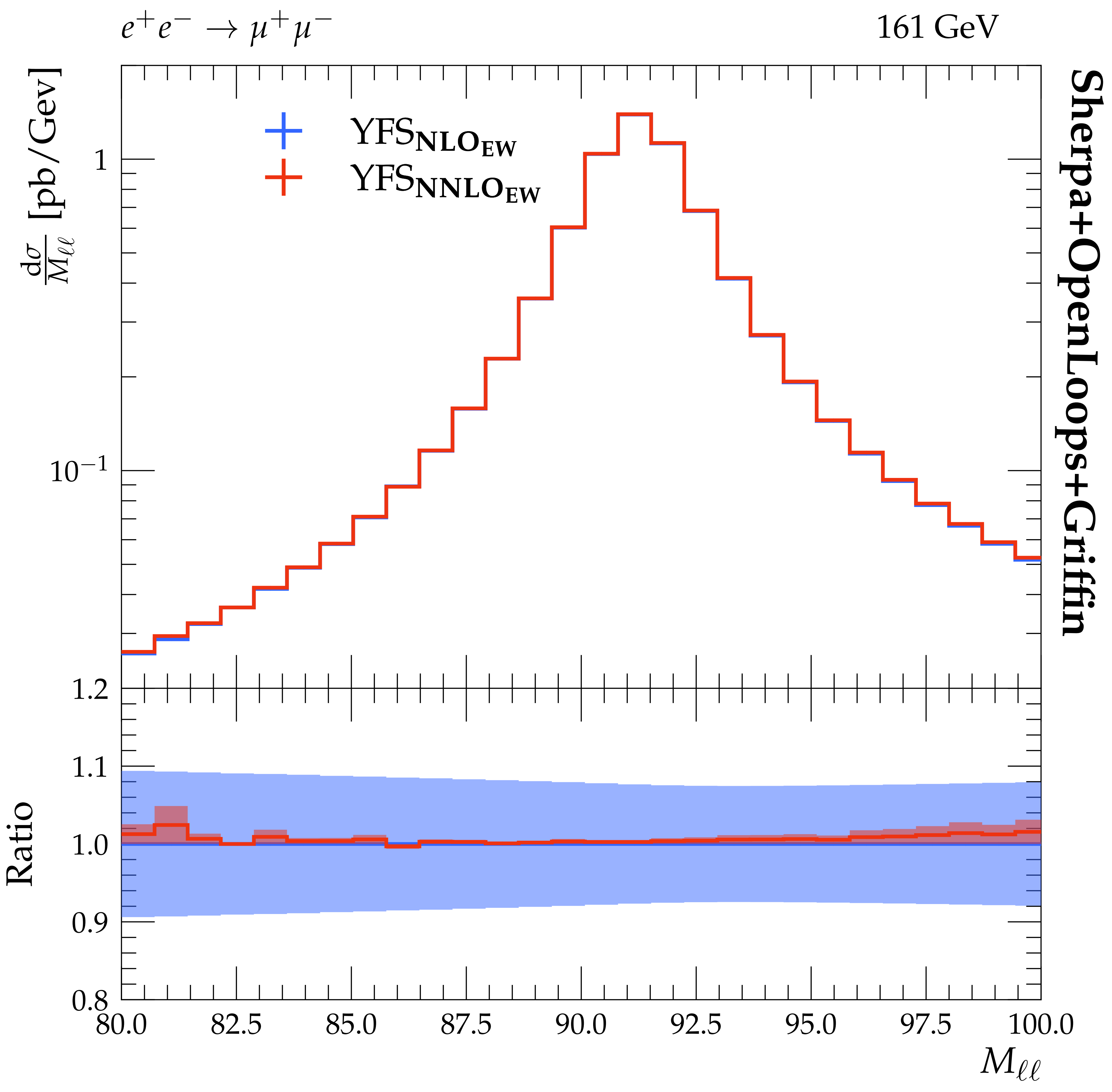
- ~2% at NLO and ~0.2% at NNLO
- Perturbative uncertainties estimated from difference w.r.t lower order. Probably too conservative
- Does not include parametric uncertainties

[Phys. Rev. D 113, 073008 A.P, F.Krauss]



Differential Results





[Phys. Rev. D 113, 073008 A.P, F.Krauss]

Wishlist

- A matched QED Shower similar e.g MC@NLO or POWHEG. New approaches?
  - Some early progress already [2603.05585, L.Flower. M. Schonherr] [Eur. Phys. J. C 86, 362 (2026), Belloni, F. et al]
  - Port electron structure functions and PDF's to LHAPDF (*A.P, M. Schönherr)
- YFS Approach:
  - Ideally another independent implementation
  - Two-loop amplitudes including non-zero fermion masses
  - How to systematically include QCD Parton Shower?
  - Collinear Enhanced YFS Resummation [Phys.Lett.B 848 (2024) 138361]

Backup

NLO: Real Contributions

$$\tilde{\beta}_1^1\left(\Phi_{n+1}\right) = \mathscr{R}\left(\Phi_{n+1}\right) - \sum_{ij} \tilde{\mathscr{D}}_{ij}\left(\Phi_{ij+1} \otimes \Phi_n\right)$$

The YFS theorem provides a formula for calculating real contributions to an arbitrary process.

NLO: Real Contributions

$$\tilde{\beta}_1^1\left(\Phi_{n+1}\right) = \mathscr{R}\left(\Phi_{n+1}\right) - \sum_{ij} \tilde{\mathscr{D}}_{ij}\left(\Phi_{ij+1} \otimes \Phi_n\right)$$

Tree level amplitude

- Calculated using standard approaches
- Fully automated

NLO: Real Contributions

$$\tilde{\beta}_1^1\left(\Phi_{n+1}\right) = \mathscr{R}\left(\Phi_{n+1}\right) - \sum_{ij} \tilde{\mathscr{D}}_{ij}\left(\Phi_{ij+1} \otimes \Phi_n\right)$$

Tree level amplitude

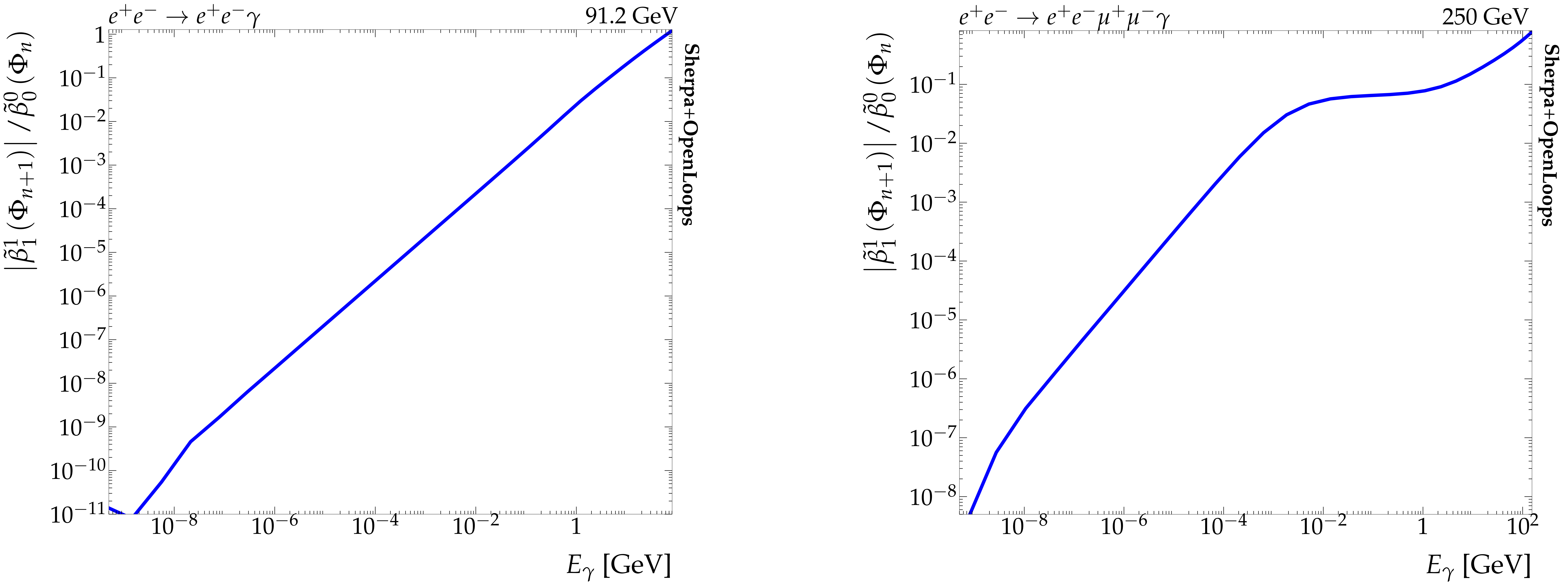
- Calculated using standard approaches
- Fully automated

Local Subtraction Term

- Contributions already resummed
- Constructed automatically from LO configuratio
- Summed over all charged dipoles

$$\tilde{\mathscr{D}}_{ij}\left(\Phi_{ij+1} \otimes \Phi_n\right) = \tilde{S}_{ij}\left(k\right)\left|\mathscr{M}_0^0\left(\Phi_n\right)\right|^2$$

NLO: Real Contributions



$\tilde{\beta}_1^1\left(\Phi_{n+1}\right)$ is finite as $E_\gamma \to 0$

All photon emissions in an event are matched with this

NLO: Virtual Contributions

$$\tilde{\beta}_0^1(\Phi_n) = \mathscr{V}(\Phi_n) - \sum_{ij} \mathscr{D}_{ij}(\Phi_{ij})$$

The YFS theorem provides a formula for calculating virtual contributions to an arbitrary process.

NLO: Virtual Contributions

$$\tilde{\beta}_0^1(\Phi_n) = \mathscr{V}(\Phi_n) - \sum_{ij} \mathscr{D}_{ij}(\Phi_{ij})$$

One-Loop amplitude

- Calculated with one-loop providers
- OpenLoops,Recola, MadLoop,GoSam…

Eur.Phys.J.C 79 (2019) 10, 866

Comput.Phys.Commun. 214 (2017) 140-173

JHEP 05 (2011) 044

Eur.Phys.J.C 74 (2014) 8, 3001

I can't calculate one-loop amplitudes but I can interface them

- **Anonymous MC Author**

NLO: Virtual Contributions

$$\tilde{\beta}_0^1(\Phi_n) = \mathscr{V}(\Phi_n) - \sum_{ij} \mathscr{D}_{ij}(\Phi_{ij})$$

One-Loop amplitude

- ❖ Calculated with one-loop providers
- ❖ OpenLoops,Recola, MadLoop,GoSam...

Local Subtraction Term

- ❖ Contributions already resummed
- ❖ Constructed automatically from LO configuration

$$\mathscr{D}_{ij}\left(\Phi_{ij} \otimes \Phi_n\right) = \tilde{\beta}_0^0\left(\Phi_n\right) \mathscr{Re}\, \mathscr{B}_{ij}(\Phi_n)$$

$$\mathscr{Re}\, \mathscr{B}_{ij}(\Phi_n) = -\frac{i}{8\pi^3} Z_i Z_j \theta_i \theta_j \int \frac{d^4k}{k^2} \left(\frac{2p_i\theta_i - k}{k^2 - 2(k \cdot p_i)\theta_i} + \frac{2p_j\theta_j + k}{k^2 + 2(k \cdot p_j)\theta_j} \right)^2$$

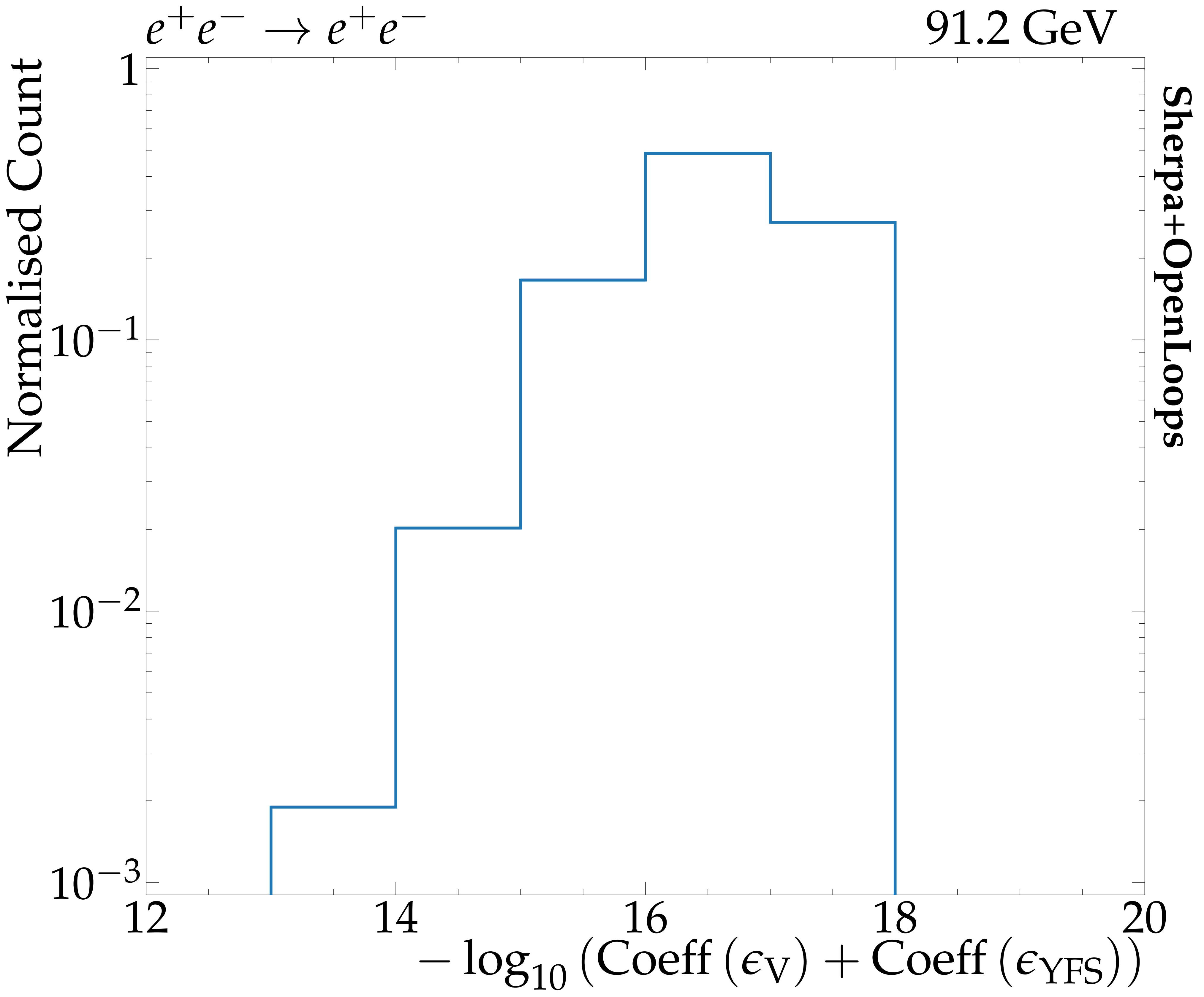
Eur.Phys.J.C 79 (2019) 10, 866

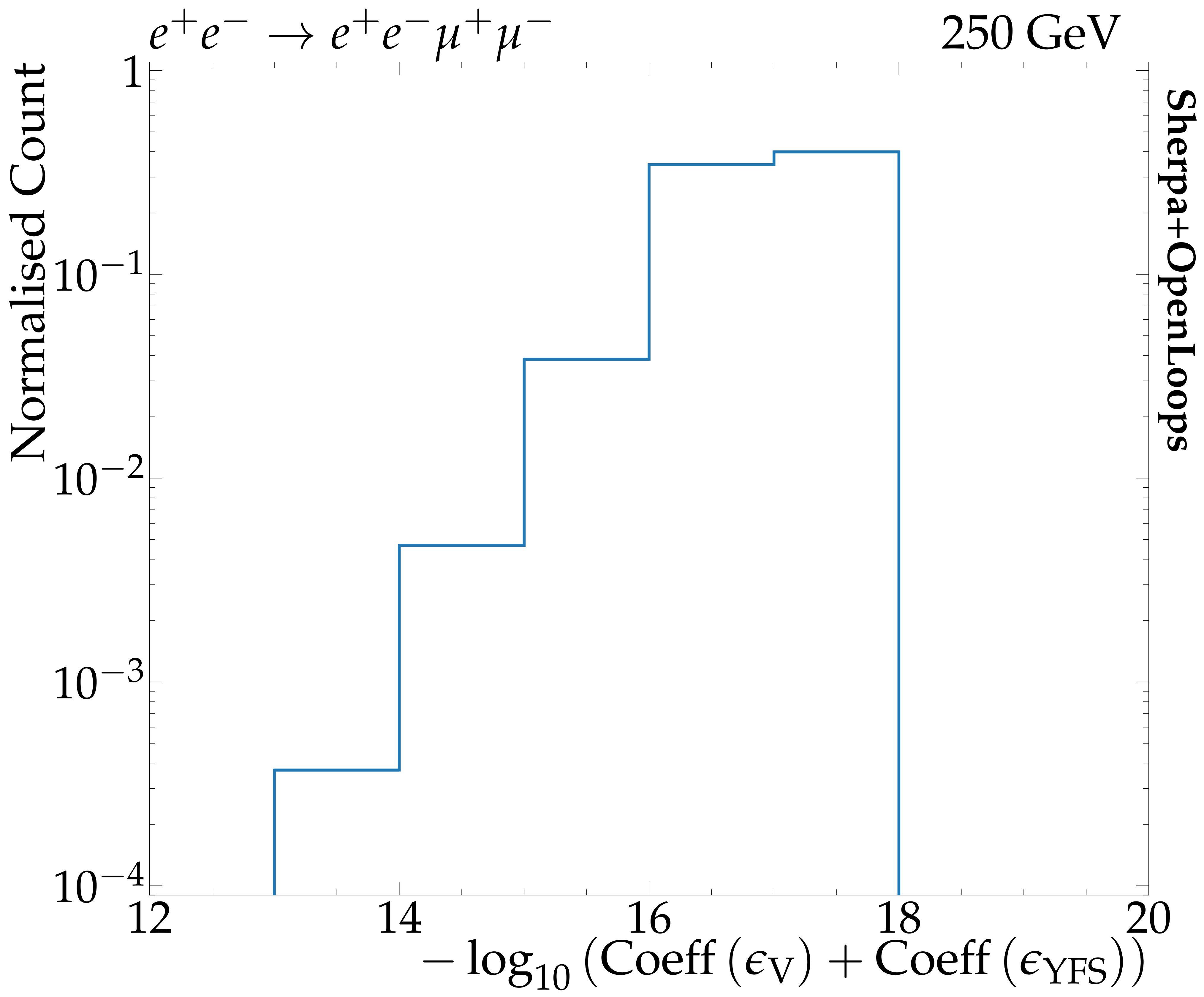
Comput.Phys.Commun. 214 (2017) 140-173

JHEP 05 (2011) 044

Eur.Phys.J.C 74 (2014) 8, 3001

NLO: Real Contributions





YFS subtraction removes **IR,** ϵ^{-1} poles
from one-loop contributions

NNLO Contributions

Real-Virtual

$$\tilde{\beta}_1^2\left(\Phi_{n+1}\right) = \mathscr{R}\mathscr{V}\left(\Phi_{n+1}\right) - \sum_{ij} \mathscr{D}_{ij}^{(1)}\left(\Phi_{ij+1} \otimes \Phi_n\right)$$

Constructed from **NLO** contribution

One-Loop amplitude

Local Subtraction Term

$$\mathscr{D}_{ij}^{(1)}\left(\Phi_{ij+1} \otimes \Phi_n\right) = \tilde{S}_{ij}\left(k\right)\tilde{\beta}_0^1\left(\Phi_n\right)$$

NNLO Contributions

Double-Virtual

$$\tilde{\beta}_0^2\left(\Phi_n\right) = \mathcal{V}\mathcal{V}\left(\Phi_n\right) - \sum_{ij} \mathcal{D}_{ij}\left(\Phi_{ij} \otimes \Phi_n\right) \tilde{\beta}_0^1\left(\Phi_n\right) - \frac{1}{2}\left(\sum_{ij} \mathcal{D}_{ij}\left(\Phi_{ij} \otimes \Phi_n\right)\right)^2 \tilde{\beta}_0^0\left(\Phi_n\right)$$

Two-Loop amplitude

- Mostly limited to $2 \rightarrow 2$
- Recently a lot of work on two loop calculations but we are far away from automation

Phys.Rev.Lett. 132 (2024) 23

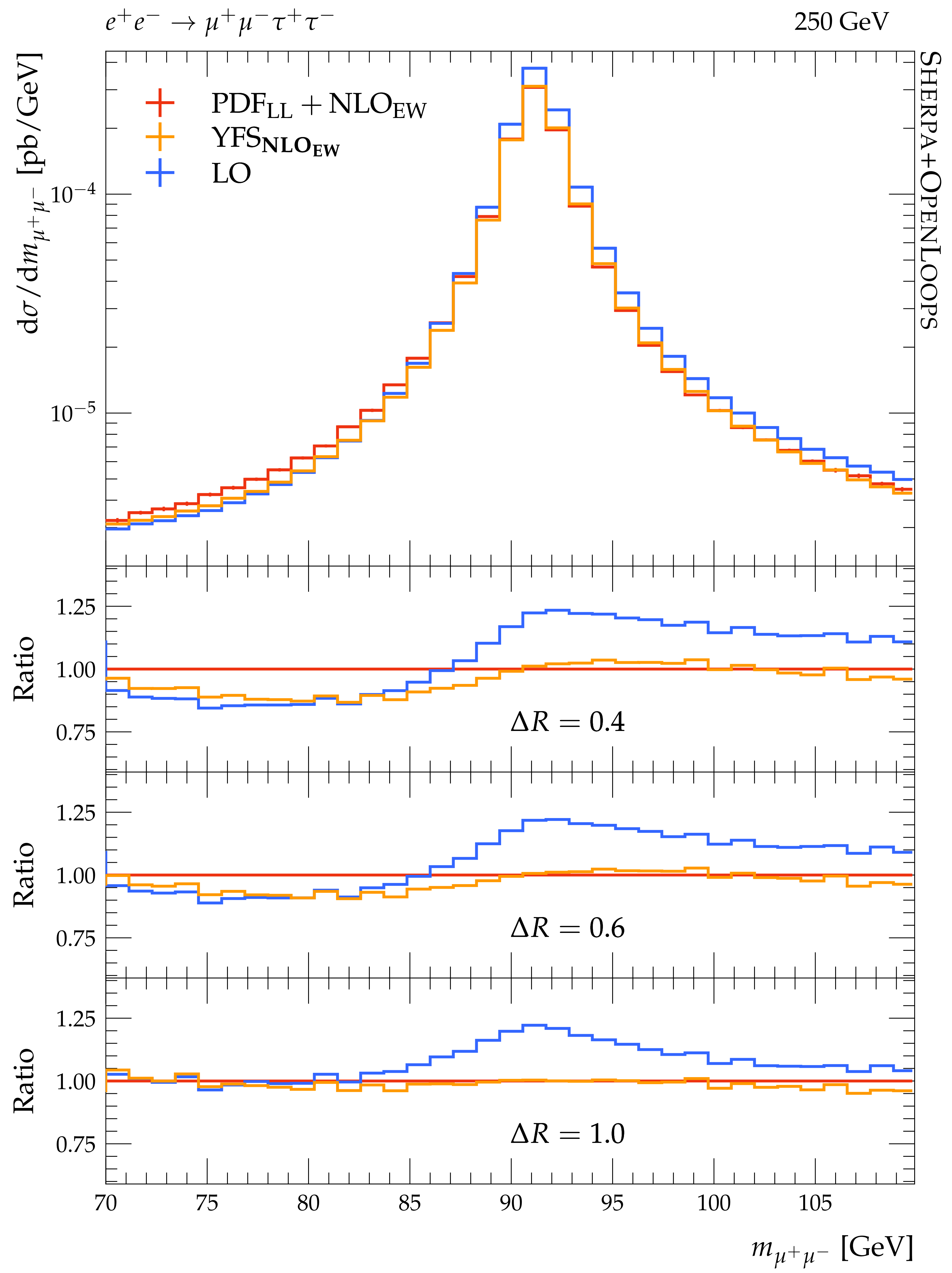
JHEP 11 (2023) 148

JHEP 11 (2023) 041

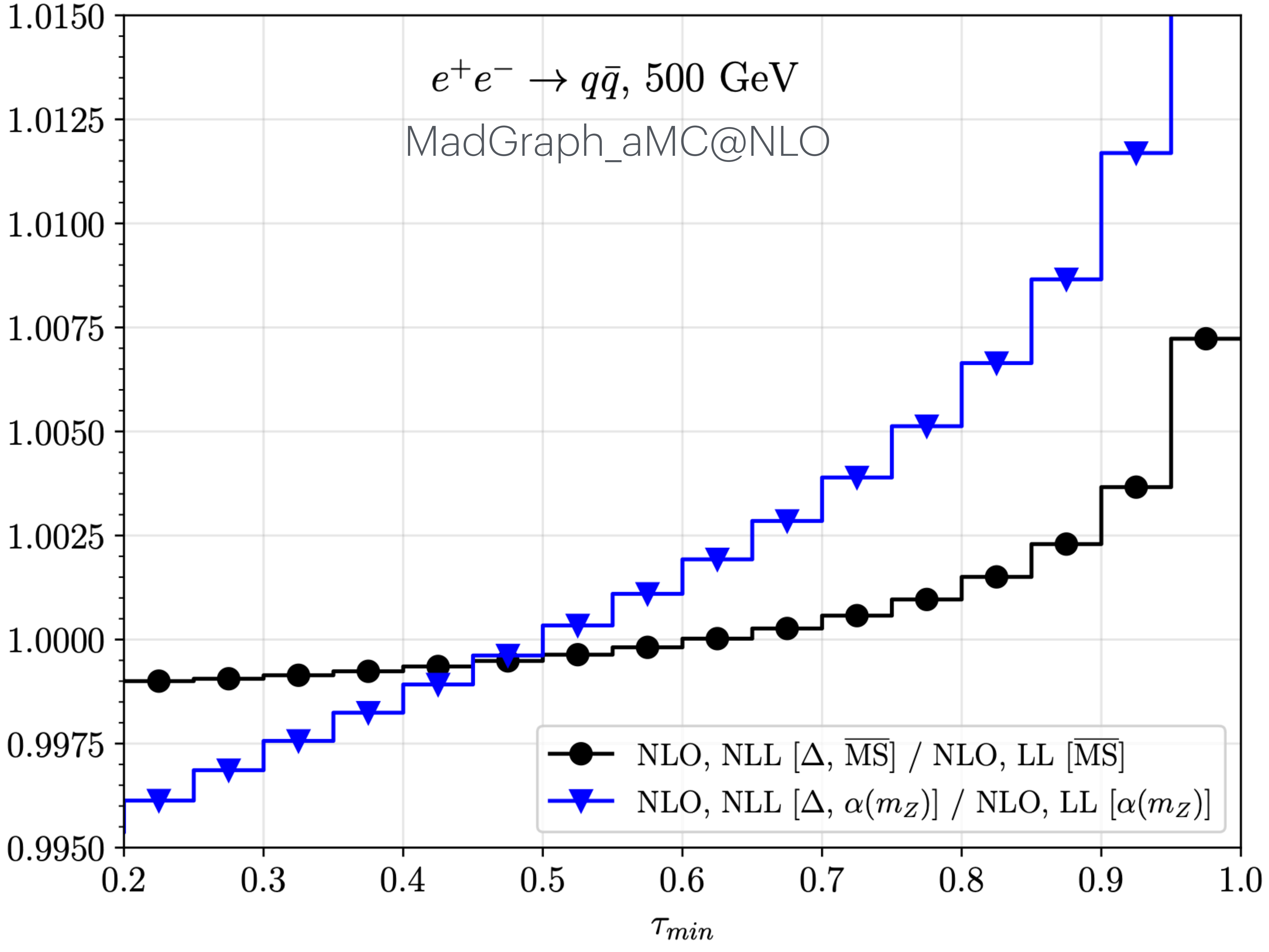
Phys.Rev.Lett. 130 (2023) 3

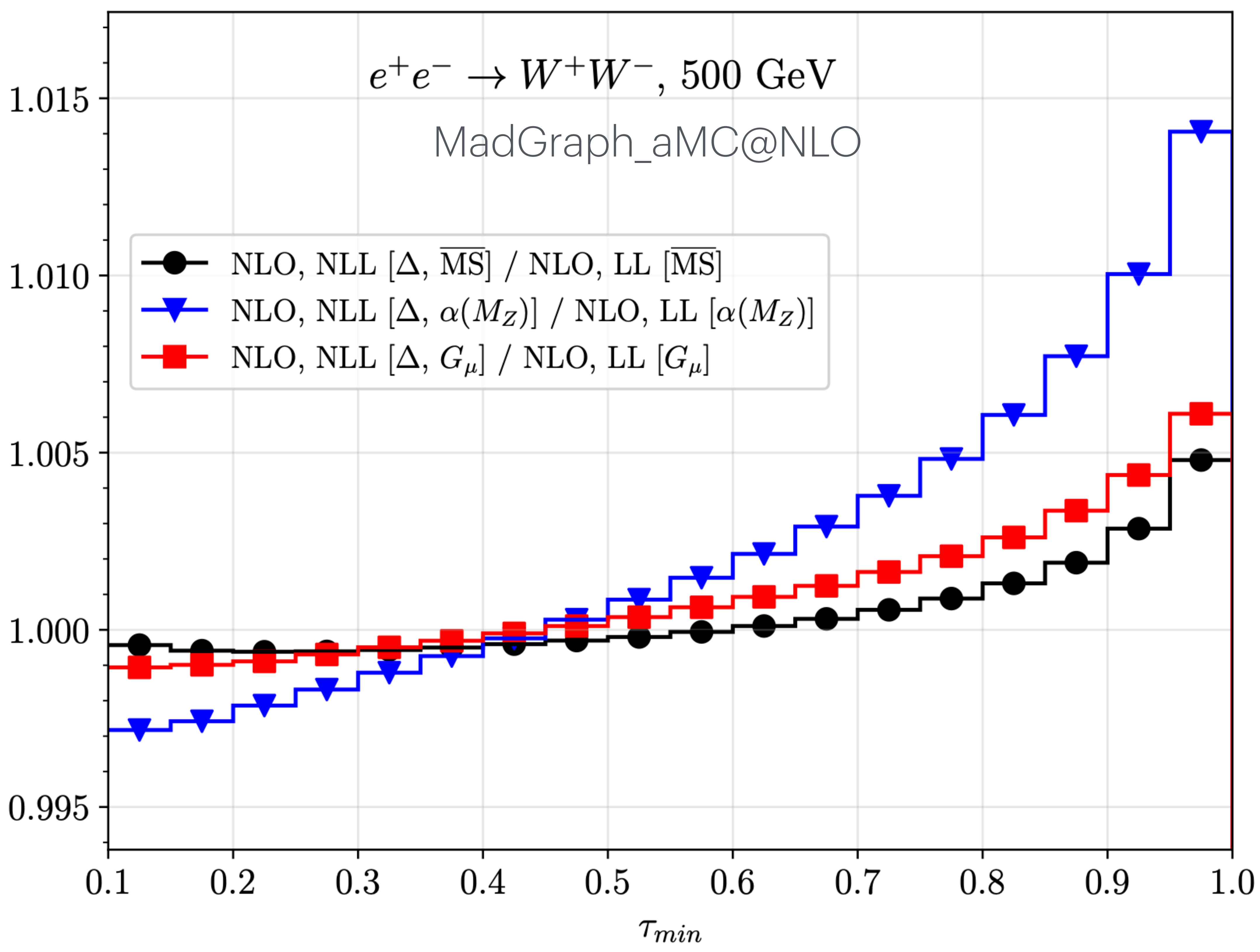
Collinear vs YFS

- ❖ Currently no publicly available QED Shower, but some recent progress
  - 2603.05585 [L.Flower, M.Schoneherr]
  - 2601.19530 [C. M. Carloni Calame et al]
  - 2602.16029 [Belloni et al]
- ❖ With aggressive clustering of photons we see that the **YFS** converges to the Fixed-Order prediction
- ❖ This will need to be redone with NLO/NLL PDFs + Parton Shower



Collinear Factorisation

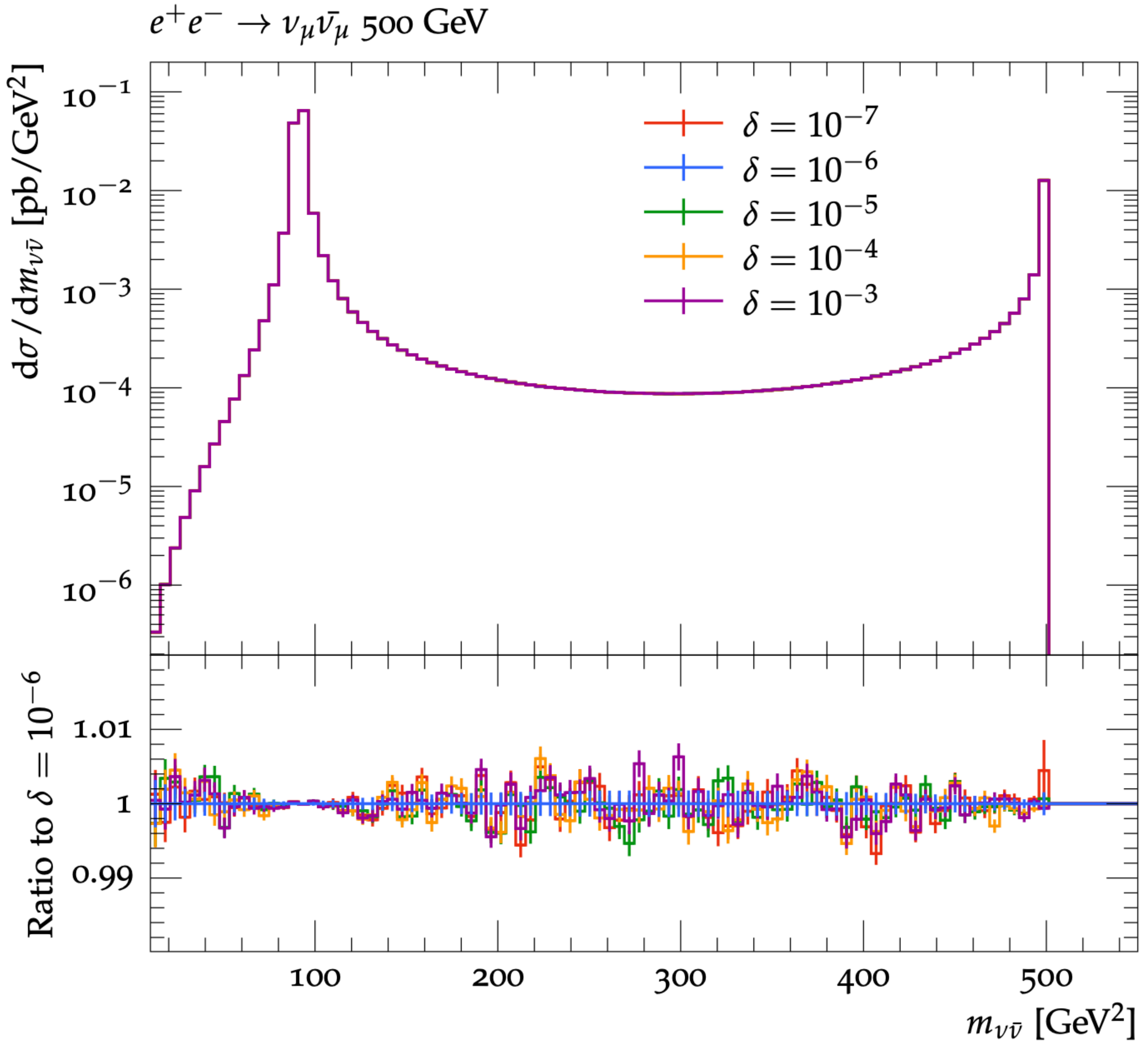


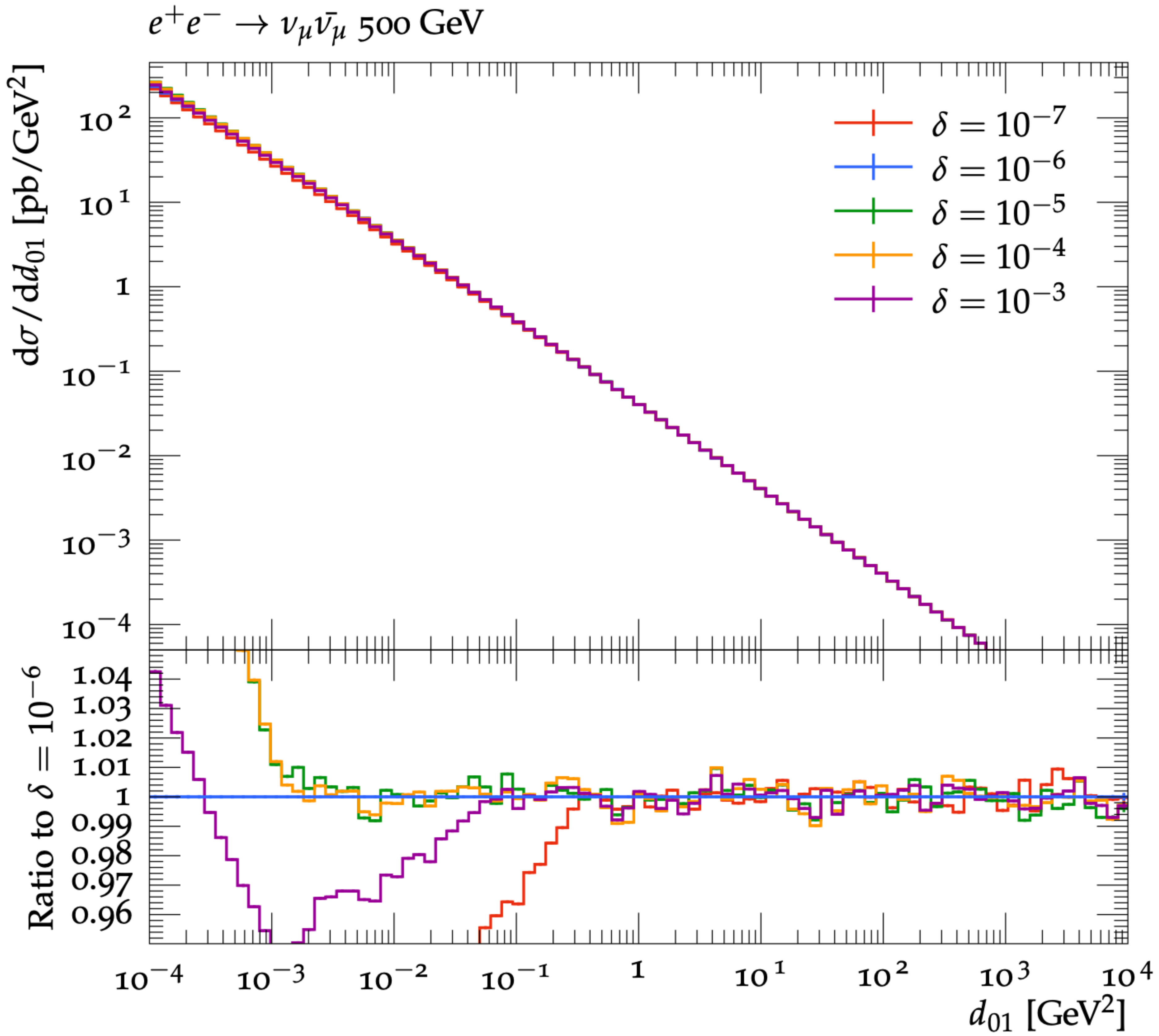


$$\sigma\left(\tau_{\text{min}}\right) = \int d\sigma\left(\tau_{\text{min}} \leq \frac{M^2}{s}\right)$$

[JHEP 10 (2022) 089, V. Bertone et al]

QED Shower

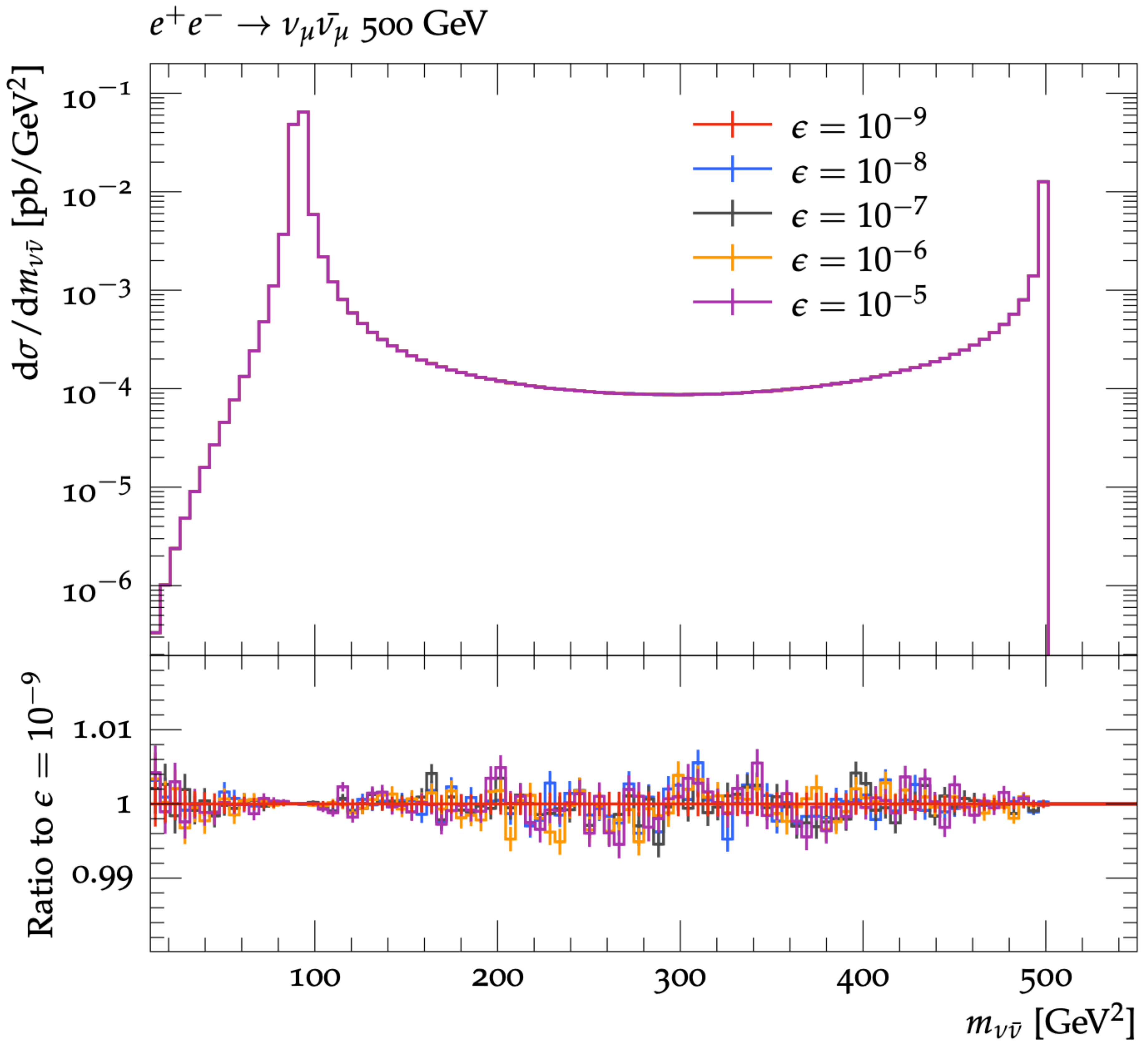


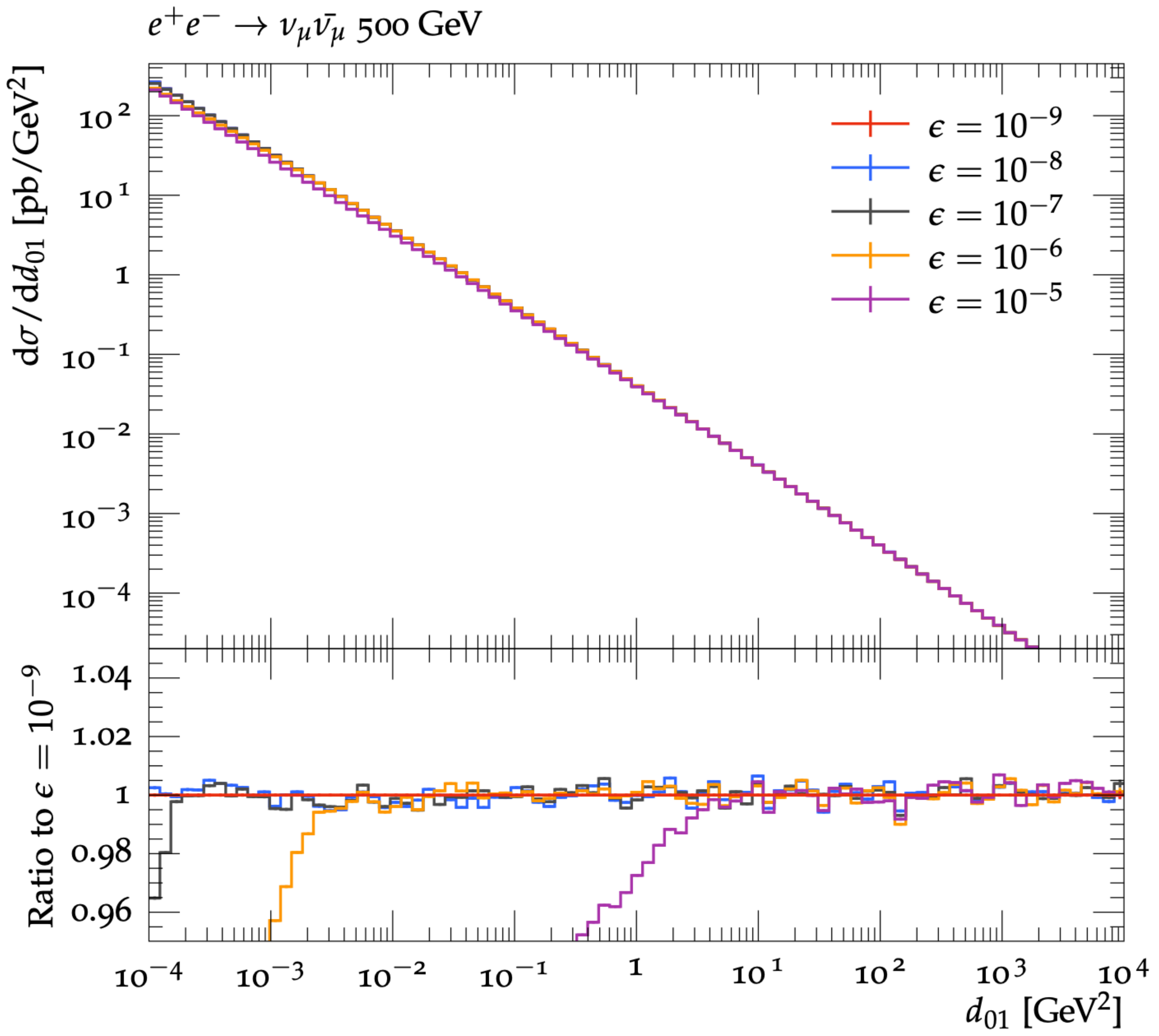


Dependency is negligible if δ chosen small enough

[2603.05585, L.Flower, M. Schönherr]

QED Shower





[2603.05585, L.Flower, M. Schönherr]

Dependency is negligible if ϵ chosen small enough

Outlook

- Future e^+e^- will require **unprecedented precision** from theory
  - $\text{NNLO}_{\textbf{EW}}$ will become mandatory and we may need to go beyond this
- Both the **PDF** and **YFS** approaches should be pursued
  - With **two independent** approaches we can cross check out results
  - Genuine differences can be used to **estimate theory uncertainties**
- Ideally a SCET based approach were the two resummations can be combined
- A matched QED parton shower for collinear approach
- How to combine YFS with QCD corrections?

We have a lot of work to do to ensure our theory calculations do not hold back the precision of a future e^+e^- collider

I can see a path for the theory community to achieve this but it will take huge effort by the community

But we have plenty of time !