Analytic methods for massive multi-loop integrals

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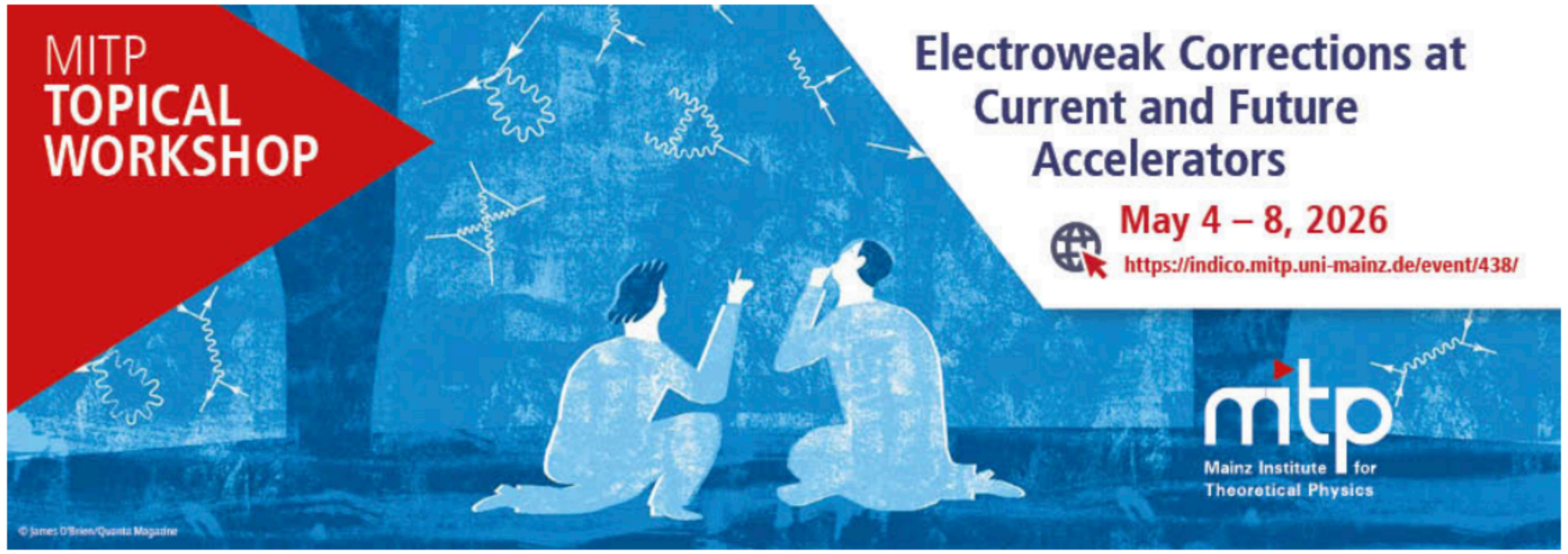
University of Bonn

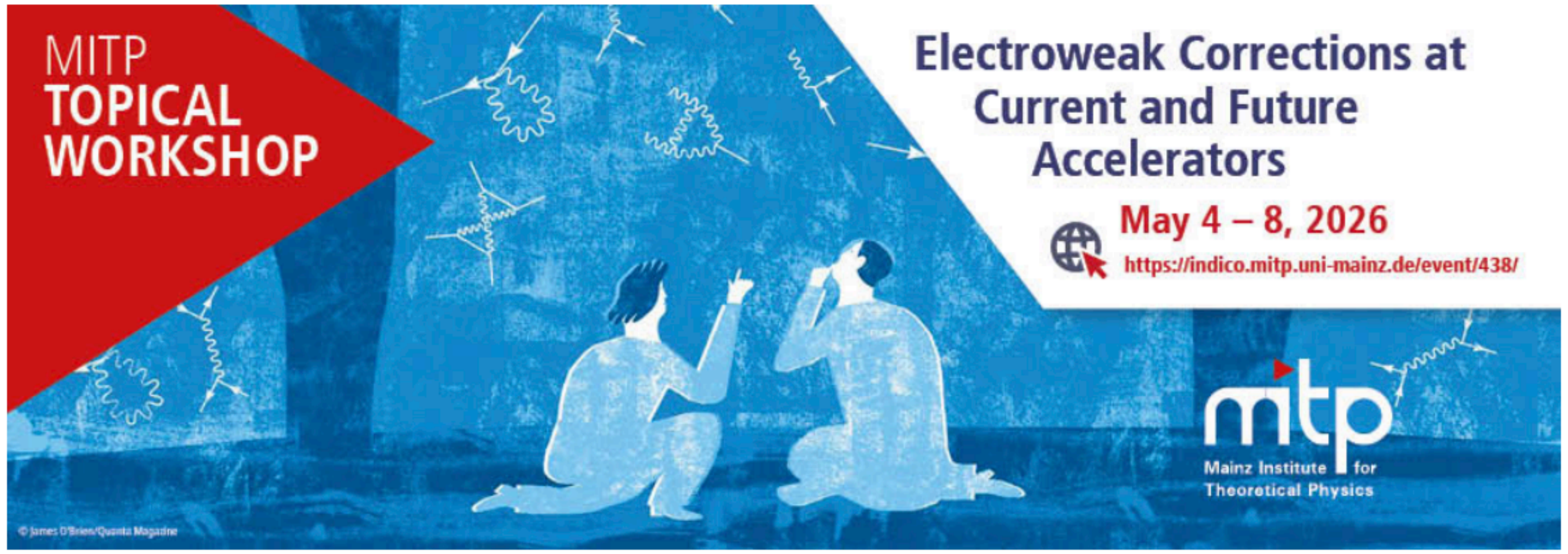
6th May 2026

Electroweak Corrections at Current and Future Accelerators

MITP, Mainz



Multi-loop electroweak corrections crucial and difficult perturbative input for the future lepton colliders!



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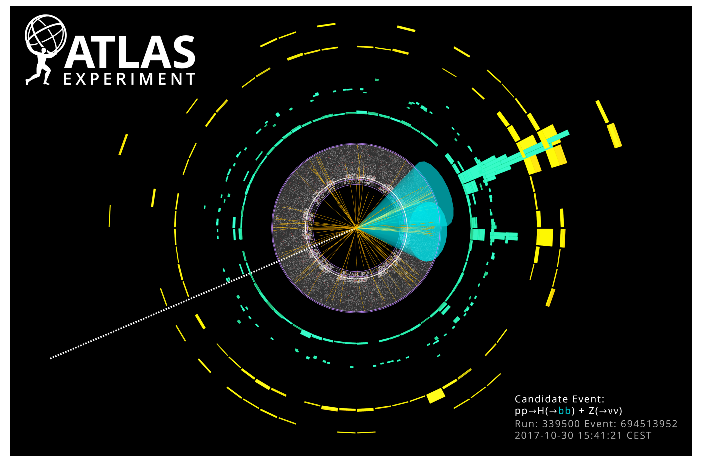


Scattering Amplitudes

QCD, QED, EW

↑

$$observables \propto |\mathcal{M}|^2$$

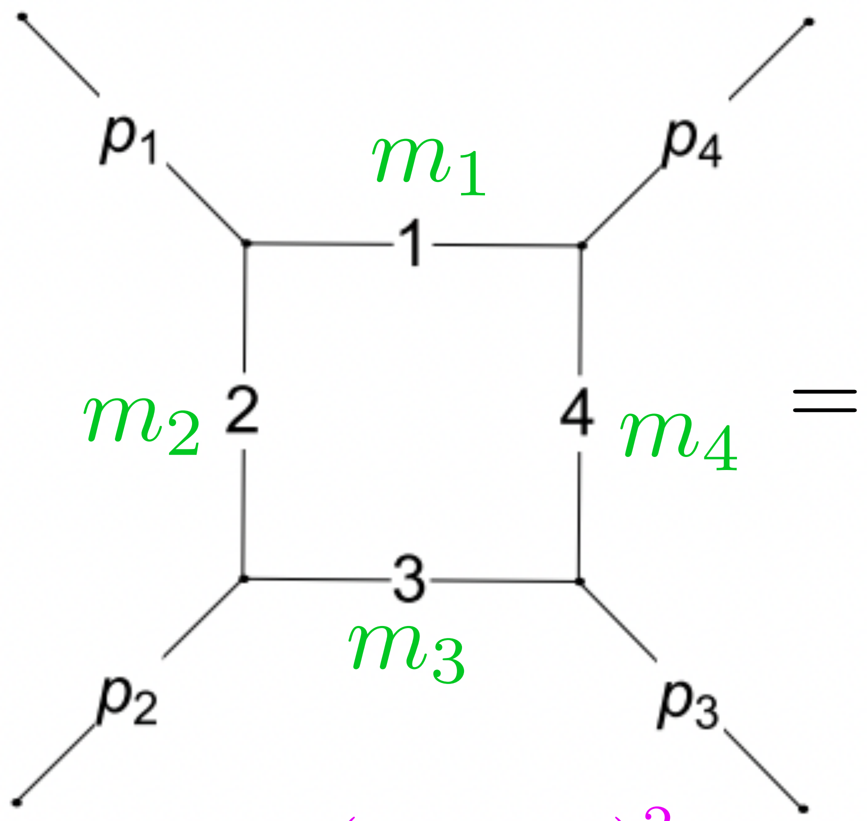
Computation of **higher-order loop corrections!**

Feynman Integrals

$$I = \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n} \frac{1}{(-q_j^2 + m_j^2)^{\nu_j}}$$

$$q_j = k + p_j$$

Textbook example



$$= \int d^4k \frac{1}{[k^2 - m_1^2][(k+p_1)^2 - m_2^2][(k+p_1+p_2)^2 - m_3^2][(k-p_4)^2 - m_4^2]}$$

$s = (p_1 + p_2)^2$

$t = (p_1 + p_3)^2$

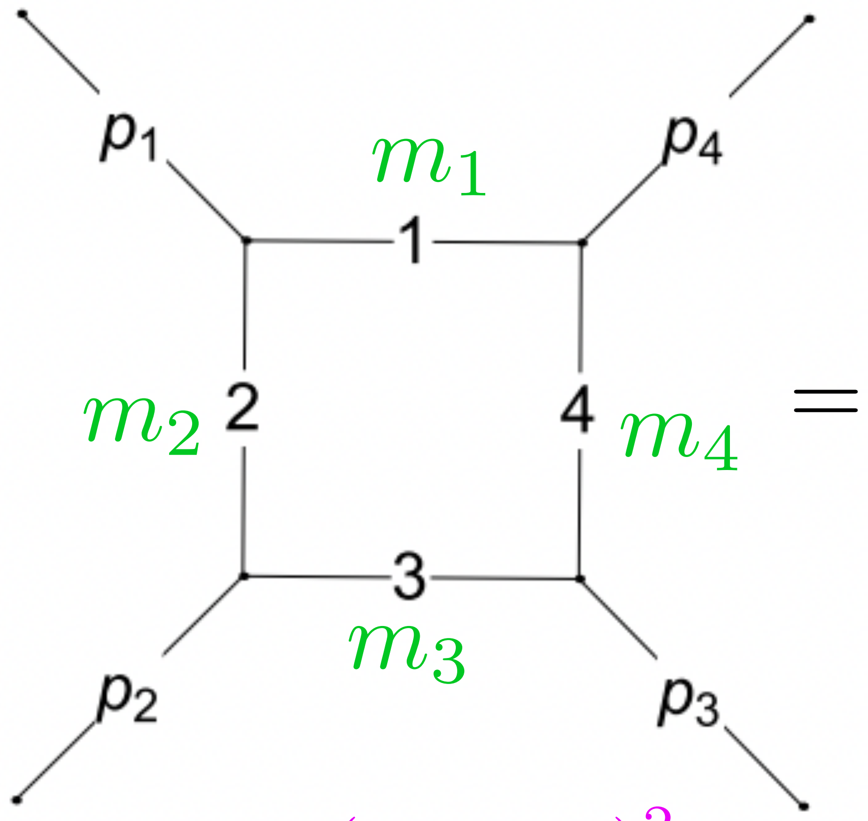
$\{s, t, m_1^2, m_2^2, m_3^2, m_4^2\}$

- Loops
- Legs
- Scales

Feynman Integrals

Feynman Integrals 

Textbook example



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$s = (p_1 + p_2)^2$

$t = (p_1 + p_3)^2$

$D = 4 - 2\epsilon$

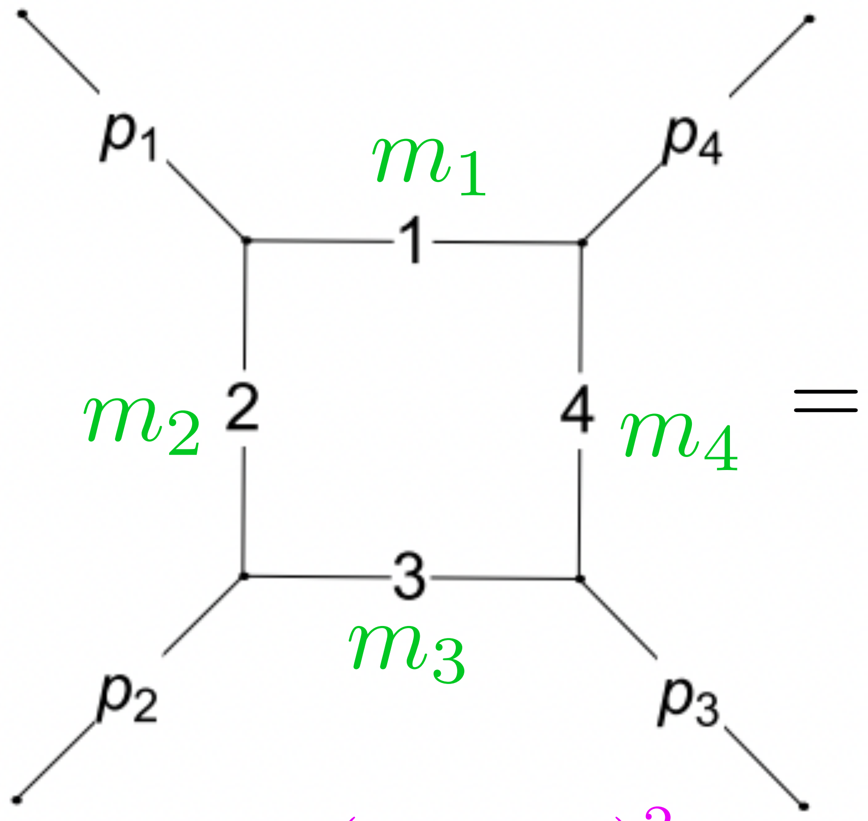
↓

Dimensional regularization parameter

Feynman Integrals

Feynman Integrals

Textbook example



m_1, m_2, m_3, m_4

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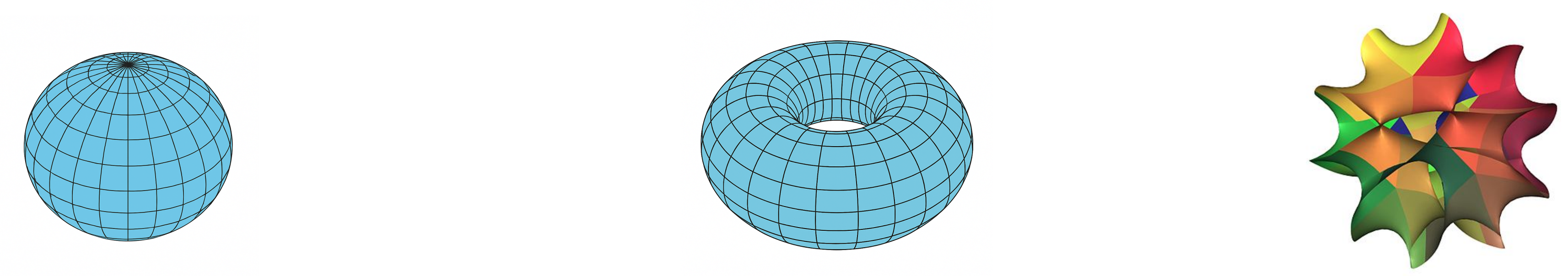
Dimensional regularization parameter

Feynman integrals and geometry



Computing the Feynman integrals

Analytic methods

- Direct integration in Feynman, Schwinger,... parametrizations
- The method of differential equations (DE)

Semi-analytic methods

Numerical methods

- Numerical integration: Sector Decomposition [Carter, Heinrich '11] [Borowka et al. '13]
- Mellin-Barnes integrations [Gluza, Kajda, Riemann '07] [Freitas, Huang '10] + others

- Semi numerical approaches: generalized series expansion (GSE), DiffExp, SeaSyde,.. [Moriello '19] [Hidding, '21] [Armadillo, Bonciani, Devoto, Rana, Vicini '26]
- Auxiliary mass flow (AMFlow), LINE [Liu, Ma '22] [Prisco, Ronca, Tramontano '25]
- Numerical integration of the DE [Rosàs, Bobadilla '25]

The method of differential equations

The method of DE for computing Feynman integrals

A basis for all Feynman integrals (MI): $\vec{I}$

Set up DE wrt all kinematics

$$\boxed{d\vec{I} = A\vec{I}}$$

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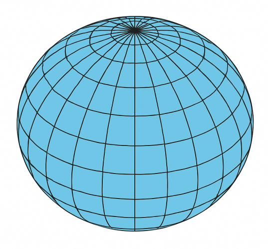
Set up DE wrt all kinematics

$$d\vec{I} = A\vec{I}$$

Change basis

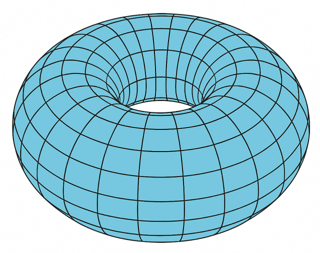
$$\vec{J} = T\vec{I}$$
$$d\vec{J} = \epsilon\tilde{A}\vec{J}$$

[Henn '13]
[Lee '14]
[Prausa '17]
[Meyer '18]
[Dlapa, Henn, Yan '20]
[Wasser, Mistlberger, Smirnov, Wasser '20]
[+ many more]



Canonical-form

Eps-form



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Mathematical Structure dependent

(Algebraic, Transcendental)

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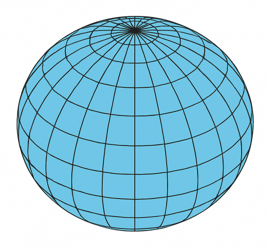

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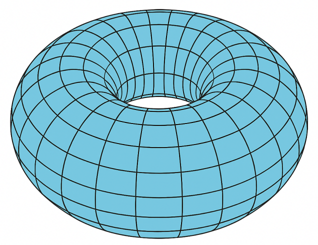

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Complexity level

kinematic scales
mass scales
loops

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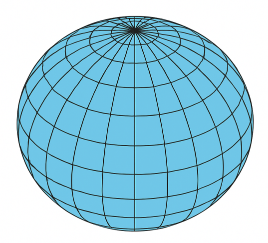

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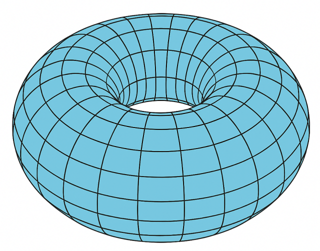

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Solve order-by-order in ϵ

$$J_k = \sum_{j=0} \epsilon^j J_k^{(j)}$$

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kinematic scales
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Analytic representations

Analytic representation

Of the coefficients in the Laurent expansion

Chen's iterated integrals [Chen '71]

Let $\mathcal{M}$ be the manifold, and $\omega' s$ be the one-forms.

Pull-back of the differential one-forms to the unit interval $[0,1]$ $\quad \gamma^*(\omega_i) = f_i(t)dt$

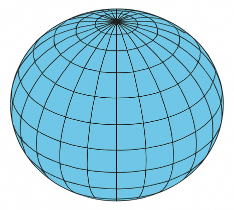
Iterated integrals of $\omega' s$ along γ is defined by

Independent of geometric structure

$$\int_\gamma I(\omega_1 ... \omega_n; \lambda) = \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) ... \int_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k)$$

Multiple Polylogarithms (MPLs)[Kummer '74][Poincare '84][Goncharov '98][Borwein, Bradley, Broadhust, Lisonek '01]

$$G(z_1, z_2, ..., z_k; y) = \int_0^y \frac{dy_1}{y_1 - z_1} G(z_2, ... z_k; y_1)$$



Analytic representation

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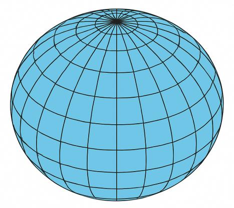
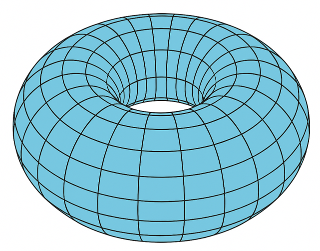


Elliptic polylogarithms (eMPLs) [Adams, Bogner, Weinzierl '14,'15] [Bloch, Vanhove '13] [Broedel, Duhr, Dulat, Tancredi '17] [Brown, Levin'11]

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 ... n_k \\ z_1 ... z_k \end{smallmatrix}; z; \tau\right) = \int_0^z dz' g^{(n_1)}(z' - z_1, \tau) \tilde{\Gamma}\left(\begin{smallmatrix} n_2 ... n_k \\ z_2 ... z_k \end{smallmatrix}; z; \tau\right)$$



Numerical Evaluation

(Highly dependent on the analytic structure)

Polylogarithmic cases

Expressible as $d\log$ one-forms

Using DE: Use GSE on the DE (ϵ- factorised) (contains square roots)

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Using AR: MPLs: GINAC, FastGPL, HandyG,..

[Vollinga '07][Wang, Yang, Zhou '21][Naterop, Signer, Ulrich '19]

Expressed as iterated integrals with $d\log$ one-forms:

1. Local series expansions
2. Construct a basis of iterated integrals, later use GSE

Elliptic (and beyond) cases

Not expressible as $d\log$ one-forms

Using DE: Use GSE on the algebraic DE (not ϵ-factorized)

[epsilon collaboration '25][EC, Sotnikov '25][Chen, Yang, Zhang '25]

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Using AR:

eMPLs: GINAC [Walden, Weinzierl ’20] [Duhr, Lorkowski, Marzucca, Mauc, Weinzierl ’26]

Iterated integrals with generic (periods and other transcendental) one-forms:

$$\int_\gamma I(\omega_1 \ldots \omega_n; \lambda)$$

Could depend on transcendental objects,
like periods of an elliptic curve or its derivative

Local series expansions

Numerical evaluation suffers:
Series expansions around special kinematic points → often insufficient
Series expansions in the whole phase-space region → often too slow

Outlook

- Multi-loop computations challenging but the analytic insights fun!
- Clearly a lot of progress will be needed to fully utilize the mathematical understanding for phenomenological applications.
- Despite the challenges the rich mathematical structure of multi-loop integrals interesting by themselves!

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Thanks!