Formalism for open EFTs with gauge degrees of freedom

Gauging open EFTs from the top down: arXiv:2512.17089

NEW!! Schwinger-Keldysh Path Integral for Gauge theories arXiv:2604.26941

Maria Mylova

in collaboration with

Greg Kaplanek and Andrew Tolley

Kavli-IPMU, Tokyo U.

MITP Program
Open Quantum Systems

30 April '26

Quantum Crossroads Workshop
3-7 Aug 2026
Kavli-IPMU, Japan

https://indico.ipmu.jp/event/519/

Quantum crossroads: Interdisciplinary, hybrid workshop at the interface of open quantum systems, high-energy physics, and quantum science

OQS in cosmology

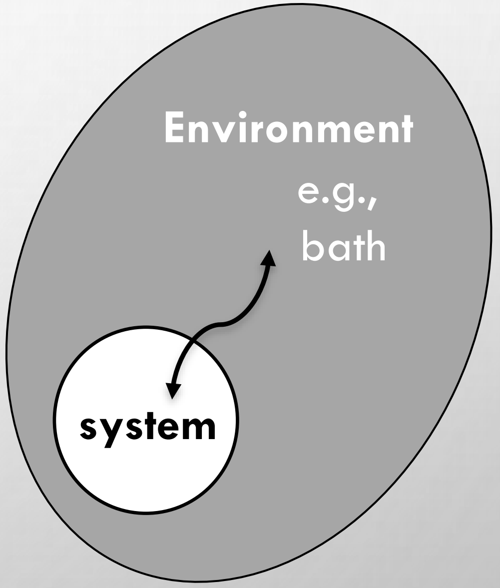
In cosmology there are several
out of (or near) equilibrium processes

open system effects may become relevant when

i) light degrees of freedom interact with the system.
ii) strongly coupled or otherwise non-perturbative regimes (e.g., reheating/preheating, electroweak phase transitions).
iii) horizons (e.g. cosmological/black-hole horizons)
iv) unresolved matter sectors
etc.

Open quantum systems

- Two interacting sectors $\hat{H} = \hat{H}_{\mathcal{S}} \otimes \mathbb{I}_{\mathcal{E}} + \hat{H}_{\mathcal{E}} \otimes \mathbb{I}_{\mathcal{S}} + g\hat{H}_{\rm int}$ (subsystems).
- Scale separation of system and environment not always feasible.
- Averaging scheme to "integrate out the environment".
- Non-Hamiltonian evolution.
- Study the effect of dissipation, noise, decoherence etc.



Density matrix

The density matrix encodes all the statistical and quantum information about the state of a system.

Schrödinger Equation

The SE: $i\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$ is satisfied by pure states $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$,
meaning: $\rho^2 = \rho, \rho^\dagger = \rho$.

von Neumann equation

Differentiate ρ and substitute for SE to get: $i\frac{d}{dt}\rho(t) = [H, \rho(t)]$.
Can accomodate for mixed states $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.
Can extend to open systems, e.g. Nakajima-Zwanzig.

The Schwinger-Keldysh path integral gives a formal solution for the time evolution of the reduced density matrix.

Density matrix

Evolution of the full density matrix

$$\rho(t_\mathrm{f}) = \hat{U}(t_\mathrm{f}, t_\mathrm{i})\ \rho_\mathrm{i}\ \hat{U}^\dagger(t_\mathrm{f}, t_\mathrm{i})\,.$$

Matrix elements

$$\rho_{mn}(t) \equiv \langle \mathbf{f}_m | \rho(t) | \mathbf{f}_n \rangle$$

- **Probability (Diagonal):** $\rho_{nn} = \langle \mathbf{f}_n | \rho | \mathbf{f}_n \rangle$.
- **Coherence (Off-Diagonal):** $\rho_{mn} = \langle \mathbf{f}_m | \rho | \mathbf{f}_n \rangle\ (m \neq n)$.
- **Expectation Value:** $\langle O \rangle = \mathrm{Tr}[O\rho]$.
- **Purity and Mixedness:** $\mathrm{Tr}\left[\rho^2\right] \leq 1$.
- **Entanglement:** e.g. von Neunmann entropy $S(\rho_A) = -\,\mathrm{Tr}[\rho_A \ln \rho_A] > 0$.

Properties: $\mathrm{Tr}\,\hat{\rho} = 1, \quad \hat{\rho}^\dagger = \hat{\rho}$, positivity ?

In-in formalism

CTP/Schwinger-Keldysh/real-time formalism

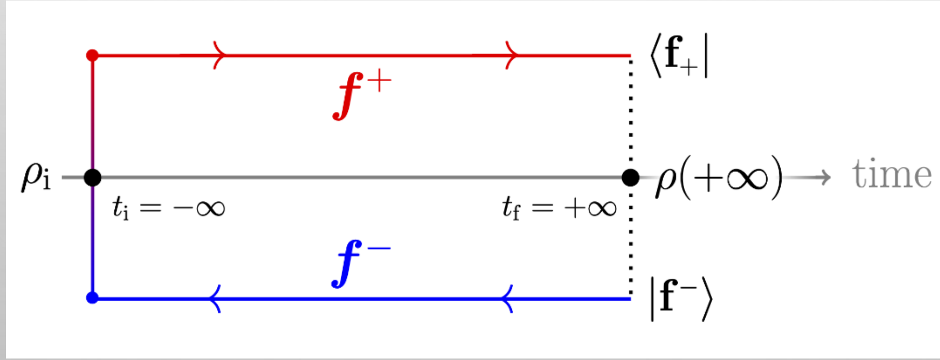
- Insert two resolutions of the identity: $\mathbb{I} = \int \mathcal{D}\mathbf{f}\, |\mathbf{f}\rangle\langle\mathbf{f}|$

$$
\begin{aligned}
\langle \mathbf{f}_+ | \rho(t_\mathrm{f}) | \mathbf{f}_- \rangle &= \langle \mathbf{f}_+ | \hat{U}(t_\mathrm{f}, t_\mathrm{i}) \rho_\mathrm{i} \hat{U}^\dagger(t_\mathrm{f}, t_\mathrm{i}) | \mathbf{f}_- \rangle \\
&= \int \mathrm{d}[\mathbf{f}_{+\mathrm{i}}, \mathbf{f}_{-\mathrm{i}}]\; \underbrace{\langle \mathbf{f}_+ | \hat{U}(t_\mathrm{f}, t_\mathrm{i}) | \mathbf{f}_{+\mathrm{i}} \rangle}_{\substack{\textbf{Path Integral} \\ \textbf{Identity} \\ \text{(+ Branch)}}} \underbrace{\langle \mathbf{f}_{+\mathrm{i}} | \rho_\mathrm{i} | \mathbf{f}_{-\mathrm{i}} \rangle}_{\substack{\textbf{Sum over} \\ \textbf{initial states}}} \underbrace{\langle \mathbf{f}_{-\mathrm{i}} | \hat{U}^\dagger(t_\mathrm{f}, t_\mathrm{i}) | \mathbf{f}_- \rangle}_{\substack{\textbf{Path Integral} \\ \textbf{Identity} \\ \text{(- Branch)}}}
\end{aligned}
$$

- Use the path integral identity:

$$
\langle \mathbf{f}_+ | \rho(t_\mathrm{f}) | \mathbf{f}_- \rangle = \int \mathrm{d}[\mathbf{f}_{+\mathrm{i}}, \mathbf{f}_{-\mathrm{i}}]\; \langle \mathbf{f}_{+\mathrm{i}} | \rho_\mathrm{i} | \mathbf{f}_{-\mathrm{i}} \rangle \int_{\mathbf{f}_{+\mathrm{i}}}^{\mathbf{f}_+} \mathcal{D}[\boldsymbol{f}_+] \int_{\mathbf{f}_{-\mathrm{i}}}^{\mathbf{f}_-} \mathcal{D}[\boldsymbol{f}_-]\; \mu\, e^{iS[\boldsymbol{f}_+] - iS[\boldsymbol{f}_-]}
$$

[L. V. Keldysh (1964)
R. P. Feynman & F. L. Vernon Jr (1963)]



$\boldsymbol{f} = (A, c, \bar{c}, \Phi, \Phi^*)$

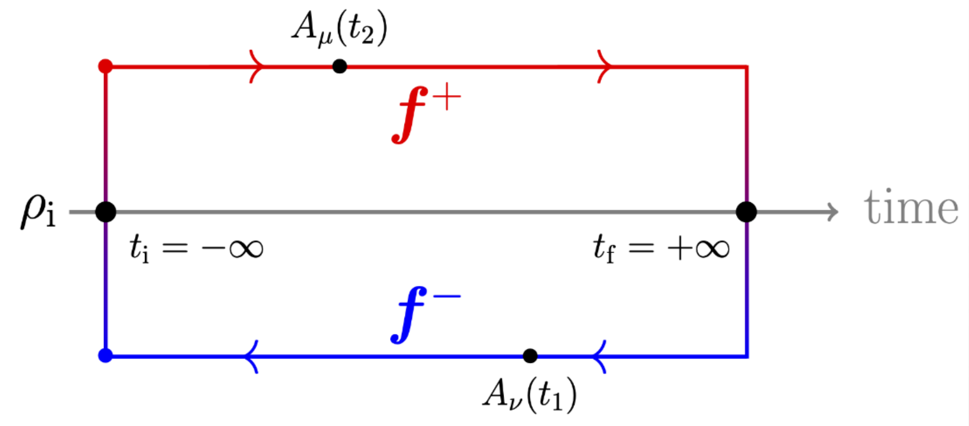
$\mathrm{f}_j = (\mathrm{A}_j, \mathrm{c}_j, \bar{\mathrm{c}}_j, \Phi_j, \Phi_j^*)$

In-in formalism

the in-in contour generates correlation functions for general initial states

Physical observables correspond to traces:

$\mathbf{f} = \mathbf{f}_+ = \mathbf{f}_-$ $\longrightarrow$



Summary of B.C.'s:

- Ensure matching between the branches at the final time t_f
- impose initial conditions at time t_i

Introducing sources $J_\pm$, we can define the generating functional, where $\{J^\pm_{\mathfrak{a}}\} = (J^\pm_A, J_c, J^\pm_{\bar{c}}, J^\pm_\Phi, J^\pm_{\Phi^*})$

e.g., differentiating with respect to sources on **different branches** yields the Wightman function of the photon

$$\left.\frac{\delta^2 Z_{\text{in-in}}}{\delta J_A^{-\mu}(x)\,\delta J_A^{+\nu}(y)}\right|_{J^\pm_{\mathfrak{a}}=0} = \mathrm{Tr}\big(\hat{A}_\mu(x)\hat{A}_\nu(y)\rho_{\rm i}\big).$$

Keldysh r/a basis

$$\boldsymbol{f}_\pm = \boldsymbol{f}_\mathrm{r} \pm \frac{1}{2}\boldsymbol{f}_\mathrm{a} \qquad \longrightarrow \qquad \underbrace{\boldsymbol{f}_\mathrm{r} = \frac{\boldsymbol{f}_+ + \boldsymbol{f}_-}{2}}_{\text{“classical field”}} \quad \text{and}\,, \quad \underbrace{\boldsymbol{f}_\mathrm{a} = \boldsymbol{f}_+ - \boldsymbol{f}_-}_{\text{“quantum field”}}\,.$$

Semi-classical expansion

$$i\Big(S[\boldsymbol{f}_+] - S[\boldsymbol{f}_-]\Big) = i\left(S\left[\boldsymbol{f}_\mathrm{r} + \frac{1}{2}\boldsymbol{f}_\mathrm{a}\right] - S\left[\boldsymbol{f}_\mathrm{r} - \frac{1}{2}\boldsymbol{f}_\mathrm{a}\right]\right) \simeq i\int \mathrm{d}^4x \left[\boldsymbol{f}_\mathrm{a}\cdot\frac{\delta S[\boldsymbol{f}_\mathrm{r}]}{\delta \boldsymbol{f}_\mathrm{r}} + \mathcal{O}(\boldsymbol{f}_\mathrm{a}^3)\right].$$

Ex: Scalar QED:

$$S \;=\; -\int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^* D^\mu\Phi + m^2\Phi^*\Phi + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{c}\Box c\right]$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $D_\mu = \partial_\mu - ieA_\mu$, $c(x)$ and $\bar{c}(x)$ are the Grassmann fields.

Rotate basis

$$S_{\mathrm{in-in}} = \int \mathrm{d}^4x \left[\partial_\nu F_r^{\nu\mu} - ie\left(\Phi_r^* D_r^\mu \Phi_r - (D_r^\mu\Phi_r)^*\Phi_r\right) + \frac{1}{\xi}\partial^\mu\partial_\nu A_r^\nu\right] A_\mu^a$$
$$+ \left[D_\mu^r D_r^\mu \Phi_r - m^2\Phi_r\right]\Phi_a^* + \left[(D_\mu^r D_r^\mu\Phi_r)^* - m^2\Phi^*\right]\Phi_a - \left(\Box\bar{c}_r\, c_a + \Box c_r\,\bar{c}_a\right)$$

Introduction to BRST gauge-fixing

In the Abelian Higgs action there is a local *U(1)* redundancy.

$$\Phi(x) \to e^{iq\lambda(x)}\Phi(x)\,, \qquad \text{and} \qquad A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x).$$

The path integral "overcounts" physically equivalent configurations.

To quantize the theory, we must first eliminate this redundancy.

$$S \;=\; -\int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^* D^\mu\Phi + m^2\Phi^*\Phi + \underbrace{\frac{1}{2\xi}(\partial_\mu A^\mu)^2}_{\text{gauge fixing term}} + \bar{c}\Box c\right]$$

The transformation is controlled by a single *global* Grassmann parameter η, with $\eta^2 = \eta$.
The local x-dependence is carried by the ghost field $c(x)$.

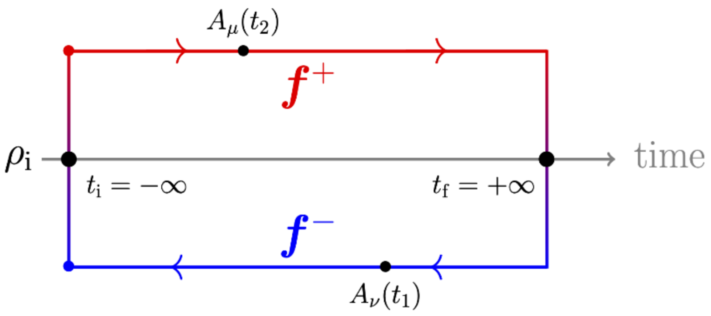
$$A_\mu(x) \to A_\mu(x) + \eta\partial_\mu c(x)\,, \qquad c(x) \to c(x)\,,$$
$$\Phi(x) \to \Phi(x) + iq\eta c(x)\Phi(x)\,, \qquad \bar{c}(x) \to \bar{c}(x) - \tfrac{\eta}{\xi}\partial_\mu A^\mu(x).$$

These transformations are generated by the BRST charge Q_{BRST}.
Physical states are those annihilated by the charge, $Q_{\mathrm{BRST}}|\psi\rangle = 0$.

$$Q_{\mathrm{BRST}}\rho = \rho Q_{\mathrm{BRST}} = 0$$

BRST gauge-fixing in the in-in formalism

Match fields at final B.C.'s $\boldsymbol{f}_{\pm}(t_f, \mathbf{x}) = \mathbf{f}(\mathbf{x})$.



$$
\begin{aligned}
A_{\pm\mu}(x) &\to A_{\pm\mu}(x) + \eta_\pm \partial_\mu c_\pm(x)\,, \\
\Phi_\pm(x) &\to \Phi_\pm(x) + iq\eta_\pm c_\pm(x)\Phi_\pm(x)\,, \\
c_\pm(x) &\to c_\pm(x)\,, \\
\bar{c}_\pm(x) &\to \bar{c}_\pm(x) - \eta_\pm \partial_\mu A^\mu_\pm(x)\,,
\end{aligned}
$$

Global BRST transf.
Preserved by the
diagonal subgroup

$$\eta_\pm = \eta$$

BRST transformations as $\mathbf{f} \to \mathbf{f} + \overbrace{\eta\hat{s}}^{\delta_{\mathrm{BRST}}}\mathbf{f}$

- Gauge symmetry, breaks down to the diagonal subgroup **only at the boundary**.
- **Two independent gauge transf's (retarded & advanced) remain in the dynamics.**

Retarded

$$
\begin{aligned}
\hat{s}A_r\mu(x) &= \partial_\mu c_r(x) \\
\hat{s}\Phi_r(x) &= iqc_r(x)\Phi_r(x) \\
\hat{s}c_r(x) &= 0 \\
\hat{s}\bar{c}_r(x) &= -\partial_\mu A^\mu_r(x)
\end{aligned}
$$

Advanced

$$
\begin{aligned}
\hat{s}A^a_\mu(x) &= \partial_\mu c_a(x) \\
\hat{s}\Phi_a(x) &= i\left[\Phi_a(x)c_r(x) + c_a(x)\Phi_r(x)\right] \\
\hat{s}c_r(x) &= 0 \\
\hat{s}\bar{c}_a(x) &= -\partial_\mu A^\mu_a(x)
\end{aligned}
$$

Decoupling limit

In the decoupling limit we recover the global symmetry.

In the local case $A'_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x), \Phi'(x) = e^{iq\lambda(x)}\Phi(x),$

split gauge transformation into a global part θ and a local part $\xi(x)$: $\lambda(x) = \frac{1}{q}\theta + \xi(x)$

$q \to 0$ the original local symmetry separates into a decoupled global transformation and a local transformation $A'_\mu(x) = A_\mu(x) + \partial_\mu \xi(x)\,, \quad \Phi'(x) = e^{i\theta}\Phi(x)$

BRST case – recover the global diagonal *U(1)*.

split ghost c into a constant vev and a fluctuation $c_\pm(x) = \frac{1}{q}c_0 + C_\pm(x)$

The in-in boundary conditions require the same vev on each branch: $c_+(t_\mathrm{f}) = c_-(t_\mathrm{f}) \implies C_+(t_\mathrm{f}) = C_-(t_\mathrm{f})$

$$q \to 0 \qquad \delta_{\mathrm{BRST}} A_{\pm\mu} = \eta \partial_\mu C_\pm\,, \qquad \delta_{\mathrm{BRST}} C_\pm = 0\,, \qquad \delta_{\mathrm{BRST}} \bar{c}_\pm = -\eta \partial_\mu A^\mu_\pm\,, \qquad \delta_{\mathrm{BRST}} \Phi^\pm = i\eta c_0 \Phi^\pm$$

the diagonal BRST transf. on Φ reduces to a diagonal global $U(1)$ transf. $\theta^+ = \theta^- = \eta c_0$

iϵ-prescription

For an initial Gaussian state with translation invariance we can encode the boundary conditions on the path integral.

Ex: iϵ for a real massive free scalar field: $\hat{\phi}(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \theta(k^0) 2\pi \delta(k^2 + m^2) \left(e^{ik.x} \hat{a}_{\mathbf{k}} + e^{-ik.x} \hat{a}^\dagger_{\mathbf{k}} \right)$

The free CTP propagator contains information about the state ρ: $\mathbf{G_0} = \begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix}$

$$G_{++}(x,y) = \mathrm{Tr}\left[\rho \mathcal{T} \hat{\phi}(x)\hat{\phi}(y)\right],$$
$$G_{+-}(x,y) = \mathrm{Tr}\left[\rho \hat{\phi}(y)\hat{\phi}(x)\right],$$
$$G_{-+}(x,y) = \mathrm{Tr}\left[\rho \hat{\phi}(x)\hat{\phi}(y)\right],$$
$$G_{--}(x,y) = \mathrm{Tr}\left[\rho \bar{\mathcal{T}} \hat{\phi}(x)\hat{\phi}(y)\right].$$

$\Downarrow$

We can repackage the state data into the iε-prescription

$$S_{\text{in-in}} = \frac{i}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \begin{pmatrix} \phi_+(-k) & \phi_-(-k) \end{pmatrix} \mathbf{G_0}^{-1}(k) \begin{pmatrix} \phi_+(k) \\ \phi_-(k) \end{pmatrix}$$
$$= \int \mathrm{d}^4 x \left(-\frac{1}{2}(\partial\phi_+)^2 - \frac{1}{2} m^2 \phi_+^2 + \frac{1}{2}(\partial\phi_-)^2 + \frac{1}{2} m^2 \phi_-^2 \right) + S_{i\epsilon}.$$

$i\epsilon$-prescription

rotate to r/a basis

$$G^{r/a}(x,y) = \begin{pmatrix} G_K(x,y) & G_R(x,y) \\ G_A(x,y) & 0 \end{pmatrix}, \qquad G_K = G_R + G_A$$

normalisation condition

$$\mathrm{Tr}(\rho) = 1 \longrightarrow S_{\text{in-in}}[f_r, f_a = 0] = 0\,.$$

$$S_{\text{in-in}} = \int \mathrm{d}^4x\, \phi_{\mathrm{a}}(x)[\Box - m^2]\phi_{\mathrm{r}}(x) + S_{i\epsilon}.$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{ik.(x-y)}\,(1+2\mathfrak{n}(k))\,\omega_k, \qquad \omega_k = \sqrt{\mathbf{k}^2+m^2}$$

with $\quad S_{i\epsilon} = -2\epsilon \int d^4x\, \phi_a(x)\, \partial_t\, \phi_r(x) + i\epsilon \int d^4x \int d^4y\, \phi_a(x)\, K_\phi(x,y)\, \phi_a(y).$

$S_{i\epsilon}$ is split into a state-dependent and a state-independent part

$$S_{i\epsilon}[\phi_r, \phi_a; \rho] \equiv S^{RA}_{i\epsilon}[\phi_r, \phi_a] + S^{K}_{i\epsilon}[\phi_a; \rho].$$

Enforces the $i\epsilon$ prescription for G_R and G_A

Encodes initial data and determines the Keldysh / Haramard propagator G_K

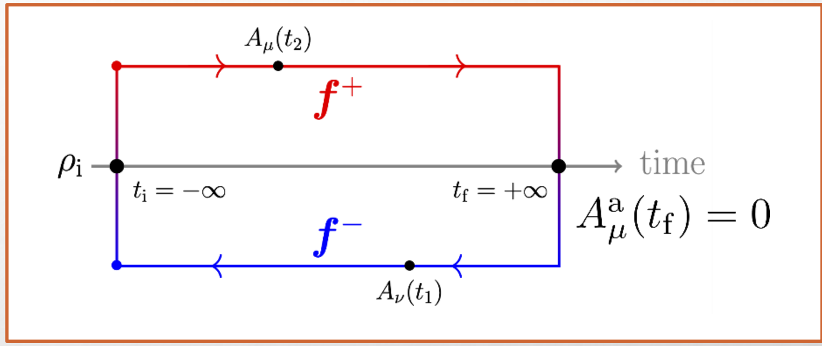
KAVLI IPMU

$i\epsilon$-prescription

Ex: massive complex scalar Φ charged under a $U(1)$ gauge field

Global case

$$S_{i\epsilon} = \int \mathrm{d}^4x \left(\Phi^*_{\mathrm{a}}(x)[\Box - m^2 - 2\epsilon\partial_t]\Phi_{\mathrm{r}}(x) + \Phi^*_{\mathrm{r}}(x)[\Box - m^2 + 2\epsilon\partial_t]\Phi_{\mathrm{a}}(x)\right)$$
$$+ 2i\epsilon \int \mathrm{d}^4x \int \mathrm{d}^4y\; \Phi^*_{\mathrm{a}}(x) K_\Phi(x,y) \Phi_{\mathrm{a}}(y)).$$



What about the gauge theory?

Not gauge invariant?

$$\Phi^*_-(x)\Phi_+(x) \to \underbrace{e^{iq\lambda_+(x) - iq\lambda_-(x)}}_{e^{iq\lambda_a(x)}}\Phi^*_-(x)\Phi_+(x).$$

Not BRST invariant

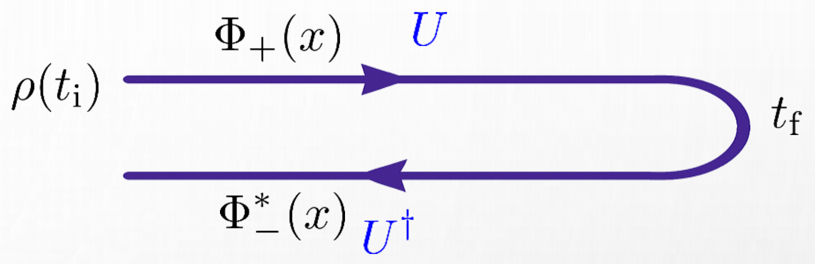
$$\hat{s}\left(\Phi^*_-(x)\Phi_+(x)\right) = iq(c_+(x) - c_-(x))\Phi^*_-(x)\Phi_+(x).$$

☑ **Invariance under the retarded (diagonal) gauge symmetry.**

☒ **Invariance under advanced gauge symmetry.**

Wilson lines

Although these fields have the **same spacetime point**, they are at **different branches** of the contour.



Introduce a Wilson line:

$$e^{\mp\frac{iq}{2}\int_x^{x_\mathrm{f}} \mathrm{d}z^\mu A_\mu^\mathrm{a}(z)}\Phi_\pm(x).$$

- Ends at the final time surface $x_\mathrm{f}^0 = t_\mathrm{f}$
- Trivially satisfies the in-in B.C's: $\Phi_+(t_\mathrm{f}) = \Phi_-(t_\mathrm{f})$
- $\Phi_\pm(x) \to e^{iq\lambda_\mathrm{r}(x)}\Phi_\pm(x)$ transforms covariantly under retarded gauge transf.
- $\Phi_\pm(x) \to e^{\pm i\frac{1}{2}q\lambda_\mathrm{a}(t_\mathrm{f})}\Phi_\pm(x) = \Phi_\pm(x)$ is invariant under advanced gauge transf.

Putting this into a BRST language

The same Wilson line ensures $\Phi_-^\dagger \Phi_+$ is BRST invariant
(because $\lambda_a(t_f) = 0 \implies c_a(t_\mathrm{f}) = 0$)

$$\hat{s}A_\mu^\mathrm{r} = \partial_\mu c_\mathrm{r}(x) \ , \qquad \hat{s}\Phi_\pm(x) = iqc_\mathrm{r}(x)\Phi_\pm(x) \ , \qquad \hat{s}\Phi_\pm^*(x) = -iqc_\mathrm{r}(x)\Phi_\pm^*(x).$$

Why we care?

Recall, we implemented the initial conditions via the iε-prescription.
For the state to be **physical** in a BRST-quantised gauge theory, it must be **BRST invariant**.

$$S_{i\epsilon} = -2\epsilon \int \mathrm{d}^4x \ (\Phi_\mathrm{a}^*(x) D_t[A_\mathrm{r}]\Phi_\mathrm{r}(x) - \Phi_\mathrm{r}^*(x) D_t[A_r]\Phi_\mathrm{a}(x)) + 2i\epsilon \int \mathrm{d}^4x \int \mathrm{d}^4y \ \Phi_\mathrm{a}^*(x) K_\Phi(x,y) \Phi_\mathrm{a}(y).$$

$$K_\Phi(x,y) \to e^{-iq\lambda_\mathrm{r}(x) + iq\lambda_\mathrm{r}(y)} K_\Phi(x,y).$$

Invariant under retarded gauge transf.

Wilson lines in practice

When integrating out charged matter, the resulting nonlocal kernels remain gauge invariant **because the Wilson lines are already built into the background-field propagators.**

- We compute the propagator of the charged field being integrated out, in a fixed gauge-field background A_μ.
- Because the charged field obeys a covariant equation, its Green's function must *parallel-transport* charge between y and x.
- Thus the kernel can be *dressed* as $D(x, y) = W(x, y)H(x, y)$.
- The Wilson line is therefore naturally built in the charged-field propagator in the gauge-field background.

Here $W(x, y)$ is the Wilson line carrying the endpoint phases, while $H(x, y)$ is gauge invariant. Thus $\Phi^\dagger(x)D(x, y)\Phi(y)$ is ensured gauge invariant.

Zinn-Justin / Master equation

$$\mathcal{S}(\Gamma) \equiv \mathcal{S}_+(\Gamma) - \mathcal{S}_-(\Gamma) = 0,$$

$$\mathcal{S}_\pm(\Gamma) \equiv \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}^4x \left[\frac{\delta\Gamma}{\delta A^a_{\pm\mu}}\frac{\delta\Gamma}{\delta K^\mu_{\pm a}} + \frac{\delta\Gamma}{\delta c^a_\pm}\frac{\delta\Gamma}{\delta K_{\pm a}} + B^a_\pm \frac{\delta\Gamma}{\delta \bar{c}^a_\pm} + \frac{\delta\Gamma}{\delta\Phi_\pm}\frac{\delta\Gamma}{\delta K^\dagger_{\pm\Phi}} + \frac{\delta\Gamma}{\delta\Phi^\dagger_\pm}\frac{\delta\Gamma}{\delta K_{\pm\Phi}} + \frac{\delta\Gamma}{\delta\psi_\pm}\frac{\delta\Gamma}{\delta K^\dagger_{\pm\psi}} + \frac{\delta\Gamma}{\delta\bar{\psi}_\pm}\frac{\delta\Gamma}{\delta K_{\pm\psi}}\right]$$

- The in-out 1PI action Γand connected generating functions W give vevs of operators in the presence of an external source (with the 1PI effective action understood as an integral over all spacetime).
- The in-in generating functions apply for an **arbitrary physical initial state**, including mixed states, and apply to the **finite time** effective action.
- You can see this by introducing sources localized at the initial time surface to probe the initial state.

$$\hat{s}\left(S_{\mathrm{NL}} - i\ln\left(\langle(-1)^{\hat{N}_\psi}\mathbf{f}_\mathrm{i}^+|\hat{\rho}(t_\mathrm{i})|\mathbf{f}_\mathrm{i}^-\rangle\right)\right) = 0$$

- ZJ equ encodes the way in which the BRST transformations are modified at the quantum level, so that the full 1PI effective action remains BRST invariant.

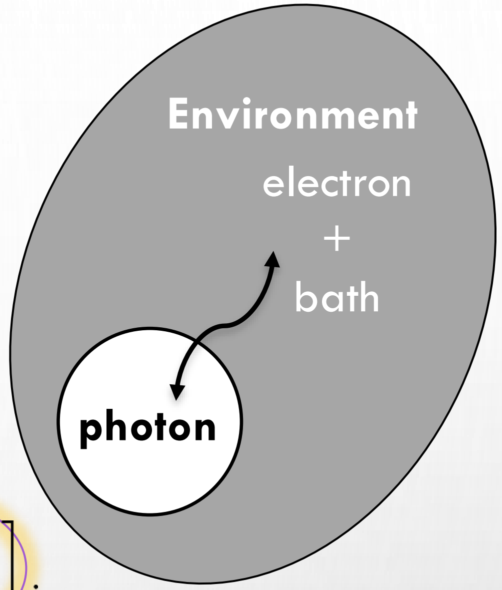
Open EFT matching

If you have access to the closed system 1PI action

- Start from the closed system: $\phi =$ system, $\chi =$ environment.
- Turn off the sources. Solve the 1PI equations: $\delta\Gamma/\delta\chi^{\pm} = 0$.
- Get $\chi^{\pm} = \chi^{\pm}_{\rm sol}[\phi^+, \phi^-]$.
- Substitute back into the closed-system 1PI action: $\Gamma_{\rm EFT}[\phi^+, \phi^-] = \Gamma[\phi^+, \phi^-, \chi^+_{\rm sol}, \chi^-_{\rm sol}]$.
- The result defines the Open EFT: $\Gamma_{\rm EFT}[\phi^+, \phi^-]$, hence $S_{\rm IF}[\phi^+, \phi^-]$ and $\rho_{\rm EFT}(\phi_i^+, \phi_i^-)$.

- The initial density matrix $\rho(\boldsymbol{\varphi}_{\rm i}^+, \boldsymbol{\varphi}_{\rm i}^-, \boldsymbol{\chi}_{\rm i}^+, \boldsymbol{\chi}_{\rm i}^-)$ need not bed factorisable and may couple the two branches.
- The initial system-environment correlations can contribute to the influence functional.

Influence Functional



TRACE OUT ENVIRONMENT → REDUCED DENSITY MATRIX

$$\varrho_{\mathcal{S}} \equiv \operatorname*{Tr}_{\mathcal{E}}[\rho].$$

Assumed the initial state factorizes: $\rho_{\mathrm{i}} = \varrho_{\mathcal{S}\mathrm{i}} \otimes \varrho_{\mathcal{E}\mathrm{i}}$

$$\left[\langle \mathrm{A}_+|\varrho_{\mathcal{S}}(t_{\mathrm{f}})|\mathrm{A}_-\rangle = \int \mathrm{d}[\mathrm{A}_{\pm\mathrm{i}}]\,\langle \mathrm{A}_{+\mathrm{i}}|\varrho_{\mathcal{S}\mathrm{i}}|\mathrm{A}_{-\mathrm{i}}\rangle \int_{A_{+i}}^{\mathrm{A}} \mathcal{D}[A_+] \int_{A_{-i}}^{\mathrm{A}} \mathcal{D}[A_-]\; e^{iS_{\mathcal{S}}[A_+]-iS_{\mathcal{S}}[A_-]+iS_{\mathrm{IF}}[A_+,A_-]}\right].$$

The Influence Functional

$$\tilde{\mu}\, e^{iS_{\mathrm{IF}}[A_+,A_-]} = \int \mathrm{d}[\Phi,\Phi_{\pm\mathrm{i}}]\,\langle \Phi_{+\mathrm{i}}|\varrho_{\mathcal{E}\mathrm{i}}|\Phi_{-\mathrm{i}}\rangle \int_{\Phi_{+\mathrm{i}}}^{\Phi} \mathcal{D}[\Phi_+] \int_{\Phi_{-\mathrm{i}}}^{\Phi} \mathcal{D}[\Phi_-]\; \mu\, e^{iS_{\mathcal{E}}[\Phi_+]+iS_{\mathrm{int}}[A_+,\Phi_+]-iS_{\mathcal{E}}[\Phi_-]-iS_{\mathrm{int}}[A_-,\Phi_-]}.$$

and similarly for Φ^*.

- Generates cross-branch mixings
- Used to study entanglement
- Obtain the generating functional by introducing sources $Z_{\mathcal{S}}[J_A^+, J_c^+, J_{\bar{c}}^+, J_A^-, J_c^-, J_{\bar{c}}^-]$

It's BRST all the way down

Recall:

- The in-in action for the closed/fundamental theory is BRST invariant under the diagonal subgroup.
- We obtain the Influence Functional by tracing out the environment DOFs.

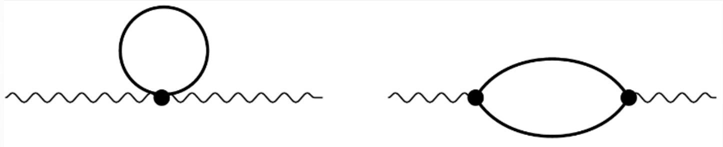
$$\text{As long as the state is BRST invariant} \implies \hat{s} S_{\mathrm{IF}}[A_+, A_-, \dots] = 0.$$

$$\text{Split: } \hat{s} = \hat{s}_{\mathcal{S}} + \hat{s}_{\mathcal{E}}. \longrightarrow \hat{s} S_{\mathrm{IF}}[A_+, A_-] = -\int \mathrm{d}^4 x \left[c_+ \partial_\mu \frac{\delta S_{\mathrm{IF}}}{\delta A_{+\mu}} + c_- \partial_\mu \frac{\delta S_{\mathrm{IF}}}{\delta A_{-\mu}} \right] = 0.$$

Ward identities $\quad \partial_\mu \frac{\delta S_{\mathrm{IF}}}{\delta A_{+\mu}} = 0\,, \quad$ and $\quad \partial_\mu \frac{\delta S_{\mathrm{IF}}}{\delta A_{-\mu}} = 0.$

The influence functional is **gauge invariant under two independent copies of gauge transformations**, one on each branch

Integrating out the electron in a thermal bath



Influence Functional (quadratic)

$$S_{\mathrm{IF}}[A_+, A_-] \simeq \frac{q^2}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\Big[- A_+^\mu(k)\Pi^\beta_{\mu\nu}(k)A_+^\nu(-k) - iA_+^\mu(k)\mathcal{N}^\beta_{\mu\nu}(-k)A_-^\nu(-k) -iA_-^\mu(k)\mathcal{N}^\beta_{\mu\nu}(k)A_+^\nu(-k) + A_-^\mu(k)\Pi^{\beta *}_{\mu\nu}(k)A_-^\nu(-k)\Big] + \mathcal{O}(q^4)\,,$$

Non-local Kernels

$$\Pi^\beta_{\mu\nu}(k) \equiv 2\eta_{\mu\nu}\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\left[\mathcal{F}^\beta(\ell)\right] - 2i\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\left[\mathcal{F}^\beta(\ell)\right]\left[\mathcal{F}^\beta(-\ell-k)\right](2\ell_\mu + k_\mu)\ell_\nu\,,$$

$$\mathcal{N}^\beta_{\mu\nu}(k) \equiv 2\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\mathcal{W}^\beta(\ell)\mathcal{W}^\beta(-\ell-k)(2\ell_\mu + k_\mu)\ell_\nu\,,$$

Vacuum divs. - absorb into field-strength renormalization ~$q^2\ F^2$

Feynman **Wightman**

$$\mathcal{F}^\beta(k) = \frac{-i}{k^2+m^2-i\epsilon} + \frac{2\pi\delta(k^2+m^2)}{e^{\beta|k_0|}-1} \quad \text{and} \quad \mathcal{W}^\beta(k) = 2\pi\delta(k^2+m^2)\left[\theta(k^0) + \frac{1}{e^{\beta|k_0|}-1}\right].$$

Gauge invariance

Satisfy the Ward identity	$k^\mu \Pi^\beta_{\mu\nu}(k) = 0$ $k^\mu \mathcal{N}^\beta_{\mu\nu}(k) = 0$	Both, **diagonal and advanced gauge** symmetries are preserved.

Decompose transverse and longitudinal components

$$S_{\rm IF}[A_{\rm r}, A_{\rm a}] \simeq q^2 \sum_{\mathfrak{p}={\rm L,T}} \int \frac{{\rm d}^4 k}{(2\pi)^4} \mathcal{P}^{\mathfrak{p}}_{\mu\nu}(k) \left[-A^\mu_{\rm a}(k) \mathcal{D}^\beta_{\mathfrak{p}}(k) A^\nu_{\rm r}(-k) + \frac{i}{2} A^\mu_{\rm a}(k) \mathcal{S}^\beta_{\mathfrak{p}}(k) A^\nu_{\rm a}(-k) \right],$$

gauge invariance is encoded in the index structure

Non-local kernels

$$\mathcal{D}^\beta_{\mathfrak{p}}(k) \equiv \underbrace{{\rm Re}[\Pi^\beta_{\mathfrak{p}}(k)]}_{\text{Modifies dispersion relation}} - \underbrace{\frac{i[\mathcal{N}^\beta_{\mathfrak{p}}(k) - \mathcal{N}^\beta_{\mathfrak{p}}(-k)]}{2}}_{\text{Dissipation}}, \quad \mathcal{S}^\beta_{\mathfrak{p}}(k) \equiv \underbrace{\frac{1}{2}{\rm Im}[\Pi^\beta_{\mathfrak{p}}(k)] - \frac{\mathcal{N}^\beta_{\mathfrak{p}}(k) + \mathcal{N}^\beta_{\mathfrak{p}}(-k)}{4}}_{\text{noise}},$$

Locality in weakly coupled theories?

Locality

- ☑ The classical Boltzmann contributions survive in the high-temperature limit, but not space-time local.
- ☑ Photon is weakly coupled to thermal environment.
- ☑ Memory effects.
- ☑ Kernels non-analytic in ω, $\implies$ the retarded propagator exhibit branch cuts near $\omega \simeq 0$.
- ☒ Gradient expansions allowed.

Wilsonian limit $m \gg \{|k|, 1/\beta\}$

- ☑ Open effects are exponentially surpressed $e^{-\beta m}$.
- ☑ Standard Wilsonian (vacuum) EFT
- ☑ Gradient expansions allowed below the branch cut $k^2 = 4m^2$,
- ☒ Does integrating heavy DOFs in weakly coupled EFTs lead to open system effects?

In cosmological scales, gravity and other fields interact only weakly at low temperatures.

In cosmology, we expect non-local kernels parametrise open-system effects.

“Open Electromagnetism” (bottom-up)

Assumptions: Translation invariant system, with an isotropic medium. There is a preffered direction: background frame field n_μ.

$$S_{\text{in-in}} = \int \mathrm{d}^4x \left[-\frac{1}{2} F^{\mathrm{r}}_{\alpha\beta} F^{\alpha\beta}_{\mathrm{a}} + \theta\, F^{\alpha\beta}_{\mathrm{r}} \tilde{F}^{\mathrm{a}}_{\alpha\beta} \right] + S_{\mathrm{IF}}, \quad S_{\mathrm{IF}} \equiv S_{\mathrm{diss}} + S_{\mathrm{noise}}.$$

$$\begin{aligned} S_{\mathrm{diss}} = \int \mathrm{d}^4x \int \mathrm{d}^4y \Big\{ & f_1(x-y) \left[E^{\mathrm{r}}_i(x)\, E^{\mathrm{a}}_i(y) - B^{\mathrm{r}}_i(x)\, B^{\mathrm{a}}_i(y) \right] + f_2(x-y)\, E^{\mathrm{r}}_i(x)\, E^{\mathrm{a}}_i(y) \\ & + f_3(x-y) \left[E^{\mathrm{r}}_i(x)\, B^{\mathrm{a}}_i(y) + B^{\mathrm{r}}_i(x)\, E^{\mathrm{a}}_i(y) \right] + f_4(x-y)\, E^{\mathrm{r}}_i(x)\, B^{\mathrm{a}}_i(y) - f_5(x-y)\, B^{\mathrm{r}}_i(x)\, A^{\mathrm{a}}_i(y) \\ & - f_6(x-y)\, \epsilon_{ijk} B^{\mathrm{r}}_k(x)\, \partial_i B^{\mathrm{a}}_j(y) + f_8\,(x-y)\, \partial_i E^{\mathrm{r}}_i(x)\, \partial_j E^{\mathrm{a}}_j(y) \Big\} \end{aligned}$$

$$\begin{aligned} S_{\mathrm{noise}} = \frac{i}{2} \int \mathrm{d}^4x \int \mathrm{d}^4y \Big\{ & g_1(x-y) \left[E^{\mathrm{a}}_i(x)\, E^{\mathrm{a}}_i(y) - B^{\mathrm{a}}_i(x)\, B^{\mathrm{a}}_i(y) \right] + g_2(x-y)\, E^{\mathrm{a}}_i(x) E^{\mathrm{a}}_i(y) \\ & + g_3(x-y) \left[E^{\mathrm{a}}_i(x)\, B^{\mathrm{a}}_i(y) + B^{\mathrm{a}}_i(x)\, E^{\mathrm{a}}_i(y) \right] + g_4(x-y)\, B^{\mathrm{a}}_i(x) E^{\mathrm{a}}_i(y) - g_5(x-y)\, B^{\mathrm{a}}_i(x) A^{\mathrm{a}}_i(y) \\ & - g_6(x-y)\, \epsilon_{ijk} B^{\mathrm{a}}_k(x) \partial_i B^{\mathrm{a}}_j(y) + g_7(x-y)\, \partial_i E^{\mathrm{a}}_i(x) \partial_j E^{\mathrm{a}}_j(y) \Big\} \end{aligned}$$

“Open Electromagnetism” (matching)

Matching to the top-down action

Open EM Influence Functional

$$S_{\mathrm{IF}} = q^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left[- A_{\mathrm{a}}^{\alpha}(-k) \mathcal{D}_{\alpha\beta}(k) A_{\mathrm{r}}^{\beta}(k) + \frac{i}{2} A_{\mathrm{a}}^{\alpha}(-k) \mathcal{S}_{\alpha\beta}(k) A_{\mathrm{a}}^{\beta}(k) \right]$$

$$\begin{aligned} -q^2 \mathcal{D}_{\alpha\beta}(k) &= \quad 2 f_1(-k)(k^2 \eta_{\alpha\beta} - k_\alpha k_\beta) + f_2(-k)\Big((n\cdot k)^2 \eta_{\alpha\beta} + k^2 n_\alpha n_\beta - (k\cdot n)[k_\alpha n_\beta + k_\beta n_\alpha]\Big)\,, \\ q^2 \mathcal{S}_{\alpha\beta}(k) &= \quad 2 g_1(-k)(k^2 \eta_{\alpha\beta} - k_\alpha k_\beta) + g_2(-k)\Big((n\cdot k)^2 \eta_{\alpha\beta} + k^2 n_\alpha n_\beta - (k\cdot n)[k_\alpha n_\beta + k_\beta n_\alpha]\Big) \end{aligned}$$

Decompose into longitudinal and transverse components

$$\begin{aligned} \mathcal{D}_{\alpha\beta} &= \mathcal{D}_{\mathrm{L}}(k)\, \mathcal{P}^{\mathrm{L}}_{\alpha\beta} + \mathcal{D}_{\mathrm{T}}(k)\, \mathcal{P}^{\mathrm{T}}_{\alpha\beta}\,, \\ \mathcal{S}_{\alpha\beta} &= \mathcal{S}_{\mathrm{L}}(k)\, \mathcal{P}^{\mathrm{L}}_{\alpha\beta} + \mathcal{S}_{\mathrm{T}}(k)\, \mathcal{P}^{\mathrm{T}}_{\alpha\beta} \end{aligned}$$

Matching carefully to the above expressions

$$\begin{aligned} -q^2 \mathcal{D}_{\mathrm{L}}(k) &= 2 f_1(-k) - f_2(-k)\,, \\ -q^2 \mathcal{D}_{\mathrm{T}}(k) &= \frac{2k^2}{|\mathbf{k}|^2} f_1(-k) + \frac{k_0^2}{|\mathbf{k}|^2} f_2(-k)\,, \end{aligned}$$

$$\begin{aligned} q^2 \mathcal{S}_{\mathrm{L}}(k) &= 2 g_1(-k) - g_2(-k)\,, \\ q^2 \mathcal{S}_{\mathrm{T}}(k) &= \frac{2k^2}{|\mathbf{k}|^2} g_1(-k) + \frac{k_0^2}{|\mathbf{k}|^2} g_2(-k)\,. \end{aligned}$$

Match the coefficients to the kernels we derived from the top down

Abelian Higgs-Kibble Model a la Caldeira-Leggett

Couple the abelian Higgs-Kibble model coupled to a charged bath

$$S = -\int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D^\mu[A]\Phi|^2 + \lambda\left(v^2 - \Phi^*\Phi\right)^2 + \underbrace{\sum_I \left(-|D_0[A]\varphi_I|^2 + \Gamma_I^2|\varphi_I|^2 - g_I(\Phi^*\varphi_I + \varphi_I^*\Phi)\right)}_{\text{a non-relativistic bath of charged scalars } \varphi_I}\right].$$

SSB vev is shifted and the bath fields acquire a vev

$$\langle\Phi\rangle = v_0 = \sqrt{v^2 + \frac{1}{2\lambda}\sum_I \frac{g_I^2}{\Gamma_I^2}}, \quad \langle\varphi_I\rangle = v_I = \frac{g_I}{\Gamma_I^2}v_0 18$$

unitary gauge

$\Phi = (v_0 + \zeta/\sqrt{2})$ with ζ real and
$\varphi_I = v_I + \frac{1}{\sqrt{2}}(\alpha_I + i\beta_I)$ with α_I and β_I real.

There is only one gauge freedom to remove one phase so the φ_I remain complex.

Introducing the Stückelberg field

$$\Phi \to (v_0 + \frac{1}{\sqrt{2}}\zeta)e^{\frac{i\chi}{\sqrt{2}v_0}}, \quad \varphi_I \to (v_I + \frac{1}{\sqrt{2}}(\alpha_I + i\beta_I))\, e^{\frac{i\chi}{\sqrt{2}v_0}},$$

with $\mathscr{D}_\mu\chi \equiv \partial_\mu\chi - \sqrt{2}qv_0A_\mu$.

α_I and β_I are are gauge-invariant bath fluctuations.

Abelian Higgs-Kibble Model a la Caldeira-Leggett

*we don't integrate out the Higgs ζ.

Tree level interaction

$$S_{\mathcal{E}}[\alpha,\beta] = \int \mathrm{d}^4x \sum_I \left[\frac{1}{2}(\partial_0\alpha_I)^2 + \frac{1}{2}(\partial_0\beta_I)^2 - \frac{1}{2}\Gamma_I^2\left(\alpha_I^2+\beta_I^2\right)\right]$$

Integrate out the bath and the Higgs

$$\begin{aligned} S_{\text{in-in}}[A_+,\chi_+,A_-,\chi_-] \;=\; & \int d^4x \left(-\frac{1}{4}F^+_{\mu\nu}F^{+\mu\nu} - \frac{1}{2}(\mathscr{D}^+_\mu\chi^+)^2 + \frac{1}{4}F^-_{\mu\nu}F^{-\mu\nu} + \frac{1}{2}(\mathscr{D}^-_\mu\chi^-)^2\right) \\ & -C\int d^4x \left[(\mathscr{D}^+_0\chi^+(x))^2 - (\mathscr{D}^-_0\chi^-(x))^2\right] \\ & -\frac{1}{2}\int d^4x \int d^4y \, \left(\mathscr{D}^+_0\chi^+(x) \quad -\mathscr{D}^-_0\chi^-(x)\right)\mathbf{d}(x-y)\begin{pmatrix}\mathscr{D}^+_0\chi^+(y)\\ -\mathscr{D}^-_0\chi^-(y)\end{pmatrix} + \dots . \end{aligned}$$

Dispersion mod

Noise/dissipation

Abelian Higgs-Kibble Model a la Caldeira-Leggett

*we don't integrate out the Higgs ζ.

Bottom-up

$$S_{\text{in-in}} = \int \mathrm{d}^4x \left[-\frac{1}{2} F^{\mathrm{r}}_{\alpha\beta} F^{\alpha\beta}_{\mathrm{a}} + \theta\, F^{\alpha\beta}_{\mathrm{r}} \tilde{F}^{\alpha\beta}_{\mathrm{a}} - \mathscr{D}^{\mathrm{r}}_{\mu} \chi_{\mathrm{r}}(x) \mathscr{D}^{\mu}_{\mathrm{a}} \chi_{\mathrm{a}}(x) \right] + S_{\mathrm{IF}}\,,$$

$$\begin{aligned}\Delta S^{\text{diss}} = \int \mathrm{d}^4x \int \mathrm{d}^4y \Big\{ & f_{10}(x-y)\, \mathscr{D}^{\mathrm{r}}_{0} \chi_{\mathrm{r}}(x) \mathscr{D}^{\mathrm{a}}_{0} \chi_{\mathrm{a}}(y) + f_{11}(x-y)\, \eta^{ij} \mathscr{D}^{\mathrm{r}}_{i} \chi_{\mathrm{r}}(x) \mathscr{D}^{\mathrm{a}}_{j} \chi_{\mathrm{a}}(y) \\ & + h_1(x-y)\, F^{\mathrm{r}}_{0i}(x) \mathscr{D}^{\mathrm{a}}_{i} \chi_{\mathrm{a}}(y) + h_2(x-y)\, \mathscr{D}^{\mathrm{r}}_{i} \chi_{\mathrm{r}}(x) F^{\mathrm{a}}_{0i}(y) + h_3(x-y)\, \epsilon_{ijk} F^{\mathrm{r}}_{ij}(x) \mathscr{D}^{\mathrm{a}}_{k} \chi_{\mathrm{a}}(y) \\ & + h_4(x-y)\, \epsilon_{ijk} \mathscr{D}^{\mathrm{r}}_{k} \chi_{\mathrm{r}}(x) F^{\mathrm{a}}_{ij}(y) \Big\}\end{aligned}$$

$$\begin{aligned}\Delta S^{\text{noise}} = \int \mathrm{d}^4x \int \mathrm{d}^4y \Big\{ & g_9(x-y)\, \mathscr{D}^{\mathrm{a}}_{0} \chi_{\mathrm{a}}(x) \mathscr{D}^{\mathrm{a}}_{0} \chi_{\mathrm{a}}(y) + g_{10}(x-y)\, \eta^{ij} \mathscr{D}^{\mathrm{a}}_{i} \chi_{\mathrm{a}}(x) \mathscr{D}^{\mathrm{a}}_{j} \chi_{\mathrm{a}}(y) \\ & + c_1(x-y)\, F^{\mathrm{a}}_{0i}(x) \mathscr{D}^{\mathrm{a}}_{i} \chi_{\mathrm{a}}(y) + c_2(x-y)\, \epsilon_{ijk} F^{\mathrm{a}}_{ij}(x) \mathscr{D}^{\mathrm{a}}_{k} \chi_{\mathrm{a}}(y) \Big\}\,.\end{aligned}$$

Conclusions

The total action respects the diagonal BRST symmetry because the influence functional is invariant under two copies of gauge symmetries by virtue of the Wilson lines which emerge naturally from the background field propagators.

More to do

- Understand how to **systematically** construct gravitational EFTs and address technical challenges

- Make progress on open questions about causality, unitarity.

- Phenomenology (gravitational waves, quantum gravity and quantum properties of matter etc)

High Temperature Limit $\beta \to 0$

In general non-local for system weakly coupled to bath.

$$\mathrm{Re}[\Pi_{\mathrm{L}}^{\beta}(k)] \simeq -\frac{1}{3|\mathbf{k}|^2\beta^2}\left[\frac{k_0}{2|\mathbf{k}|}\log\left|\frac{k_0/|\mathbf{k}|+1}{k_0/|\mathbf{k}|-1}\right|-1\right]+\mathcal{O}\left(\frac{1}{\beta}\right),$$

$$\mathrm{Im}[\Pi_{\mathrm{L}}^{\beta}(k)] \simeq \frac{\theta(k^2)\pi}{3|\mathbf{k}|^3\beta^3}-\frac{\theta(k^2)+\theta(-k^2-4m^2)}{2\pi|\mathbf{k}|^2\beta^2}\left[\sqrt{1+\frac{4m^2}{k^2}}-\frac{|k_0|}{2|\mathbf{k}|}\log\left|\frac{\Omega_+}{\Omega_-}\right|\right]+\mathcal{O}\left(\frac{1}{\beta}\right),$$

$$\mathcal{N}_{\mathrm{L}}^{\beta}(k) \simeq -\frac{\theta(k^2)\pi}{3|\mathbf{k}|^3\beta^3}-\frac{\theta(k^2)\pi k_0}{12|\mathbf{k}|^3\beta^2}+\frac{\theta(k^2)+\theta(-k^2-4m^2)}{2\pi|\mathbf{k}|^2\beta^2}\left[\sqrt{1+\frac{4m^2}{k^2}}-\frac{|k_0|}{2|\mathbf{k}|}\log\left|\frac{\Omega_+}{\Omega_-}\right|\right],$$

$$\mathrm{Re}[\Pi_{\mathrm{T}}^{\beta}(k)] \simeq \frac{1}{6|\mathbf{k}|^2\beta^2}\left[\frac{k_0^2}{|\mathbf{k}|^2}+\left(1-\frac{k_0^2}{|\mathbf{k}|^2}\right)\frac{k_0}{2|\mathbf{k}|}\log\left|\frac{k_0/|\mathbf{k}|+1}{k_0/|\mathbf{k}|-1}\right|-1\right]+\mathcal{O}\left(\frac{1}{\beta}\right),$$

$$\begin{aligned}\mathrm{Im}[\Pi_{\mathrm{T}}^{\beta}(k)] \simeq{}& \frac{2k^2}{|\mathbf{k}|^2}\Bigg\{-\frac{\theta(k^2)\pi}{12|\mathbf{k}|^3\beta^3}, \\ &+\frac{\theta(k^2)+\theta(-k^2-4m^2)}{8\pi|\mathbf{k}|^2\beta^2}\left[\sqrt{1+\frac{4m^2}{k^2}}+\left(1+\frac{4m^2}{k^2}-\frac{k_0^2}{|\mathbf{k}|^2}\right)\frac{|\mathbf{k}|}{2|k_0|}\log\left|\frac{\Omega_+}{\Omega_-}\right|\right]\Bigg\}+\mathcal{O}\left(\frac{1}{\beta}\right),\end{aligned}$$

$$\begin{aligned}\mathcal{N}_{\mathrm{T}}^{\beta}(k) \simeq{}& \frac{2k^2}{|\mathbf{k}|^2}\Bigg\{+\frac{\theta(k^2)\pi}{12|\mathbf{k}|^3\beta^3}-\frac{\theta(k^2)\pi k_0}{24|\mathbf{k}|^3\beta^2}, \\ &-\frac{\theta(k^2)+\theta(-k^2-4m^2)}{8\pi|\mathbf{k}|^2\beta^2}\left[\sqrt{1+\frac{4m^2}{k^2}}+\left(1+\frac{4m^2}{k^2}-\frac{k_0^2}{|\mathbf{k}|^2}\right)\frac{|\mathbf{k}|}{2|k_0|}\log\left|\frac{\Omega_+}{\Omega_-}\right|\right]\Bigg\}+\mathcal{O}\left(\frac{1}{\beta}\right),\end{aligned}$$

Wilsonian limit

Wilsonian limit dominates; open effects are very suppressed.

$$
\begin{aligned}
\mathrm{Re}[\Pi^\beta_{\mathrm{L}}(k)] &\simeq -\frac{1}{32}\left[-\frac{k^2}{15m^2}+\mathcal{O}\left(m^{-4}\right)\right]+e^{-\beta m}\frac{\sqrt{m}\left(\frac{k^2}{k_0^2}-1\right)}{\sqrt{2}(\pi\beta)^{3/2}|\mathbf{k}|^2}+\mathcal{O}\left(m^{-\frac{1}{2}}\right)\\
\mathrm{Im}[\Pi^\beta_{\mathrm{L}}(k)] &\simeq \frac{\theta(k^2)}{8\pi|\mathbf{k}|^3}\left(1+e^{-\beta|k_0|}\right)e^{-\beta m\frac{|\mathbf{k}|}{|k|}}\left[\frac{4|\mathbf{k}|^2m^2}{\beta k^2}+\frac{8m\mathbf{k}}{\beta^2|k|}+\mathcal{O}\left(m^0\right)\right]\\
\mathrm{Re}[\Pi^\beta_{\mathrm{T}}(k)] &\simeq \frac{k^2}{32\pi^2|\mathbf{k}|^2}\left[-\frac{k^2}{15m^2}+\mathcal{O}\left(m^{-4}\right)\right]+e^{-\beta m}\left[\frac{\sqrt{m}\left(k^2+5|\mathbf{k}|^2\right)}{4\sqrt{2}\pi^{3/2}\beta^{3/2}|\mathbf{k}|^4}+\mathcal{O}\left(m^{-\frac{1}{2}}\right)\right]\\
\mathrm{Im}[\Pi^\beta_{\mathrm{T}}(k)] &\simeq -\frac{\theta(k^2)}{16\pi|\mathbf{k}|^5}\left(1+e^{-\beta|k_0|}\right)e^{-\beta m\frac{|\mathbf{k}|}{|k|}}\left[\frac{8m|\mathbf{k}||k|}{\beta^2}+\mathcal{O}\left(m^0\right)\right]\\
\mathcal{N}^\beta_{\mathrm{L}}(k) &\simeq -\frac{\theta(k^2)}{4\pi|\mathbf{k}|^3}e^{-\beta m\frac{|\mathbf{k}|}{|k|}}\left[\theta(k^0)+e^{\beta|k_0|}\theta(-k^0)+1\right]\left[\frac{4|\mathbf{k}|^2m^2}{\beta k^2}+\frac{8|\mathbf{k}|m}{\beta^2|k|}+\mathcal{O}\left(m^0\right)\right]\\
\mathcal{N}^\beta_{\mathrm{T}}(k) &\simeq \frac{\theta(k^2)}{8\pi|\mathbf{k}|^5}e^{-\beta m\frac{|\mathbf{k}|}{|k|}}\left[\theta(k^0)+e^{\beta|k_0|}\theta(-k^0)+1\right]\left[\frac{8m|\mathbf{k}||k|}{\beta^2}+\mathcal{O}\left(m^0\right)\right]
\end{aligned}
$$

Quantisation of gauge theories

Insist on positive-energy states but allow for states with negative norm

Preserve Lorentz invariance

Example, canonically quantise the photon, the timelike 1p state picks up a a negative norm

$$\langle 0|\hat{a}_0(\mathbf{k})\hat{a}_0^\dagger(\mathbf{k}')|0\rangle = -2\omega_k(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')$$

→ Vacuum wavefunctional is not normalisable

Indefinite Hilbert space → eigenvalues of Hermitian operators need not be real

$$\hat{A}_0|A_4(\mathbf{x})\rangle = iA_4|A_4(\mathbf{x})\rangle \implies \hat{A}_0 = i\hat{A}_4$$

$\hat{A}_4$ anti-Hermitian
but self-adjoint wrt a redefined inner product

$$\langle\psi|\psi\rangle = \int \mathcal{D}A_4\, \psi^*(-A_4)\psi(A_4)$$

vacuum wavefunctional is normalisable

in **SK,** unitarity *i.e.* conservation of probability → $A_0^\pm \to iA_4^\pm$