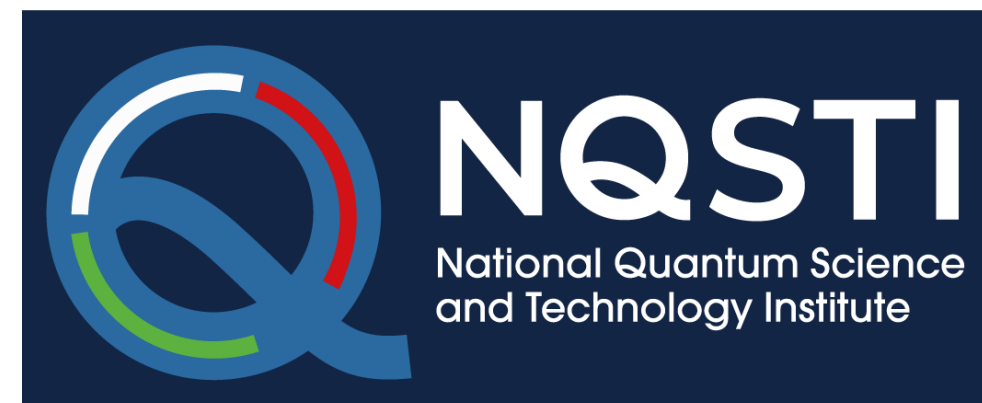


Quantum Collision Models: a versatile tool for open system dynamics

Stefano Cusumano

Mainz, 21 April 2026

Open Quantum Systems: dissipation and decoherence form subatomic to cosmic scales



Outline

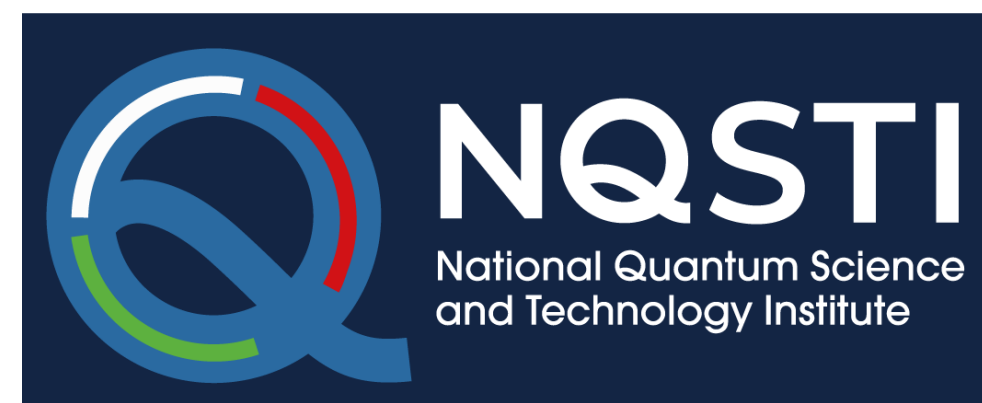
Introduction to Quantum Collision Models

- Time discretization and environment discretization
- Repeated Interactions
- The discrete Markovian Master Equation
- The time continuous limit

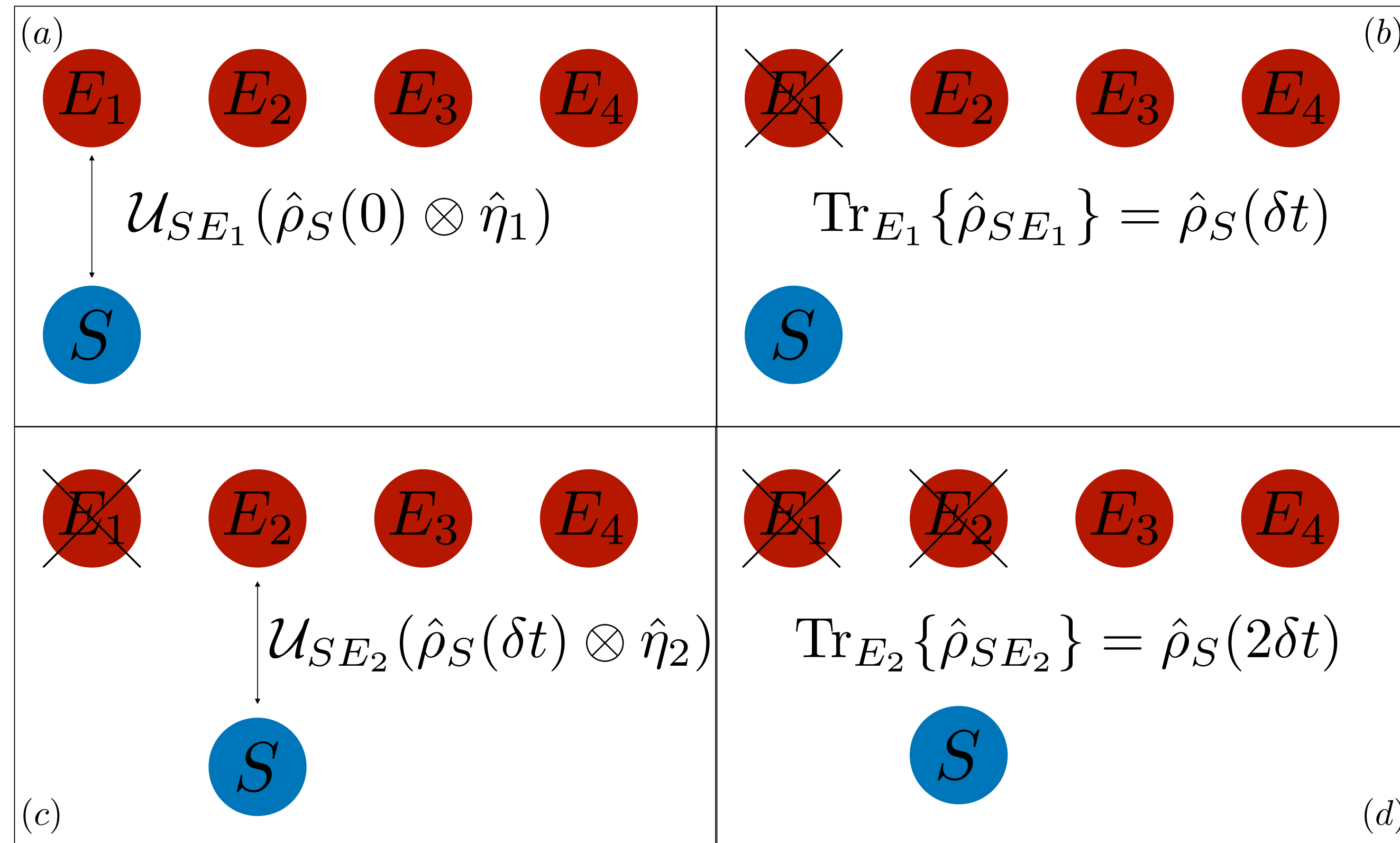
- Non-Markovian collision models
- Non-trivial geometries
- Structured Ancillae
- Metrology

Nice reviews: Phys. Rep. 1, 954 (2022)

Nice tutorial: Entropy, 24(9) (2022)



The basic setting



Model the dynamics

The most general Hamiltonian can be written as

$$H_{\text{tot}} = H_S + H_{\mathcal{E}} + H_{S\mathcal{E}} \quad H_{\mathcal{E}} = \sum_i H_{E_i} \quad H_{S\mathcal{E}} = \sum_i H_{SE_i}$$

Over a single time step one has the Hamiltonian

$$H_{\text{tot}}(n\delta t) = H_S + H_{E_n} + H_{SE_n}$$

The system density matrix evolves as:

$$\rho_S(n) \otimes \eta_{E_n} \rightarrow U_{n+1}(\rho_S(n) \otimes \eta_{E_{n+1}})U_{n+1}^\dagger$$



Tracing away the environment

The unitary operator generating the dynamics is $U_n = \exp[-i\delta t H_{\text{tot}}(n)] = \sum_{j=0}^{\infty} \frac{(-i\delta t)^j}{j!} H_{\text{tot}}^j(n)$

Tracing away the ancilla, one gets $\rho_S(n+1) = \left[I + \sum_{j=1}^{\infty} P_j \right] \rho_S(n)$

Each P_j describes the dynamics of S at the j-th perturbative order

$$P_j[\rho_S(n-1)] = \frac{1}{m!} \left(-\frac{i\delta t}{\hbar} \right) \langle [H_{\text{tot}}, [H_{\text{tot}}, [\dots, [H_{\text{tot}}, \rho_S(n-1) \otimes \eta_{E_n}]]]] \rangle$$

See also New J. Phys. 19, 013035 (2017) for a nice treatment of arbitrarily high perturbative orders



The discrete master equation

One assumes an interaction Hamiltonian of the form $H_{\text{int}}(n) = g_n(t)A_S \otimes B_{E_n}$

Integrating over one time step $\int_{t_{n-1}}^{t_n} H_{\text{tot}}(n) = H_S + H_{E_n} + \bar{g}A_S \otimes B_{E_n}$ $\bar{g} = \frac{1}{\delta t} \int_{t_{n-1}}^{t_n} g(t)dt$

We can write the full discrete master equation:

$$\begin{aligned} \rho_S(n+1) = & \rho_S(n-1) - i\frac{\delta t}{\hbar}[H_S + \bar{g}\langle B_{E_n} \rangle A_S, \rho_S(n)] + \frac{i\delta t^2}{2\hbar^2}\bar{g}\langle i[B_{E_n}, H_{E_n}] \rangle \\ & - \frac{\delta t^2}{2\hbar^2} \left([H_S, [H_S, \rho_S(n)]] + \bar{g}\langle B_{E_n} \rangle ([H_S, [A_S, \rho_S(t)]] + [A_S, [H_S, \rho_S(t)]] \right) \\ & - \frac{\delta t^2 \bar{g}^2}{2\hbar^2} \langle B_{E_n} B_{E_n} \rangle [A_S, [A_S, \rho_S(n)]] + \dots \end{aligned}$$

The continuous time limit

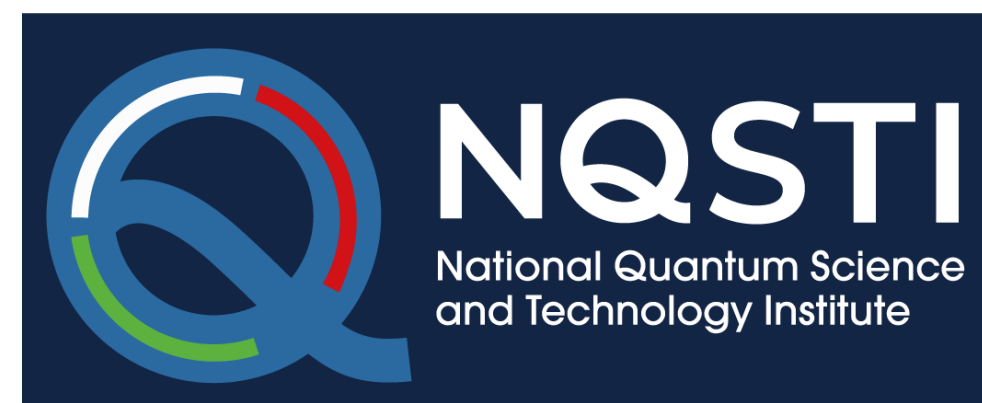
One can obtain a time continuous time limit of the master equation by sending to zero the time step $\delta t \rightarrow 0$, while sending to infinity the number of collisions $n \rightarrow \infty$

$$\frac{d\rho_S(t)}{dt} = \lim_{\delta t \rightarrow 0, n \rightarrow \infty} \frac{\rho_S(n) - \rho_S(n-1)}{\delta t}$$

The next step is to check which terms of the expansion survive, that is:

$$\lim_{\delta t \rightarrow 0} \frac{\delta t^k \bar{g}^k \langle B_{E_n}^k \rangle}{\delta t} \neq 0$$

These terms can be non null, as the $\langle B_{E_n}^k \rangle$ are moments of an (in principle) arbitrary probability distribution.



Effective unitary dynamics

A condition for effective unitary dynamics:

$$\lim_{\delta t \rightarrow 0} \frac{\delta t^k \bar{g}^k \langle B_{E_n}^k \rangle}{\delta t \bar{g} \langle B_{E_n} \rangle} = 0 \quad \forall k \geq 2$$

The time continuous dynamics reads: $\frac{d\rho_S(t)}{dt} = -\frac{i}{\hbar} [H_S + \Theta A_S, \rho_S(t)]$, $\Theta = \lim_{\delta t \rightarrow 0} \delta t \bar{g} \langle B_{E_n} \rangle$

The unitary dynamics induced by the environment is null when the environmental state is a stationary state of $H_{\mathcal{E}}$.

The one point correlation function of the environment can always be made null by appropriately redefining the operators B_{E_n} .

The price to pay is a renormalization of H_S .



Second order terms: decoherence and dissipation

When the second order term stays relevant, one obtains the usual Born-Markov ME

$$\frac{d\rho_S(t)}{dt} = -\frac{i}{\hbar} [H_S + (\Theta - \tilde{\Theta})A_S, \rho_S(t)] - \frac{\Gamma}{2\hbar^2} [A_S, [A_S, \rho_S]]$$

$$\tilde{\Theta} = \lim_{\delta t \rightarrow 0} \frac{i\bar{g} \langle [B_{E_n}, H_{E_n}] \rangle}{2\hbar} \quad \Gamma = \lim_{\delta t \rightarrow 0} \delta t \bar{g}^2 \langle B_{E_n}^2 \rangle$$

The double commutator multiplying Γ can be expanded to obtain the usual dissipator in Lindblad form. The Born-Markov ME describes decoherence and dissipation happening at rate Γ .



Example 1: a qubit interacting with a qubit environment

$$H_S = \frac{\omega}{2} \sigma_S^z \quad H_{E_n} = \frac{\omega}{2} \sigma_{E_n}^z \quad H_{SE_n} = g(\sigma_S^+ \sigma_{E_n}^- + \sigma_S^- \sigma_{E_n}^+) \quad \eta_{E_n} = \frac{e^{-\beta H_{E_n}}}{\mathcal{Z}}$$

$$U_{SE_n} \simeq I - ig\delta t H_{SE_n} - \frac{g^2 \delta t^2}{2} H_{SE_n}^2$$

$$\langle \sigma_{E_n}^\pm \rangle = \text{Tr}[\sigma_{E_n}^\pm \eta_{E_n}] = 0 \quad \langle \sigma_{E_n}^+ \sigma_{E_n}^- \rangle = \text{Tr}[\sigma_{E_n}^+ \sigma_{E_n}^- \eta_{E_n}] = \gamma_+ \quad \langle \sigma_{E_n}^- \sigma_{E_n}^+ \rangle = \text{Tr}[\sigma_{E_n}^- \sigma_{E_n}^+ \eta_{E_n}] = \gamma_-$$

One obtains the usual Markovian ME, with a stimulated absorption and the spontaneous emission terms.

$$\rho_S(n+1) = \rho_S(n) + \frac{g^2 \delta t^2 \gamma_+}{2} (\sigma_S^+ \rho_S(n) \sigma_S^- - \{\sigma_S^- \sigma_S^+, \rho_S(n)\}) + \frac{g^2 \delta t^2 \gamma_-}{2} (\sigma_S^- \rho_S(n) \sigma_S^+ - \{\sigma_S^+ \sigma_S^-, \rho_S(n)\})$$

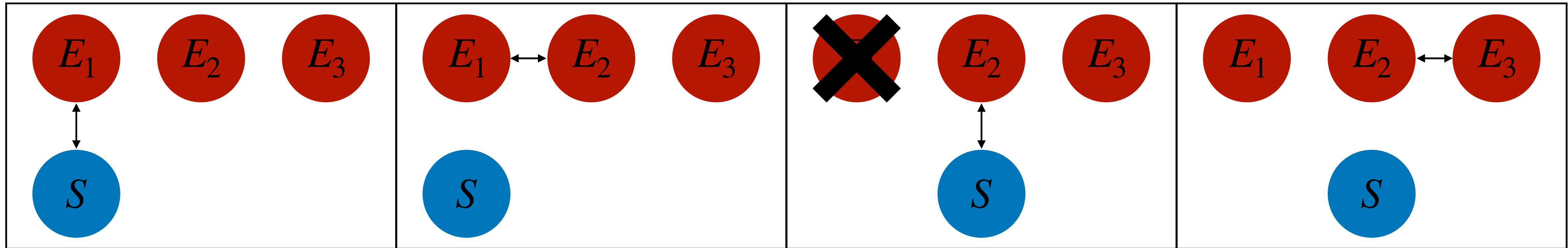


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Non-Markovian Collision Models

Let us introduce some memory. This is easily done with collision models.



One let two ancillary interact after the first interacted with the system and before the second one interacts with the system. This effectively introduces memory!

Further details in PRA 87, 040103(R) (2013) and Phys. Scr. T153, 014010 (2013)

Introducing memory: intra-ancilla interactions

A common (and simple) choice for the intra-ancilla interaction is the partial swap interaction:

$$U_{E_n E_{n+1}} = \cos \theta I - i \sin \theta S_{E_n E_{n+1}} \quad S_{E_n E_{n+1}} (\eta_{E_n}^{\text{out}} \otimes \eta_{E_{n+1}}) S_{E_n E_{n+1}} = \eta_{E_{n+1}} \otimes \eta_{E_n}^{\text{out}}$$

Coherent Swap

$$U_{E_n E_{n+1}} (\eta_{E_n}^{\text{out}} \otimes \eta_{E_{n+1}}) U_{E_n E_{n+1}}^\dagger = \cos^2 \theta \eta_{E_n}^{\text{out}} \otimes \eta_{E_{n+1}} + \sin^2 \theta \eta_{E_{n+1}} \otimes \eta_{E_n}^{\text{out}} - i \frac{\sin 2\theta}{2} [\eta_{E_n}^{\text{out}} \otimes \eta_{E_{n+1}}, S_{E_n E_{n+1}}]$$

Incoherent Swap

$$U_{E_n E_{n+1}} (\eta_{E_n}^{\text{out}} \otimes \eta_{E_{n+1}}) U_{E_n E_{n+1}}^\dagger = \cos^2 \theta \eta_{E_n}^{\text{out}} \otimes \eta_{E_{n+1}} + \sin^2 \theta \eta_{E_{n+1}} \otimes \eta_{E_n}^{\text{out}}$$

The second case allows analytical solution!



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The non-Markovian ME

Initially: $R_0 = \rho_S(0) \otimes \eta_{E_1} \otimes \eta_{E_2} \otimes \dots \otimes \eta_{E_n}$

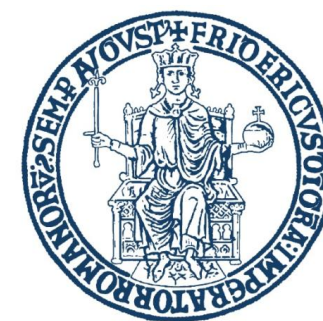
At the n-th step: $R_n = \left(\mathcal{U}_{S,n} \mathcal{U}_{n,n-1} \mathcal{U}_{S,n-1} \dots \mathcal{U}_{2,1} \mathcal{U}_{S,1} \right) R_0$ $\mathcal{U}R = URU^\dagger$

Recast the equation in recursive form $R_n = (1-p) \sum_{j=1}^{n-1} p^{j-1} \mathcal{U}_{S,n}^j R_{n-j} + p^{n-1} \mathcal{U}_{S,n}^n R_0$

Tracing away the environment $\rho_n = (1-p) \sum_{j=1}^{n-1} p^{j-1} \mathcal{E}_j \rho_{n-j} + p^{n-1} \mathcal{E}_n \rho_0$

The map \mathcal{E}_j only depends on $\mathcal{U}_{S,n}$ and the initial state of the environment

$$\mathcal{E}_j \rho = \text{Tr}_E \left[\mathcal{U}_{S,n}^j \left(\rho \otimes_{i=1}^n \eta_{E_n} \right) \right]$$



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The time continuous limit and exact solution

Taking the time continuous limit:
$$\dot{\rho}(t) = \Gamma \int_0^t dt' e^{-\Gamma t'} \mathcal{E}(t') \dot{\rho}(t-t') + e^{-\Gamma t} \dot{\mathcal{E}}(t) \rho_0$$

Γ is connected with p via
$$p = e^{-\Gamma \delta t}$$

One can solve the equation above exactly, and find the super operator describing the evolution of $\rho(t)$

$$\rho(t) = \Lambda(t) \rho(0) \quad \Lambda(t) = \sum_{k=1}^{\infty} \Gamma^{k-1} \mathcal{L}^{-1}[\tilde{\mathcal{E}}^k(s + \Gamma)](t)$$

$$\mathcal{L}^{-1}[\tilde{\mathcal{E}}^k(s + \Gamma)](t) = e^{-\Gamma t} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{k-2}} dt_{k-1} \mathcal{E}(t_{k-1}) \mathcal{E}(t_{k-2}-t_{k-1}) \dots \mathcal{E}(t-t_1)$$



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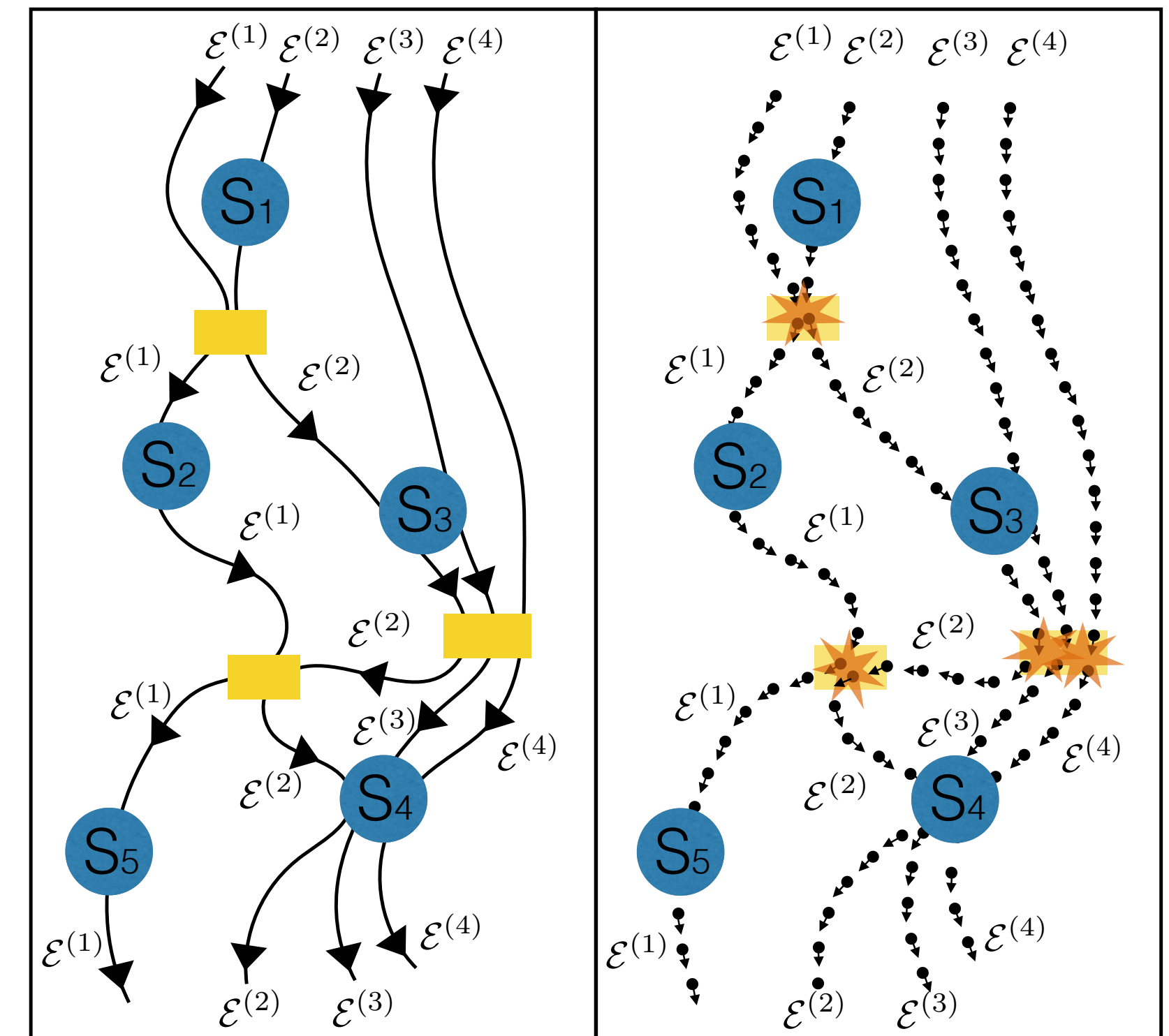


Cascaded collision model

Using collision models it is also possible to treat multipartite quantum systems in non trivial geometries

An example is given by networks of quantum system where a signal propagates among the subsystems, diffusing correlations

This happens in a variety of situations: collective emission in optical cavities, atoms in a waveguide, quantum networks, quantum state engineering



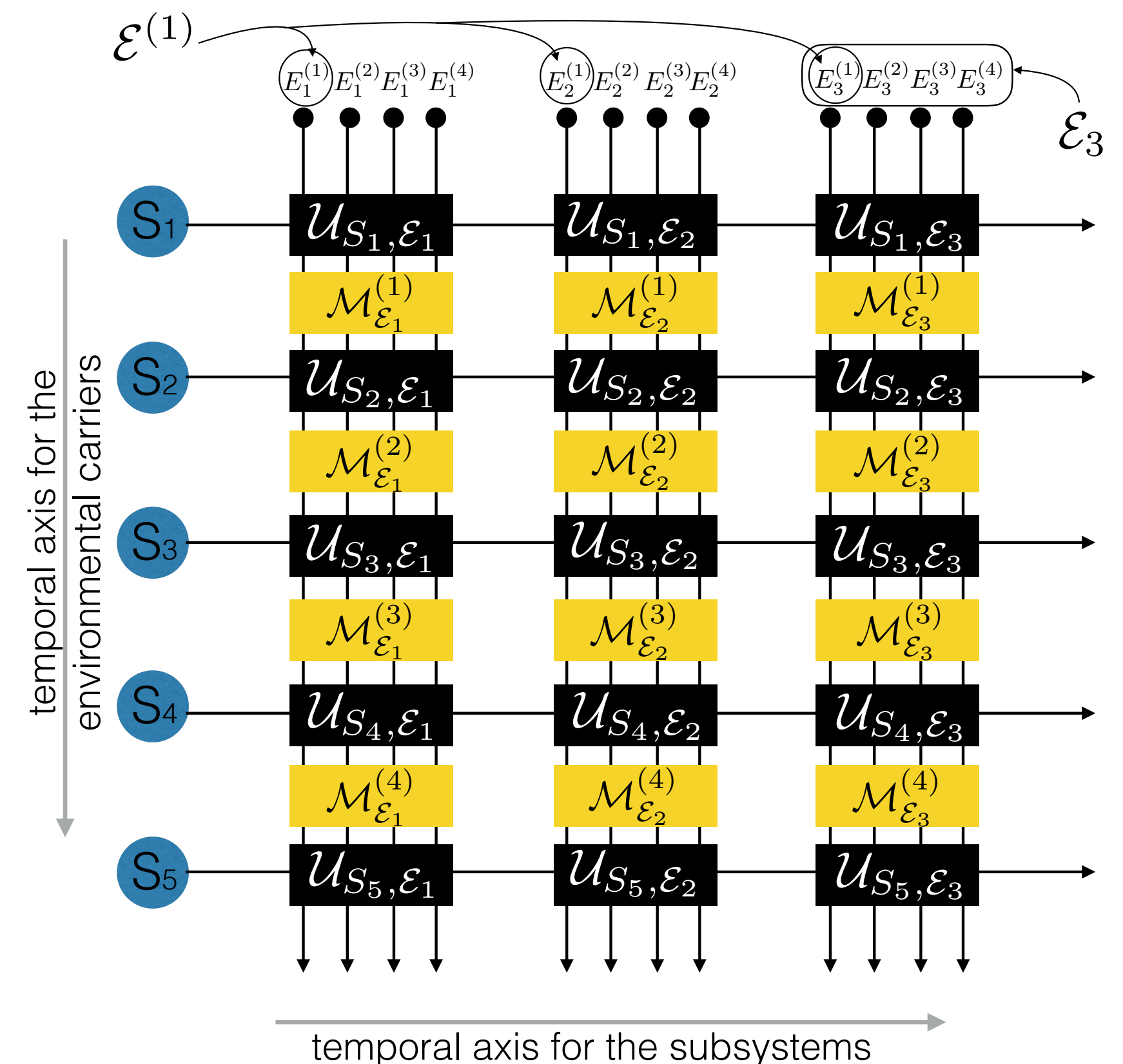
The cascaded collision model

One can recast the dynamics in a quantum circuit like fashion, where each of the subsystem interact with the appropriate ancillas.

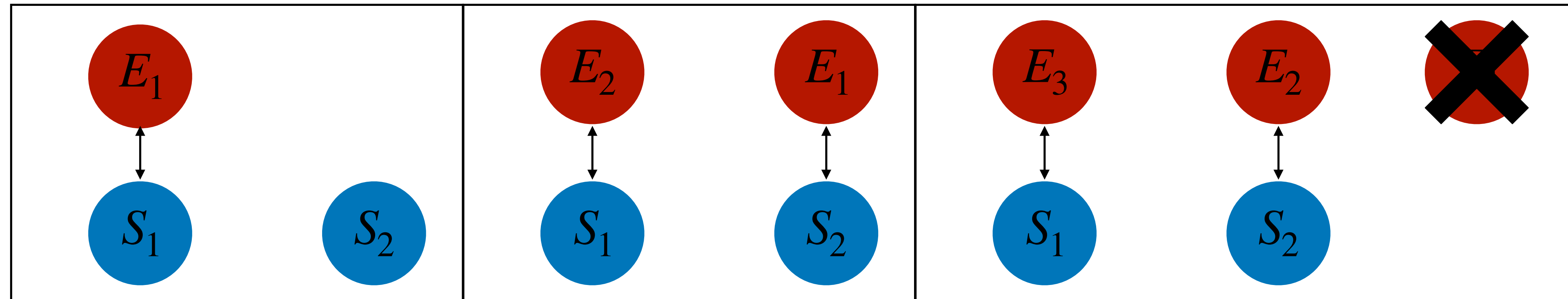
In spite of the fact the all ancillas interact with more than one subsystem, the model is Markovian when one considers the whole system.

However, non-Markovianity is observed when one tries to trace away subsystems that are causally connected with the remaining ones.

For further details PRA 95, 053838 (2017) and PRA 97, 053811 (2018)



Example 3: two qubit in a waveguide

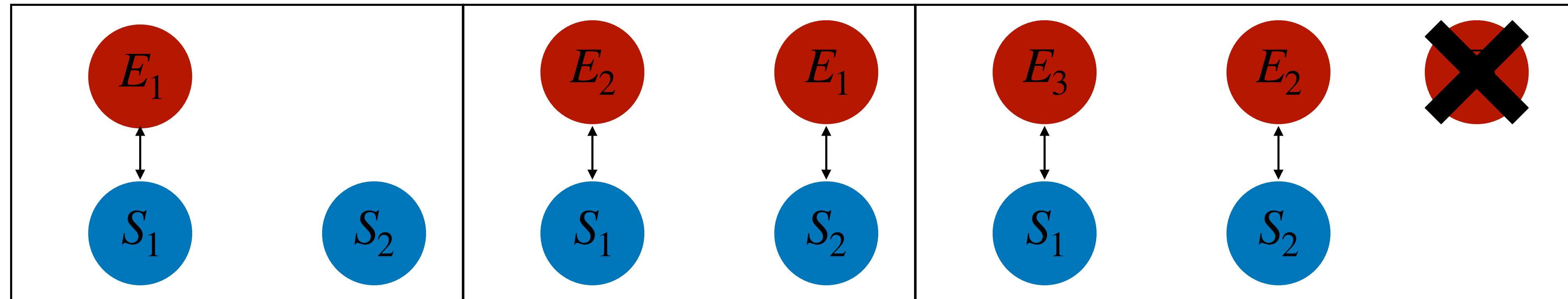


$$\dot{\rho}_S(t) = \sum_{m=1}^2 \mathcal{L}_m \rho_S(t) + \mathcal{D}_{1 \rightarrow 2} \rho_S(t) \quad \mathcal{D}_{1 \rightarrow 2} \rho_S(t) = \zeta_{12} A_{S_1} [\rho_S(t), A_{S_2}] - \xi_{12} [\rho_S(t), A_{S_2}] A_{S_1}$$

This master equation includes an effective interaction term between the two subsystems due to the common interaction with the environment

See also New J. Phys. 14, 063014 (2012) for a quantum state engineering perspective

Example 3: two qubit in a waveguide



One can recast the ME in the usual Lindblad form, where however the Lindblad operators are non local and the Hamiltonian is renormalized by a chiral term

$$\dot{\rho}_S(t) = -i[H_{12}, \rho_S(t)] + \sum_i \tilde{\mathcal{L}}_i \rho_S(t) \quad H_{12} = \text{Im}[\xi_{12}] A_1 \otimes A_2$$

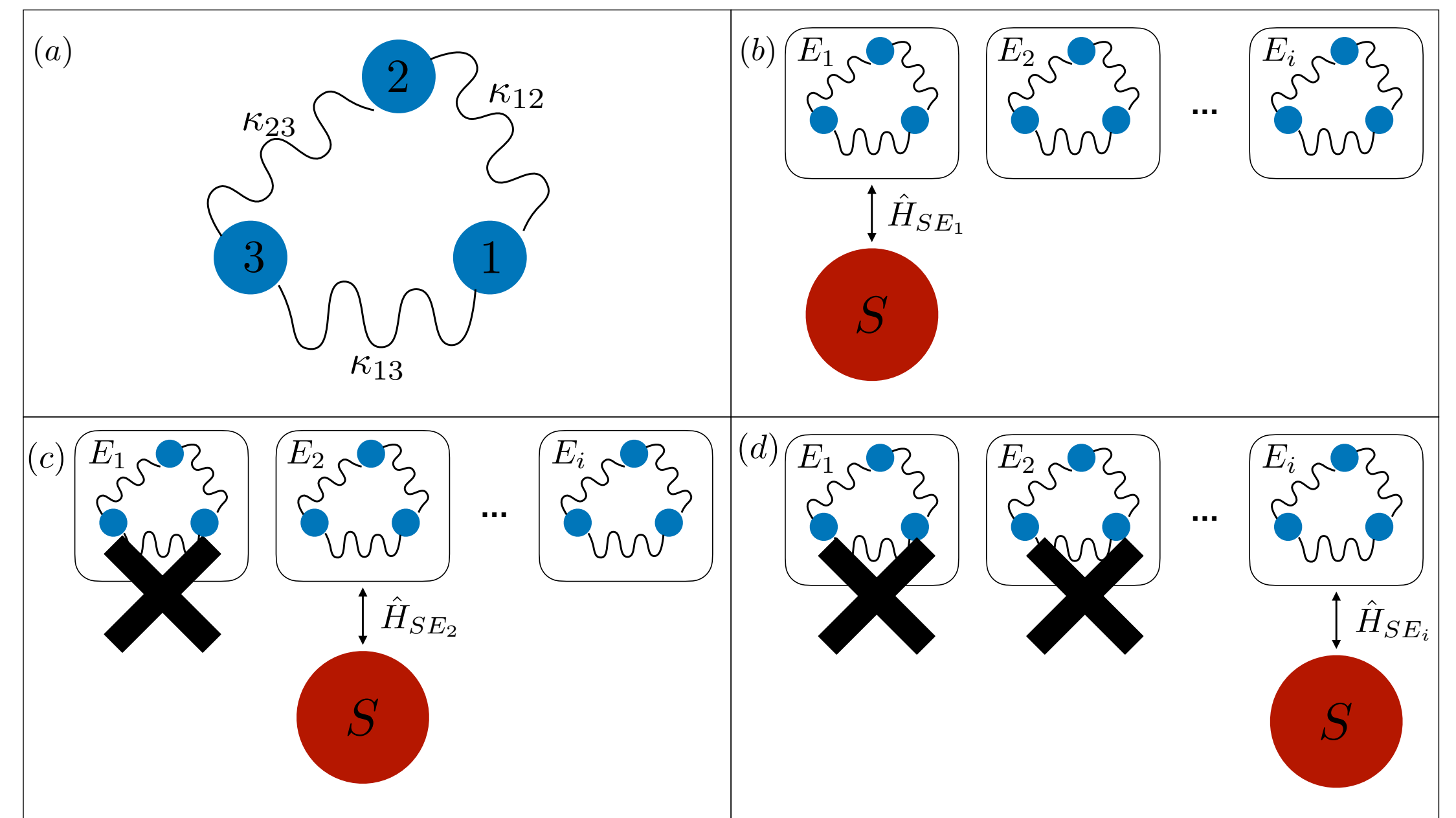
Non trivial environments: structured collision models

One can also consider non-trivial ancillas in order to represent the features of structured environments, e.g. environments possessing coherence or out of equilibrium

The main idea is to consider multipartite ancillae

Depending on the interaction between the system and the ancillae one can get different effects

See New J Phys. 26, 023001 (2024) and Quantum Sci. Technol. 10, 045056 (2025)



The interacting ancilla Hamiltonian

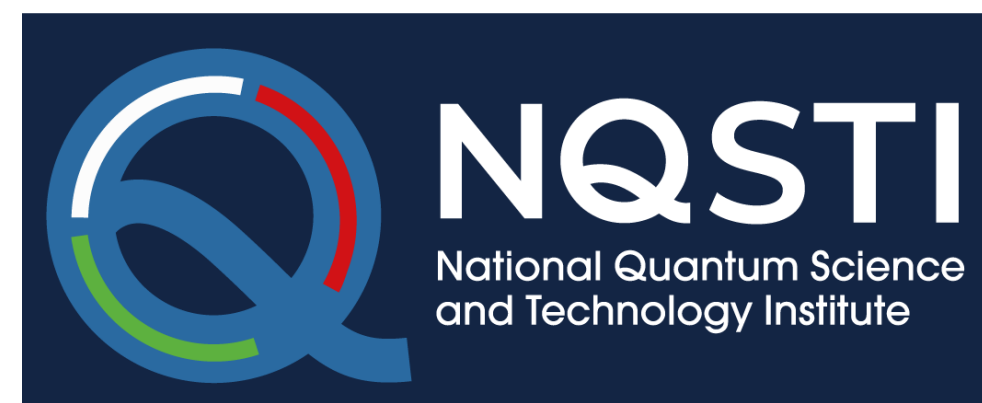
One assumes an ancilla Hamiltonian of the form $H_{E_n} = H_{E_n}^{\text{free}} + H_{E_i}^{\text{tot}}$

One can define two different orthogonal basis $\{ |E_i\rangle \}, \{ |E'_i\rangle \}$

The main point here is that if the system S interact with the dressed basis $\{ |E'_i\rangle \}$, this might give rise to an induced unitary dynamics, i.e. generating coherence in S.

We are going to consider as an example the case where the ancillae are made out of two qubits, with Hamiltonian:

$$H_{E_i} = \frac{\omega_1}{2} \sigma_1^z + \frac{\omega_2}{2} \sigma_2^z + \kappa_{12} (\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+)$$



Local vs global coupling

We can consider two different coupling Hamiltonian. The first couples the qubit S locally with each of the ancillary qubits:

$$H_{SE_n} = \alpha_1(\sigma_S^+ \sigma_1^- + \sigma_S^- \sigma_1^+) + \alpha_2(\sigma_S^+ \sigma_2^- + \sigma_S^- \sigma_2^+)$$

In this case one simply obtains the usual Markovian master equation we have already seen.

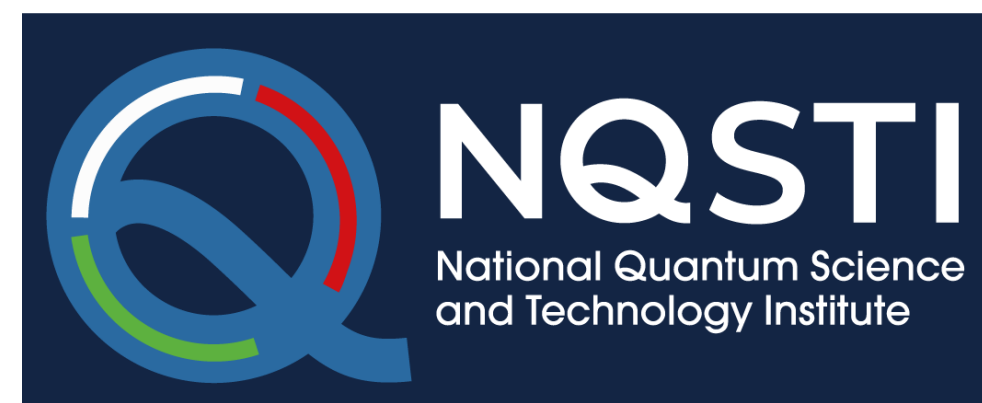
Consider instead:

$$H_{SE_n} = \alpha \sigma_S^+ \sigma_{E_i}^- + \alpha^* \sigma_S^- \sigma_{E_i}^+$$

$$\sigma_{E_i}^+ = |E_3\rangle\langle E_2| = (\sigma_{E_i}^-)^\dagger$$

This induces an effective unitary dynamics on S, described by the Hamiltonian

$$H_S^{\text{eff}} = \text{Re}[\text{Tr}[\sigma_{E_i}^+ \eta_{E_n}]] \sigma_S^x + \text{Im}[\text{Tr}[\sigma_{E_i}^+ \eta_{E_n}]] \sigma_S^y$$

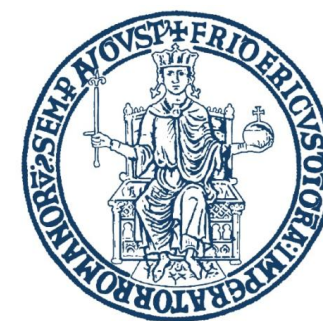
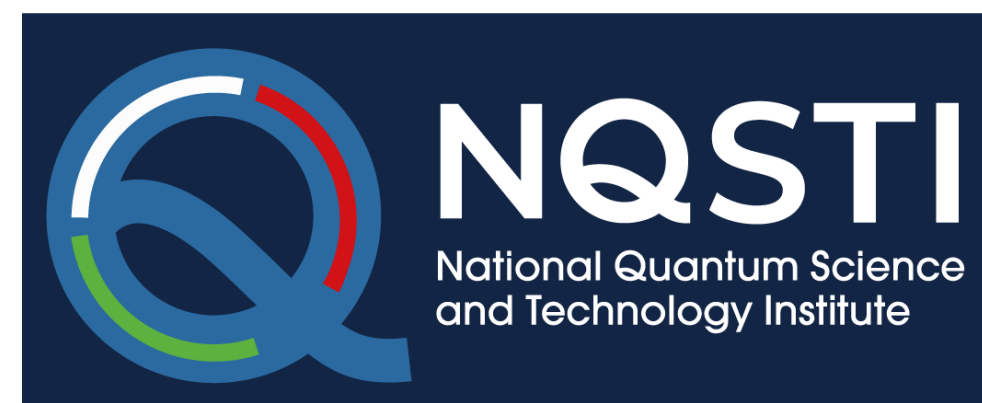
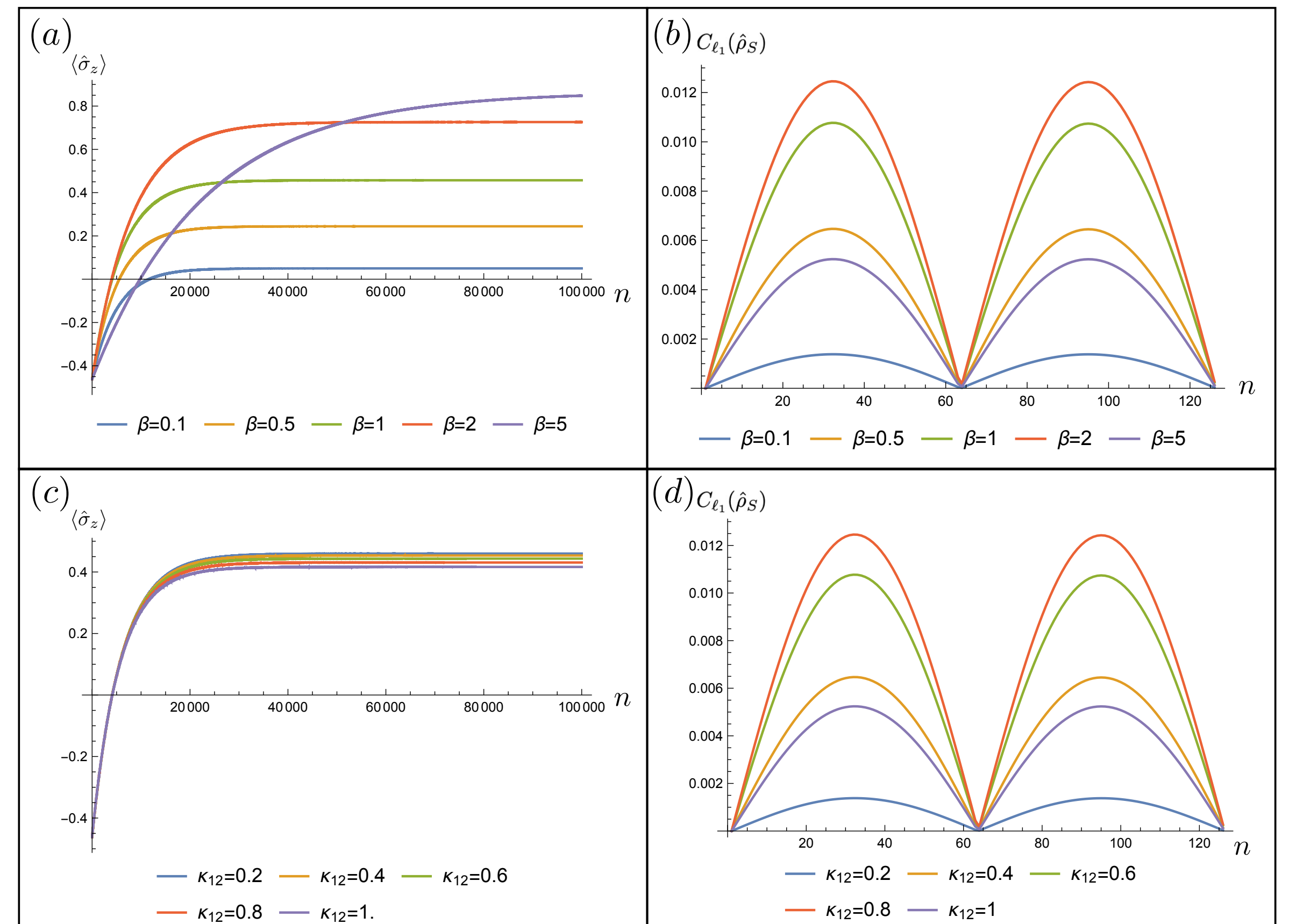


Coherence transfer and thermodynamic cost

The presence of the extra Hamiltonian term leads to the generation of steady state coherence and equilibrium populations different from the thermal ones

Naturally, this comes at the cost of introducing energy into the system, as the interaction between system and environment is not energy conserving.

This allowed to assign a work cost to the steady state coherence generation.



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One more application: quantum metrology

As last application, let us consider the use of collision model to represent metrological procedures.

In this model several ancillae interact with a probe in contact with a thermal environment. The task is to measure the temperature T .

If we were to look at the probe S , the best we could do would be to achieve the thermal Fisher information

$$\mathcal{F}_{\text{th}} = \frac{C}{k_B T^2}$$

$$C = \frac{\langle H_A^2 \rangle - \langle H_A \rangle^2}{k_B T^2}$$

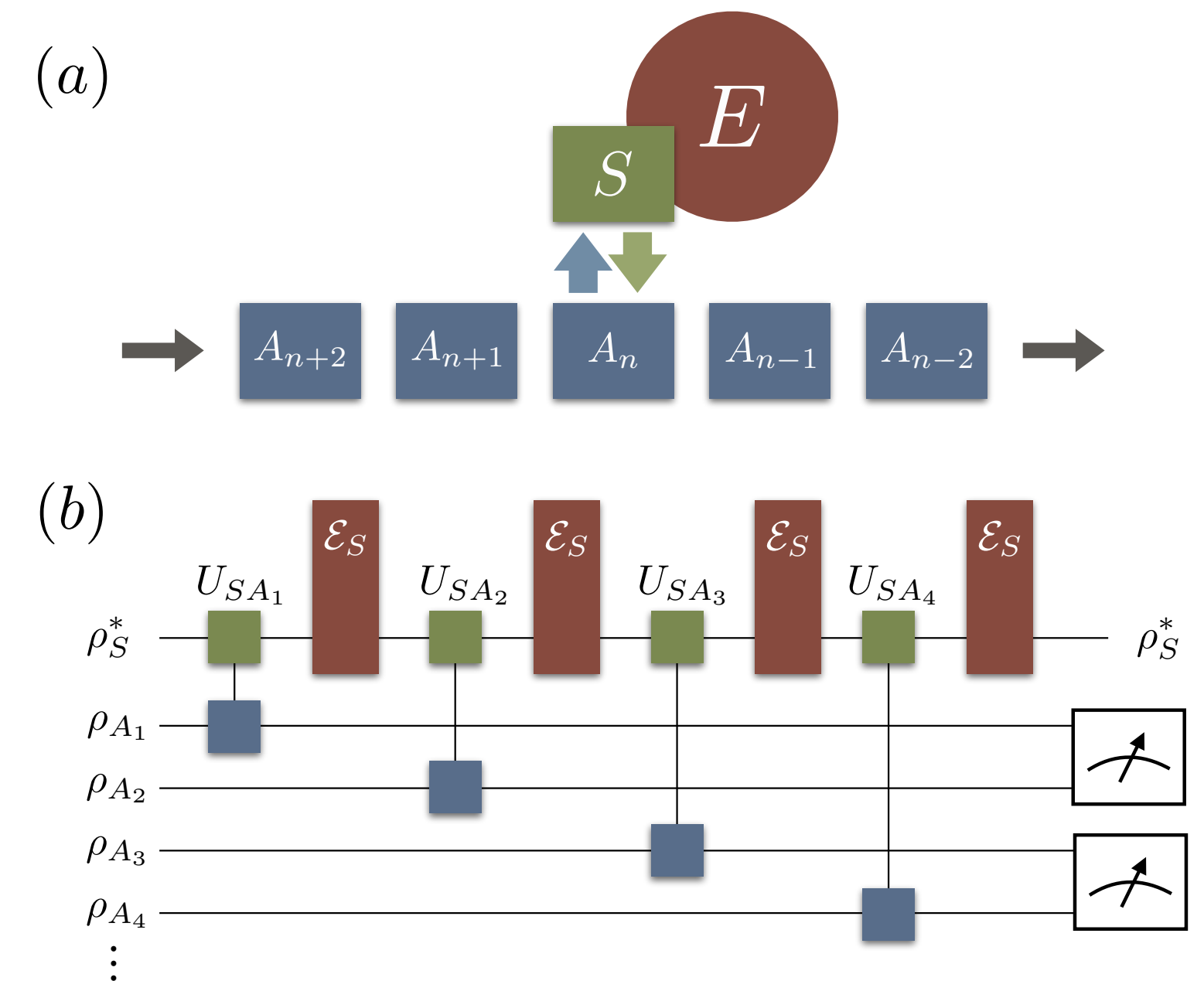


Figure from PRL 123, 180602 (2019)

One more application: quantum metrology

Using the ancillae to extract information from S one can actually do better!

After the interaction of the ancillae with S one gets a state ρ_{A_1, \dots, A_N} (a)

One can find a POVM $\{\Pi_x\}$ and define the probability distribution

$$p(x) = \text{Tr}[\Pi_x \rho_{A_1, \dots, A_N}]$$

Then one constructs the Fisher information:

$$F_N(\{\Pi_x\}, T, \rho_{A_1, \dots, A_N}) = \sum_x p(x) (\partial_T \ln p(x))^2$$

The Quantum Fisher information is defined as:

$$\mathcal{F}_N = \max_{\{\Pi_x\}} F_N(\{\Pi_x\}, T, \rho_{A_1, \dots, A_N}) = \text{Tr}[\rho_{A_1, \dots, A_N} \Lambda^2]$$

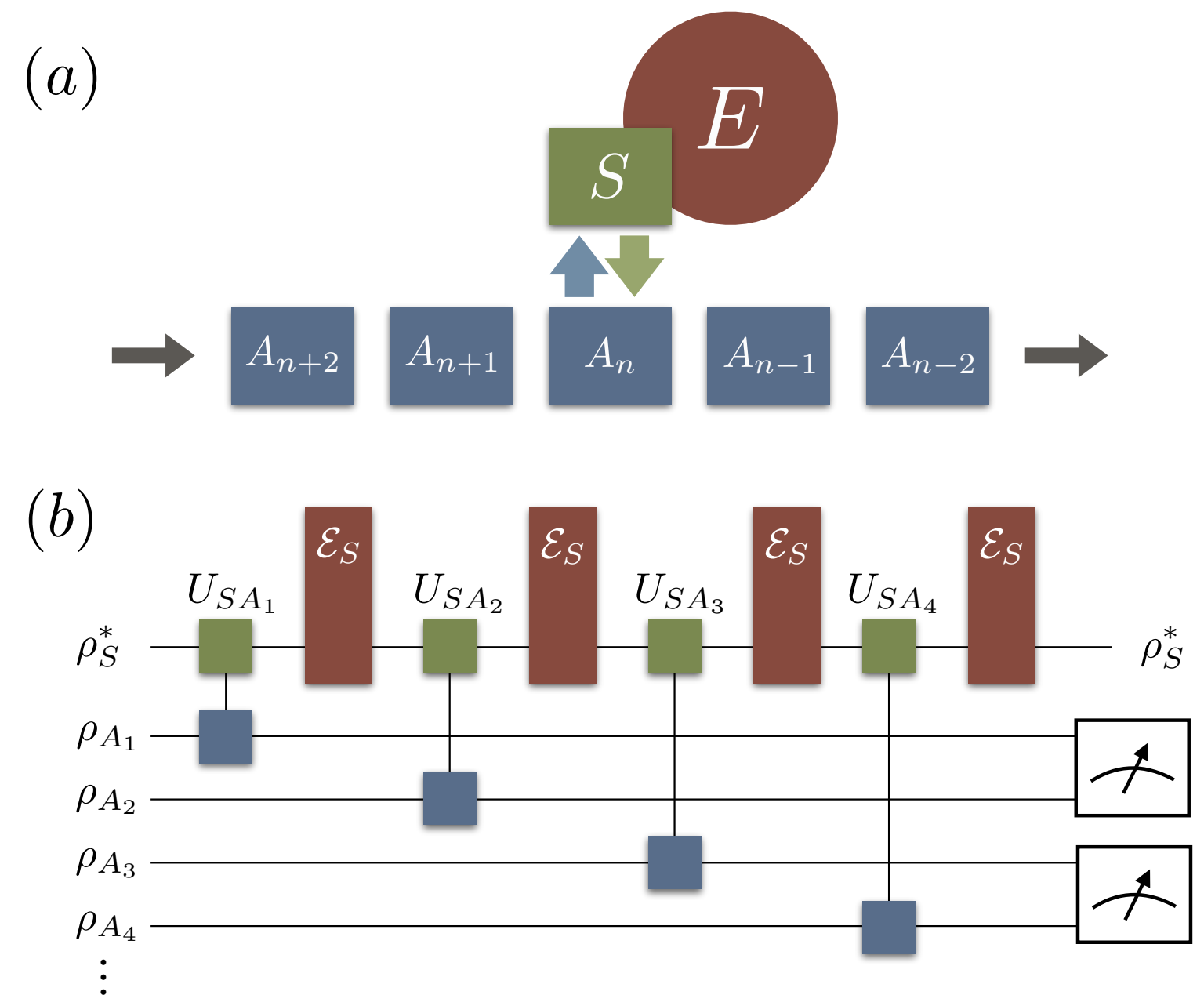


Figure from PRL 123, 180602 (2019)



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One more application: quantum metrology

$$\rho_{SA_1, \dots, A_N} = \mathcal{T}_S \mathcal{U}_{SA_N} \mathcal{T}_S \dots \mathcal{U}_{SA_1} (\rho_S \otimes \rho_A^{\otimes N}) \quad \rho_S(n) = \text{Tr}[\mathcal{T} \mathcal{U}_{SA_n} (\rho_S(n-1) \otimes \rho_A)]$$

$$\dot{\rho}_S = \mathcal{E}_S(\rho_S) = \gamma(\bar{n} + 1) \mathcal{L}_{\sigma_S^-} + \gamma\bar{n} \mathcal{L}_{\sigma_S^+} \quad \mathcal{T}_S = e^{\mathcal{E}_S(\rho_S)\tau} \quad H_{SA_n} = \hbar g (\sigma_S^+ \sigma_{A_n}^- + \sigma_S^- \sigma_{A_n}^+)$$

Because of the joint action of the bath and the ancillas, ρ_S equilibrates to a non thermal state

$$\frac{\mathcal{F}_1}{\mathcal{F}_{\text{th}}} = \frac{(\bar{n} + 1)(e^\Gamma - 1 + 2\bar{n}\Gamma)^2}{e^{2\Gamma}(\bar{n} + 1) - \bar{n} - e^\Gamma} \quad \Gamma = \gamma(2\bar{n} + 1)\tau_{SE}$$

Already for single ancilla measurements one gets an advantage!

$$\frac{\mathcal{F}_2}{2\mathcal{F}_1} = 1 + \frac{(\bar{n}\Gamma)^2}{e^\Gamma - 1} + \mathcal{O}(g\tau_{SA})^2$$

Joint measurements of two ancillas lead to a greater advantage!

See also PRA 102, 042417 for further optimisation

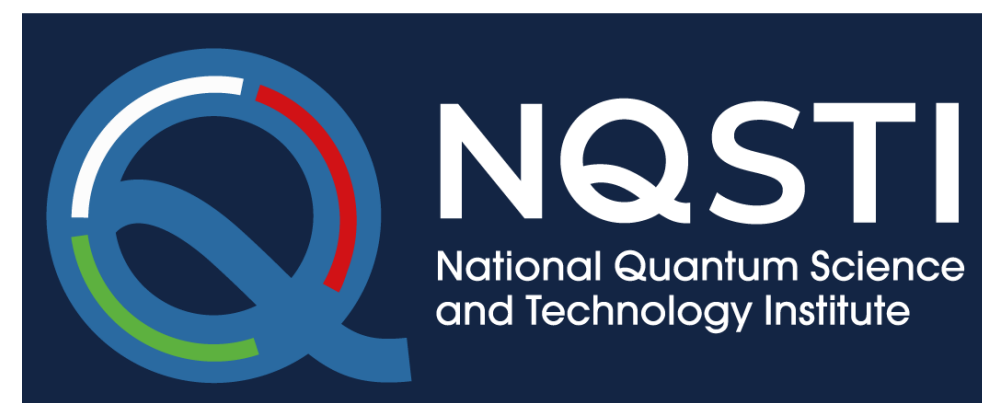


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Conclusions

- Quantum collision models allow for an easy and intuitive description of open system dynamics
- One can easily derive non-Markovian ME, tuning the degree of non-Markovianity
- One can consider non trivial geometries in system S , leading to non local dissipative channels
- One can consider structured environments, e.g. non thermal ones
- Description of metrological protocols over performing with respect to classical and sequential approaches.
- Applications in QCD, cosmic fluctuations or other?



Thanks for your attention!

