Towards a universal Lindblad equation for in-QGP quarkonia dynamics

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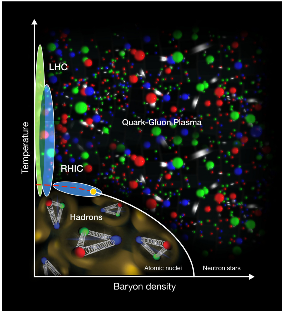
Outline

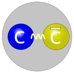
- Introduction & motivation.
- Quantum Brownian-optical transient regime.
- Towards a regime-independent description of in-QGP quarkonia dynamics.

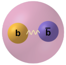
Motivation

- Quark-gluon plasma (QGP): is a hot and dense color-deconfined medium that existed in the early universe and is reproduced in heavy ion collisions (HIC).
  We know little about this state of matter !



- Quarkonia: are bound states of heavy quark-antiquark pairs ($Q\bar{Q}$), charmonium and bottomonium.





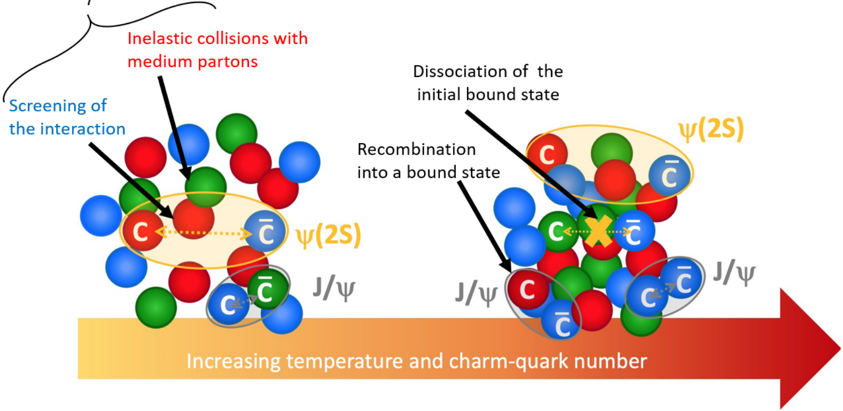
- In-QGP quarkonia: A composite system of interacting quarkonia and QGP whose study deepens our understanding of QGP. e.g., quarkonia suppression.

Quarkonium as a QGP probe

T. Matsui & H. Satz, PLB 178 (1986) 416.
F. Karsch, M. T. Mehr, and H. Satz
Phys. C 37 (1988) 617

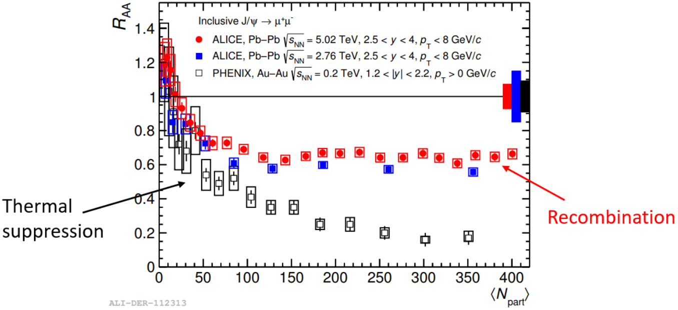
M. Laine et al., JHEP 03 (2007) 054.
A. Beraudo, J. P. Blaizot, C. Ratti,
Nucl. Phys. A 806 (2008) 312.
N. Brambilla, J. Ghiglieri, A. Vairo and P. Petreczky,
Phys. Rev. D 78, 014017 (2008)

$$V(r) = -\frac{A}{r} e^{-m_D r} - i\phi(m_D r)$$



Quarkonium in heavy ion collisions

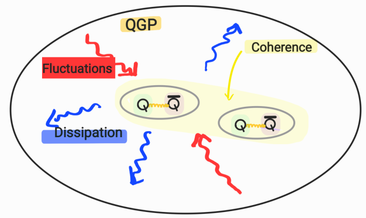
The in-QGP suppression (and regeneration) of quarkonia can be quantified through the nuclear modification factor R_{AA}.



Is there a formalism that enables the construction of a transport theory for quarkonium from first principles?

Open quantum systems + EFT approach

- Quarkonia interaction with QGP involves the exchange of several physical properties, such as quantum coherence and energy, between the two systems $\Rightarrow$ in-QGP quarkonia should be studied as an open quantum system.



- The main idea of the open quantum systems (OQS) formalism is to use the hierarchy in dynamical time scales inherent to the constituents of the global system, quarkonia-QGP, and split it as follows:
  Total system = Quarkonia (our system of interest) + QGP (medium)
- The work boils down into deriving a master equation that describes the dynamics of quarkonia under the influence of QGP.

A master equation for in-QGP quarkonia

The master equation is derived from the von-Neumann equation of the total system

$$\frac{\mathrm{d}\hat{\rho}_{\mathrm{tot}}(t)}{\mathrm{dt}} = -i\left[\hat{H}_{\mathrm{tot}}, \hat{\rho}_{\mathrm{tot}}(t)\right], \;\; \hat{\rho}_{\mathrm{tot}} = |\Psi_{\mathrm{tot}}\rangle\langle\Psi_{\mathrm{tot}}| \tag{1}$$

with

$$\hat{H}_{\mathrm{tot}} = \hat{H}_{Q\bar{Q}} \otimes \hat{I}_{QGP} + \hat{I}_{Q\bar{Q}} \otimes \bar{H}_{QGP} + g\,\hat{H}_{int} \tag{2}$$

Our system of interest is quarkonia $\rightarrow$ We can trace out the QGP degrees of freedom

$$\hat{\rho}_{Q\bar{Q}} = \mathrm{Tr}_{\mathrm{QGP}}\left(\hat{\rho}_{\mathrm{tot}}\right) \tag{3}$$

and implement a set of approximations to end up with an equation of the form

$$\frac{\mathrm{d}\hat{\rho}_{Q\bar{Q}}(t)}{\mathrm{dt}} = \hat{\mathcal{L}}\left[\hat{\rho}_{Q\bar{Q}}(t)\right] \tag{4}$$

Characteristic time scales of an open quantum system

The quarkonium-QGP system involves a set of a hierarchically ordered energy scales, $M, Mv, Mv^2, T, m_D, \Lambda_{QCD}$, which translates into a hierarchy between the main time scales that characterize the in-QGP quarkonia dynamics:

1. The system's intrinsic time scale: $\tau_S \sim \frac{1}{Mv^2} \sim \frac{1}{E}$.
2. The environment, or QGP, correlation time: $\tau_E \sim \frac{1}{gT}, \frac{1}{T}$.
3. The relaxation time scale: $\tau_R \sim \frac{M}{g^4 T^2}$

The hierarchy between these times scales justifies the use of a set of approximations in deriving a given master equation.
e.g. $M \gg T \implies \tau_E \ll \tau_R$, thus, one can use the Born-Markov approximation.

Constraints on the master equation

The master equation, or dynamical map, which governs the time evolution of the quarkonia density matrix, must satisfy two fundamental requirements:

- Given that probabilities can only take on positive values, it follows that the map must preserve this positivity

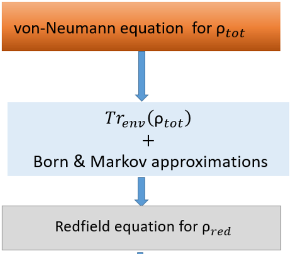
$$\langle \phi | \hat{\rho}(t=0) | \phi \rangle \geq 0 \rightarrow \langle \phi | \hat{\rho}(t) | \phi \rangle \geq 0, \; \forall \; |\phi\rangle \tag{5}$$

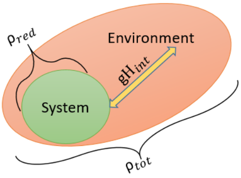
- The trace of the initial normalized density matrix must be preserved during its time evolution, i.e. probabilities should always sum up to one. It follows that the map must be trace preserving

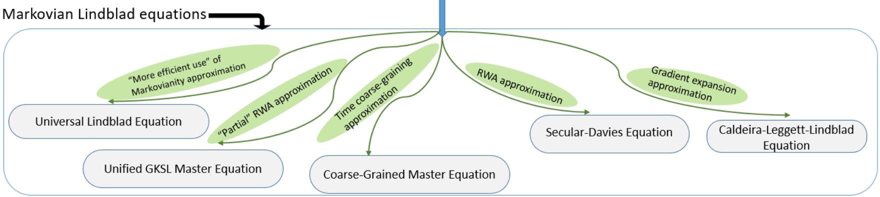
$$\mathrm{Tr}\left(\hat{\rho}(t)\right) = \mathrm{Tr}\left(\hat{\rho}(0)\right) \tag{6}$$

In order for a Markovian master equation to be "complete positive and trace preserving (CPTP)" it must be of Lindblad form.

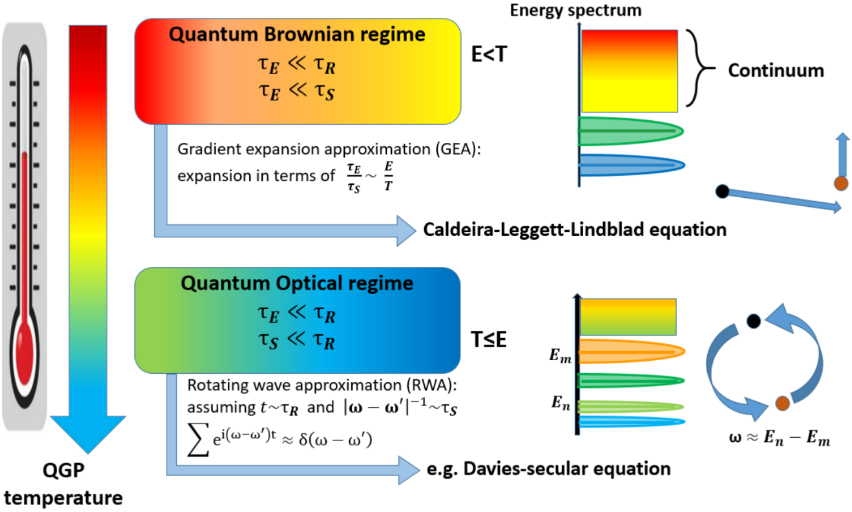
"All roads" lead to a quantum master equation







Quantum Brownian vs. quantum optical regimes



The master equations of the two regimes

Having $H_{\text{int}} = \sum_i A^{(i)} \otimes B^{(i)}$, the quantum Brownian regime is described by the Caldeira-Leggett-Lindblad equation

$$\frac{\mathrm{d}\hat{\rho}_{Q\bar{Q}}(t)}{\mathrm{dt}} = -i\left[\hat{H}_{Q\bar{Q}} + \hat{H}_{\mathrm{LS}}, \hat{\rho}_{Q\bar{Q}}(t)\right] + \sum_{i,j} \gamma_{ij}(0)\left(\bar{A}^{(j)}\hat{\rho}_{Q\bar{Q}}(t)\bar{A}^{(i)\dagger} - \frac{1}{2}\left\{\bar{A}^{(i)\dagger}\bar{A}^{(j)}, \hat{\rho}_{Q\bar{Q}}(t)\right\}\right), \tag{7}$$

with

$$\bar{A}^{(i)} \equiv \hat{A}^{(i)} + \frac{i}{4T}\dot{\hat{A}}^{(i)},\ \dot{\hat{A}}^{(i)} \equiv i\left[\tilde{H}_{Q\bar{Q}}, \hat{A}^{(i)}\right]. \tag{8}$$

The quantum optical regime is described e.g. by the Davies secular equation

$$\frac{\mathrm{d}\hat{\rho}_{Q\bar{Q}}(t)}{\mathrm{dt}} = -i\left[\hat{H}_{Q\bar{Q}} + \hat{H}_{\mathrm{LS}}, \hat{\rho}_{Q\bar{Q}}(t)\right] + \sum_{\omega}\sum_{i,j} \gamma_{ij}(\omega)\left(\hat{A}^{(j)}(\omega)\hat{\rho}_{Q\bar{Q}}(t)\hat{A}^{(i)\dagger}(\omega) - \frac{1}{2}\left\{\hat{A}^{(i)\dagger}(\omega)\hat{A}^{(j)}(\omega), \hat{\rho}_{Q\bar{Q}}(t)\right\}\right), \tag{9}$$

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Toward a unified quantum description of the dynamics

- Due to the dynamical aspect of the QGP, the dynamics of quarkonia covers both the quantum Brownian and quantum optical regimes.
- The two regimes are described by two different master equations.

$\longrightarrow$ Can the two master equations be related ?

$\longrightarrow$ Is it possible to describe the two regimes with a single & regime-independent ***Lindblad*** equation?

Quarkonium-QGP dynamics (NRQCD ??)

$$H_{\text{tot}} = H_{Q\bar{Q}} \otimes I_{QGP} + I_{Q\bar{Q}} \otimes H_{QGP} + gH_{int}, \tag{10}$$

The system intrinsic Hamiltonian $H_{Q\bar{Q}}$ corresponds to the free case

$$H_{Q\bar{Q}} = \frac{\boldsymbol{p}_Q^2}{2M} + \frac{\boldsymbol{p}_{\bar{Q}}^2}{2M}, \tag{11}$$

while the QGP Hamiltonian corresponds to the light particle sector of NRQCD

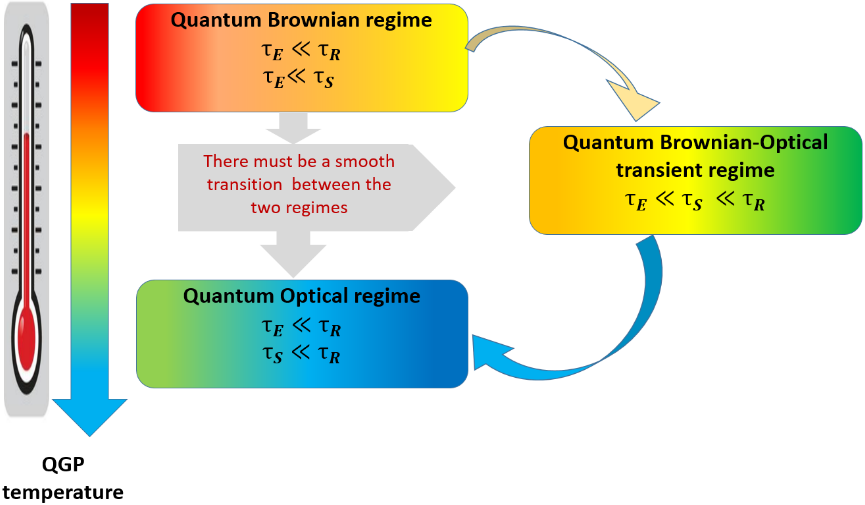
$$H_{QGP} = H_{q+A}. \tag{12}$$

The interaction quarkonium-QGP is given by

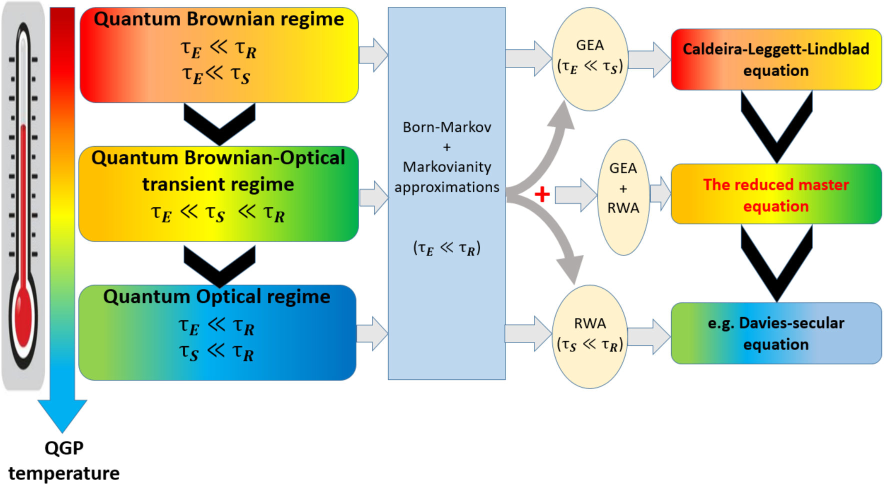
$$H_{int} = -g \int_{\boldsymbol{x}} n^a(\boldsymbol{x})\, A_0^a(\boldsymbol{x}), \tag{13}$$

$n^a(\boldsymbol{x})$ is the color charge density, $n^a(\boldsymbol{x}) = \delta(\boldsymbol{x} - \hat{\boldsymbol{r}})t^a \otimes I - I \otimes \delta(\boldsymbol{x} - \hat{\boldsymbol{r}})\tilde{t}^a$.

Quantum Brownian-Optical transient regime



Derivation of a reduced master equation



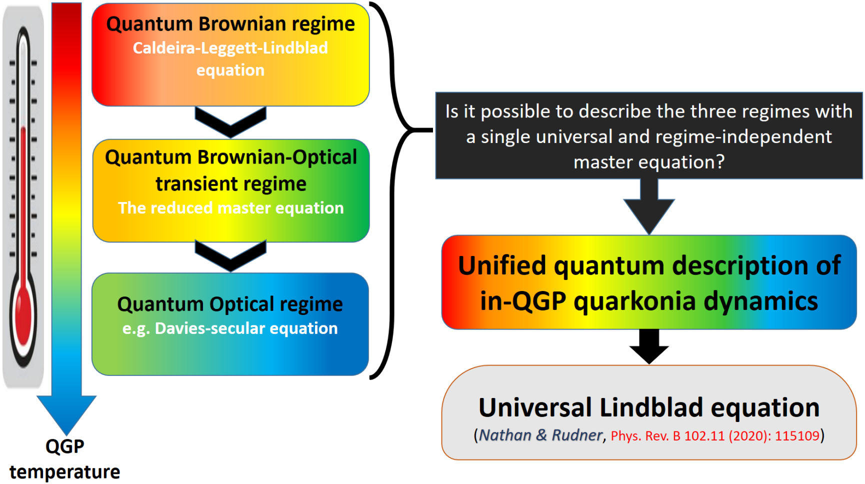
The reduced master equation

Starting from NRQCD, we derived the reduced master equation given by

$$\frac{d\rho_{Q\bar{Q}}(t)}{dt} = -i\left[H_{Q\bar{Q}} + H_{LS}, \rho_{Q\bar{Q}}(t)\right] - \frac{1}{2}\sum_{\omega}\int_{\boldsymbol{x}\boldsymbol{x}'} W(0, \boldsymbol{x}-\boldsymbol{x}')\left(1+\frac{\omega}{2T}\right) \times \left(\left\{n^{\dagger}(\omega,\boldsymbol{x})\, n(\omega,\boldsymbol{x}'), \rho_{Q\bar{Q}}(t)\right\} - 2n(\omega,\boldsymbol{x}')\,\rho_{Q\bar{Q}}(t)\, n^{\dagger}(\omega,\boldsymbol{x})\right). \tag{14}$$

- The equation takes a Lindblad form without the need for the second order term in the gradient expansion which would contribute with a factor $\propto \frac{\omega^2}{T^2} \rightarrow$ This additional term helps to ensure a better accuracy as the temperature decreases.
- The imaginary potential was found in J-P.Blaizot et al.Phy.Rev D, 98(7), 074007 to be dependent on the transition energy at low temperature. The reduced equation illustrates how this dependence emerges "dynamically" as the dynamics shifts from the quantum Brownian to the quantum optical regime.
- The reduced equation can be helpful to understand the deviations from the Gibbs-Boltzmann distribution when the system approaches equilibrium in the quantum Brownian regime.

The universal Lindblad equation



Main steps toward the universal Lindblad equation (ULE)

Starting from the time-dependent Redfield equation:

$$\frac{d\rho^I(t)}{dt} = - \int_{t_0}^{t} dt' \int_{\boldsymbol{x}\boldsymbol{x}'} \left[n^a(t,\boldsymbol{x}), n^a(t',\boldsymbol{x}')\rho^I(t) \right] \Delta^{>}(t-t',\boldsymbol{x}-\boldsymbol{x}') + h.c. \tag{15}$$

The main steps to derive the NRQCD ULE are:

1. The Markovianity approximation $\int_{t_0}^{t} dt' \to \int_{-\infty}^{+\infty} dt' \theta(t-t')$
2. Define the " jump correlator function" $g(t,\boldsymbol{x})$ as :

$$\Delta^{>}(t-t',\boldsymbol{x}-\boldsymbol{x}') = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}v\mathrm{d}\boldsymbol{y}\, g(t-v,\boldsymbol{x}-\boldsymbol{y})\, g(v-t',\boldsymbol{y}-\boldsymbol{x}'). \tag{16}$$

   In Fourier space, this relation has a simple expression

$$\Delta^{>}(q_0,\boldsymbol{q}) = \left[g(q_0,\boldsymbol{q})\right]^2, \tag{17}$$

3. A change of the time integration domains based on the Markov approximation, $\tau_E \ll \tau_R$.

It is worth to note that neither the GEA nor the RWA are used !

The QCD ULE

The NRQCD ULE expression is obtained as

$$\frac{d\rho(t)}{dt} = -i\left[H_Q + \Lambda, \rho(t)\right] + \int_{\boldsymbol{y}} \left(L(\boldsymbol{y})\, \rho(t)\, L^{\dagger}(\boldsymbol{y}) - \frac{1}{2}\left\{ L^{\dagger}(\boldsymbol{y})\, L(\boldsymbol{y}), \rho(t) \right\} \right) \quad (18)$$

where Λ is the Lamb-shift term and the Lindblad operator is given by

$$L(\boldsymbol{y}) = \int_{-\infty}^{+\infty} \mathrm{d}v \int_{\boldsymbol{x}} g(t - v, \boldsymbol{y} - \boldsymbol{x})\; U(v, t)\; n^{a}_{\boldsymbol{x}}\, U^{\dagger}(v, t)\,, \quad (19)$$

where Λ is the Lamb-shift term.

Singlet-octet coupled ULEs

By tracing out the center of mass and projecting on color and energy bases, we obtain the singlet-octet ULEs

$$\frac{d\tilde{\rho}(t)}{dt} = -i\left[H_Q + \tilde{\Lambda}, \tilde{\rho}(t)\right] + \int_{\boldsymbol{y}} \sum_{n \in J} \gamma_n \Bigg(\tilde{L}_n(\boldsymbol{y})\, \tilde{\rho}(t)\, \tilde{L}_n^{\dagger}(\boldsymbol{y}) - \frac{1}{2}\left\{ \tilde{L}_n^{\dagger}(\boldsymbol{y})\, \tilde{L}_n(\boldsymbol{y}), \tilde{\rho}(t) \right\} \Bigg), \tag{20}$$

$\sum_{n \in J}$ refers to the sum over the set of quantum jumps $\{\mathrm{so}, \mathrm{os}, \mathrm{oo}^{(1)}, \mathrm{oo}^{(2)}\}$.

$$\tilde{\rho}(t) = \begin{pmatrix} \tilde{\rho}_s(t) & 0 \\ 0 & \tilde{\rho}_o(t) \end{pmatrix}, \quad H_Q = \begin{pmatrix} H_s & 0 \\ 0 & H_o \end{pmatrix}, \quad \tilde{\Lambda} = \begin{pmatrix} \tilde{\Lambda}_s & 0 \\ 0 & \tilde{\Lambda}_o \end{pmatrix},$$

$$\gamma_{\mathrm{so}} = \gamma_{\mathrm{os}} = \begin{pmatrix} N_c^2 - 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_{\mathrm{oo}^{(1)}} = \gamma_{\mathrm{oo}^{(2)}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

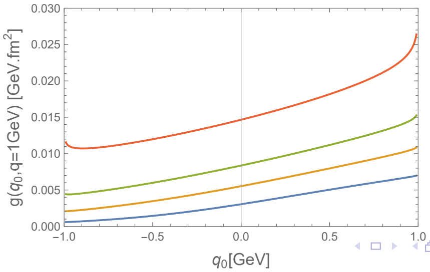
Illustrations 3D (preliminary results)

The dissociation rate of in-QGP quarkonia (singlet bound states) through singlet-to-octet transitions represents a pivotal component of their dynamics.

$$\Gamma^{\mathrm{s}}_{m,n} = \langle m|\mathcal{L}_{\mathrm{s}}|n\rangle := \frac{4\pi\alpha_S C_F}{\hbar c}\int_{\boldsymbol{k}}\int_{\boldsymbol{q}} \tilde{g}\left(E^{(\mathrm{s})}_m - E^{(\mathrm{o})}_k, q\right)\langle \boldsymbol{k}|S_{q\,s}|m\rangle^{\star} \times \tilde{g}\left(E^{(\mathrm{s})}_n - E^{(\mathrm{o})}_k, q\right)\langle \boldsymbol{k}|S_{q\,s}|n\rangle. \tag{21}$$

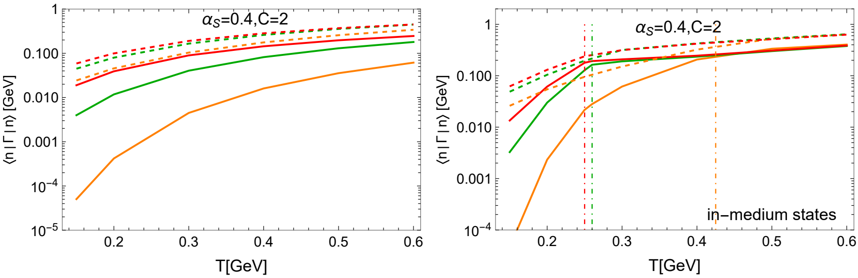
Using the HTL approximation, we can get an expression for the jump correlator function

$$\tilde{g}\left(q_0, \boldsymbol{q}\right) = \sqrt{e^{\beta q_0}\Delta^{<}\left(q_0, \boldsymbol{q}\right)} = \sqrt{e^{\beta q_0} N\left(q_0\right)\sigma\left(q_0, \boldsymbol{q}\right)} \tag{22}$$



Decay widths $\Gamma^{s}_{n,n}$

- One observes a natural increase with T as well as with the hierarchy of excited states.
- The absence of the energy gap facilitates transitions, thus, the decay rate becomes larger.
- As expected, the increase is faster in the case of in-medium states.



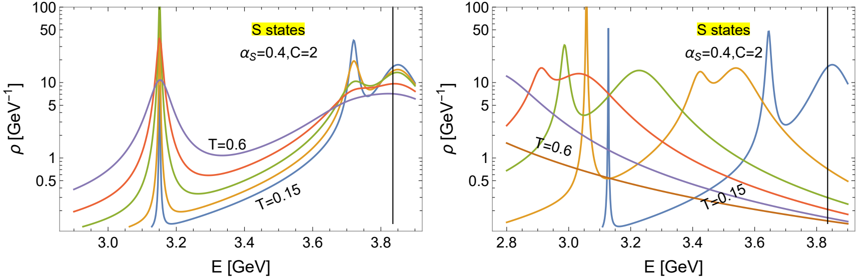
J/ψ (orange), χ (green), ψ (red)

Spectral densities

Further insight into the dissociation of in-QGP quarkonia bound states can be obtained through the spectral normalized density

$$\rho_{\mathrm{s}}(E) := \frac{1}{\pi}\Im\left(\mathrm{tr}\frac{1}{\hat{H}_{\mathrm{s}}^{\mathrm{eff}} - E\hat{I}}\right), \tag{23}$$

with $\hat{H}_{\mathrm{s}}^{\mathrm{eff}} := \hat{H}_{\mathrm{s}} - \frac{i}{2}\hat{\Gamma}_{\mathrm{s}}$.



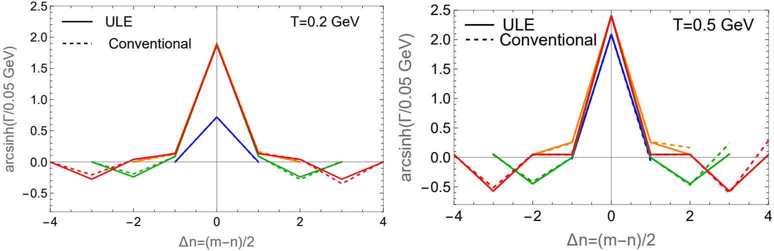
vaccum states in-medium states

→ These results illustrate, in a preliminary way, how it is possible to encompass both QO and QBM regimes within a unique positive definite ULE.

Redfield vs. ULE decay rate

The ULE decay rate can be compared with that obtained from the Redfield Equation in J-P.Blaizot et al.Phy.Rev D, 98(7), 074007 by Blaizot & Escobedo

$$\langle m|\hat{\Gamma}_{\mathrm{s}}^{\mathrm{BE}}|n\rangle = \frac{4\pi\alpha_S C_F}{\hbar c}\int_{\boldsymbol{k}}\int_{\boldsymbol{q}} \Delta^{>}\left(E_n^{(\mathrm{s})} - E_k^{(\mathrm{o})}, q\right)\langle \boldsymbol{k}|S_{q\,s}|m\rangle^{\star}\langle \boldsymbol{k}|S_{q\,s}|n\rangle. \qquad (24)$$



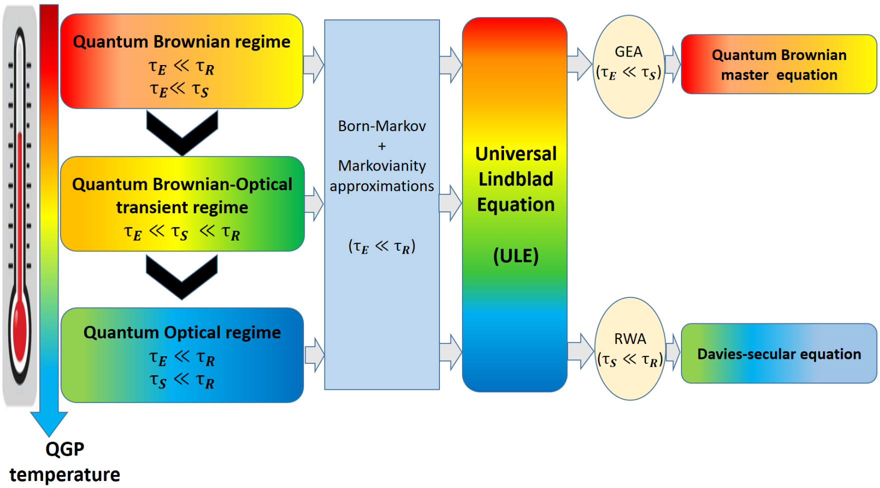
Each line correspond to given fixed even sum $m+n$; 4 (blue), 6 (orange), 8 (green) and 10 (red)

→ The ULE succeeds to reproduce the results of the Redfield equation with a high accuracy. But Should the Redfield equation be used as a benchmark? To be discussed !

Contact with previous works

Can this universal Lindblad equation lead to the previous regime-dependent master equations in the corresponding limit of each regime?

Contact with previous master equations



The quantum Brownian motion limit of the ULE

In the Lindblad operator expression

$$L(t, \boldsymbol{y}) = \int_{-\infty}^{+\infty} \mathrm{d}v \int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{x} g(t - v, \boldsymbol{y} - \boldsymbol{x}) \, U(v, t) \, n_{\boldsymbol{x}}^{a} U^{\dagger}(v, t), \qquad (25)$$

we can consider $\tau_E \sim |t - v|$ and $\tau_S \sim H_Q^{-1}$ and exploit the hierarchy $\tau_E \ll \tau_S \longleftrightarrow H_Q(t - v) \ll 1$ to implement a gradient expansion of the evolution operator $U(v, t) = e^{-iH_Q(t-v)}$ as:

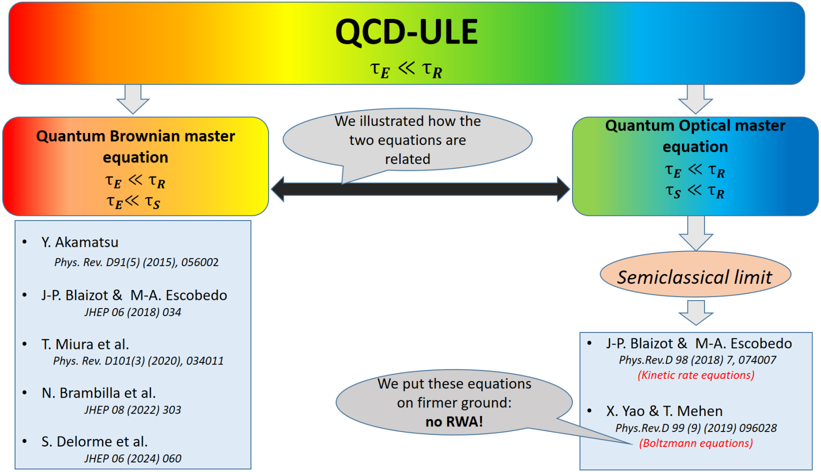
$$U(v, t) = e^{-iH_Q(t-v)} \simeq 1 - iH_Q(t - v), \qquad (26)$$

This gives

$$L(t, \boldsymbol{y}) \simeq \int_{-\infty}^{+\infty} \mathrm{d}v \int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{x} g(t - v, \boldsymbol{y} - \boldsymbol{x}) \left[n_{\boldsymbol{x}}^{a} + (t - v) \, \dot{n}_{\boldsymbol{x}}^{a} \right] \qquad (27)$$

Substituting this expression in the ULE and expressing the jump correlator functions in terms of the thermal correlator, we recover the expression of Caldeira-Leggett-Lindblad equation.

More on the contact with previous works: summary



- We showed that the QCD-ULE encompasses several of the previously employed master equations. So these equations can be systematically rederived within a unified framework.

Conclusions

- Deriving a master equation for the quantum Brownian-optical transient regime can be helpful to understand the transition between the two regimes and explore the associated change in the corresponding steady state.
- The application of the universal Lindblad equation to the study of in-QGP quarkonia dynamics has provided a foundation for a unified quantum description that encompasses both dynamical regimes within a CPTP master equation.

Perspectives

Some of the main perspectives:

- Perform a numerical resolution of the coupled singlet-octet ULEs that we derived. The primary focus of this study will be to explore the transition between the two regimes, as well as to investigate the steady state.
- Derivation of a time dependent universal Lindblad equation which is expected to be more accurate than the time-independent version.
- Study the impact of memory effects on the dynamics of in-QGP quarkonia, especially, for charmonia.

Thank you for your attention !

Back up

Various regimes of quarkonium in QGP

- Several scales are involved in quarkonium-QGP system: $M, Mv, Mv^2, T, m_D, \Lambda_{QCD}$.
- The temperatures attained so far in heavy ion collisions satisfy $M \gg T$, and since $M \gg \Lambda_{QCD}$ $\rightarrow$ Heavy quark mass is larger than thermal and quantum fluctuations $\rightarrow$ We can adopt a non relativistic description.
- Different hierarchies between the scales Mv, Mv^2, T, m_D leads to different regimes.
- In case of high QGP temperature, $T \gg E \sim Mv^2$ we are in the quantum Brownian motion regime and a gradient expansion is required to get a CPTP master equation.
- The appropriate basis of this regime is phase space variables.
- Effective field theories are a suitable starting point to derive a Lindblad equation in our case. The hierarchy between the soft scale $p \sim Mv$ and temperature T dictates the use of NRQCD or pNRQCD.

Derivation of master equation

The main approximations usually adopted are

- QGP is assumed to be at thermal equilibrium all along the evolution time.
- The quarkonia-QGP total density matrix assumed to be initially and remain factorisable all along the time evolution → Born approximation.

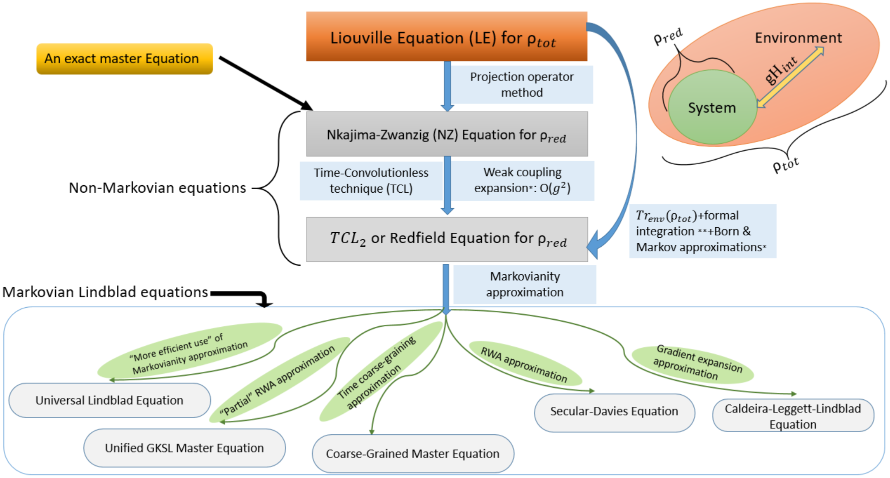
$$\hat{\rho}_{\text{tot}}(t) \simeq \hat{\rho}_{Q\bar{Q}}(t) \otimes \hat{\rho}_{\text{QGP}}(0) \tag{28}$$

- The quarkonia state time evolution depends only on its current state, i.e the dynamical map is local in time → Markovian approximation ($\tau_R >> \tau_E$).

$$\frac{\mathrm{d}\hat{\rho}_{Q\bar{Q}}(t)}{\mathrm{dt}} = \mathcal{L}(t)\,\hat{\rho}_{Q\bar{Q}}(t) \tag{29}$$

Depending on whether we are in quantum optical or Brownian regime, additional approximations as gradient expansion or rotating wave approximation are needed to get complete positive trace preserving (CPTP) master equation !

More on the derivation of Lindblad equations



* Only main approximations are mentioned. ** The formal solution is substituted back into LE.

Lindblad operators

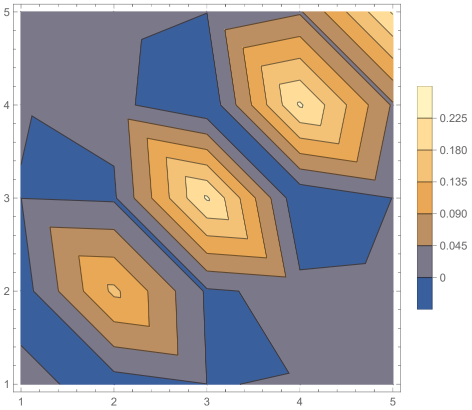
$$\tilde{L}_{\mathrm{so}}\left(n,\boldsymbol{k}\right)=-i\sqrt{\frac{1}{2N_c}}\int_{\boldsymbol{q}}e^{i\boldsymbol{q}\cdot\boldsymbol{z}}g\left(\frac{k^2}{M}-E_n,\boldsymbol{q}\right)\langle n|S_{\boldsymbol{q}\cdot\hat{\boldsymbol{s}}}|\boldsymbol{k}\rangle\begin{pmatrix}0&1\\0&0\end{pmatrix},\tag{30}$$

$$\tilde{L}_{\mathrm{os}}\left(n,\boldsymbol{k}\right)=-i\sqrt{\frac{1}{2N_c}}\int_{\boldsymbol{q}}e^{i\boldsymbol{q}\cdot\boldsymbol{z}}g\left(E_n-\frac{k^2}{M},\boldsymbol{q}\right)\langle \boldsymbol{k}|S_{\boldsymbol{q}\cdot\hat{\boldsymbol{s}}}|n\rangle\begin{pmatrix}0&0\\1&0\end{pmatrix},\tag{31}$$

$$\tilde{L}^{(1)}_{\mathrm{oo}}\left(\boldsymbol{k},\boldsymbol{k}'\right)=-i\sqrt{\frac{N_c^2-4}{4N_c}}\int_{\boldsymbol{q}}e^{i\boldsymbol{q}\cdot\boldsymbol{z}}g\left(\frac{k'^2}{M}-\frac{k^2}{M},\boldsymbol{q}\right)\langle \boldsymbol{k}|S_{\boldsymbol{q}\cdot\hat{\boldsymbol{s}}}|\boldsymbol{k}'\rangle\begin{pmatrix}0&0\\0&1\end{pmatrix},\tag{32}$$

$$\tilde{L}^{(2)}_{\mathrm{oo}}\left(\boldsymbol{k},\boldsymbol{k}'\right)=\sqrt{\frac{N_c}{4}}\int_{\boldsymbol{q}}e^{i\boldsymbol{q}\cdot\boldsymbol{z}}g\left(\frac{k'^2}{M}-\frac{k^2}{M},\boldsymbol{q}\right)\langle \boldsymbol{k}|C_{\boldsymbol{q}\cdot\hat{\boldsymbol{s}}}|\boldsymbol{k}'\rangle\begin{pmatrix}0&0\\0&1\end{pmatrix},\tag{33}$$

Contour plot of the $\Gamma^{s}_{m,n}$ elements for T= 0.4 GeV



$\rightarrow$ we notice small contribution from the off-diagonal elements.

Caldeira-Leggett-Lindblad equation from NRQCD

In the quantum Brownian regime, the Lindblad equation is given by:

$$\frac{\mathrm{d}\hat{\rho}_{Q\bar{Q}}}{\mathrm{dt}} \equiv \sum_{i=0}^{4} \hat{\mathcal{L}}_i \hat{\rho}_{Q\bar{Q}} = -i\left[\hat{H}_{Q\bar{Q}} + \Delta\hat{H}_{Q\bar{Q}}, \hat{\rho}_{Q\bar{Q}}(t)\right] + \int_{\boldsymbol{x},\boldsymbol{y}} W(\boldsymbol{x}-\boldsymbol{y})\left(\tilde{n}^a_{\boldsymbol{x}}\hat{\rho}_{Q\bar{Q}}\tilde{n}^{a\dagger}_{\boldsymbol{y}} - \frac{1}{2}\left\{\tilde{n}^{a\dagger}_{\boldsymbol{y}}\tilde{n}^a_{\boldsymbol{x}}, \hat{\rho}_{Q\bar{Q}}\right\}\right) \tag{34}$$

with

$$\tilde{n}^a_{\boldsymbol{x}} = n^a_{\boldsymbol{x}} - \frac{i}{4T}\dot{n}^a_{\boldsymbol{x}} \tag{35}$$

The Liouville superoperators $\mathcal{L}_i$ capture different aspects of the dynamics:

Unitary dynamics:
- $\mathcal{L}_0$: Kinetic term
- $\mathcal{L}_1$: Static potential (with shift/screening)

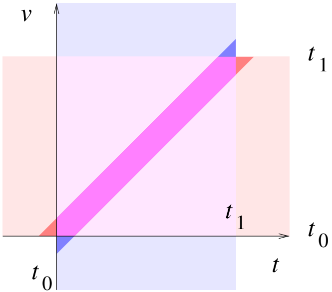
Non-unitary dynamics:
- $\mathcal{L}_2$: Fluctuations (decoherence)
- $\mathcal{L}_3$: Dissipation
- $\mathcal{L}_4$: Preserve positivity $(sub-dominant)$

Change of integration domains- ULE

$$\rho^I(t_1)-\rho^I(t_0) = -\int_{t_0}^{t_1}\mathrm{dt}\int_{-\infty}^{+\infty}\mathrm{dv}\int_{t_0}^{t}\mathrm{t}'\,\mathrm{g}(\mathrm{t}-\mathrm{v})\,\mathrm{g}(\mathrm{v}-\mathrm{t}')\left[\mathrm{n^a}(\mathrm{t}),\mathrm{n^a}(\mathrm{t}')\,\rho^{\mathrm{I}}(\mathrm{t}')\right]+\mathrm{h.c.},$$

becomes

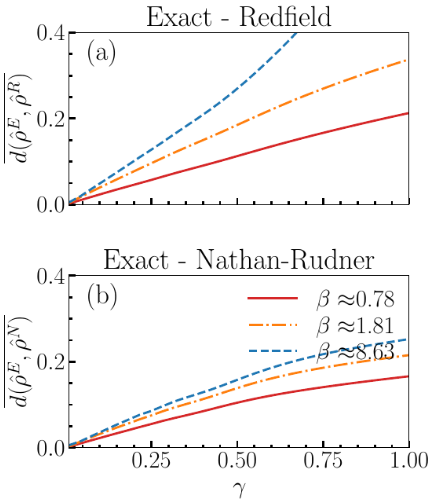
$$\rho^I(t_1)-\rho^I(t_0) = -\int_{t_0}^{t_1}\mathrm{dv}\int_{-\infty}^{+\infty}\mathrm{dt}\int_{t_0}^{t}\mathrm{t}'\,\mathrm{g}(\mathrm{t}-\mathrm{v})\,\mathrm{g}(\mathrm{v}-\mathrm{t}')\left[\mathrm{n^a}(\mathrm{t}),\mathrm{n^a}(\mathrm{t}')\,\rho^{\mathrm{I}}(\mathrm{t}')\right]+\mathrm{h.c.}$$



The band corresponding to $|t-v| \simeq \tau_E$ is shown with darker colors.

Benchmarking of ULE- Damped HO system

Santiago et al. did a comparative study in PhysRevB.110.064319 between the exact HU-Paz-Zhang master equation, the Redfield equation, and the ULE.



Singlet-octet coupled ULEs

By performing the projections in color space, we get a set of singlet-octet coupled ULEs for $\rho(t) = \begin{pmatrix} \rho_s(t) & 0 \\ 0 & \rho_o(t) \end{pmatrix}$ of the form:

$$\frac{d\rho(t)}{dt} = -i\left[H_Q + \Lambda(t), \rho(t)\right] + \int_{\boldsymbol{z}} \sum_{n=0}^{2} \gamma_n \left[L_n(t,\boldsymbol{z})\,\rho(t)\,L_n^{\dagger}(t,\boldsymbol{z}) - \frac{1}{2}\left\{L_n^{\dagger}(t,\boldsymbol{z})\,L_n(t,\boldsymbol{z}), \rho(t)\right\}\right], \tag{36}$$

This explicit Lindblad form is required to apply the quantum trajectories method. Alternatively, following J-P.Blaizot et al.Phy.Rev D, 98(7), 074007, the set of singlet-octet coupled ULEs can also be written as follows:

$$\frac{d\rho_s(t)}{dt} = -i\left[H_s + \Lambda_s(t), \rho_s(t)\right] + \mathcal{L}_{ss}(t)\,\rho_s(t) + \mathcal{L}_{so}(t)\,\rho_o(t), \tag{37}$$

$$\frac{d\rho_o(t)}{dt} = -i\left[H_o + \Lambda_o(t), \rho_o(t)\right] + \mathcal{L}_{os}(t)\,\rho_s(t) + \mathcal{L}_{oo}(t)\,\rho_o(t), \tag{38}$$

pNRQCD ULE

$$\frac{d\rho(t)}{dt} = -i\left[H_Q + \Lambda(t), \rho(t)\right] + \int_{\boldsymbol{z}} \left[L_i(t,\boldsymbol{z})\,\rho(t)\,L_i^\dagger(t,\boldsymbol{z}) - \frac{1}{2}\left\{L_i^\dagger(t,\boldsymbol{z})\,L_i(t,\boldsymbol{z}),\rho(t)\right\}\right] \tag{39}$$

The Lindblad operator is given by

$$L_i(t,\boldsymbol{z},\boldsymbol{R}) = \int_{-\infty}^{+\infty} \mathrm{d}v g(t-v,\boldsymbol{z}-\boldsymbol{R})\,U(v,t)\,V_i^a U^\dagger(v,t) \tag{40}$$

The jump correlators $g(t,\boldsymbol{x})$ that appears in the pNRQCD ULE is defined by [1]

$$\begin{aligned} &g^2\,Tr_{\mathrm{QGP}}\left(\rho_{\mathrm{QGP}}\,\tilde{E}_i^a(t,\boldsymbol{R})\,\tilde{E}_j^b(t',\boldsymbol{R}')\right) \\ &\qquad = \delta_{ij}\delta^{ab}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}v\mathrm{d}\boldsymbol{z}\,g(t-v,\boldsymbol{R}-\boldsymbol{z})\,g\left(v-t',\boldsymbol{z}-R'\right). \end{aligned} \tag{41}$$

NRQCD vs. pNRQCD ULEs

- The ULE can also be derived starting from pNRQCD and be compared to the small dipole limit of the NRQCD ULE.
- For example, the small dipole limit of the singlet-singlet matrix element in the NRQCD case is given by

$$\mathcal{L}_{ss}^{\mathrm{NRQCD}}(t)\,\rho_s(t) = \frac{-C_F}{2}\int_{vv'q_0q_0'\boldsymbol{q}} e^{-i\left(q_0-q_0'\right)t}e^{-iq_0v+iq_0'v'}g\left(q_0,\boldsymbol{q}\right)g\left(q_0',\boldsymbol{q}\right) \times\left\{U_s(t,v)\,\boldsymbol{q}.\hat{\boldsymbol{r}}\,U_o^\dagger(t,v)\,U_o(t,v')\,\boldsymbol{q}.\hat{\boldsymbol{r}}'\,U_s^\dagger(t,v'),\rho_s(t)\right\}, \tag{42}$$

- While, in the pNRQCD case, this element is given by

$$\mathcal{L}_{ss}^{\mathrm{pNRQCD}}(t)\,\rho_s(t) = \frac{-C_F}{2}\int_{vv'q_0q_0'\boldsymbol{q}} e^{-i\left(q_0-q_0'\right)t}e^{-iq_0v+iq_0'v'}\mathcal{G}\left(q_0,\boldsymbol{q}\right)\mathcal{G}\left(q_0',\boldsymbol{q}\right) \times\left\{U_s(t,v)\,r_i\,U_o^\dagger(t,v)\,U_o(v',t)\,r_i'\,U_s^\dagger(t,v'),\rho_s(t)\right\}, \tag{43}$$

The two theories yield identical results because of the following relation:

$$\delta_{ij}\int_{\boldsymbol{q}}\mathcal{G}\left(q_0,\boldsymbol{q}\right)\mathcal{G}\left(q_0',\boldsymbol{q}\right) = \int_{\boldsymbol{q}} q_iq_j\,g\left(q_0,\boldsymbol{q}\right)g\left(q_0',\boldsymbol{q}\right).$$

NRQCD vs. pNRQCD ULEs

- By a similar comparison for the rest of terms in the singlet ULE, we can show that the two theories yield the same singlet equation.
- Regarding the octet ULE, the singlet-octet matrix element $\mathcal{L}_{os}$ is identical in the two theories, while the octet-octet $\mathcal{L}_{oo}$ and Lamb-shift Λ_o terms show a slight disagreement which is due to NLO order terms in the dipole size when using

$$C_{\boldsymbol{q}.\hat{\boldsymbol{r}}} \equiv 2\cos(\frac{\boldsymbol{q}.\hat{\boldsymbol{r}}}{2}) \approx 2\hat{\boldsymbol{I}} - \frac{\boldsymbol{q}^2\hat{\boldsymbol{r}}^2}{4}, \tag{44}$$

in expressions such as:

$$U_o\,(t,v)\,C_{\boldsymbol{q}.\hat{\boldsymbol{r}}}\,U_s^\dagger\,(t,v)\,\tilde{U}_s\,(t,v')\,C_{\boldsymbol{q}.\hat{\boldsymbol{r}}'}\,U_o^\dagger\,(t,v')\,\rho_o\,(t)\,. \tag{45}$$

- In particular, the cross terms $2\hat{\boldsymbol{I}} \times \frac{\boldsymbol{q}^2\hat{\boldsymbol{r}}'^2}{4}$ and $\frac{\boldsymbol{q}^2\hat{\boldsymbol{r}}^2}{4} \times 2\hat{\boldsymbol{I}}$ are not found in the pNRQCD expressions.

Kinetic rate equations (singlet equation)

$$\frac{\mathrm{d}p_n^s}{\mathrm{dt}} = C_F \int_{\boldsymbol{qp}} \left(p_{\boldsymbol{p}}^o - e^{-\beta\left(\frac{p^2}{M} - E_n\right)} p_n^s \right) \Delta^> \left(\frac{p^2}{M} - E_n, \boldsymbol{q} \right) \left|\langle n \left| S_{\boldsymbol{q}.\hat{\boldsymbol{s}}} \right| \boldsymbol{p} \rangle\right|^2$$

$$- C_F \sum_{m \neq n} \int_{\boldsymbol{qp}} \mathrm{P} \int_{q_0} \frac{g\left(q_0 + E_n - E_m, \boldsymbol{q}\right) g\left(q_0, \boldsymbol{q}\right)}{q_0 - E_m + \frac{k^2}{M}}$$

$$\times \langle n \left| S_{\boldsymbol{q}.\hat{\boldsymbol{s}}} \left|\boldsymbol{p}\right\rangle \langle \boldsymbol{p} \right| S_{\boldsymbol{q}.\hat{\boldsymbol{s}}'} \left|m\right\rangle \langle m | \rho_s(t) \right| n \rangle$$

$$+ C_F \sum_{m \neq n} \int_{\boldsymbol{qp}} \mathrm{P} \int_{q_0} \frac{g\left(q_0 + E_m - E_n, \boldsymbol{q}\right) g\left(q_0, \boldsymbol{q}\right)}{q_0 - E_n + \frac{k^2}{M}}$$

$$\times \langle n | \rho_s(t) | m \rangle \langle m | S_{\boldsymbol{q}.\hat{\boldsymbol{s}}} | \boldsymbol{p} \rangle \langle \boldsymbol{p} | S_{\boldsymbol{q}.\hat{\boldsymbol{s}}'} | n \rangle$$

$$\frac{-C_F}{2} \sum_{m \neq n} \int_{\boldsymbol{qp}} g\left(E_n - \frac{p^2}{M}, \boldsymbol{q}\right) g\left(E_m - \frac{p^2}{M}, \boldsymbol{q}\right)$$

$$\times \langle n | S_{\boldsymbol{q}.\hat{\boldsymbol{s}}} | \boldsymbol{p} \rangle \langle \boldsymbol{p} | S_{\boldsymbol{q}.\hat{\boldsymbol{s}}'} | m \rangle \langle m | \rho_s(t) | n \rangle$$

$$\frac{-C_F}{2} \sum_{m \neq n} \int_{\boldsymbol{qp}} g\left(E_m - \frac{p^2}{M}, \boldsymbol{q}\right) g\left(E_n - \frac{p^2}{M}, \boldsymbol{q}\right)$$

$$\times \langle n | \rho_s(t) | m \rangle \langle m | S_{\boldsymbol{q}.\hat{\boldsymbol{s}}} | \boldsymbol{p} \rangle \langle \boldsymbol{p} | S_{\boldsymbol{q}.\hat{\boldsymbol{s}}'} | n \rangle$$

$$+ C_F \int_{\boldsymbol{qpp}', \boldsymbol{p} \neq \boldsymbol{p}'} g\left(\frac{p^2}{M} - E_n, \boldsymbol{q}\right) g\left(\frac{p'^2}{M} - E_n, \boldsymbol{q}\right)$$

$$\times \langle n | S_{\boldsymbol{q}.\hat{\boldsymbol{s}}'} | \boldsymbol{p}' \rangle \langle \boldsymbol{p}' | \rho_o(t) | \boldsymbol{p} \rangle \langle \boldsymbol{p} | S_{\boldsymbol{q}.\hat{\boldsymbol{s}}} | n \rangle$$

Coupled Boltzmann equations (singlet)- Derivation scheme

$$\frac{d\rho_s(t)}{dt} = -i\left[H_s + \Lambda_s(t), \rho_s(t)\right] + \mathcal{L}_{ss}(t)\rho_s(t) + \mathcal{L}_{so}(t)\rho_o(t), \tag{46}$$

Projecting we get,

$$\begin{aligned}\frac{d\langle n, \boldsymbol{P}_1|\rho_s(t)|n', \boldsymbol{P}_2\rangle}{dt} =& -i\langle n, \boldsymbol{P}_1|\left[H_s + \Lambda_s(t), \rho_s(t)\right]|n', \boldsymbol{P}_2\rangle \\ &+ \langle n, \boldsymbol{P}_1|\mathcal{L}_{ss}(t)\rho_s(t)|n', \boldsymbol{P}_2\rangle + \langle n, \boldsymbol{P}_1|\mathcal{L}_{so}(t)\rho_o(t)|n', \boldsymbol{P}_2\rangle\end{aligned}, \tag{47}$$

More on the derivation

$$\langle n, \boldsymbol{P}_1|\mathcal{L}_{ss}(t)\rho_s(t)|n', \boldsymbol{P}_2\rangle \propto \frac{-C_F}{2}\sum_m \int_{\boldsymbol{qp}} g\left(E_n - \frac{p^2}{M}, \boldsymbol{q}\right) g\left(E_m - \frac{p^2}{M}, \boldsymbol{q}\right) \times \langle n|S_{\boldsymbol{q}.\hat{\boldsymbol{s}}}|\boldsymbol{p}\rangle\langle \boldsymbol{p}|S_{\boldsymbol{q}.\hat{\boldsymbol{s}}'}|m\rangle\langle m, \boldsymbol{P}_1|\rho_s(t)|n', \boldsymbol{P}_2\rangle ,$$

$$\langle n, \boldsymbol{P}_1|\mathcal{L}_{so}(t)\rho_o(t)|n', \boldsymbol{P}_2\rangle = C_F\int_{\boldsymbol{qpp}'} g\left(\frac{p^2}{M} - E_{n'}, \boldsymbol{q}\right) g\left(\frac{p'^2}{M} - E_n, \boldsymbol{q}\right) \times \langle n|S_{\boldsymbol{q}.\hat{\boldsymbol{s}}'}|\boldsymbol{p}'\rangle\langle \boldsymbol{p}', \boldsymbol{P}_1|\rho_o(t)|\boldsymbol{p}, \boldsymbol{P}_2\rangle\langle \boldsymbol{p}|S_{\boldsymbol{q}.\hat{\boldsymbol{s}}}|n'\rangle .$$

Wigner transform is given by

$$f_n(t, \boldsymbol{R}, \boldsymbol{P}) = \int_{\Delta\boldsymbol{P}} e^{i\Delta\boldsymbol{P}.\boldsymbol{R}}\langle \boldsymbol{P}_1, n|\rho_s(t)|\boldsymbol{P}_{,2}, n\rangle$$

$$f_o(t, \boldsymbol{R}, \boldsymbol{P}, \boldsymbol{r}, \tilde{\boldsymbol{p}}) = \int_{\Delta\boldsymbol{P}\Delta\boldsymbol{p}} e^{i\Delta\boldsymbol{p}.\boldsymbol{R}+i\Delta\boldsymbol{p}.\boldsymbol{r}}\langle \boldsymbol{P}_1, \boldsymbol{p}|\rho_o(t)|\boldsymbol{P}_2, \boldsymbol{p}'\rangle, \tag{48}$$

We use also the semiclassical, or gradient, expansion of the octet Wigner function

$$f_o(t, \boldsymbol{R}, \boldsymbol{P}, \boldsymbol{r}, \tilde{\boldsymbol{p}}) = f_o(t, \boldsymbol{R}, \boldsymbol{P}, \boldsymbol{r}_0, \tilde{\boldsymbol{p}}) + (\boldsymbol{r} - \boldsymbol{r}_0).\nabla_{\boldsymbol{r}_0} f_o(t, \boldsymbol{R}, \boldsymbol{P}, \boldsymbol{r}_0, \tilde{\boldsymbol{p}}) + \cdots , \tag{49}$$

Boltzmann equation

$$\int_{\Delta \boldsymbol{P}} e^{i\Delta \boldsymbol{P}.\boldsymbol{R}} \langle \boldsymbol{P}, n | \mathcal{L}_{ss} \rho_s(t) | \boldsymbol{P} + \Delta \boldsymbol{P}, n \rangle = - C_F \int_{\boldsymbol{qp}} \left[g\left(E_n - \frac{p^2}{M}, \boldsymbol{q} \right) \right]^2, \times |\langle n | S_{\boldsymbol{q}.\hat{\boldsymbol{s}}} | \boldsymbol{p} \rangle|^2 f_n(t, \boldsymbol{R}, \boldsymbol{P}) \tag{50}$$

$$\int_{\Delta \boldsymbol{P}} e^{i\Delta \boldsymbol{P}.\boldsymbol{R}} \langle \boldsymbol{P}, n | \mathcal{L}_{so} \rho_o(t) | \boldsymbol{P} + \Delta \boldsymbol{P}, n \rangle = C_F \int_{\boldsymbol{qp}} \left[g\left(\frac{p^2}{M} - E_n, \boldsymbol{q} \right) \right]^2 \times |\langle n | S_{\boldsymbol{q}.\hat{\boldsymbol{s}}} | \boldsymbol{p} \rangle|^2 f_o(t, \boldsymbol{R}, \boldsymbol{P}, 0, \boldsymbol{p}) \tag{51}$$

Singlet Boltzmann equation

$$
\begin{aligned}
\left[\frac{\partial}{\partial t}+\frac{\boldsymbol{P}}{2M}.\nabla_{\boldsymbol{R}}\right] f_n\left(t,\boldsymbol{R},\boldsymbol{P}\right) = &- C_F\int_{\boldsymbol{qp}}\left[g\left(E_n-\frac{p^2}{M},\boldsymbol{q}\right)\right]^2 |\langle n|S_{\boldsymbol{q}.\hat{\boldsymbol{s}}}|\boldsymbol{p}\rangle|^2 f_n\left(t,\boldsymbol{R},\boldsymbol{P}\right) \\
&+ C_F\int_{\boldsymbol{qp}}\left[g\left(\frac{p^2}{M}-E_n,\boldsymbol{q}\right)\right]^2 |\langle n|S_{\boldsymbol{q}.\hat{\boldsymbol{s}}}|\boldsymbol{p}\rangle|^2 f_o\left(t,\boldsymbol{R},\boldsymbol{P},0,\boldsymbol{p}\right)
\end{aligned} \tag{52}
$$

The quantum correction to this equation is given by

$$
\begin{aligned}
&-i\frac{C_F}{2}\int_{\boldsymbol{qp}}\left[g\left(\frac{p^2}{M}-E_n,\boldsymbol{q}\right)\right]^2 |\langle n|S_{\boldsymbol{q}.\hat{\boldsymbol{s}}}|\boldsymbol{p}\rangle|^2 \\
&\times\left(\frac{\nabla_{\boldsymbol{p}}\langle n|S_{\boldsymbol{q}.\hat{\boldsymbol{s}}}|\boldsymbol{p}\rangle}{\langle n|S_{\boldsymbol{q}.\hat{\boldsymbol{s}}}|\boldsymbol{p}\rangle}-\frac{\nabla_{\boldsymbol{p}}\langle \boldsymbol{p}|S_{\boldsymbol{q}.\hat{\boldsymbol{s}}}|n\rangle}{\langle \boldsymbol{p}|S_{\boldsymbol{q}.\hat{\boldsymbol{s}}}|n\rangle}\right).\nabla_{\boldsymbol{r_0}} f_o\left(t,\boldsymbol{R},\boldsymbol{P},\boldsymbol{r}_0,\boldsymbol{p}\right)|_{\boldsymbol{r_0}=0}.
\end{aligned} \tag{53}
$$