Non-Markovian dynamics of bottomonia in the QGP

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The system $(\mathbb{S})$: Quarkonium states

1. Quarkonium states are non-relativistic bound states of heavy quarks Q whose $M \gg \frac{1}{r} \gg \Delta E$
2. At $T = 0$ bottomonium: $M_b \sim 5$ GeV, $1/r \sim 1$ GeV, $\Delta E \sim 500$ MeV
3. At $T = 0$ charmonium: $M_b \sim 1.5$ GeV, $1/r \sim 700$ MeV, $\Delta E \sim 500$ MeV
4. Can be described using the non-relativistic EFT called pNRQCD *[Brambrilla, Pineada, Soto, Vairo (1999)]*

pNRQCD

1. The lagrangian is

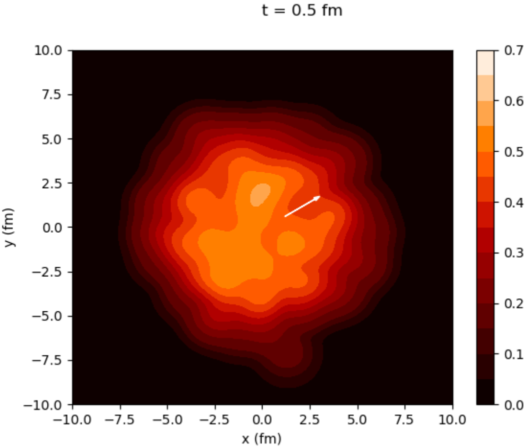
$$L_{\mathrm{pNRQCD}} = \int d^3\mathbf{r}\ \mathrm{tr}\Big(\mathcal{S}(\mathbf{r})^\dagger[i\partial_0 - h_s]\mathcal{S}(\mathbf{r}) + \mathcal{O}(\mathbf{r})^\dagger[iD_0 - h_o]\mathcal{O}(\mathbf{r})\Big) + \mathcal{O}^\dagger(\mathbf{r})\mathbf{r}\cdot g\mathbf{E}\mathcal{S}(\mathbf{r}) + \frac{1}{4}\{\mathcal{O}^\dagger(\mathbf{r})\{\mathbf{r}\cdot g\mathbf{E}, \mathcal{O}(\mathbf{r})\}\} + \ldots\Big)$$

2. $\mathbf{r}$ is the relative separation between $Q\overline{Q}$, $\mathcal{S}$ is the singlet wavefunction and $\mathcal{O}$ is the octet wavefunction
3. $\mathbf{E}$ is the chromo-electric field and $h_{o,s} = -\frac{\nabla^2}{M} + v_{o,s}(\mathbf{r})$

The environment (E): a Quark-Gluon plasma

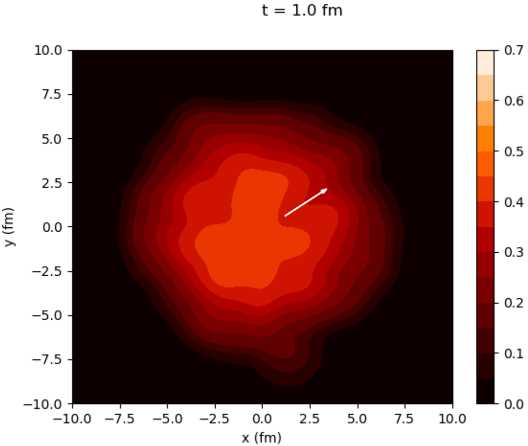
1. After the collision of two relativistic nuclei (Au+Au at RHIC at 200 GeV, Pb+Pb at LHC at $\sim$ few TeV), a hot plasma of quarks and gluons (QGP) is created
2. Circumstantial evidence suggests that this plasma is locally thermally equilibrated: at each x^ν it is described as a fluid element that moves with a four velocity $u^\mu(x^\nu)$ and has a temperature $T(x^\mu)$
3. The plasma expands and cools down very rapidly on a time scale $\sim$ 10fm/c from $T \sim 500$ MeV to freeze-out $T \sim 150$ MeV on a time scale t_{F} of $\sim$ 10 fm
4. This picture for the environment describes the yields of copious (1000) "soft" particles ($p \lesssim 3$ GeV) and its dependence on the collision geometry (N_{part}) very well

Quark-Gluon plasma evolution



1. Temperature contours from hydrodynamics *[S. Pal et. al. (2015, 2018)]*

Quark-Gluon plasma evolution



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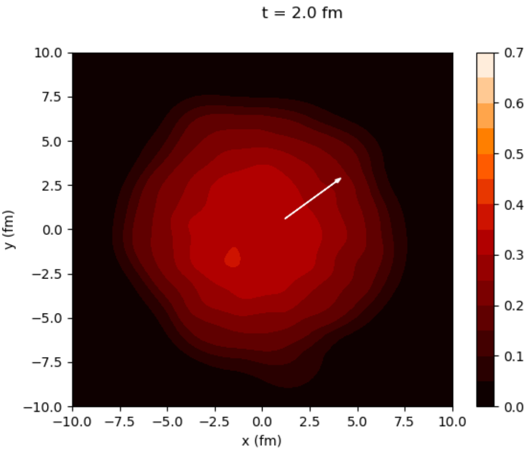

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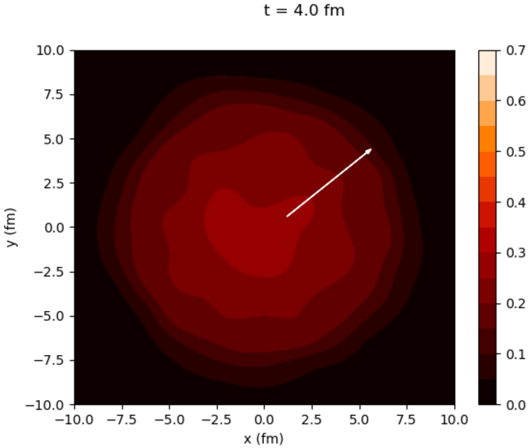


1. Temperature contours from hydrodynamics *[S. Pal et. al. (2015, 2018)]*

$\mathrm{S} + \mathrm{E}$

1. $H_{\mathrm{tot}} = H_{\mathrm{S}} + H_{\mathrm{E}} + V_{\mathrm{I}}$

$$H_{\mathrm{S}} = \left(\frac{p^2}{M} + v_s(\mathbf{r})\right)|s\rangle\langle s| + \left(\frac{p^2}{M} + v_o(\mathbf{r})\right)|o_a\rangle\langle o_a|$$

$$V_{\mathrm{I}} = -g\mathbf{r}\cdot\mathbf{E}^a\left(\frac{1}{\sqrt{2N_c}}|s\rangle\langle o_a| + \frac{1}{\sqrt{2N_c}}|o_a\rangle\langle s| + \frac{1}{2}d_{abc}|o_b\rangle\langle o_c|\right)$$

   Justified if $M \gg 1/r \gg \Delta E, T$

2. $\rho_s(t) = \langle s|\rho_{\mathrm{S}}|s\rangle$, $\rho_o(t) = \langle o_a|\rho_{\mathrm{S}}|o_a\rangle$
3. ρ_{S} is diagonal in s, o basis $\rho_{\mathrm{S}} = \mathrm{diag}\{\rho_s, \rho_o\}$. Similarly, $H_{\mathrm{S}} = \mathrm{diag}\{h_s, h_o\}$.

The master equation

1. A master equation (in the interaction picture) up to $\mathcal{O}(V_{\mathrm{I}}^3)$

$$\frac{d\rho_{\mathrm{S}}}{dt} = -\int_0^t du \, \mathrm{tr}_{\mathrm{E}}\{[V_{\mathrm{I}}(t), [V_{\mathrm{I}}(u), \rho_{\mathrm{S}}(t) \otimes \rho_{\mathrm{E}}]]\}$$

[Redfield (1957); Breuer, Petruccione (1999); Brambilla et. al. (2017)]

2. Born approximation: $\rho \approx \rho_{\mathrm{S}} \otimes \rho_{\mathrm{E}}$ (weak coupling $V_I \ll T$), and the first Markov approximation $\rho_{\mathrm{S}}(u) \approx \rho_{\mathrm{S}}(t)$ (weak interaction $V_I \ll T$)
3. Tracing over the environment gives the master equation (now in Schrödinger picture)

$$\frac{d\rho_{\mathrm{S}}}{dt} = -i[H_{\mathrm{eff}}, \rho_{\mathrm{S}}] + \int_0^t ds \sum_{n=1}^{3} \Gamma_n(t, t-s) \left\{ \mathbf{V}_n(-s)\rho_{\mathrm{S}}(t)\mathbf{V}_n^\dagger(0) + \mathrm{HC} \right\}$$

4. $H_{\mathrm{eff}} = H_{\mathrm{S}} - i \int\limits_0^t ds \sum\limits_{n=1}^{3} \Gamma_n(t, t-s)\mathbf{V}_n^\dagger(0)\mathbf{V}_n(-s)$

The jump operators and medium correlator

1. Three jump operators, $n = 1 : s \to o$, $n = 2 : o \to s$, $n = 3 : o \to o$

$$\mathbf{V}_n(t) = \begin{cases} e^{ih_o t}\mathbf{r}e^{-ih_s t}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & n = 1 \\ e^{ih_s t}\mathbf{r}e^{-ih_o t}\sqrt{\frac{1}{(N_c^2-1)}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & n = 2 \\ e^{ih_o t}\mathbf{r}e^{-ih_o t}\sqrt{\frac{N_c^2-4}{2(N_c^2-1)}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & n = 3\ . \end{cases}$$

2. Corresponding three correlators. Only showing $n = 1$,

$$\Gamma_1(t, t') = \frac{g^2}{6N_c}\mathrm{tr}_{\mathrm{E}}\Big(\mathbf{E}^a(t, 0)\mathcal{W}_{ab}(t, t')\mathbf{E}^b(t', 0)\Big)$$

Expansion in $\tau_{\rm E}/\tau_{\rm S}$ (LO, NLO expansion)

1. $\mathbf{V}_n(t)$ evolves on a time scale of $\tau_{\rm E} \sim \frac{1}{\Delta E}$
2. $\Gamma(t)$ is substantial only for $t < \tau_{\rm E} \sim \frac{1}{T}$. For example, in weak coupling, $\tau_{\rm E} \sim \frac{1}{gT}$
3. The relaxation time (inverse of the thermal width) $\tau_R \sim 1/\Gamma$
4. If $\tau_R, \tau_S \gg \tau_E$ then the correlations in the environment are lost rapidly on the system time scales, and hence the evolution equation of $\rho_S(t)$ does not depend on the history of the evolution: memoryless evolution *Akamatsu (2017, 2020), Brambilla et. al. (2017, 2023), Yao et. al. (2018, 2021), Blaizot et. al. (2018, 2024)*
5. If $\tau_{\rm E} \ll \tau_{\rm S}$ then

$$\mathbf{V}_i(t) \sim e^{ih_\alpha t}\mathbf{r}_i e^{-ih_\beta t} \approx \mathbf{r}_i + it(h_\alpha \mathbf{r}_i - \mathbf{r}_i h_\beta) + \mathcal{O}\left[\left(\frac{\tau_{\rm E}}{\tau_{\rm S}}\right)^2\right]$$

LO NLO

6. Both LO and NLO can be written in a Lindblad form (memoryless)
7. In this approximation, only $\tilde{\Gamma}(\omega = 0) = \frac{1}{2}(\kappa + i\gamma)$ needed

Expansion in $\tau_{\mathrm{E}}/\tau_{\mathrm{S}}$?

1. However, $\Delta E \sim 500\mathrm{MeV}$ for $\Upsilon(1S)$, a little smaller for $\Upsilon(2S)$. On the other hand $T \lesssim 500\mathrm{MeV}$
2. For $\Upsilon(1S)$ in particular, it is worthwhile investigating whether further corrections in $\tau_{\mathrm{E}}/\tau_{\mathrm{S}}$ can have an effect on quantum dynamics
3. Need to solve the non-Markovian master equation, which is more challenging

Going to angular momentum basis

1. In l basis *Brambilla et. al. (2017)*, $\rho^l = \sum_m \langle l, m|\rho_{\rm S}|l, m\rangle$,

$$\frac{d\rho_l(t)}{dt} = -i[h^l_{\rm eff}, \rho_l(t)] + \Big\{ \int_0^t ds \sum_{n,l'} \Gamma_n(t, t-s) \Big($$

$$T_n(l \to l', -s)\rho_{l'}(t) T_n^\dagger(l' \to l, 0)\Big) + \mathrm{HC}\Big\}$$

2. The key jump operator, $n = 1 : s \to o$,

$$T_n(l \to l', t) = C_{l,l'} e^{ih_o^{l'} t} r \; e^{-ih_s^l t} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

3. $h^l = \begin{pmatrix} h_s^l & 0 \\ 0 & h_o^l \end{pmatrix}$, $h_{s,o} = -\frac{1}{M}\Big(\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{l(l+1)}{r^2}\Big) + v_{s,o}(r)$
4. $C_{l,l'} = \delta_{l',l+1}\sqrt{\frac{l+1}{2l+1}} + \delta_{l',l-1}\sqrt{\frac{l}{2l+1}}$

Numerical challenge

1. Computing the non-Markovian jump operators
   $C_i(t) \sim \int_0^t \, ds \Gamma(t, t-s) e^{-s h_o^{l'}} r \, e^{-s h_s^{l}}$
2. $e^{-s h_o^{l'}} r \, e^{-s h_s^{l}}$ can be computed easily in the energy basis, transformed to position, and then stored for all $s \in [0, 10]$ fm
3. Convolution with $\Gamma(t, t')$ is the computationally intensive step, especially because Γ depends on T, and T is different for each trajectory of the quarkonium. (Would be much simpler in a time independent medium)
4. By parallelizing the evaluation of the convolution, $C_i(t)$ can be computed more efficiently

General stochastic unravelling

1. With $C_i(t)$ in hand, stochastic unravelling techniques can be used for general equations of the kind

$$\frac{\partial \rho_{\mathrm{S}}(t)}{\partial t} = (-iH_{\mathrm{eff}}(t))\rho_{\mathrm{S}}(t) + \rho_{\mathrm{S}}(-iH_{\mathrm{eff}}(t))^{\dagger} + \sum_i \Big(C_i(t)\rho_{\mathrm{S}}(t)D_i^{\dagger}(t) + \mathrm{H.C.} \Big) \, .$$

2. $d|\psi(t)\rangle = -i \; dt \; H_{\mathrm{eff}}(t)|\psi(t)\rangle + dN_i(t)J_i(t)|\psi(t)\rangle$.
   $J_i(t) \sim C_i(t)$
3. Implementation of jumps becomes computationally intensive due to averaging over large ensembles $dN_i(t)$

Simplification

1. A simplification motivated by *Brambilla et. al. (2022)*. For the $\Upsilon(1S)$, jumps are not very important. Evolution with h_{eff} is sufficient
2. $$\frac{\partial|\psi_l, t\rangle}{\partial t} = - ih_s^l\,|\psi_l, t\rangle - \int_0^t ds\,\Gamma(t, t-s)\left\{\left(\frac{l+1}{2l+1}\right)\, r\, e^{-ih_o^{l+1}s} r\, e^{ih_s^l s} + \left(\frac{l}{2l+1}\right)\, r\, e^{-ih_o^{l-1}s} r\, e^{ih_s^l s}\right\}\,|\psi_l, t\rangle$$
3. We leave excited states for the future

Setup

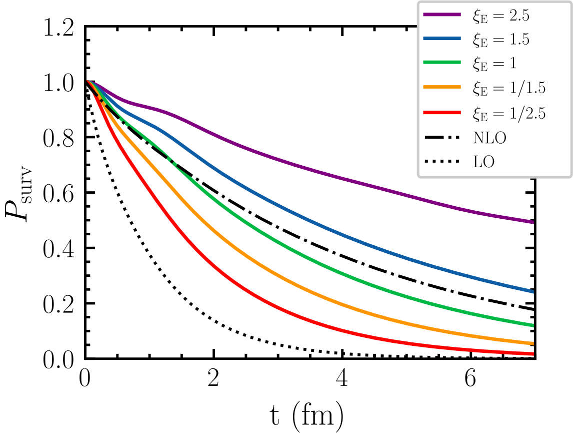
1. We want to illustrate the dependence of the observables on the hierarchy between τ_{S} and τ_{E}
2. τ_{S} is fixed $\sim 1/\Delta E$. So we vary $\tau_{\mathrm{E}} = \xi_{\mathrm{E}}/T$ by changing ξ_{E}
3. Need to generalize the correlator from zero to finite frequency. Take $\gamma = 0$ for simplicity. For the real part, we take,

$$\Gamma(t, t') = \frac{\kappa}{2\tau_{\mathrm{E}}} e^{|t-t'|/\tau_{\mathrm{E}}}$$

4. $\tau_{\mathrm{E}} \to 0$ gives a δ function and get back the Lindblad form
5. The spectral function corresponding this correlator is,

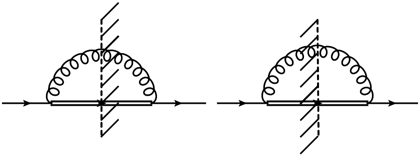
$$\frac{\rho_{\mathbf{EE}}(\omega)}{\omega} \sim \frac{\kappa\Omega^2}{T(\omega^2 + \Omega^2)}, \quad \Omega = \frac{1}{\tau_{\mathrm{E}}} \ .$$

Intuition from a constant T medium



1. *[Vyshakh, Sharma (2025)]*, $\Upsilon(1S)$ evolution with constant $T = 300$ MeV. $\kappa/T^3 = 4$
2. Early transient oscillations $\sim 1/\Delta E$

The decay-rate reduces due to finite frequency effects

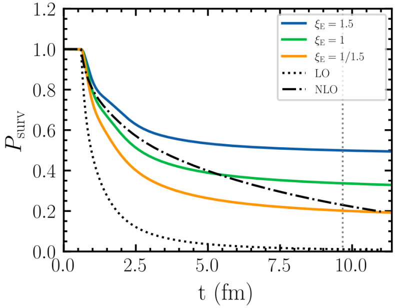


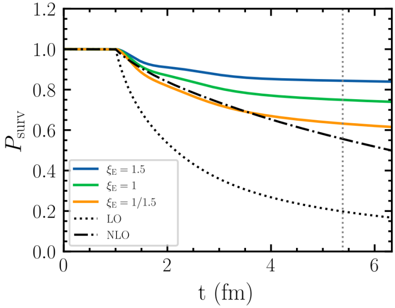
1. We can understand the slower decay of the quarkonium state in a simple way *[Sharma, Singh (2023)]*
2. The decay rate of a state ψ in a static medium, is given by

$$\frac{1}{\tau_D} = \frac{g^2}{3N_c} \sum_f |\langle f|\mathbf{r}|\psi\rangle|^2 f(k^0) \int \frac{d^3\mathbf{k}}{(2\pi)^3} \rho_{\mathbf{EE}}(k^0, \mathbf{k})|_{k^0=E_f-E_\psi}$$

3. The chromoelectric spectral function $\rho_{\mathbf{EE}}$ in the limit $k^0 \to 0$ is given by κ

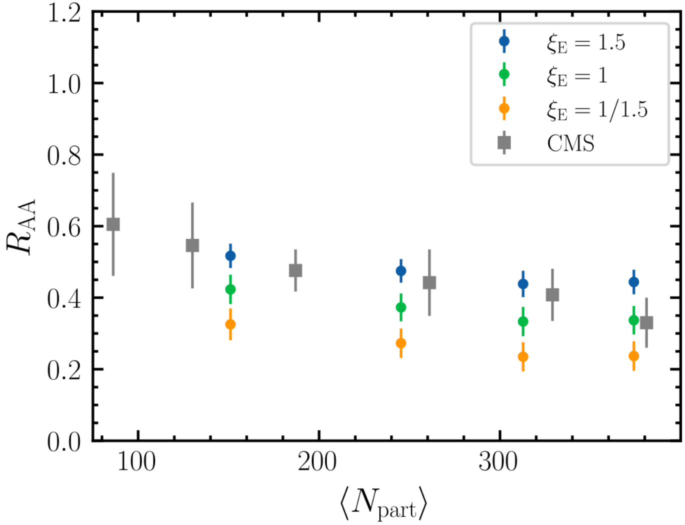
For a Bjorken expanding medium





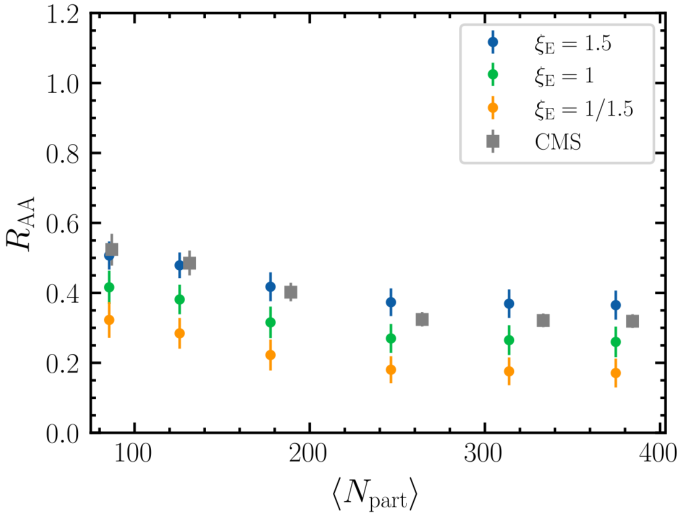
1. $T(\tau) = T(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{\frac{1}{3}}$
2. $T(\tau_0) = 480$ MeV, $\tau_0 = 0.6$ fm (CMS 5.02 TeV) and
   $T(\tau_0) = 333$ MeV, $\tau_0 = 1.0$ fm (RHIC 200 GeV)
3. Vertical line corresponds to $T = 190$ MeV used in some NLO formalisms
4. *[Vyshakh, Sharma (2025)]*

With realistic hydro background simulation (LHC)



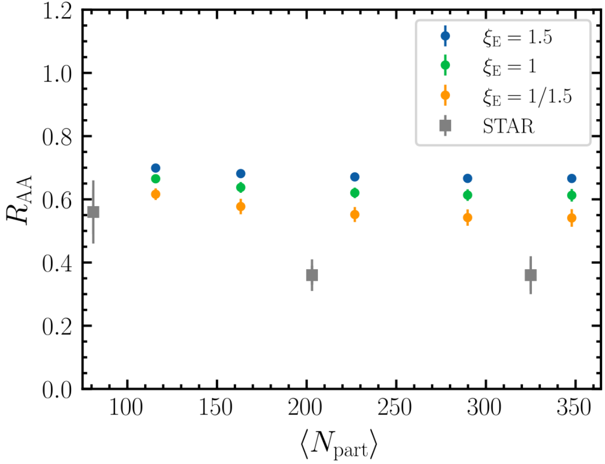
1. Data from *CMS (2017)* (2.76 TeV)

With realistic hydro background simulation (LHC)



1. Data from *CMS (2017)* (5.02 TeV). *ATLAS (2023)* (2.02 TeV) is consistent with CMS and is not shown.

With realistic hydro background simulation (RHIC)



1. Data from *STAR (2023)* (200 GeV)
2. Possibly larger nuclear matter effects at RHIC compared to LHC? Dependence of κ on T?
3. *[Strickland, Thapa (2023)]*

Non-Markovian effects in open heavy-flavour evolution

Markovian evolution of open heavy flavour

1. So far we considered bound states of heavy Q, $\bar{Q}$
2. Individual Q (and $\bar{Q}$) are also produced and move in the medium, and are observed
3. They exhibit momentum diffusion and energy loss due to collisions from the medium
4. Usually described by a classical Langevin equation,

$$\frac{d\mathbf{p}(t)}{dt} = -\eta_D \mathbf{p}(t) + \xi(t)$$

5. $\langle \xi_i(t) \xi_j(t') \rangle = \kappa \delta_{ij} \delta(t - t')$
6. Fluctuation dissipation theorem, $\eta_D = \frac{\kappa}{2MT}$. Ensures thermal equilibration at late time
7. $\kappa \propto \int_{-\infty}^{\infty} dt \, \mathrm{tr}[\langle \mathbf{E}(t,0) \phi(t,0) \mathbf{E}(0,0) \rangle]$
8. Intuition: $M \gg T$, thus energy exchange is small. This implies $\Gamma(\omega = 0)$ dominates heavy quark relaxation

Non-Markovian effects in open heavy flavour dynamics

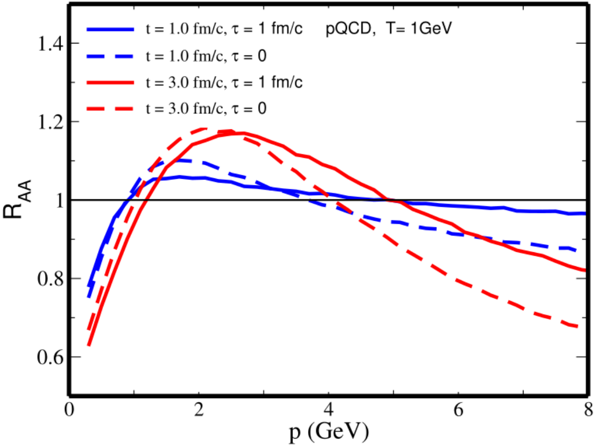
1. Suppose $k^\mu = (\omega, k)$ is exchanged with the medium
2. $\omega = \frac{(p+k)^2}{2M} - \frac{p^2}{2M}$. $k \sim T$. For p approaching M, $\omega \sim T$ and hence $\Gamma(\omega)$ can not be set to $\Gamma(0)$
3. To include this effect, people have tried non-Markovian Langevin equations

$$\frac{d\mathbf{p}(t)}{dt} = -\int \gamma(t-t')\mathbf{p}(t') + \xi(t)$$

4. $\langle \xi_i(t)\xi_j(t')\rangle = \frac{\kappa}{\tau}\delta_{ij}e^{-|t-t'|/\tau}$
5. $\gamma(t,t') = \frac{\kappa}{2MT\tau}\delta_{ij}e^{-|t-t'|/\tau}$
6. The relation between γ and the ξ correlation function assumed on the basis of fluctuation dissipation

Non-Markovian effects in open heavy flavour dynamics

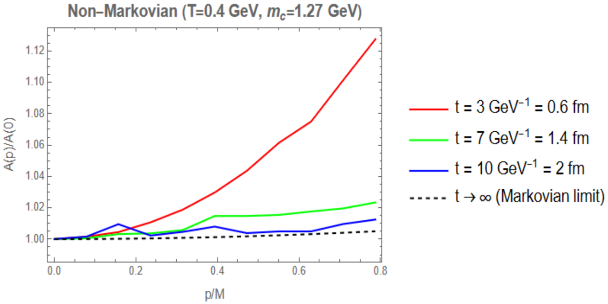
1. Affects dynamics: reduces R_{AA} and delays onset of suppression and flow *[Pooja, Das, Greco, Ruggierri (2022)]*



A more fundamental treatment?

1. The form of γ in open heavy flavour dynamics is chosen in an ad-hoc manner
2. OQS provides a clean approach to deriving Langevin-like equations with memory
3. We start from the Redfield equation for the heavy quark
4. In the Lindblad limit, classicalization using a Wigner transformation gives the usual Langevin equation without memory
5. More generally, one obtains a generalized non-Markovian Langevin like-equation, with both γ and noise correlators defined in terms of the electric field correlators at finite frequency
6. Relation between γ and the noise correlator different from standard assumptions based on the fluctuation dissipation theorem
7. *[Vyshakh, Shaikh, Sharma (in progress)]*

Dependence of η_D on p/M



1. Use the HTL form of the $\langle EE \rangle$
2. (Preliminary) See an increase in the drag coefficient with p
3. Will be interesting to see how momentum diffusion is modified

Summary

1. For bottomonium, pNRQCD leads to the factorization of the non-perturbative quantity, $\langle \mathbf{EE} \rangle$ from the dynamics of the $Q\bar{Q}$. Affects the quantum evolution of quarkonia.
2. In the hierarchy $\frac{\tau_{\rm E}}{\tau_{\rm S}} \ll 1$, Lindblad equations can be derived
3. Quantum approaches give a good description of LHC data but miss at RHIC
4. Forgoing the expansion in $\frac{\tau_{\rm E}}{\tau_{\rm S}}$ leads to a master equation with memory, and this affects quantum evolution
5. This is not a bug but a feature. Reduces sensitivity of the parameters to T_F
6. However, RHIC will need something more

Backup slides