

Hyperfine splittings of heavy quarkonium hybrids

J. Soto, S. Tomàs Valls, *Phys. rev. D* 108, 014025

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Etotic Quarkonia in Heavy-ion Collisions (EQHC2026)



Institut de Ciències del Cosmos
UNIVERSITAT DE BARCELONA



Outline

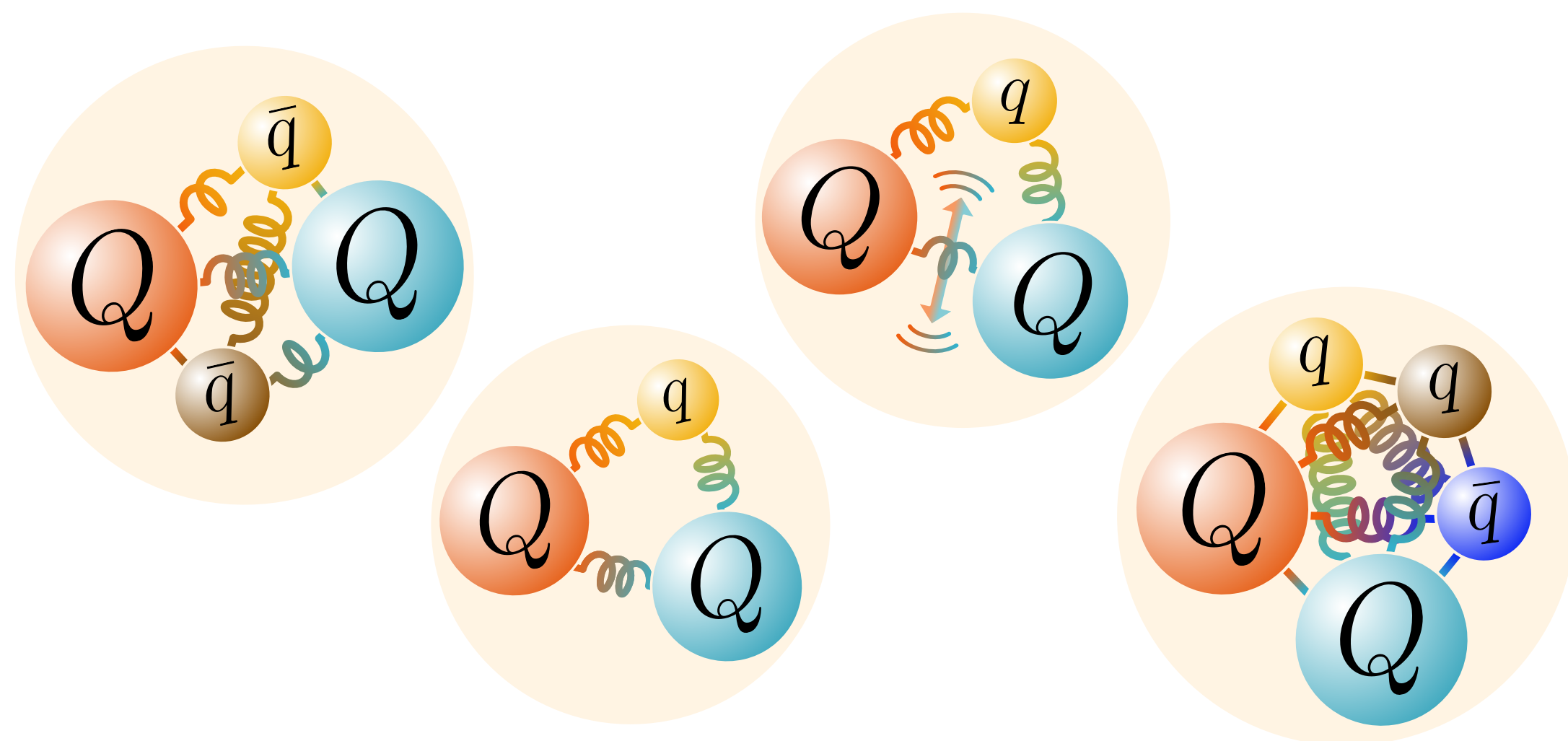
- What are heavy quarkonium hybrids?
- The Born-Oppenheimer Approximation for QCD
- Hyperfine splitting for quarkonium hybrids
- Results and Comparison
- Conclusion

What are heavy hybrids?

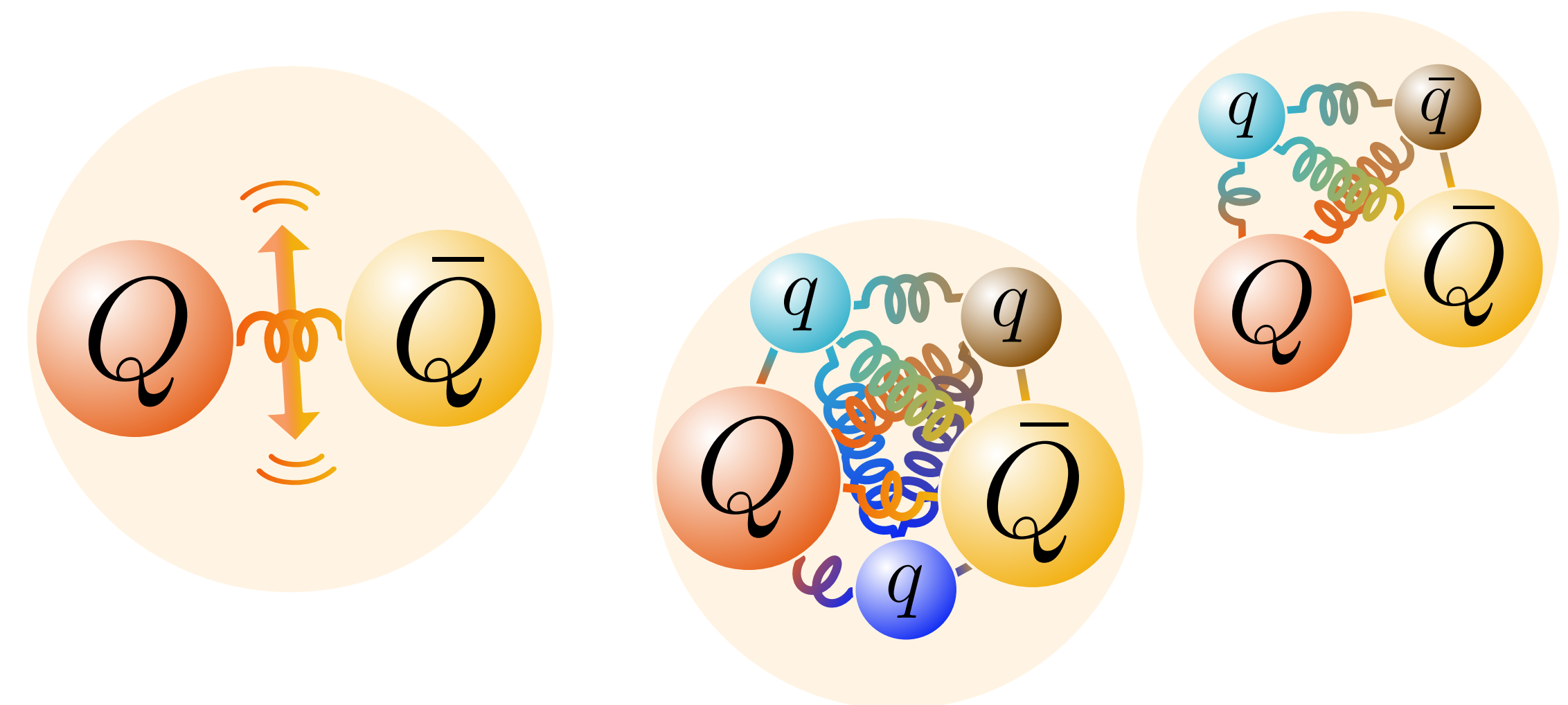
Exotic heavy hadrons

Using as heavy quarks $Q = c, b$ and as light quarks $q = u, d, s$ the hadrons with two heavy quarks can be

$QQ + \text{light quarks and gluons}$



$Q\bar{Q} + \text{light quarks and gluons}$

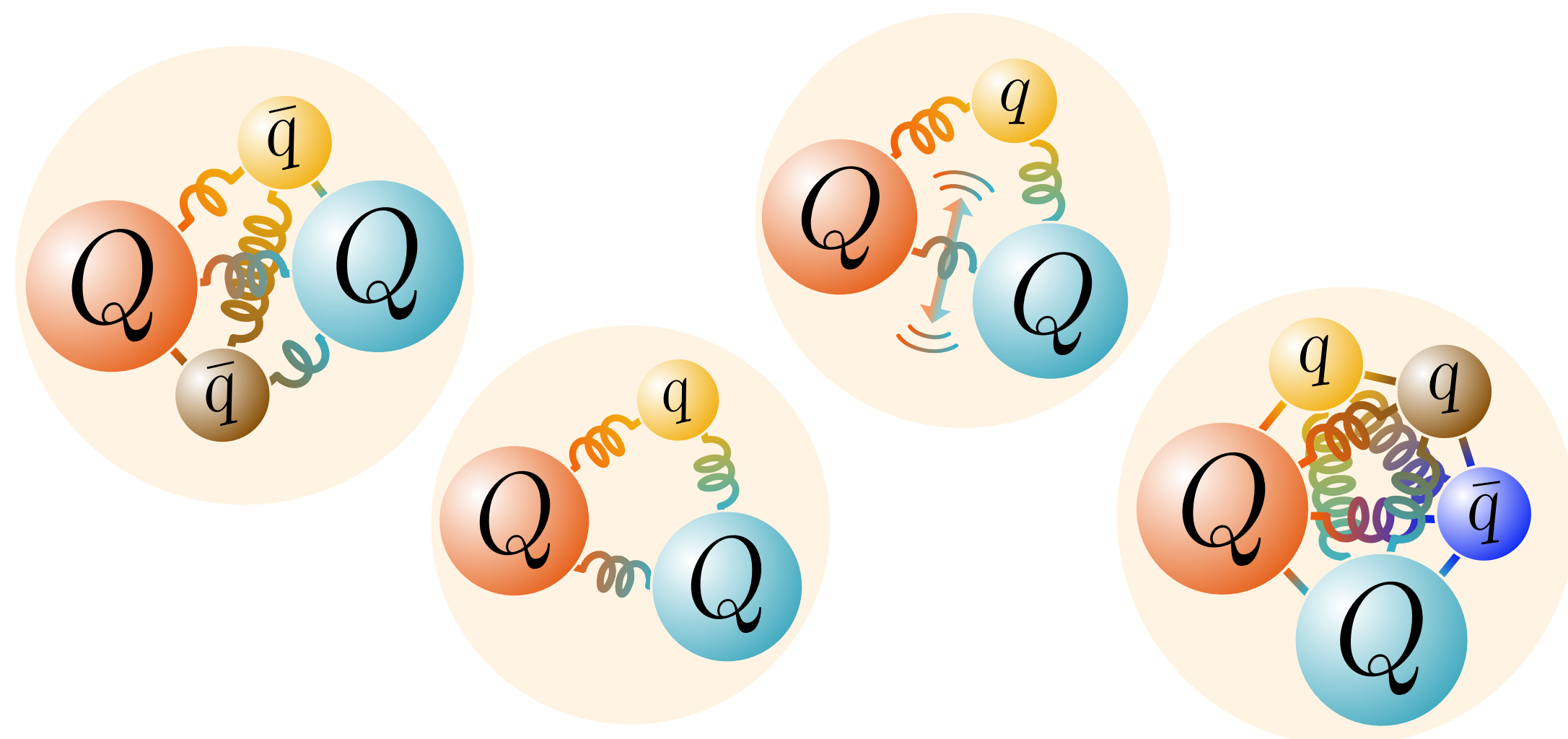


What are heavy hybrids?

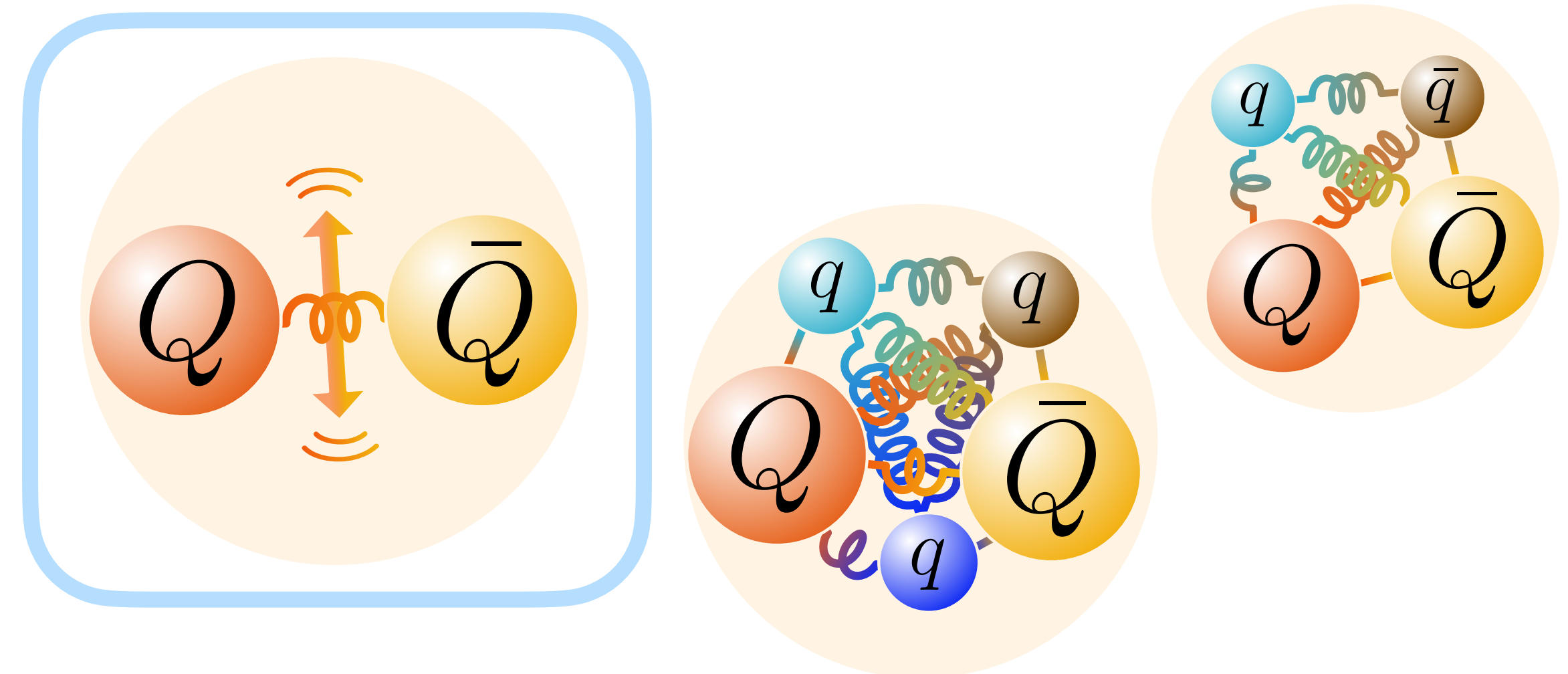
Exotic heavy hadrons

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What are heavy hybrids?

Scales in heavy hadrons

- Heavy quarks: $Q = c, b \rightarrow m_Q \gg \Lambda_{QCD}$

In the hadron rest frame the heavy quark moves slowly compared to the LDOF

- We have a **non-relativistic system** \rightarrow multiscale problem

$$m_Q \gg p \gtrsim \Lambda_{QCD} \gg E$$

- We use an **Effective Field Theory** \rightarrow

Born-Oppenheimer EFT

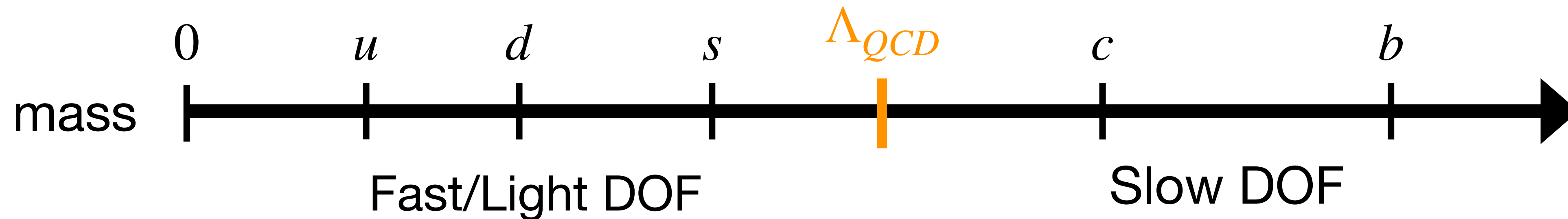
[Berwein, Braaten, Brambilla, Krein, Mohapatra, Oncala, Scirpa, Soto, Tarrús Castellà, Vairo,...]

The B-O Approx for QCD

Philosophy of the B-O Approximation

Slide: [Bruschini]

$$m_Q \gg p \gtrsim \Lambda_{QCD} \gg E$$



Two-step procedure

- Calculate the QCD potential models using data from lattice QCD
- Solve Schrödinger equation for $Q\bar{Q}$

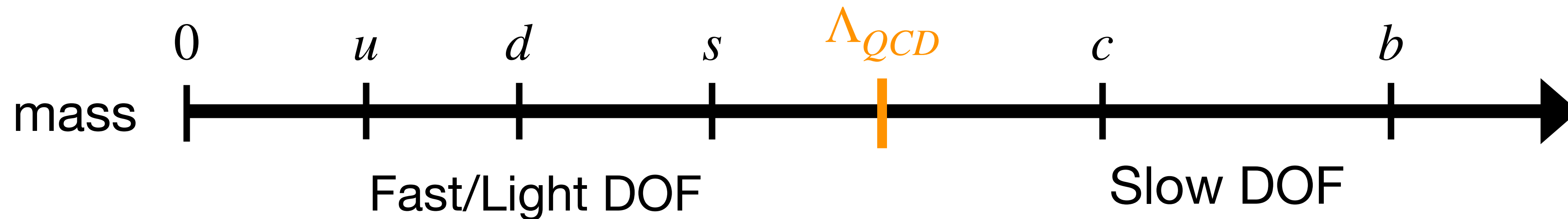
[Juge, Kuti, Morningstar]

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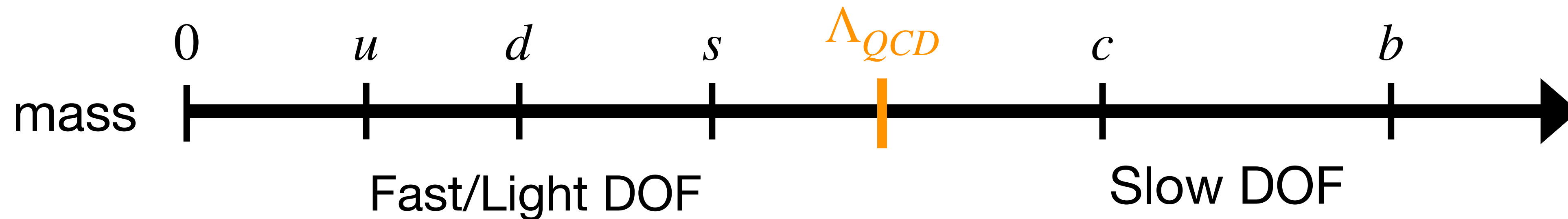
$$\rightarrow V^{(0)}(\vec{r}), V^{(1)}(\vec{r}), \dots$$

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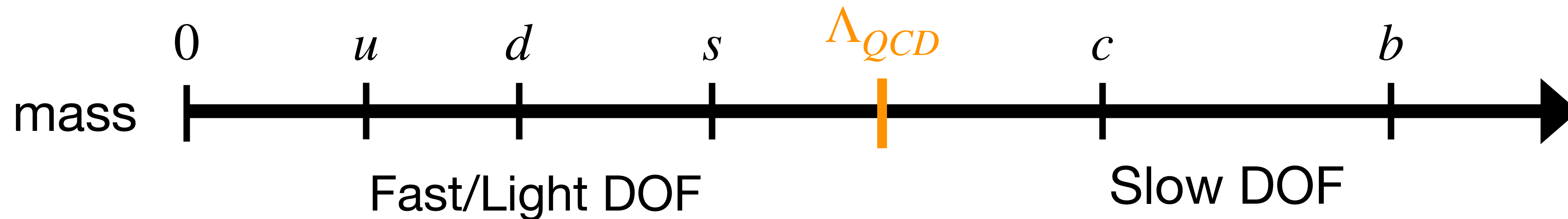
$$\rightarrow \left(-\frac{\Delta}{m_Q} + V^{(0)}(\vec{r}) + \frac{V^{(1)}(\vec{r})}{m_Q} + \dots \right) \Psi(\vec{r}) = E\Psi(\vec{r})$$

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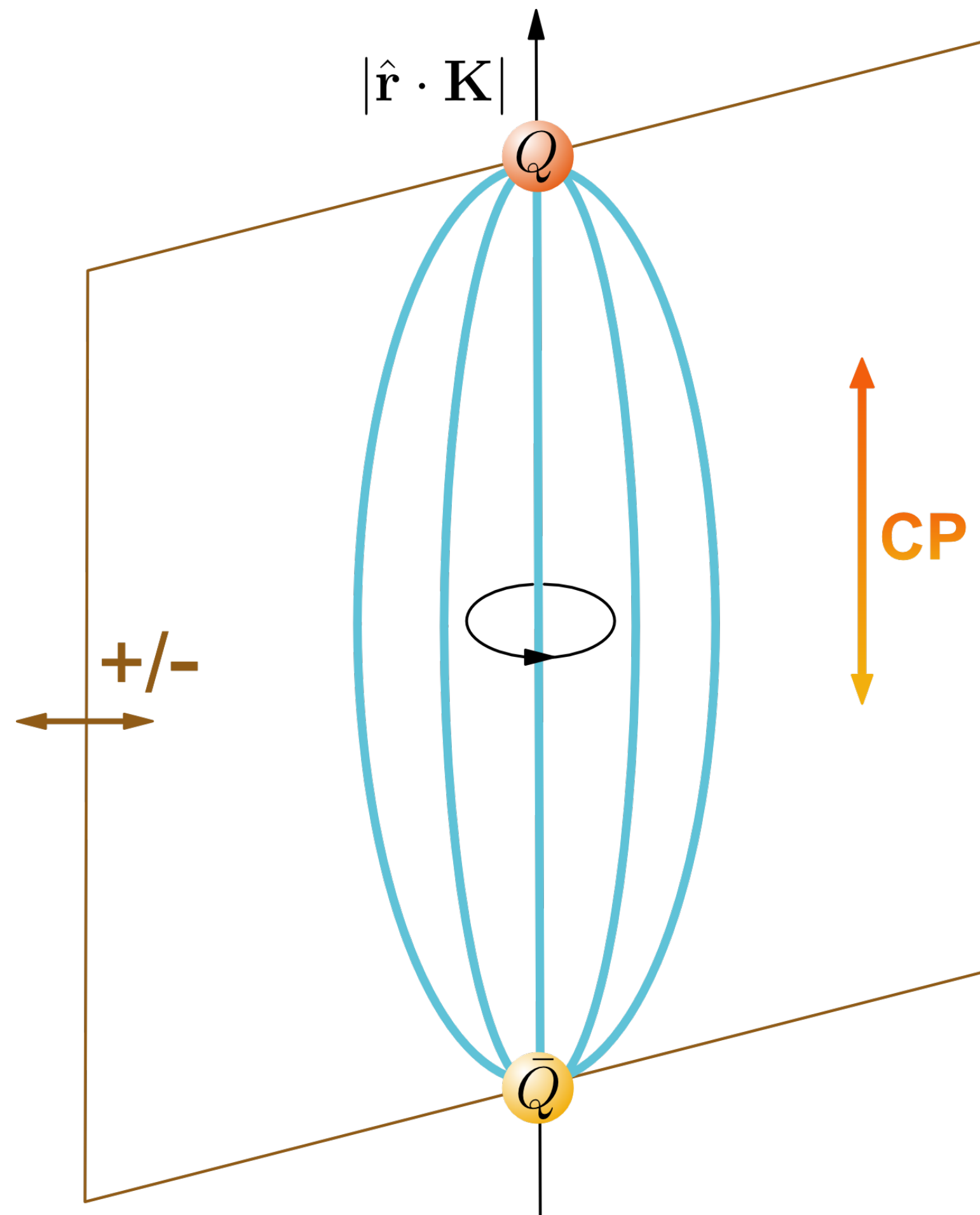
BOEFT

- At LO: $m_Q \rightarrow \infty$ (static limit)
- At NLO: $\mathcal{O}(1/m_Q)$ expansion

[Berwein, Brambilla, Krein, Mohapatra, Oncala, Scirpa, Soto, Tarrús Castellà, Vairo,...]

The B-O Approx for QCD

B-O Symmetries and Quantum numbers



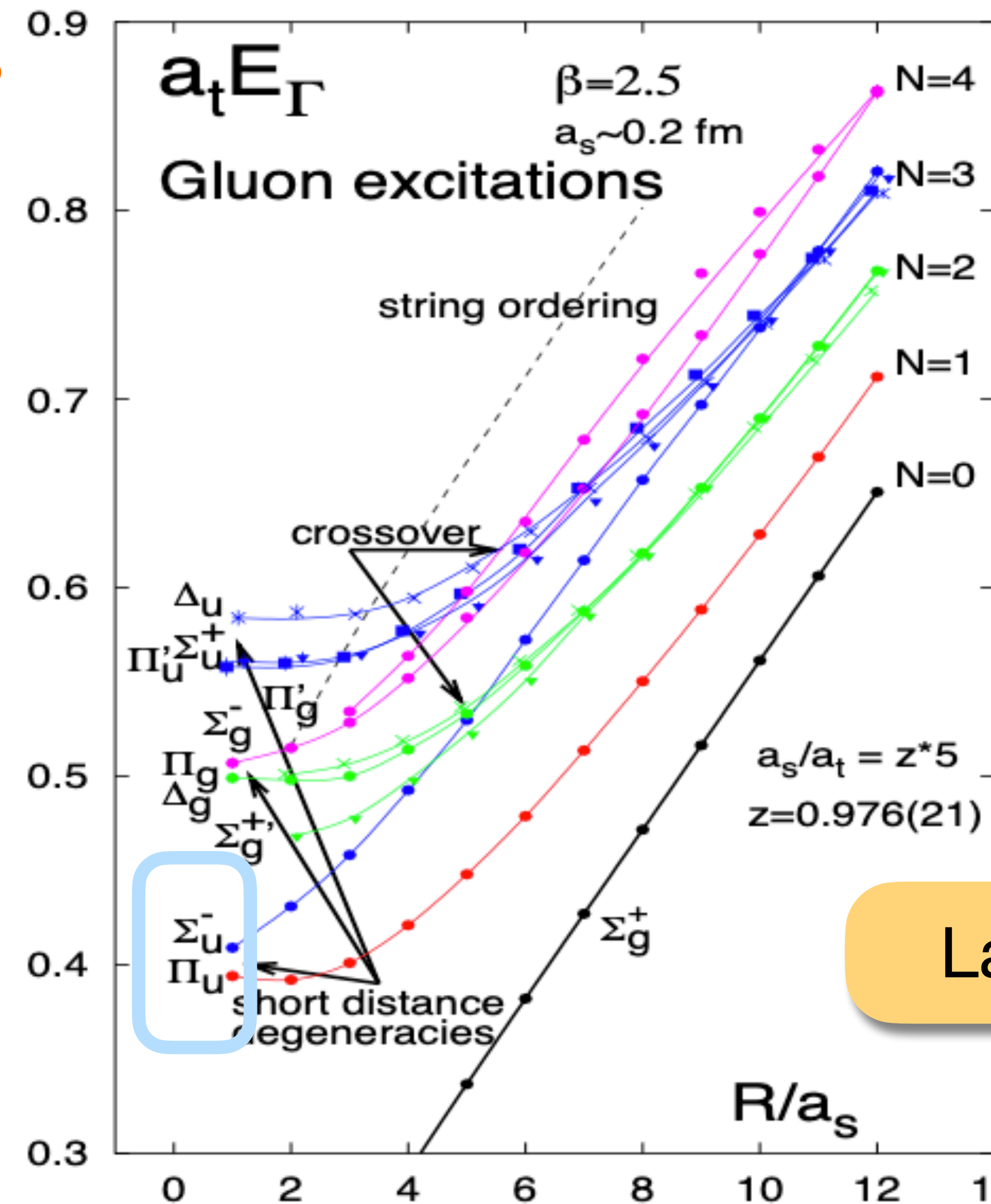
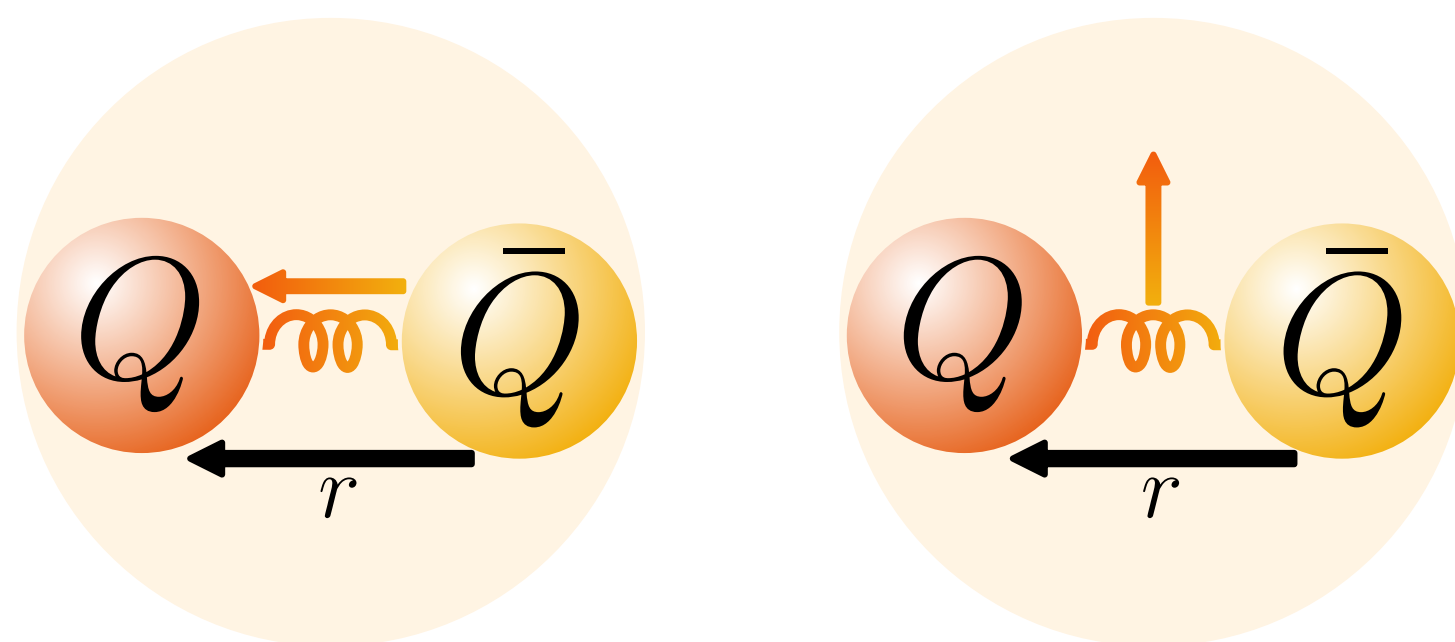
- Irreps of the $D_{\infty h}$ group $\rightarrow \Lambda_{\eta}^{\sigma}$
- Total angular momentum \mathbf{K} of LDOF with eigenvalue κ
- $|\mathbf{K} \cdot \hat{\mathbf{r}}|$ with eigenvalue λ and $\Lambda \equiv |\lambda|$
 $\Lambda = 0, 1, \dots = \Sigma, \Pi, \dots$
- CP eigenvalue $\eta = +1, -1 = g, u$
- Sign under reflection $\sigma = +, -$

The B-O Approx for QCD

B-O Hybrid Quantum numbers

- Low-lying potentials

LDOF: κ^{PC}	Λ_{η}^{σ}
1^{+-}	Σ_u^{-}, Π_u



Lattice results

Figure: [Juge, Kuti, Morningstar arXiv:hep-lat/0207004]

The B-O Approx for QCD

B-O Hybrid Potentials at $\mathcal{O}(1/m_Q)^0$

- **Obtain the potentials** with correct behaviour at short and long distances [Onocala, Soto. arXiv:1702.03900]

$$V_{\Sigma_u^-}(r) = \frac{\sigma_p}{r} + \kappa_s r + E_s^{\bar{Q}Q}$$

$$V_{\Pi_u}(r) = \frac{\sigma_s}{r} \left(\frac{1 + b_1 r + b_2 r^2}{1 + a_1 r + a_2 r^2} \right) + \kappa_p r + E_p^{\bar{Q}Q}$$

- **Fit the parameters** with lattice QCD results

Two-step procedure

- Calculate the QCD potential models using data from lattice QCD
- Solve Schrödinger equation for $Q\bar{Q}$

Heavy spin independent potentials

The B-O Approx for QCD

Hybrid spectrum with BOEFT at $\mathcal{O}(1/m_Q)^0$

- Lagrangian density:

$$\mathcal{L} = \text{tr} \left(H^{i\dagger} (\delta_{ij} i \partial_0 - h_{Hij}) H_j \right)$$

$$h_{Hij} = \left(-\frac{\nabla^2}{m_Q} + V_{\Sigma_u^-}(r) \right) \delta_{ij} \\ + (\delta_{ij} - \hat{r}_i \hat{r}_j) [V_{\Pi_u}(r) - V_{\Sigma_u^-}(r)]$$

Two-step procedure

- Calculate the QCD potential models using data from lattice QCD
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- **Solve** the eigenvalue problem to find the **degenerate** energy spectrum

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Two-step procedure

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$$\left(-\frac{\Delta}{m_Q} + V^{(0)}(\vec{r}) \right) \Psi(\vec{r}) = E \Psi(\vec{r})$$

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The B-O Approx for QCD

Hybrid spectrum with BOEFT at $\mathcal{O}(1/m_Q)^0$

- Spin average spectrum for charmonium multiplets

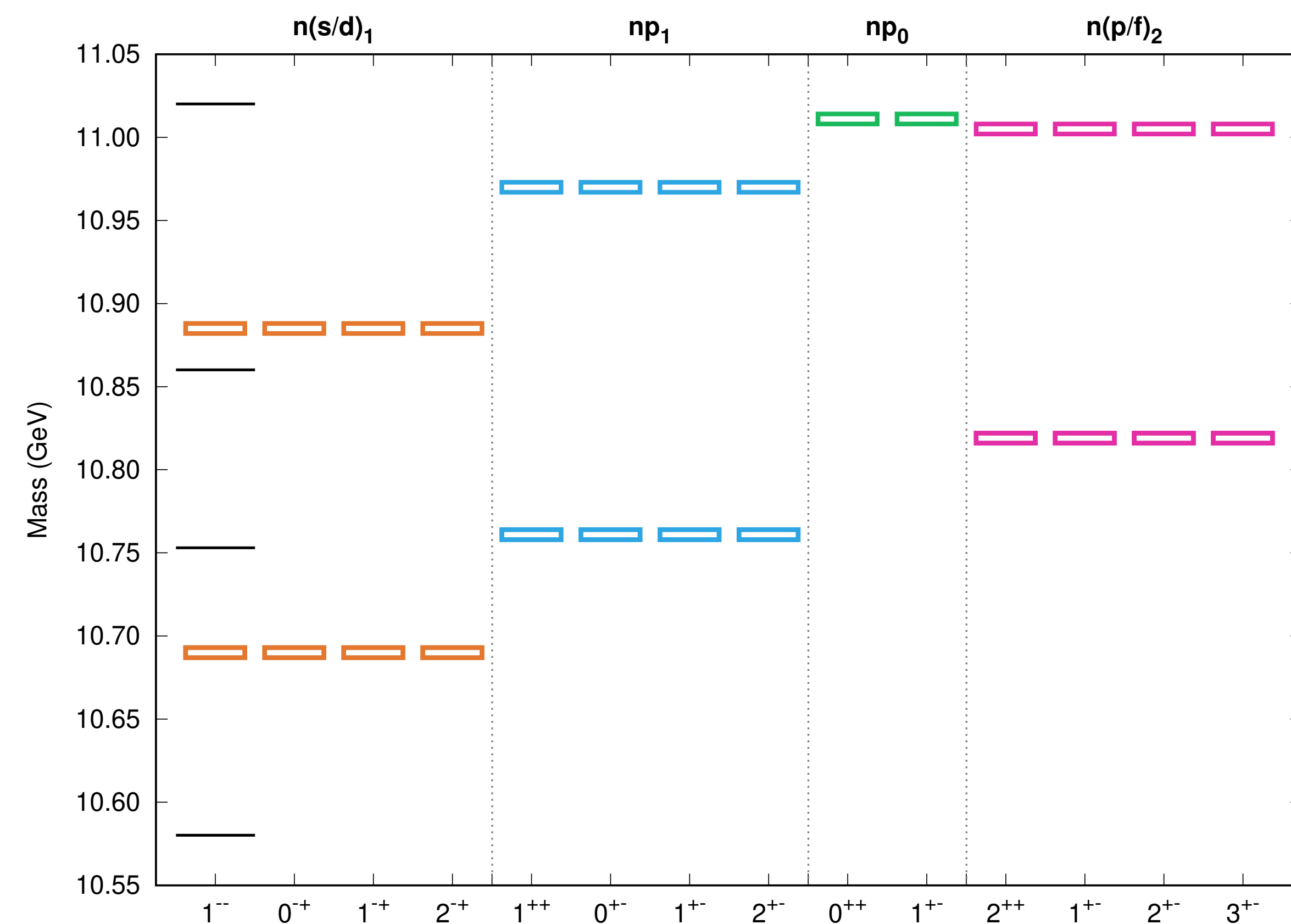
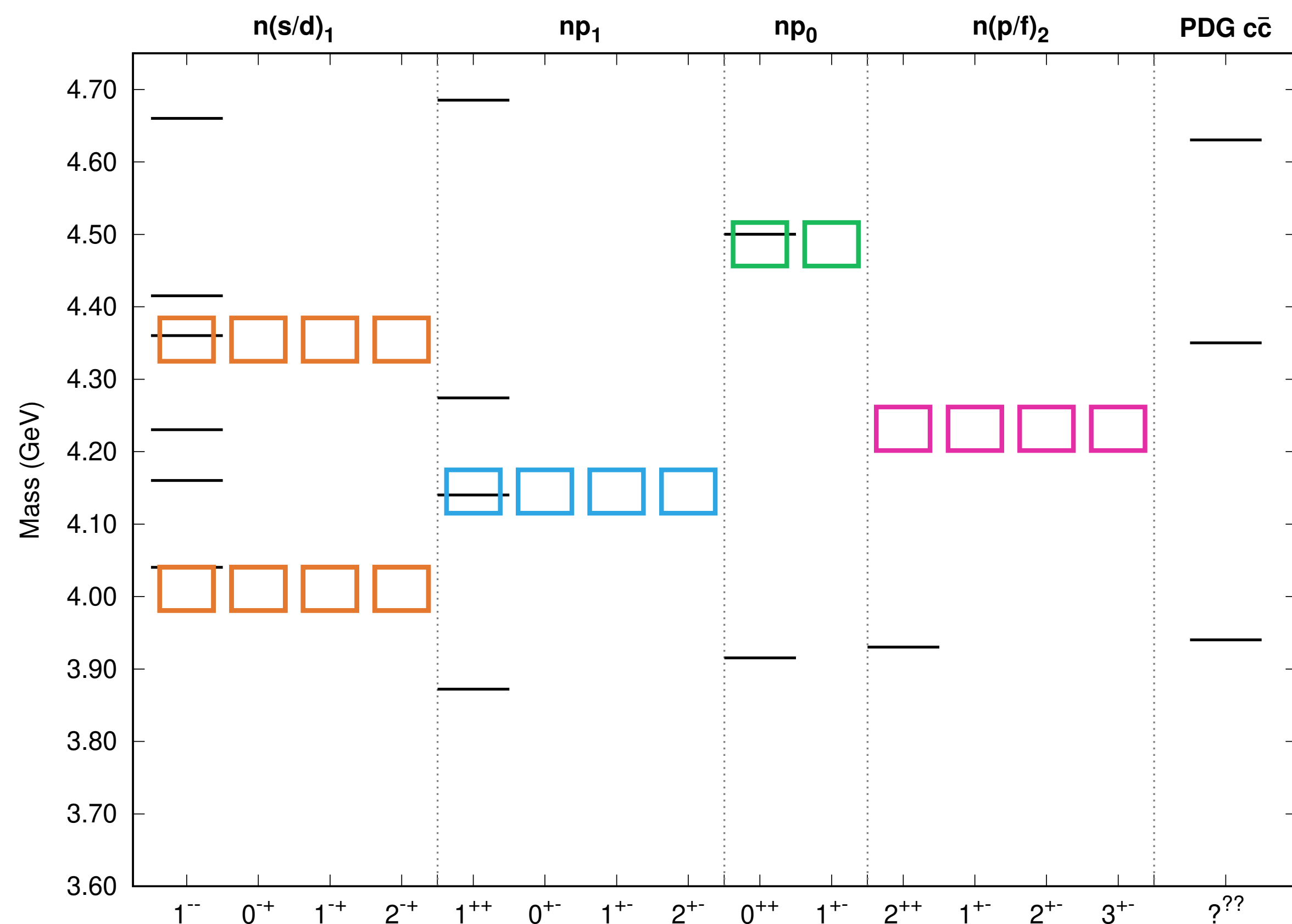
$\kappa^{PC} = 1^{+-}$	NL_J	$M_{c\bar{c}g}$	$\mathcal{J}^{PC}(S_{Q\bar{Q}} = 0)$	$\mathcal{J}^{PC}(S_{Q\bar{Q}} = 1)$	Λ_η^σ
H_1	$1(s/d)_1$	4011	1^{--}	$(0,1,2)^{-+}$	Σ_u^-, Π_u
H_2	$1p_1$	4145	1^{++}	$(0,1,2)^{+-}$	Π_u
H_3	$1p_0$	4486	0^{++}	1^{+-}	Σ_u^-
H_4	$1(p/f)_2$	4231	2^{++}	$(1,2,3)^{+-}$	Σ_u^-, Π_u

[Oncales, Soto.
arXiv:1702.03900]

- Spin effects enter at $\mathcal{O}(1/m_Q)$ in hybrids

The B-O Approx for QCD

Hybrid spectrum with BOEFT at $\mathcal{O}(1/m_Q)^0$



Hyperfine splitting for hybrids

Spin dependent potentials $\mathcal{O}(1/m_Q)$

$$\begin{aligned} \left[V_{\kappa P}^{(1)}(\mathbf{r}) \right]^{n'm';nm} &= -2V_{hf}(r) \left(\delta^{n'm} \delta^{nm'} - \delta^{n'm'} \delta^{nm} \right) + 2V_{hf2}(r) \left(\hat{r}^i \hat{r}^j - \frac{\delta^{ij}}{3} \right) \times \\ &\times \left(\delta^{jm} \delta^{n'i} \delta^{nm'} + \delta^{ni} \delta^{jm'} \delta^{n'm} - \delta^{jm'} \delta^{n'i} \delta^{nm} - \delta^{ni} \delta^{jm} \delta^{n'm'} \right) \end{aligned}$$

Two-step procedure

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We need to find the **hyperfine potentials**

$$V_{hf}, V_{hf2}$$

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Hyperfine splitting for hybrids

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- **Short distance behaviour** \rightarrow **pNRQCD**

$$\frac{V_{hf}(r)}{m_Q} = \frac{c_F(m_Q)k_A}{m_Q} + \mathcal{O}(r^2) \quad , \quad \frac{V_{hf2}(r)}{m_Q} = \frac{c_F(m_Q)k_B}{m_Q} r^2 + \mathcal{O}(r^4)$$

- **Long distance behaviour** \rightarrow **EST**

$$V_{hf2}(r) = -\frac{1}{2} (V_{1+11}^{sa}(r) + V_{1+10}^{sb}(r)) \quad \left\{ \begin{array}{l} \frac{V_{1+11}^{sa}}{m_Q} = -\frac{2c_F\pi^2 g\Lambda'''}{m_Q\kappa r^3} \equiv V_{ld}^{sa}(r) \\ \frac{V_{1+10}^{sb}}{m_Q} = \mp \frac{2c_F\pi^2 g\Lambda'}{m_Q\sqrt{\pi\kappa}r^2} \equiv V_{ld}^{sb}(r) \end{array} \right.$$

$$V_{hf}(r) = \frac{1}{6} V_{1+11}^{sa}(r) - \frac{1}{3} V_{1+10}^{sb}(r)$$

[Soto, Tarrús. arXiv:2005.00552]

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Hyperfine splitting for hybrids

Spin dependent potentials $\mathcal{O}(1/m_Q)$

Lattice results of the spectrum of $c\bar{c}g$

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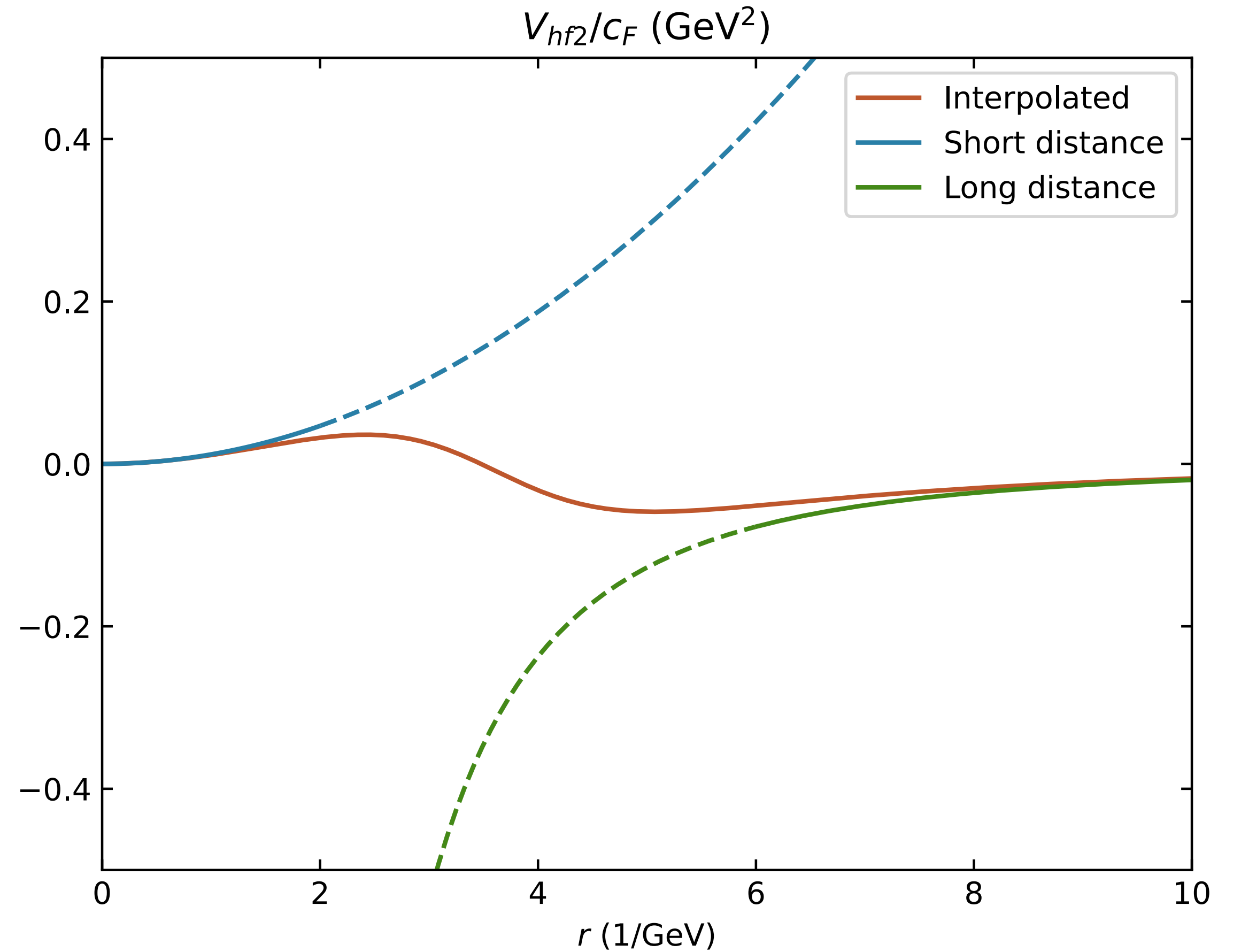
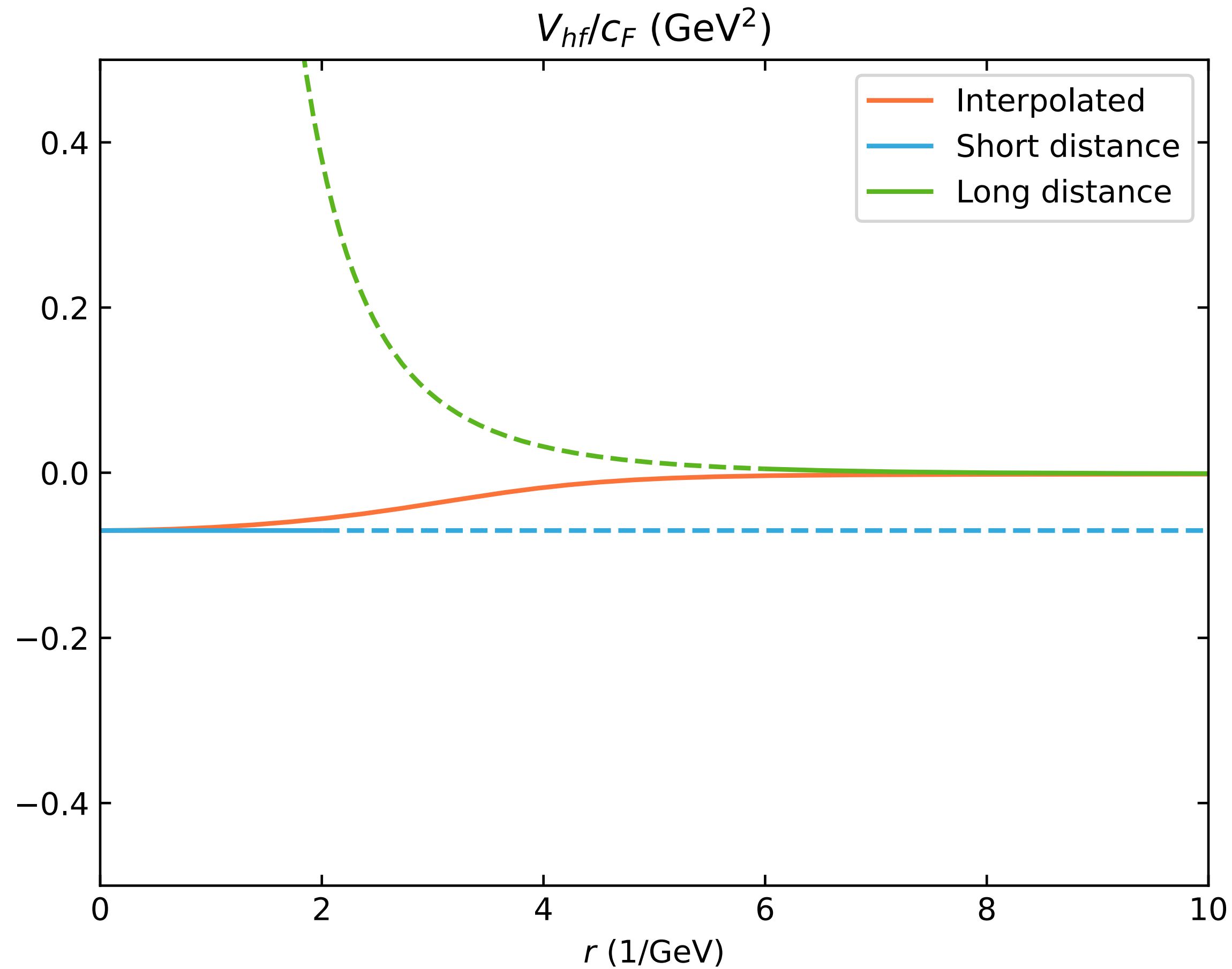
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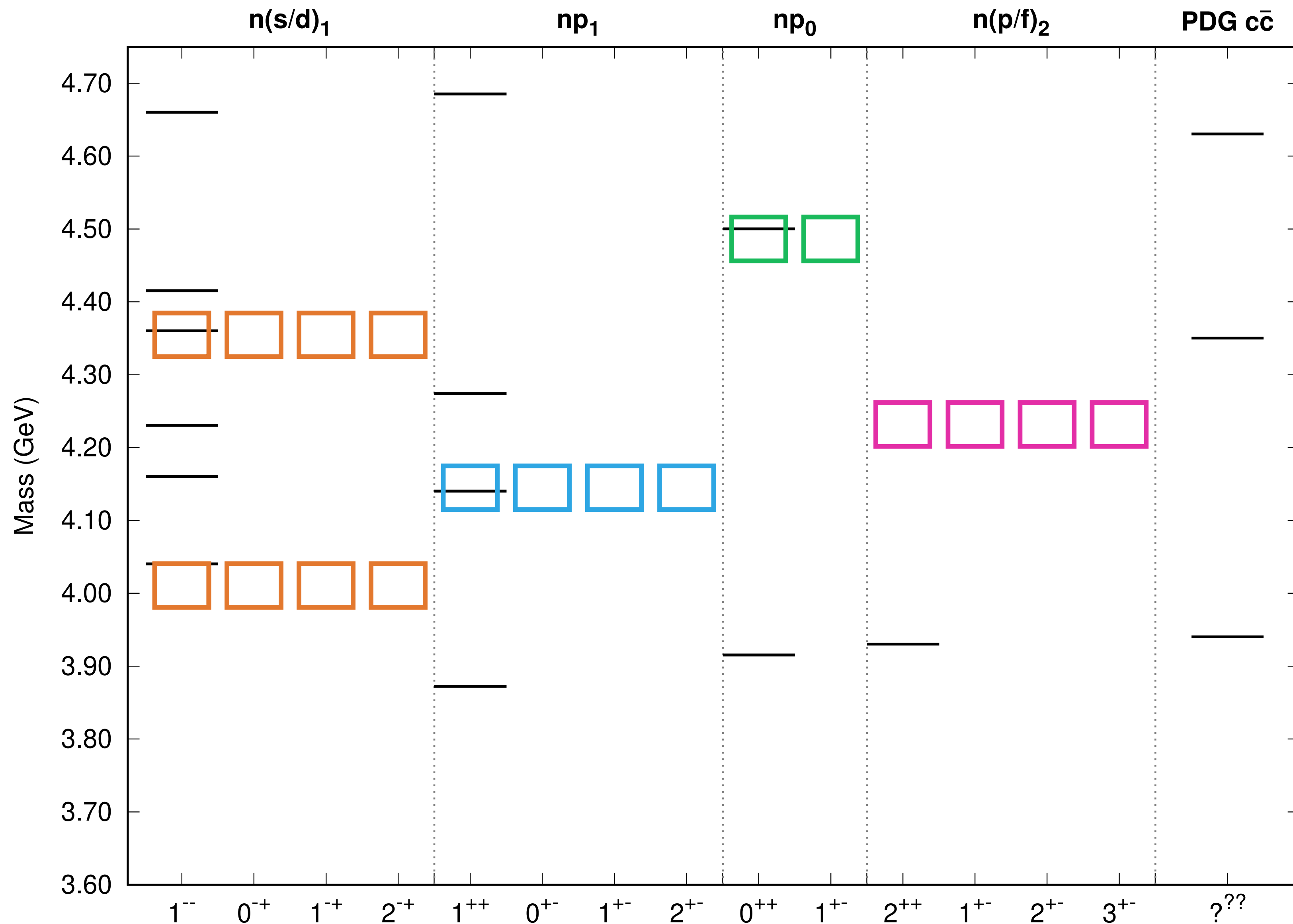
Hyperfine splitting for hybrids

Hybrid potential interpolation



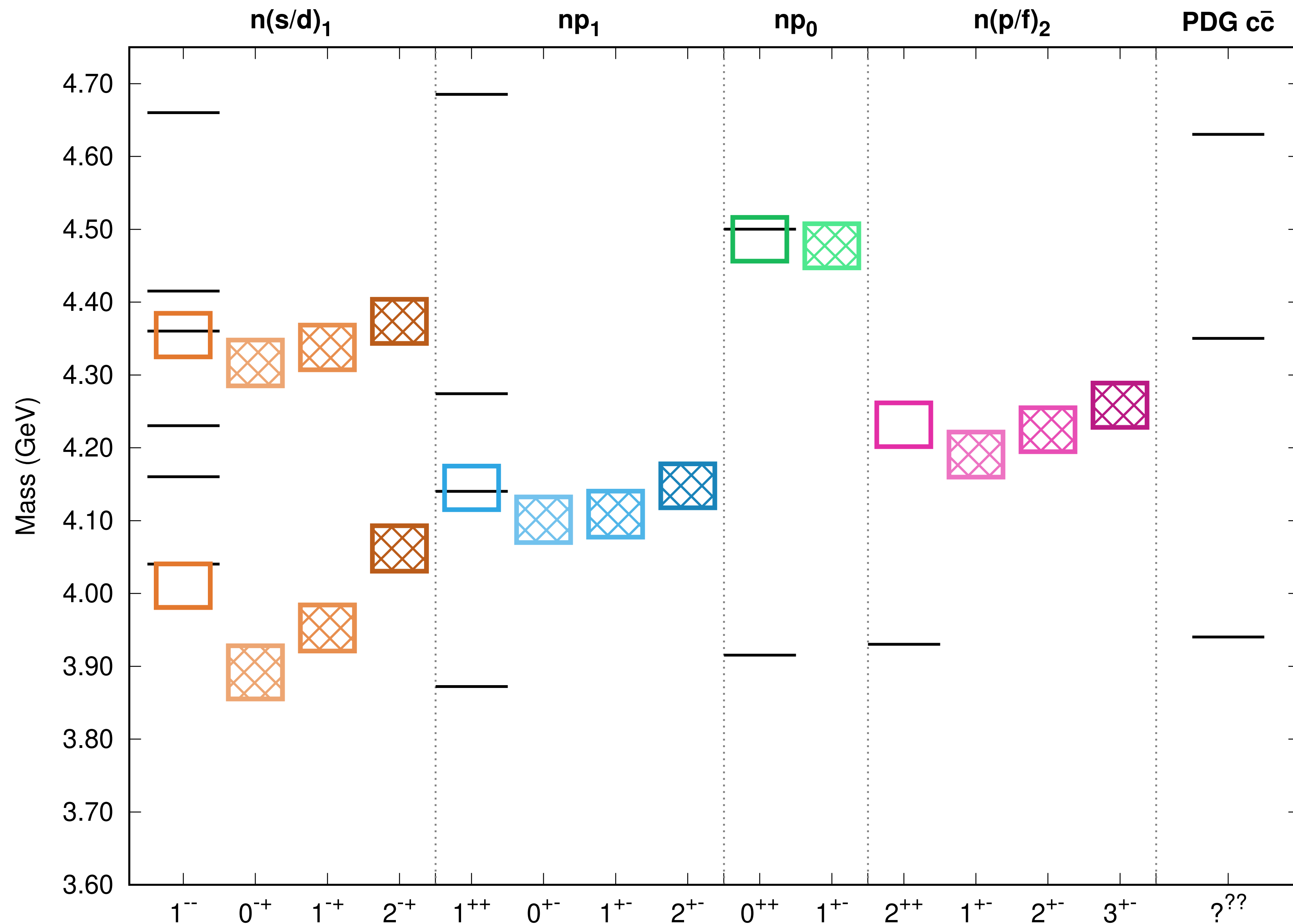
Results

Hyperfine splitting with
 $m_c = 1.477\text{GeV}$



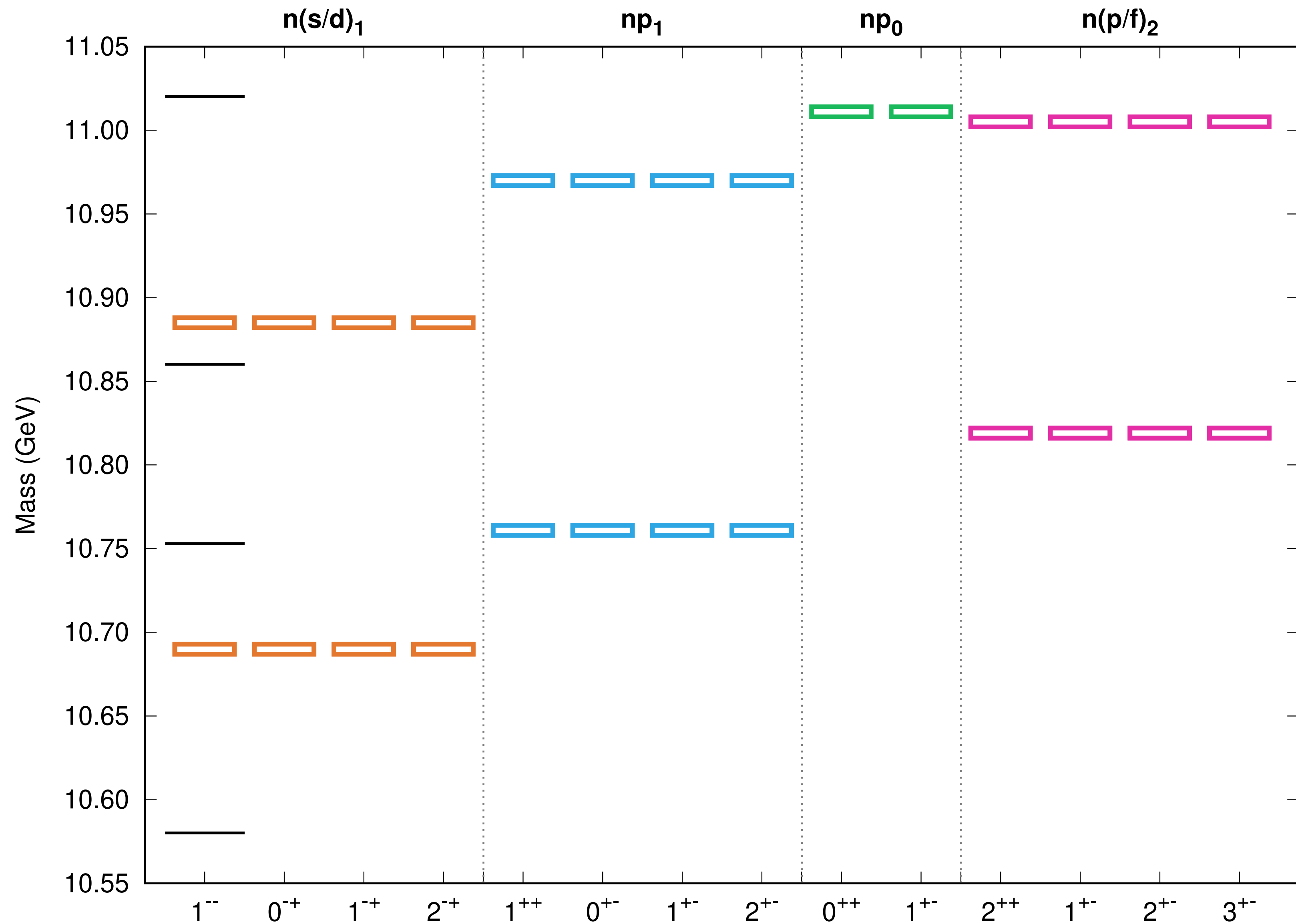
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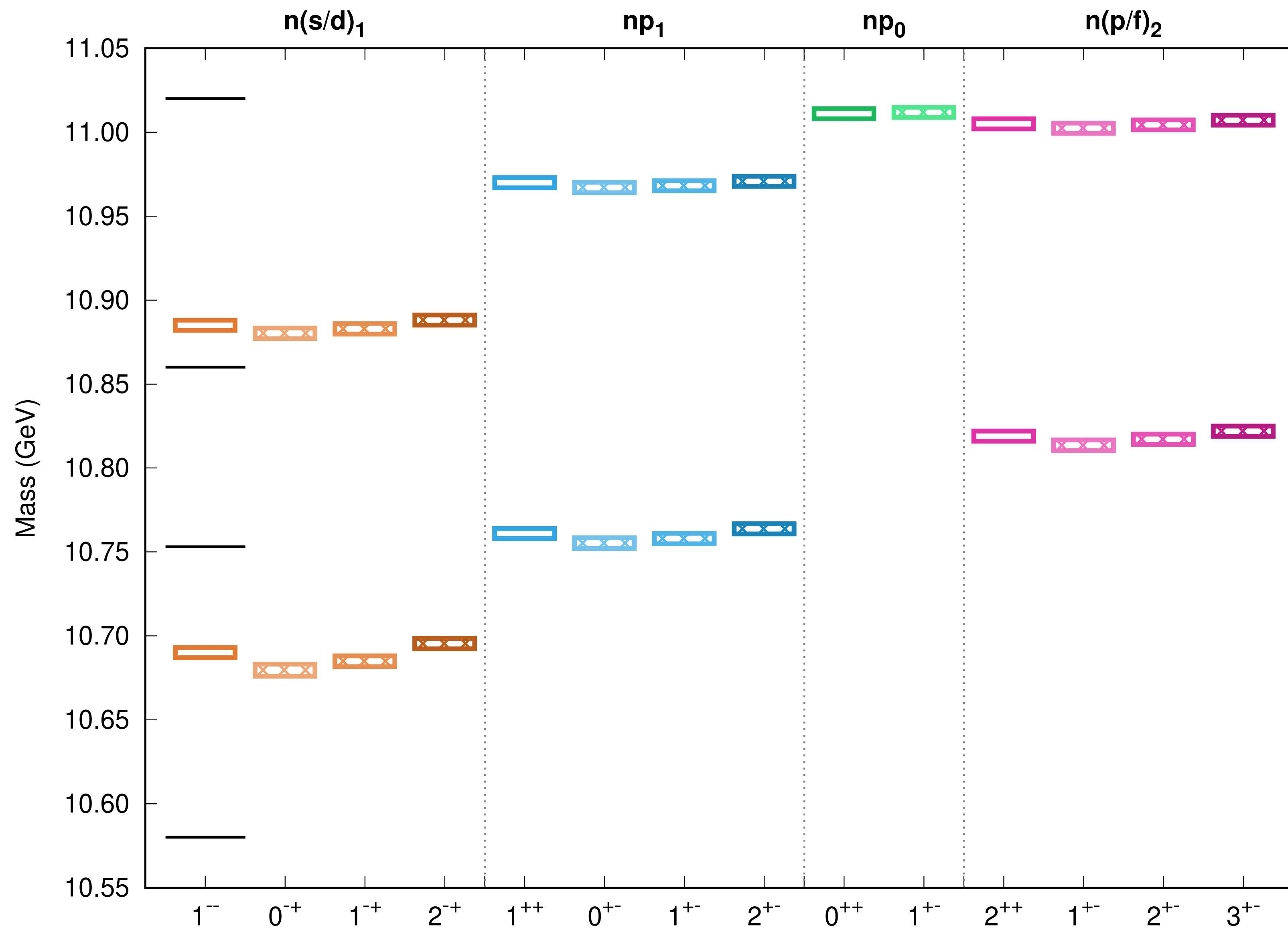
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Hyperfine splitting with
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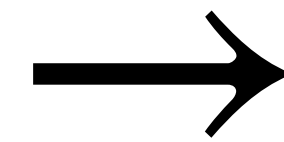


Results

Error bars discussion

- Input data error on k_A and k_B :

Assuming that the lattice data is **fully correlated**



Perform fits to the maximum, minimum and average lattice values

- **Errors in the spectrum:**

k_A and k_B errors

+

Higher orders in the $1/m_Q$ expansion

+

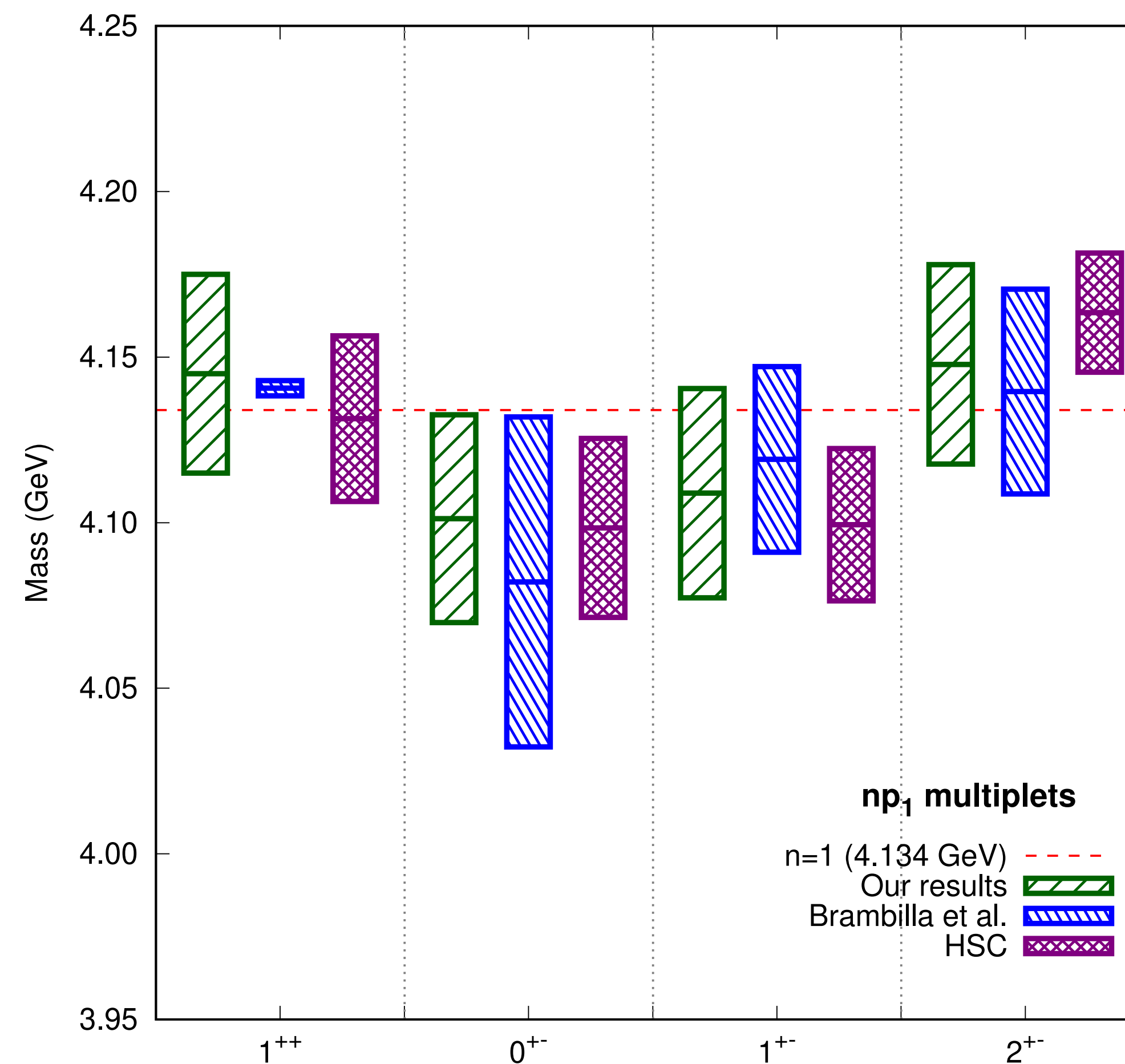
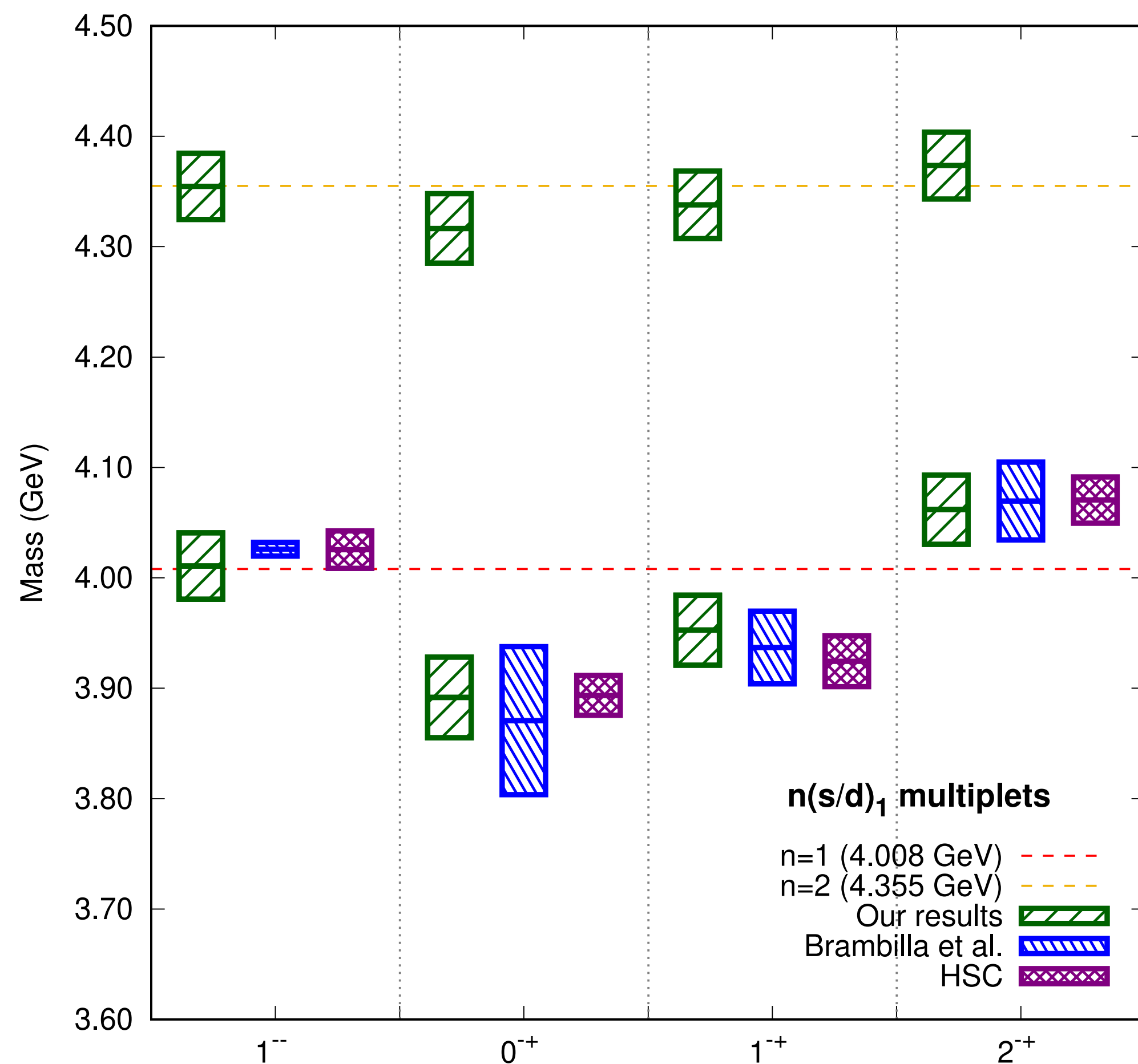
~~Uncertainty $g\Lambda'$, $g\Lambda'''$~~
 Error due to interpolation

30 MeV & 3 MeV

Comparison

Hyperfine splitting of H_1, H_2 with $m_c = 1.477\text{ GeV}$

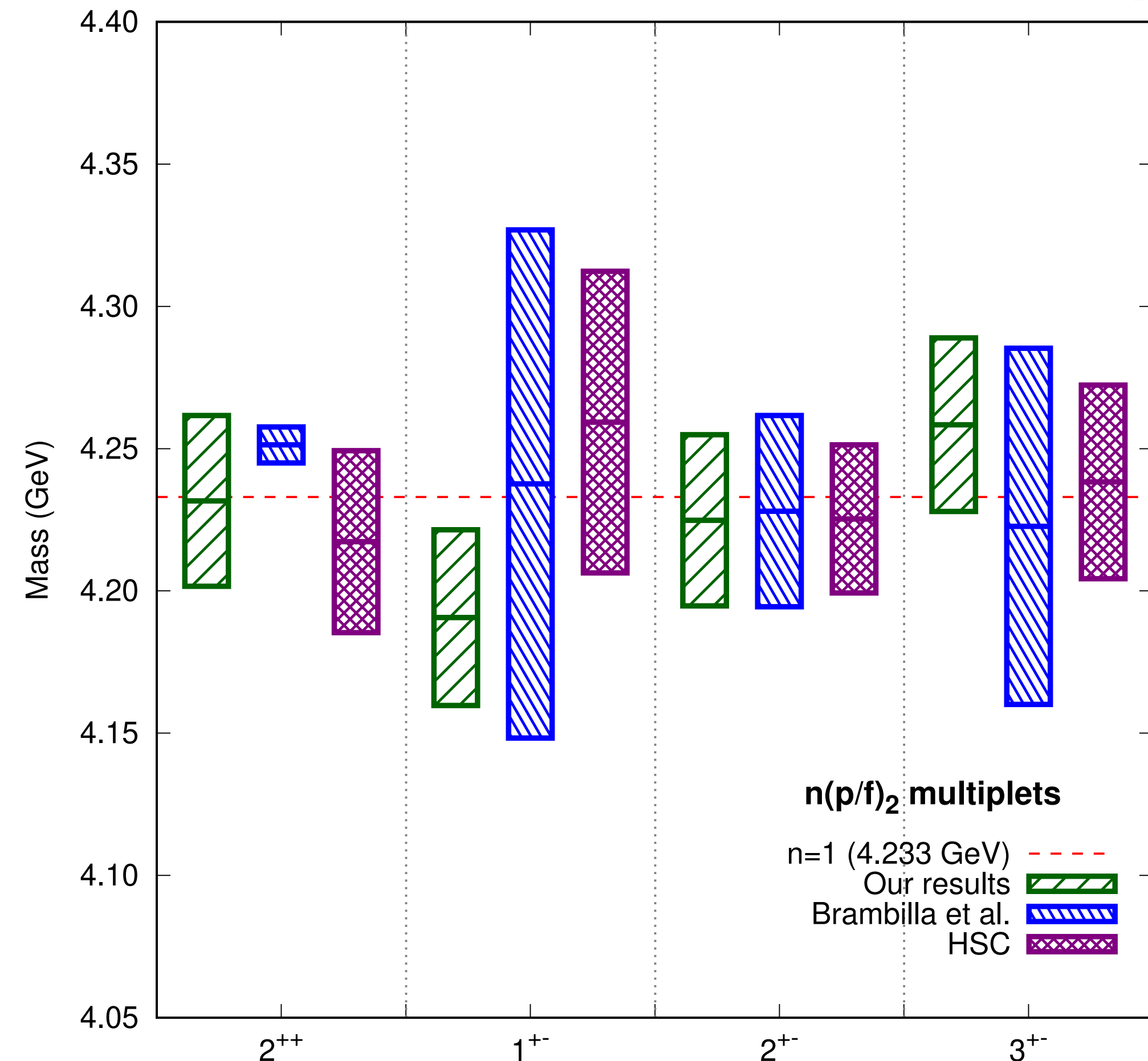
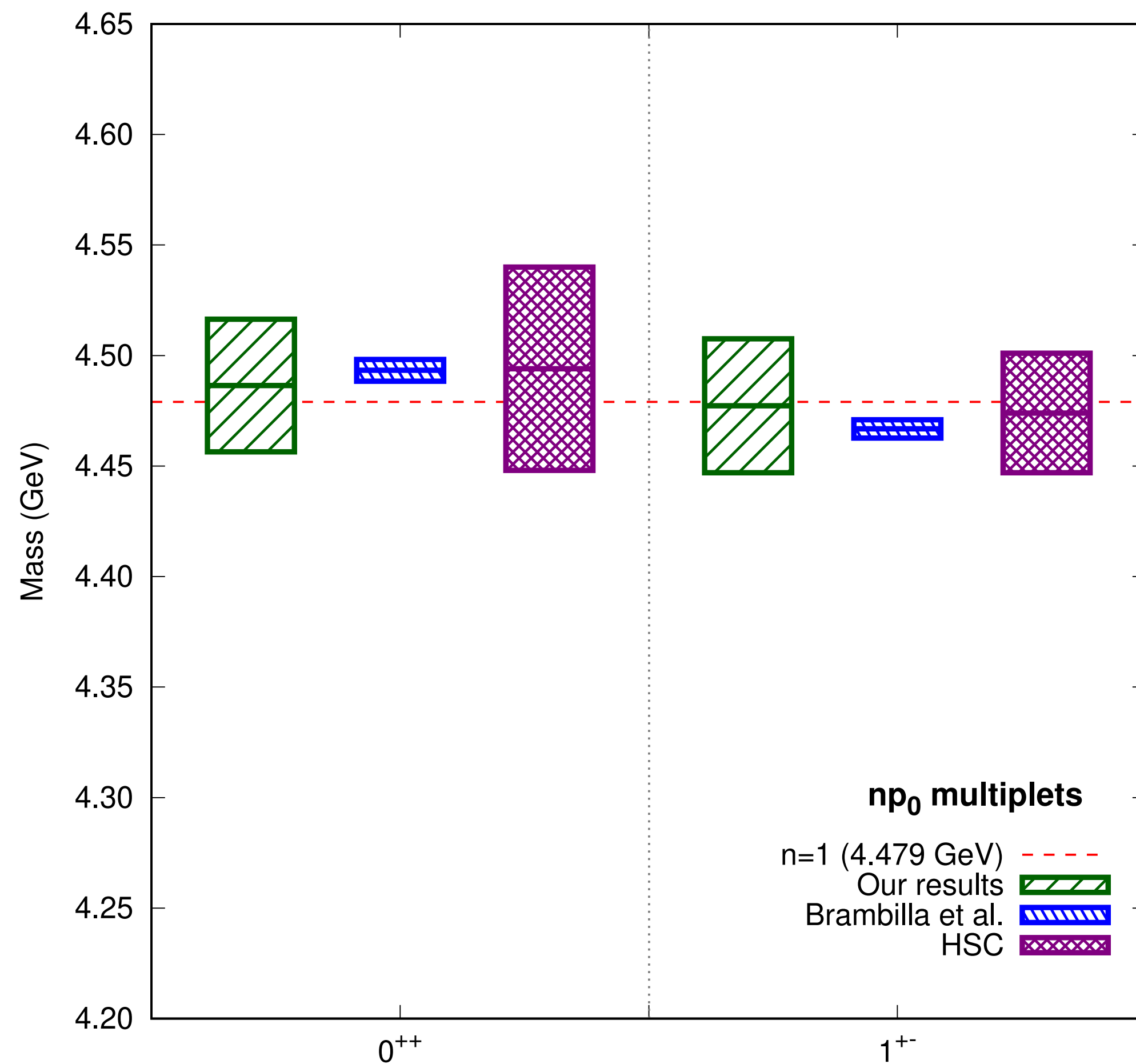
[Brambilla, Lai, Segovia, Tarrús, Vairo.
 arXiv:1805.07713 - 1908.11699]
 [Cheung, O'Hara, Moir, Peardon...
 arXiv:1610.01073]



Comparison

Hyperfine splitting of H_3, H_4 with $m_c = 1.477\text{ GeV}$

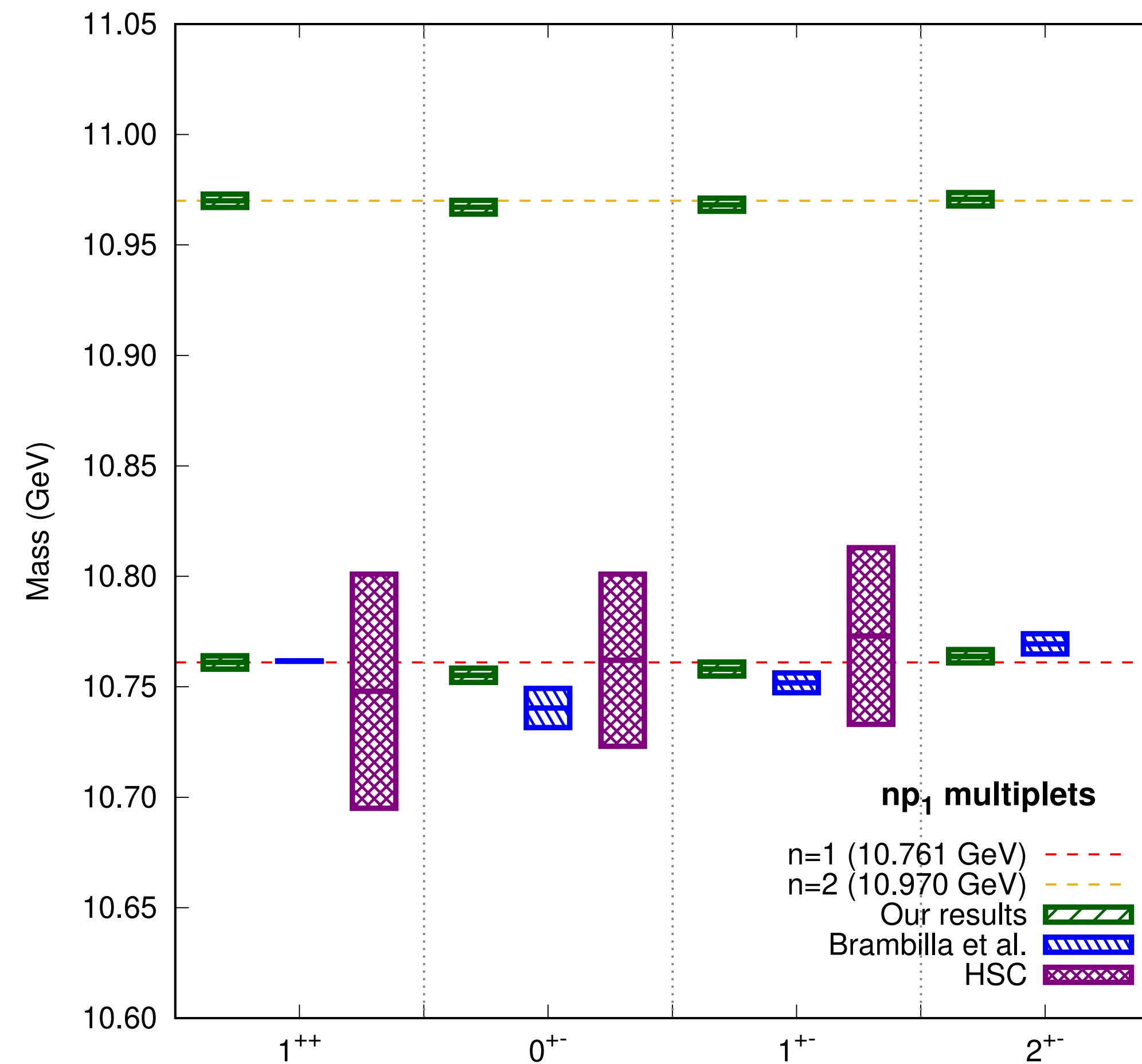
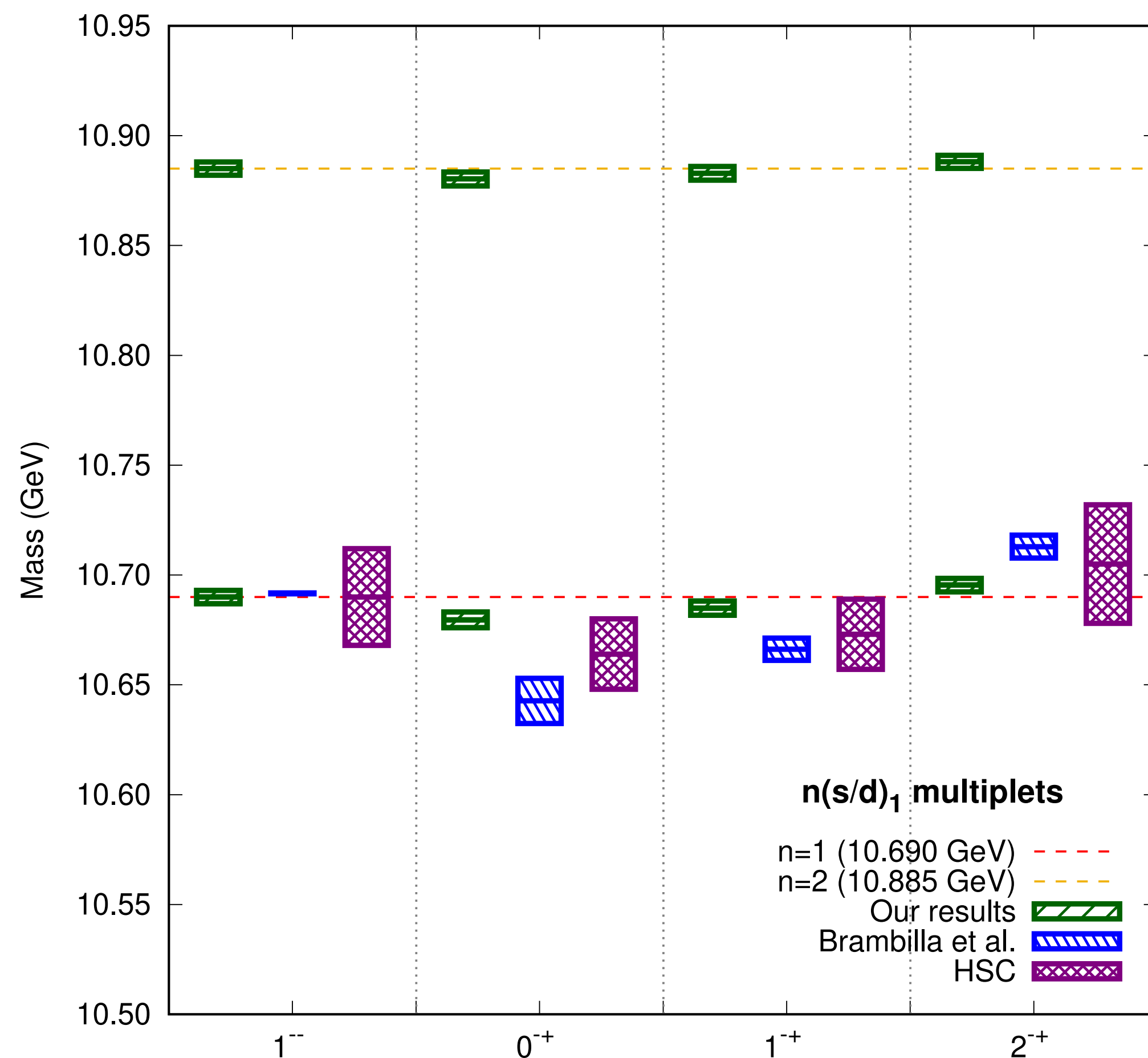
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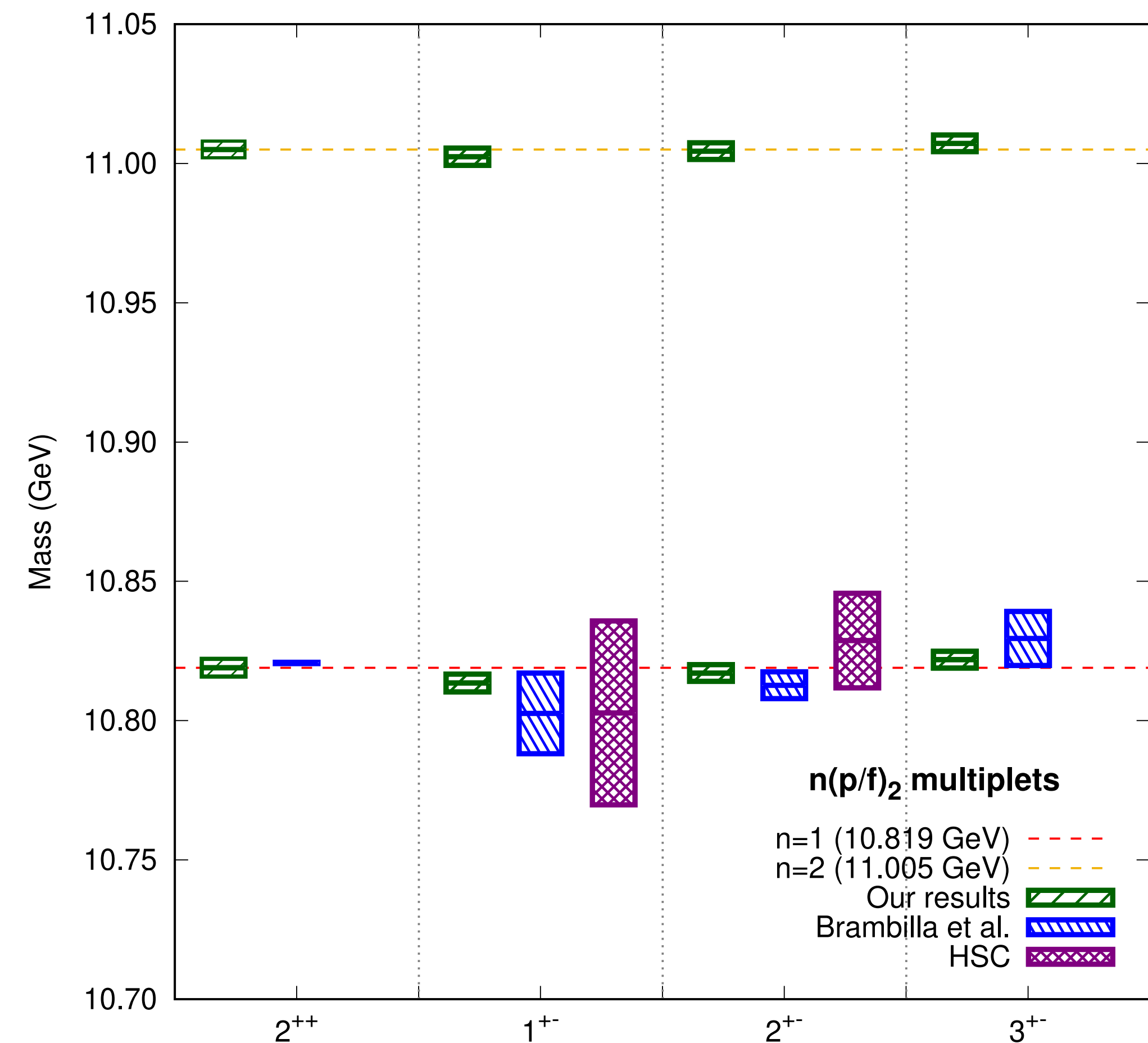
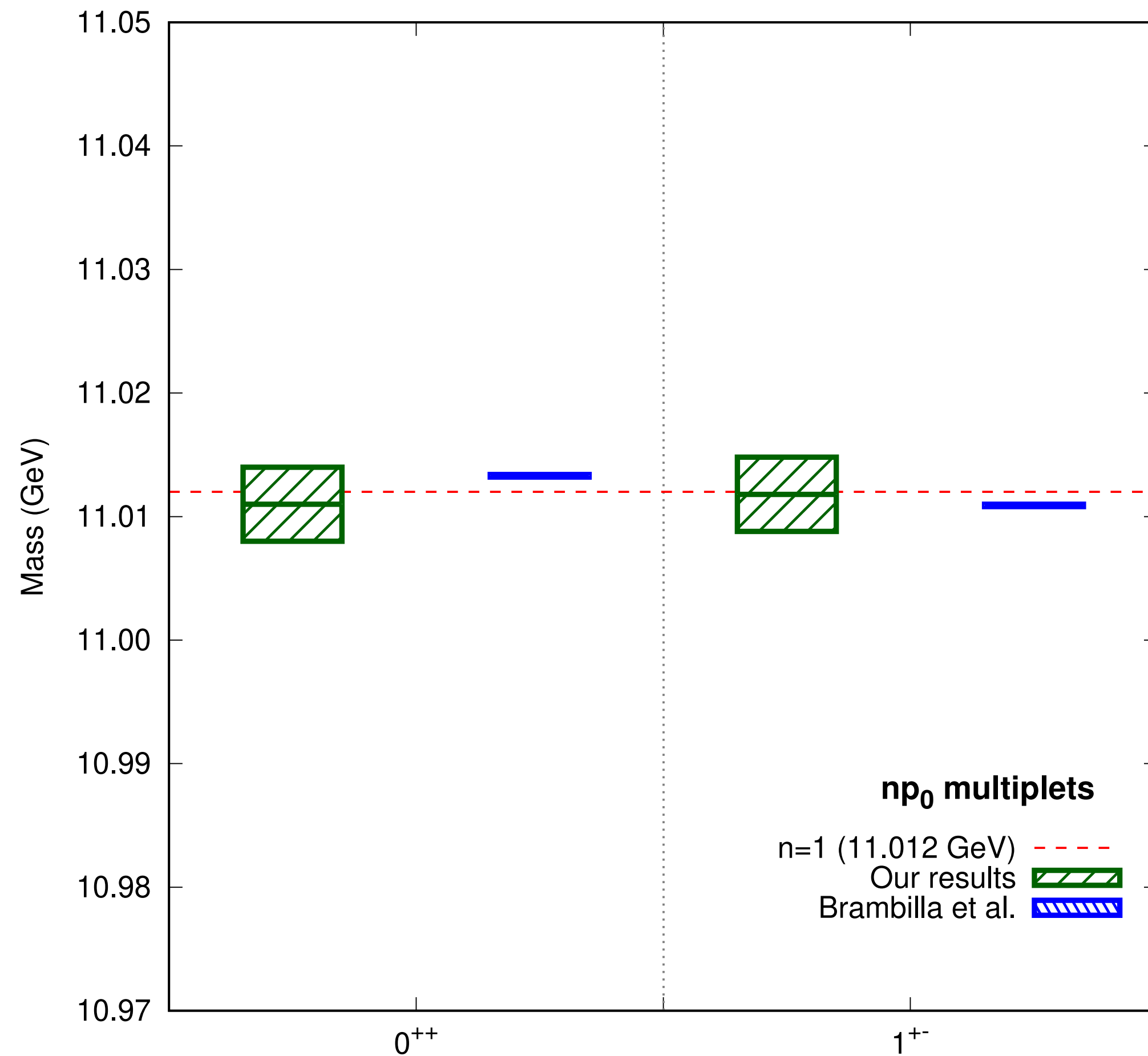
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 [Ray, Wilson. arXiv:2008.02656]



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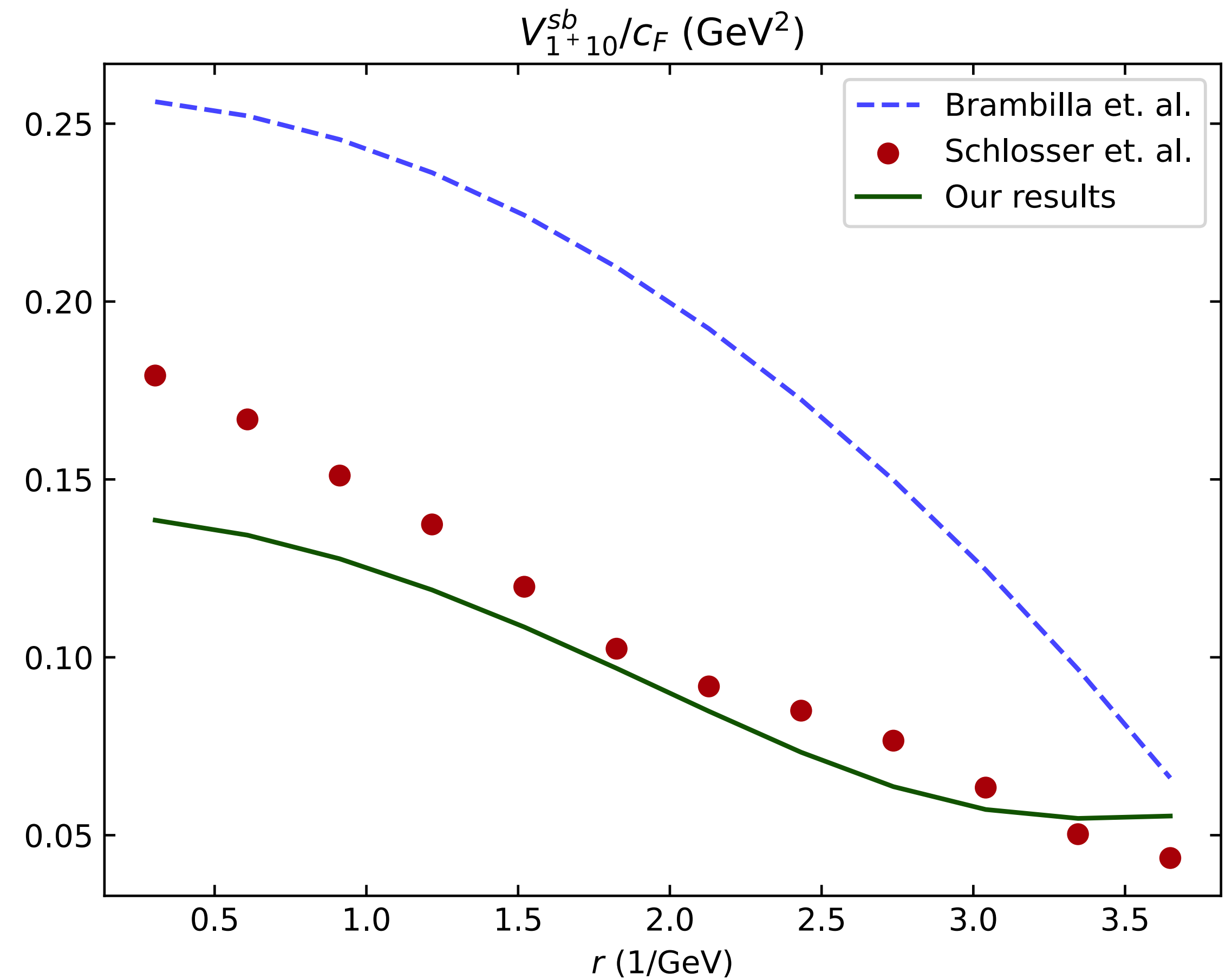
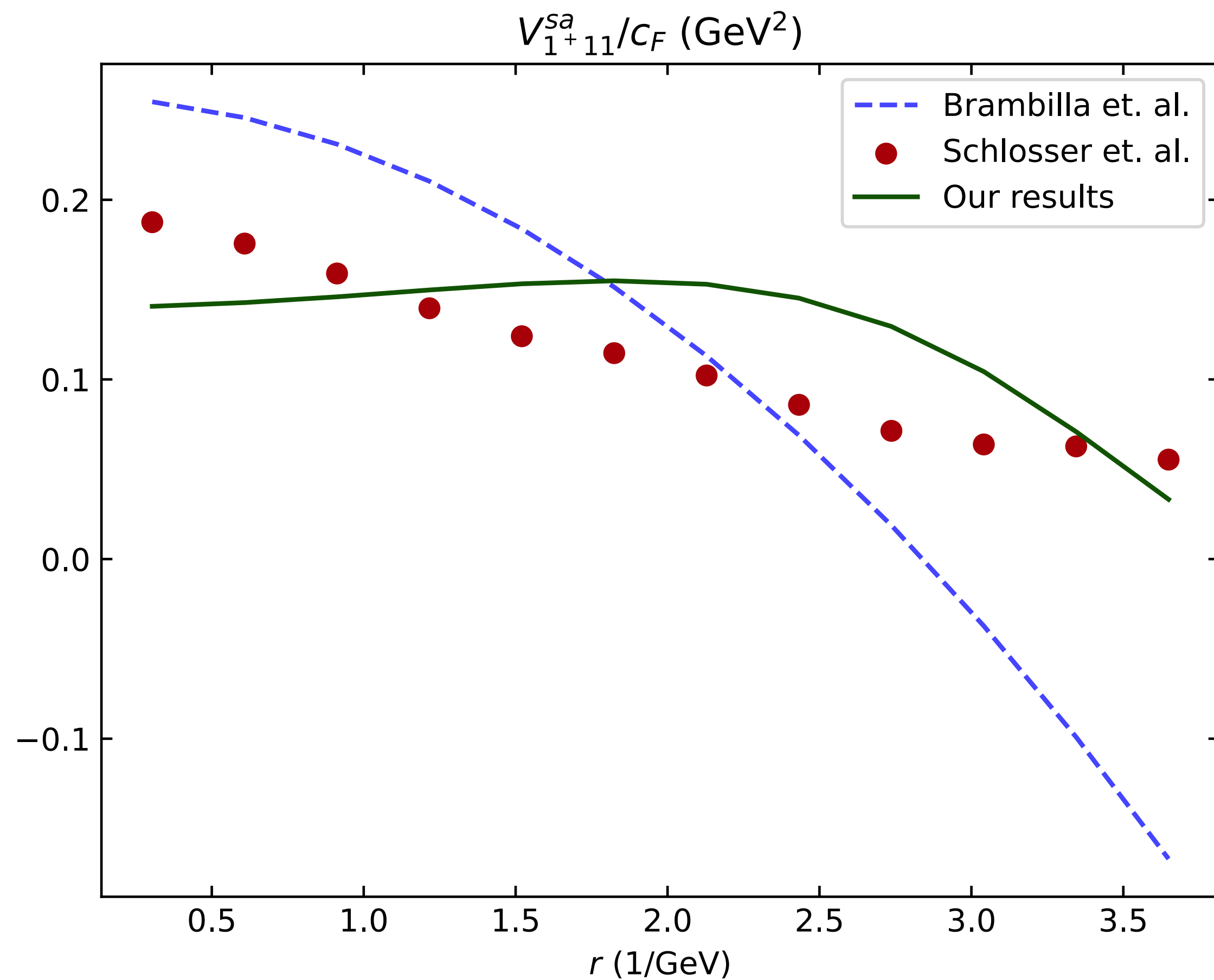


Comparison

Further analysis of the potentials

[Brambilla, Lai, Segovia, Tarrús.
arXiv:1908.11699]

[Schlosser, Wagner. arXiv:2501.08844]



Conclusions

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- If those potentials are not available → **Interpolation** between short and long distances seems to work really good

Thank you!

The B-O Approx for QCD

Coupled channel Schrödinger equation

- Equations for $J \neq 0$:

$$\left[-\frac{1}{m_Q} \frac{\partial^2}{\partial r^2} + \begin{pmatrix} \frac{(J+1)J}{m_Q r^2} & 0 \\ 0 & \frac{(J+1)(J+2)}{m_Q r^2} \end{pmatrix} + V_{\Sigma_u^-}(r) + V_q(r) \begin{pmatrix} \frac{J+1}{2J+1} & \frac{\sqrt{(J+1)J}}{2J+1} \\ \frac{\sqrt{(J+1)J}}{2J+1} & \frac{J}{2J+1} \end{pmatrix} \right] \begin{pmatrix} P_J^-(r) \\ P_J^+(r) \end{pmatrix} = E \begin{pmatrix} P_J^-(r) \\ P_J^+(r) \end{pmatrix}$$

$$\left(-\frac{1}{m_Q} \frac{\partial^2}{\partial r^2} + \frac{J(J+1)}{m_Q r^2} + V_{\Pi_u}(r) \right) P_J^0(r) = E P_J^0(r)$$

- Equations for $J = 0$:

$$\left(-\frac{1}{m_Q} \frac{\partial^2}{\partial r^2} + \frac{2}{m_Q r^2} + V_{\Pi_u}(r) \right) P_0^+(r) = E P_0^+(r)$$

Hyperfine splitting for hybrids

Long distance potential

- By solving the Wilson Loop [Soto, Tarrús]

$$\frac{V_{1+11}^{sa}(r)}{m_Q} = - \lim_{T \rightarrow \infty} \frac{g c_F}{m_Q T} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \frac{\langle B^*(\mathbf{0}, \frac{T}{2}) B^3(\frac{\mathbf{r}}{2}, t) B(\mathbf{0}, -\frac{T}{2}) - B(\mathbf{0}, \frac{T}{2}) B^3(\frac{\mathbf{r}}{2}, t) B^*(\mathbf{0}, -\frac{T}{2}) \rangle_{\square}}{\langle B(\mathbf{0}, \frac{T}{2}) B^*(\mathbf{0}, -\frac{T}{2}) + B^*(\mathbf{0}, \frac{T}{2}) B(\mathbf{0}, -\frac{T}{2}) \rangle_{\square}}$$

$$\frac{V_{1+10}^{sb}(r)}{m_Q} = \lim_{T \rightarrow \infty} \frac{g c_F}{4m_Q} \frac{V_{\Pi_u} - V_{\Sigma_u^-}}{\sin\left(\left(V_{\Pi_u} - V_{\Sigma_u^-}\right)\frac{T}{2}\right)} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \frac{\langle (B^*(\mathbf{0}, \frac{T}{2}) B(\frac{\mathbf{r}}{2}, t) - B(\mathbf{0}, \frac{T}{2}) B^*(\frac{\mathbf{r}}{2}, t)) B^3(\mathbf{0}, -\frac{T}{2}) \rangle_{\square}}{\langle B(\mathbf{0}, \frac{T}{2}) B^*(\mathbf{0}, -\frac{T}{2}) + B^*(\mathbf{0}, \frac{T}{2}) B(\mathbf{0}, -\frac{T}{2}) \rangle_{\square}^{1/2} \langle B^3(\mathbf{0}, \frac{T}{2}) B^3(\mathbf{0}, -\frac{T}{2}) \rangle_{\square}^{1/2}}$$

- With the chromomagnetic field

$$B(\mathbf{r}, t) = B^1(\mathbf{r}, t) + iB^2(\mathbf{r}, t)$$

$$B^*(\mathbf{r}, t) = B^1(\mathbf{r}, t) - iB^2(\mathbf{r}, t)$$

Hyperfine splitting for hybrids

Interpolated potentials

- Interpolation between the short and long distances

$$\frac{V_{hf}(r)}{m_Q} = \frac{A + \left(\frac{r}{r_0}\right)^2 \left(\frac{1}{6} V_{ld}^{sa}(r_0) - \frac{r}{3r_0} V_{ld}^{sb}(r_0)\right)}{1 + \left(\frac{r}{r_0}\right)^5}$$

$$\frac{V_{hf2}(r)}{m_Q} = \frac{Br^2 - \left(\frac{r}{r_0}\right)^5 \left(\frac{r_0}{2r} V_{ld}^{sa}(r_0) + \frac{1}{2} V_{ld}^{sb}(r_0)\right)}{1 + \left(\frac{r}{r_0}\right)^7}$$

- With matching scale $r_0 \simeq 3.96 \text{ GeV}^{-1} \sim 1/\Lambda_{QCD}$

Hyperfine splitting for hybrids

Parameter results

$$A = c_F(m_Q) k_A / m_Q$$

$$B = c_F(m_Q) k_B / m_Q$$

$$c_F(m_c) \equiv c_F(\nu = 1 \text{ GeV}, m_c) = 1.12155$$

$$c_F(m_b) \equiv c_F(\nu = 1 \text{ GeV}, m_b) = 0.87897$$

$$g\Lambda' \sim -59 \text{ MeV}$$

$$g\Lambda''' \sim \pm 230 \text{ MeV}$$

$$\kappa \simeq 0.187 \text{ GeV}^2$$

$$A = -0.070 \pm 0.010 \text{ GeV}$$

$$B = 0.0117 \pm 0.0003 \text{ GeV}^3$$

$$A' = A \frac{c_F(m_b)m_c}{c_F(m_c)m_b} \quad B' = B \frac{c_F(m_b)m_c}{c_F(m_c)m_b}$$

$$A' = -0.017 \pm 0.002 \text{ GeV}$$

$$B' = 0.0028 \pm 0.0001 \text{ GeV}^3$$

$$\chi^2 / \text{dof} = 0.582$$