

Investigation of heavy exotic mesons using lattice QCD static potentials

“Exotic Quarkonia in Heavy-ion Collisions” – Mainz Institute for Theoretical Physics

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Outline

- Part 1:
Hybrid spin-dependent and hybrid-quarkonium potentials, Born-Oppenheimer studies of $\bar{c}c$ and $\bar{b}b$ hybrid mesons
in collaboration with
Carolin Schlosser, Vilija de Jonge, Fabian Geiger (Goethe University Frankfurt)
Paul Pütz, Antonio Vairo (Technical University Munich)
- Part 2: (only, if there is time left ... rather unlikely)
Selected results on $\bar{b}bqq$ tetraquarks

Part 1: Hybrid spin-dependent and
hybrid-quarkonium potentials,
Born-Oppenheimer studies of $\bar{c}c$ and $\bar{b}b$
hybrid mesons

Main goals / literature

- (1) Compute **heavy hybrid potentials** (static limit as well as $1/m_Q$ corrections), i.e. potentials of a heavy quark antiquark pair ($\bar{b}b$ or $\bar{c}c$), where the flux tubes are excited with quantum numbers different from the ground state.
→ SU(3) lattice gauge theory
- (2) Use these potentials to compute the **spectra of $\bar{b}b$ and $\bar{c}c$ hybrid mesons** and to study their **mixing with ordinary quarkonium**.
→ Effective field theory, Born-Oppenheimer approximation, quantum mechanics
- Our work mostly focuses on (1)
[C. Schlosser, M.W., Phys. Rev. D **105**, no. 5, 054503 (2022) [arXiv:2111.00741]]
[C. Schlosser, M.W., Phys. Rev. D **111**, no. 7, 074504 (2025) [arXiv:2501.08844]]
but the talk is more about (2).
(my goal in this talk is to explain, why we compute heavy hybrid potentials, what one can study with them and how ... rather than explaining less interesting lattice technicalities)
- Selected further references on (1) and (2):
[K. J. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. **90**, 161601 (2003) [hep-lat/0207004]]
[S. Capitani, O. Philipsen, C. Reisinger, C. Riehl. M.W., Phys. Rev. D **99**, 034502 (2019) [arXiv:1811.11046 [hep-lat]]]
[R. Oncala, J. Soto, Phys. Rev. D **96**, no. 1, 014004 (2017) [arXiv:1702.03900]]
[J. Soto, S. T. Valls, Phys. Rev. D **108**, no. 1, 014025 (2023) [arXiv:2302.01765]]
[M. Berwein, N. Brambilla, A. Mohapatra, A. Vairo, Phys. Rev. D **110**, no. 9, 094040 (2024) [arXiv:2408.04719]]

$\bar{b}b$ and $\bar{c}c$ hybrid meson masses: BO

- Born-Oppenheimer approximation: Compute $\bar{b}b$ and $\bar{c}c$ hybrid meson masses in two steps.

- Compute potentials of two static quarks ($\bar{b}b$ or $\bar{c}c$) and corrections proportional to $1/m_Q$, $(1/m_Q)^2$, ... (m_Q is the heavy quark mass) in the presence of **excited gluons** using lattice gauge theory.
- Solve the Schrödinger equation for the relative coordinate of $\bar{b}b$ or $\bar{c}c$ using the **potentials** from (1),

$$\left(-\frac{1}{m_Q}\Delta + V^{(0)}(\mathbf{r}) + \frac{1}{m_Q}V^{(1)}(\mathbf{r}) + \dots \right) \Psi(\mathbf{r}) = E\Psi(\mathbf{r})$$

(Ψ has 16 components, $V^{(n)}$ is a 16×16 matrix). Energy eigenvalues E correspond to masses of $\bar{b}b$ and $\bar{c}c$ hybrid mesons.

[E. Braaten, C. Langmack, D. H. Smith, Phys. Rev. D **90**, 014044 (2014) [arXiv:1402.0438]]

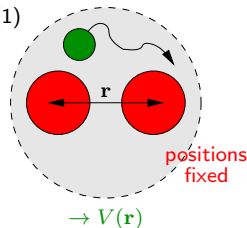
...

[R. Oncala, J. Soto, Phys. Rev. D **96**, no. 1, 014004 (2017) [arXiv:1702.03900]]

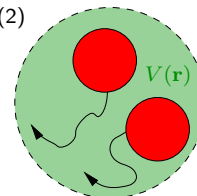
[J. Soto, S. T. Valls, Phys. Rev. D **108**, no. 1, 014025 (2023) [arXiv:2302.01765]]

[C. Schlosser, M.W., Phys. Rev. D **111**, no. 7, 074504 (2025) [arXiv:2501.08844]]

step (1)



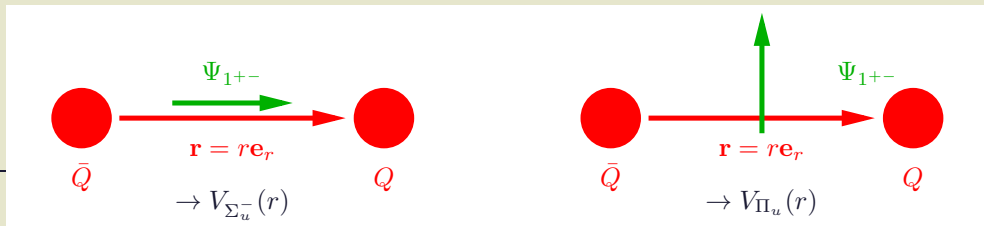
step (2)



→ mass of a $\bar{b}b$ or $\bar{c}c$ hybrid meson

Potentials in the static limit, $(1/m_Q)^0$

- Heavy hybrid meson: $\bar{Q}Q$ + (valence) gluon.
- The gluon has spin 1, represented by a 3-vector,
 - a chromomagnetic field operator $\mathbf{B}$ (in the lattice computation of hybrid potentials),
 - a 3-component wave function Ψ_{1+-} (in the Schrödinger equation; 1^{+-} denotes the quantum numbers of a gluon, i.e. indicates that Ψ_{1+-} is a hybrid wave function).
- In the static approximation (infinitely heavy quarks), quark spins are irrelevant.
- There is another 3-vector, the relative coordinate of the quarks, $\mathbf{r} = \mathbf{r}_Q - \mathbf{r}_{\bar{Q}}$.
- Two possibilities:
 - $\mathbf{B}$ (or equivalently Ψ_{1+-}) and $\mathbf{r}$ are parallel.
→ Corresponding hybrid static potential is denoted as $V_{\Sigma_u^-}(r)$.
 - $\mathbf{B}$ (or equivalently Ψ_{1+-}) and $\mathbf{r}$ are perpendicular.
→ Corresponding hybrid static potential is denoted as $V_{\Pi_u}(r)$.



Lattice computation of $V_{\Sigma_u^-}(r)$, $V_{\Pi_u}(r)$ (1)

- Two possibilities:
 - $\mathbf{B}$ and $\mathbf{r}$ are parallel.
 → Corresponding hybrid static potential is denoted as $V_{\Sigma_u^-}(r)$.
 - $\mathbf{B}$ and $\mathbf{r}$ are perpendicular.
 → Corresponding hybrid static potential is denoted as $V_{\Pi_u}(r)$.
- To determine static potentials use lattice gauge theory to compute temporal correlation functions of suitable operators $\mathcal{O}(r)$

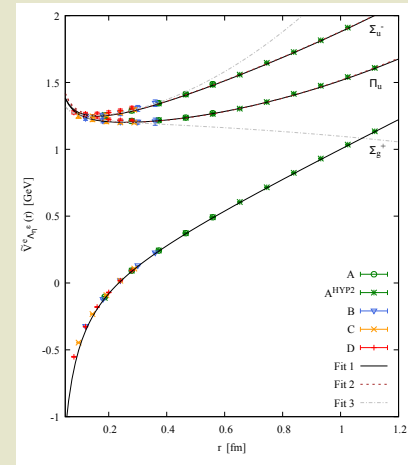
$$\lim_{t \rightarrow \infty} \langle \Omega | \mathcal{O}^\dagger(r, t) \mathcal{O}(r, 0) | \Omega \rangle \propto \exp(-V(r)t).$$

- Suitable operators are:

$$\mathcal{O}_{\Sigma_u^-}(r) = \bar{Q}(-r/2)U(-r/2; 0)B_z(0)U(0; +r/2)Q(+r/2)$$

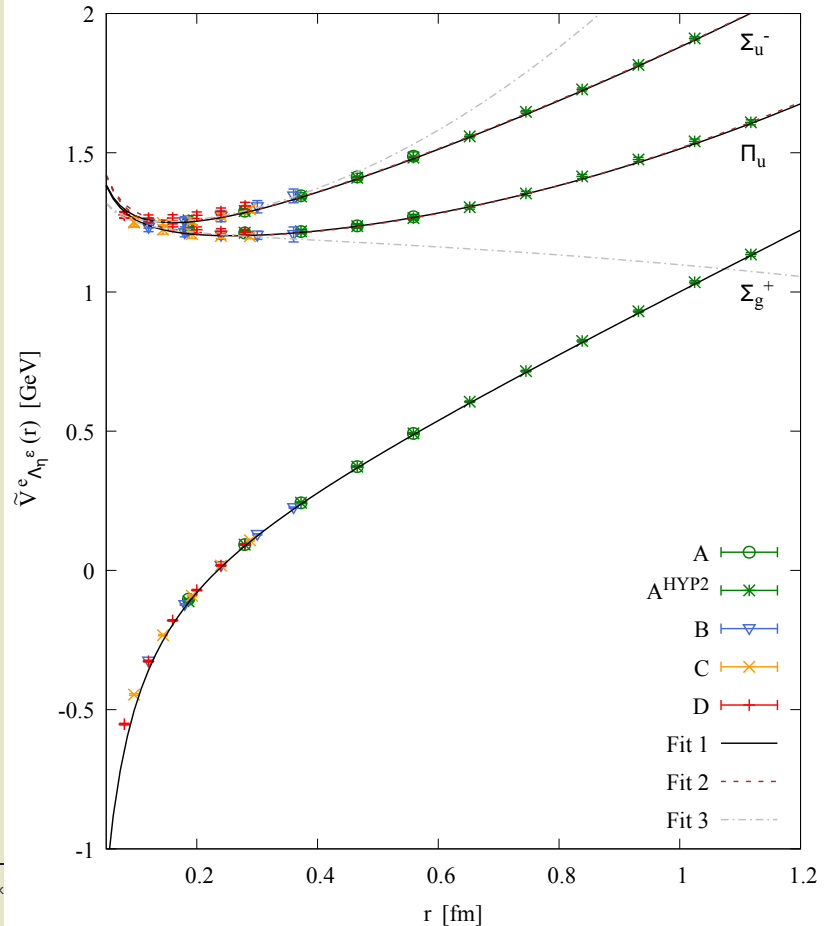
$$\mathcal{O}_{\Pi_u}(r) = \begin{cases} \bar{Q}(-r/2)U(-r/2; 0)B_x(0)U(0; +r/2)Q(+r/2) \\ \bar{Q}(-r/2)U(-r/2; 0)B_y(0)U(0; +r/2)Q(+r/2) \end{cases}.$$

- Q and $\bar{Q}$ (spinless quarks) are separated in z direction; only z coordinates are written.
- U denote straight gluonic parallel transporters (necessary for gauge invariance).



Lattice computation of $V_{\Sigma_u^-}(r)$, $V_{\Pi_u}(r)$ (2)

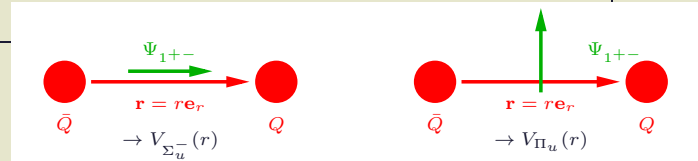
- Σ_g^+ : Ordinary static potential
($\rightarrow$ ordinary quarkonium).
- Σ_u^-, Π_u : Hybrid static potentials
($\rightarrow$ hybrid quarkonium).
- Computation on 4 ensembles with lattice spacings
 $a = 0.040 \text{ fm}, \dots, 0.096 \text{ fm}$.
[C. Schlosser, M.W., Phys. Rev. D **105**,
no. 5, 054503 (2022)
[arXiv:2111.00741]]



$(1/m_Q)^0$: Potential matrix, SE

- Two possibilities:

- Ψ_{1+-} and $\mathbf{r}$ are parallel.
→ Corresponding hybrid static potential is denoted as $V_{\Sigma_u^-}(r)$.
- Ψ_{1+-} and $\mathbf{r}$ are perpendicular.
→ Corresponding hybrid static potential is denoted as $V_{\Pi_u}(r)$.



- Schrödinger equation:

$$\left(-\frac{1}{m_Q} \Delta + V_{1+-}^{(0)}(\mathbf{r}) \right) \Psi_{1+-}(\mathbf{r}) = E \Psi_{1+-}(\mathbf{r}).$$

- Ψ_{1+-} is a 3-component wave function (the components represent the gluon spin)
→ The Hamilton operator is a 3×3 matrix.
→ 3 coupled PDEs.
- The 3×3 potential matrix is

$$V_{1+-}^{(0)}(\mathbf{r}) = V_{\Sigma_u^-}(r) (\mathbf{e}_r \otimes \mathbf{e}_r) + V_{\Pi_u}(r) (1 - \mathbf{e}_r \otimes \mathbf{e}_r).$$

[M. Berwein, N. Brambilla, J. Tarrús Castellà, A. Vairo, Phys. Rev. D **92**, no. 11, 114019 (2015)

[arXiv:1510.04299]]

[C. Schlosser, M.W., Phys. Rev. D **111**, no. 7, 074504 (2025) [arXiv:2501.08844]]

Radial SEs (1)

- The Schrödinger equation

$$\left(-\frac{1}{m_Q} \Delta + V_{1^{+-}}^{(0)}(\mathbf{r}) \right) \Psi_{1^{+-}}(\mathbf{r}) = E \Psi_{1^{+-}}(\mathbf{r})$$

(a PDE) can be decoupled into infinitely many radial equations (ODEs).

- Orbital angular momentum $L^{PC} = 0^{++}, 1^{--}, 2^{++}, \dots$ (spherical harmonics Y_{LM} in $\Psi_{1^{+-}}$)
- Gluon spin 1^{+-} . (3 components of $\Psi_{1^{+-}}$)
- Heavy spins are irrelevant/neglected in the static approximation.

→ Total angular momentum (with heavy spins excluded): $J^{PC} = L^{PC} \otimes 1^{+-}$.

$$\begin{aligned} - \quad J^{PC} = 1^{-+} &= 1^{--} \otimes 1^{+-} && (H_2 \text{ multiplet}) \\ J^{PC} = 2^{+-} &= 2^{++} \otimes 1^{+-} && (H_5 \text{ multiplet}) \\ &\dots && \\ - \quad J^{PC} = 0^{-+} &= 1^{--} \otimes 1^{+-} && (H_3 \text{ multiplet}) \\ J^{PC} = 1^{+-} &= 0^{++} \otimes 1^{+-} / 2^{++} \otimes 1^{+-} && (H_1 \text{ multiplet}) \\ J^{PC} = 2^{-+} &= 1^{--} \otimes 1^{+-} / 3^{--} \otimes 1^{+-} && (H_4 \text{ multiplet}) \\ &\dots && \end{aligned}$$

Radial SEs (2)

- Hybrid quantum numbers: $J^{PC} = 1^{-+}, 2^{+-}, \dots, 0^{-+}, 1^{+-}, 2^{-+}, \dots$
- No mixing with ordinary quarkonium, which can have quantum numbers $J^{PC} = L^{PC} = 0^{++}, 1^{--}, 2^{++}, \dots$
- Example: Radial Schrödinger equation for $J^{PC} = 0^{-+} = 1^{--} \otimes 1^{+-}$ (H_3 multiplet).

$$- \Psi_{1^{+-}}(\mathbf{r}) = \left(\underbrace{Y_{1x}(\Omega)}_{\propto x} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \underbrace{Y_{1y}(\Omega)}_{\propto y} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \underbrace{Y_{1z}(\Omega)}_{\propto z} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) R(r) \propto \mathbf{e}_r R(r).$$

- Inserting this wave function into the Schrödinger equation

$$\left(-\frac{1}{m_Q} \Delta + V_{1^{+-}}^{(0)}(\mathbf{r}) \right) \Psi_{1^{+-}}(\mathbf{r}) = E \Psi_{1^{+-}}(\mathbf{r}) \quad (3 \text{ coupled PDEs containing all } J^{PC})$$

$$V_{1^{+-}}^{(0)}(\mathbf{r}) = V_{\Sigma_u^-}(r) (\mathbf{e}_r \otimes \mathbf{e}_r) + V_{\Pi_u}(r) (1 - \mathbf{e}_r \otimes \mathbf{e}_r)$$

leads to the radial Schrödinger equation

$$\left(-\frac{1}{m_Q} \frac{d^2}{dr^2} + \frac{2}{m_Q r^2} + V_{\Sigma_u^-}(r) \right) R(r) = E R(r) \quad (1 \text{ coupled ODE for } J^{PC} = 0^{-+}).$$

Spin-dependent and mixing potentials (1)

- Currently ongoing research: **inclusion of the heavy quark spins** and **mixing with ordinary quarkonium** order by order in $1/m_Q$ (at the moment just order $1/m_Q$).
 - The $\bar{Q}Q$ pair can have spin $S = 0$ ($S_z = 0$ [$\rightarrow$ 1 possible configuration]) or $S = 1$ ($S_z = -1, 0, +1$ [$\rightarrow$ 3 possible configurations]), i.e. 4 spin configurations in total.
 - On previous slides: $\Psi_{1+-}(\mathbf{r})$ has 3 components (gluon spin 1).
 - From now on: $\Psi_{1+-}(\mathbf{r})$ has $3 \times 4 = 12$ components (gluon spin 1, quark spin 0 or 1).
 - Moreover, there can be mixing with ordinary quarkonium (represented by a 4-component wave function $\Psi_{0++}(\mathbf{r})$).

$\rightarrow$ Including both heavy quark spins and mixing leads to

- the 16-component wave function $\Psi(\mathbf{r}) = (\Psi_{0++}(\mathbf{r}), \Psi_{1+-}(\mathbf{r}))$,
- a 16×16 Hamilton operator and potential matrix $V(\mathbf{r}) = V^{(0)}(\mathbf{r}) + (1/m_Q)V^{(1)}(\mathbf{r})$,
- the Schrödinger equation

$$\left(-\frac{1}{m_Q}\Delta + V^{(0)}(\mathbf{r}) + \frac{1}{m_Q}V^{(1)}(\mathbf{r}) \right) \Psi(\mathbf{r}) = E\Psi(\mathbf{r})$$

(16 coupled PDEs).

Spin-dependent and mixing potentials (2)

- Schrödinger equation:

$$\left(-\frac{1}{m_Q} \Delta + V^{(0)}(\mathbf{r}) + \frac{1}{m_Q} V^{(1)}(\mathbf{r}) \right) \Psi(\mathbf{r}) = E \Psi(\mathbf{r})$$

with $\Psi(\mathbf{r}) = (\Psi_{0^{++}}(\mathbf{r}), \Psi_{1^{+-}}(\mathbf{r}))$.

- $V^{(0)}(\mathbf{r})$:

- 16×16 matrix, block diagonal.

- Upper left 4×4 block: ordinary static potential $V_{\Sigma_g^+}(r)$ (“quarkonium channels”).

- Lower right 12×12 block: hybrid static potentials $V_{\Sigma_u^-}(r)$, $V_{\Pi_u}(r)$ (“hybrid channels”).

- $V^{(1)}(\mathbf{r})$:

- **16×16 matrix**, introduces heavy spin effects, couples quarkonium and hybrid channels.

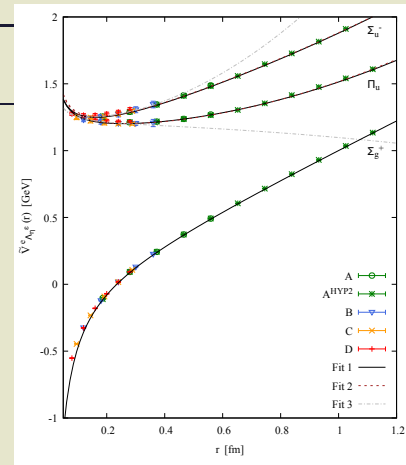
- Contains 2 hybrid spin-dependent potentials $V_{11}^{sa}(r)$, $V_{10}^{sb}(r)$.

- Contains 2 hybrid-quarkonium mixing potentials $V_{\Sigma_u^-}^{\text{mix}}(r)$, $V_{\Pi_u}^{\text{mix}}(r)$.

[R. Oncala, J. Soto, Phys. Rev. D **96**, no. 1, 014004 (2017) [arXiv:1702.03900]]

[J. Soto, S. T. Valls, Phys. Rev. D **108**, no. 1, 014025 (2023) [arXiv:2302.01765]]

- **The main focus of our work is to carry out the first lattice gauge theory computation of these four $1/m_Q$ potentials.**



Spin-dependent and mixing potentials (3)

- All J^{PC} possible and included in the Schrödinger equation. (exception 0^{--})
- For each J : three PC sectors, where ordinary quarkonium and hybrids mix, one pure hybrid PC sector ($0^{+-}, 1^{-+}, 2^{+-}, \dots$).
- For each J : suitable angular momentum projections lead to radial coupled-channel Schrödinger equations with 6, 2, 4, and 4 channels, respectively (for $J < 2$ a smaller number of channels).

[R. Oncala, J. Soto, Phys. Rev. D **96**, no. 1, 014004 (2017) [arXiv:1702.03900]]

[J. Soto, S. T. Valls, Phys. Rev. D **108**, no. 1, 014025 (2023) [arXiv:2302.01765]]

L^{PC}	gluon	J^{PC}	multiplet	S^{PC}	J^{PC}	type
0^{++}		0^{++}		0^{++}	0^{++}	ordinary
0^{++}	1^{+-}	1^{+-}	H_1	1^{--}	0^{+-}	hybrid
2^{++}	1^{+-}	1^{+-}	H_1	1^{--}	0^{+-}	hybrid
1^{--}		1^{--}		1^{--}	0^{++}	ordinary
1^{--}	1^{+-}	0^{+-}	H_3	0^{+-}	0^{++}	hybrid
1^{--}	1^{+-}	1^{+-}	H_2	1^{--}	0^{+-}	hybrid
1^{--}		1^{--}		0^{+-}	1^{+-}	ordinary
1^{--}	1^{+-}	0^{+-}	H_3	1^{--}	1^{+-}	hybrid
1^{--}	1^{+-}	1^{+-}	H_2	1^{--}	1^{+-}	hybrid
1^{--}	1^{+-}	2^{+-}	H_4	1^{--}	1^{+-}	hybrid
3^{--}	1^{+-}	2^{+-}	H_4	1^{--}	1^{+-}	hybrid
1^{--}		1^{--}		1^{--}	1^{++}	ordinary
1^{--}	1^{+-}	1^{+-}	H_2	0^{+-}	1^{++}	hybrid
0^{++}		0^{++}		1^{--}	1^{--}	ordinary
2^{++}		2^{++}		1^{--}	1^{--}	ordinary
0^{++}	1^{+-}	1^{+-}	H_1	0^{+-}	1^{--}	hybrid
2^{++}	1^{+-}	1^{+-}	H_1	0^{+-}	1^{--}	hybrid
0^{++}	1^{+-}	1^{+-}	H_1	1^{--}	1^{+-}	hybrid
2^{++}	1^{+-}	1^{+-}	H_1	1^{--}	1^{+-}	hybrid
2^{++}	1^{+-}	2^{+-}	H_5	1^{--}	1^{+-}	hybrid
2^{++}		2^{++}		0^{+-}	2^{+-}	ordinary
0^{++}	1^{+-}	1^{+-}	H_1	1^{--}	2^{+-}	hybrid
2^{++}	1^{+-}	1^{+-}	H_1	1^{--}	2^{+-}	hybrid
2^{++}	1^{+-}	2^{+-}	H_5	1^{--}	2^{+-}	hybrid
2^{++}	1^{+-}	3^{+-}		1^{--}	2^{+-}	hybrid
4^{++}	1^{+-}	3^{+-}		1^{--}	2^{+-}	hybrid
1^{--}		1^{--}		1^{--}	2^{--}	ordinary
2^{++}	1^{+-}	2^{+-}	H_5	0^{+-}	2^{--}	hybrid

Lattice computation of $V_{11}^{sa}(r)$, $V_{10}^{sb}(r)$, $V_{\Sigma_u^-}^{\text{mix}}(r)$, $V_{\Pi_u}^{\text{mix}}(r)$ (1)

- The $1/m_Q$ potentials $V_{11}^{sa}(r)$, $V_{10}^{sb}(r)$, $V_{\Sigma_u^-}^{\text{mix}}(r)$, $V_{\Pi_u}^{\text{mix}}(r)$ are related to chromomagnetic expectation values,

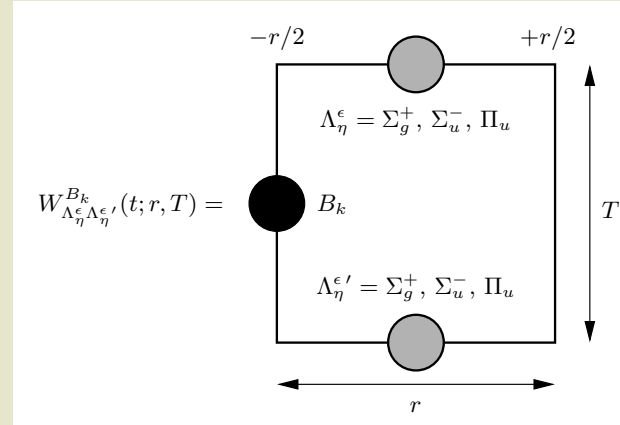
$$\begin{aligned}
 V_{11}^{sa}(r) &= +igc_F \langle 0, \Pi_u^- | B_z(-r/2) | 0, \Pi_u^+ \rangle(r) \\
 V_{10}^{sb}(r) &= +igc_F \langle 0, \Sigma_u^- | B_y(-r/2) | 0, \Pi_u^+ \rangle(r) \\
 V_{\Pi_u}^{\text{mix}}(r) &= +\frac{igc_F}{2m_Q} \langle 0, \Sigma_g^+ | B_x(-r/2) | 0, \Pi_u^+ \rangle(r) \\
 V_{\Sigma_u^-}^{\text{mix}}(r) &= +\frac{igc_F}{2m_Q} \langle 0, \Sigma_g^+ | B_z(-r/2) | 0, \Sigma_u^- \rangle(r)
 \end{aligned}$$

(for heavy quarks separated along the z axis).

- Such matrix elements can be extracted from generalized Wilson loops,

$$\langle 0, \Lambda_\eta^\epsilon | B_k(-r/2) | 0, \Lambda_\eta^{\epsilon'} \rangle(r) = \lim_{T \rightarrow \infty} \frac{i}{g} R_{\Lambda_\eta^\epsilon \Lambda_\eta^{\epsilon'}}^B(t; r, T)$$

$$\begin{aligned}
 R_{\Lambda_\eta^\epsilon \Lambda_\eta^{\epsilon'}}^B(t; r, T) &= \\
 &= W_{\Lambda_\eta^\epsilon \Lambda_\eta^{\epsilon'}}^B(t; r, T) \left(\frac{1}{W_{\Lambda_\eta^\epsilon}(r, T) W_{\Lambda_\eta^{\epsilon'}}(r, T)} \right)^{1/2} \left(\frac{W_{\Lambda_\eta^{\epsilon'}}(r, T/2 - t) W_{\Lambda_\eta^\epsilon}(r, T/2 + t)}{W_{\Lambda_\eta^\epsilon}(r, T/2 - t) W_{\Lambda_\eta^{\epsilon'}}(r, T/2 + t)} \right)^{1/2}.
 \end{aligned}$$



Lattice computation of $V_{11}^{sa}(r)$, $V_{10}^{sb}(r)$, $V_{\Sigma_u^-}^{\text{mix}}(r)$, $V_{\Pi_u}^{\text{mix}}(r)$ (2)

- Technical difficulties (compared to the computation of static [i.e. $(1/m_Q)^0$] potentials):
 - Twice as large T extents for generalized Wilson loops needed for comparable ground state domination.
 - (–) Loops with large temporal extents are needed (poor signal-to-noise ratio).
 - (–) Chromomagnetic field insertions introduce large discretization errors.
 - Chromomagnetic field insertions need renormalization.
 - (–) Corresponding renormalization constants (= matching factors) c_F are known for the gradient flow scheme, but only perturbatively to order α_s .
[N. Brambilla, X. P. Wang, JHEP **06**, 210 (2024) [arXiv:2312.05032]]
- Apply **gradient flow**, which is a technique to smear gauge link configurations in a mathematically controlled way via a single parameter called gradient flow time t_f .
 - (+) Gradient Flow reduces (certain) discretization errors as well as statistical errors.
 - (–) An extrapolation to gradient flow time t_f is necessary. This requires independent computations at several different t_f . Computing time increases accordingly.
 - (–) “Gradient flow smearing” introduces systematic errors for small $\bar{Q}Q$ separations.
 - Discard $\bar{Q}Q$ separations $r \lesssim 2\sqrt{8t_f} + a$ (in our computation $r \leq 4a$).
 - Time consuming computations at rather small values of the lattice spacing a with many lattice sites required.

Lattice computation of $V_{11}^{sa}(r)$, $V_{10}^{sb}(r)$, $V_{\Sigma_u^-}^{\text{mix}}(r)$, $V_{\Pi_u}^{\text{mix}}(r)$ (3)

- Results shown have been obtained at a single lattice spacing $a = 0.060$ fm and a single gradient flow time t_f .

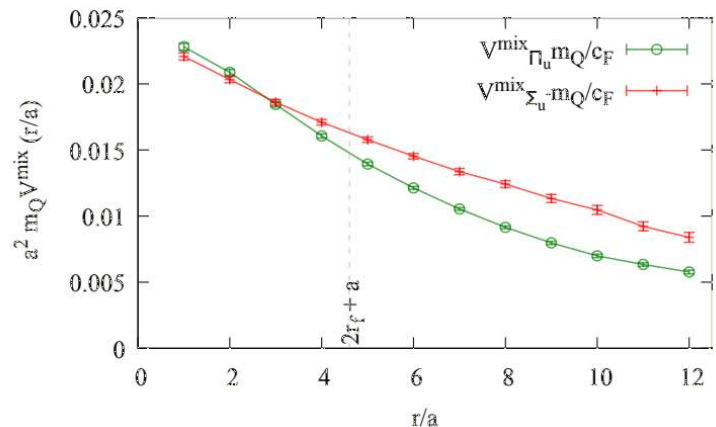
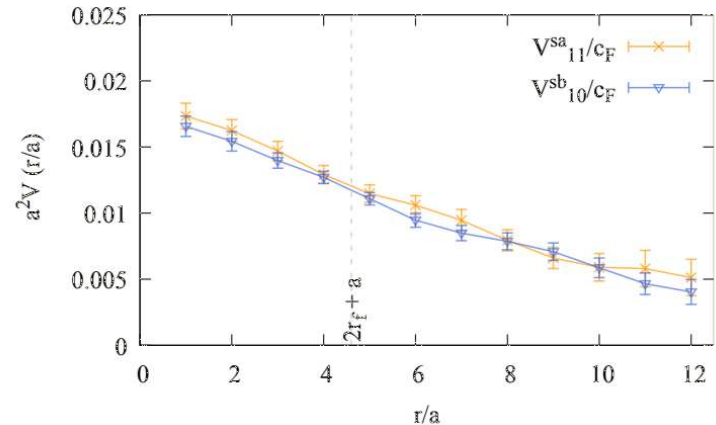
(–) No continuum extrapolation.

(–) No zero flow time extrapolation, no renormalization.

→ **Proof of principle.**

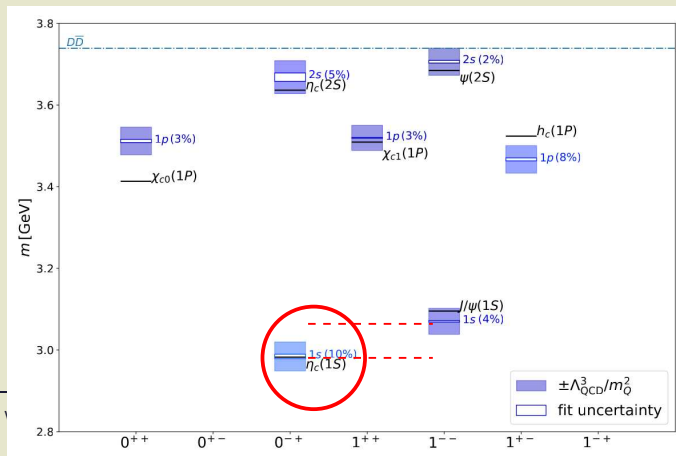
[C. Schlosser, M.W., Phys. Rev. D **111**, no. 7, 074504 (2025) [arXiv:2501.08844]]

- Ongoing work: small separations, large separations, different lattice spacings and gradient flow times, continuum extrapolation and renormalization.
[V. de Jonge, ongoing PhD project, Goethe University Frankfurt]



BO results based on these preliminary potentials

- Low-lying charmonium states in non-exotic sectors $J^{PC} = 0^{++}, 0^{-+}, 1^{++}, 1^{--}, 1^{+-}$.
- Experimental results (PDG): **black lines**.
- Theoretical results (Born-Oppenheimer with $1/m_c$ mixing potentials):
[P. Pütz, ongoing PhD project, Technical University Munich]
 - **White boxes**: statistical errors; **blue boxes**: $\Lambda_{\text{QCD}}^3/m_c^2$, crude systematic uncertainties.
 - Schrödinger equations with 2, 3, 2, 4 and 5 coupled channels, respectively.
 - Particularly interesting: η_c has a sizable hybrid component of around 10%.
→ **Conclusion: Even when just studying low-lying quarkonium states, the inclusion of hybrid structures seems to be mandatory ...?**



Summary and outlook

- **First lattice gauge theory computation of $\bar{Q}Q$ potentials describing heavy spin corrections for hybrid quarkonium and mixing with ordinary quarkonium.**
 - **At the moment only results for a single lattice spacing a and a single flow time t_f .**
 - No continuum extrapolation.
 - No zero flow time extrapolation, no renormalization.

Current results are rather a “proof of principle”.
 - **Both smaller and larger separations are needed (lattices with many sites required).**
 - **Smaller statistical errors via longer simulations and computations.**
- Corresponding large scale computations are about to be started.
- Use these potentials in coupled-channel Schrödinger equations to predict masses of $\bar{b}b$ and $\bar{c}c$ hybrid mesons and their composition (i.e. the amount of mixing with ordinary quarkonium).

Part 2: Selected results on $\bar{b}\bar{b}qq$ tetraquarks

[P. Bicudo, M.W., Phys. Rev. D **87**, 114511 (2013) [arXiv:1209.6274]]

[P. Bicudo, K. Cichy, A. Peters, B. Wagenbach, M.W., Phys. Rev. D **92**, 014507 (2015) [arXiv:1505.00613]]

[P. Bicudo, K. Cichy, A. Peters, M.W., Phys. Rev. D **93**, 034501 (2016) [arXiv:1510.03441]]

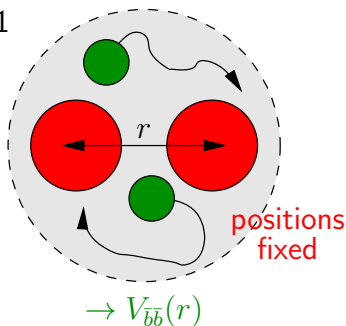
[P. Bicudo, J. Scheunert, M.W., Phys. Rev. D **95**, 034502 (2017) [arXiv:1612.02758]]

[P. Bicudo, M. Cardoso, A. Peters, M. Pflaumer, M.W., Phys. Rev. D **96**, 054510 (2017) [arXiv:1704.02383]].

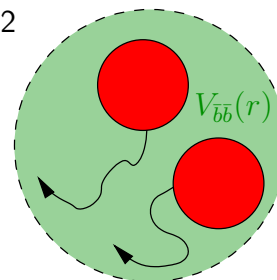
Basic idea: lattice QCD and BO

- **Basic idea:** Investigate the existence of $\bar{b}\bar{b}qq$ tetraquarks in two steps.
 - (BO1) **Compute potentials $V_{\bar{b}\bar{b}}(r)$ for the two static antiquarks ($\bar{b}\bar{b}$) in the presence of two lighter quarks ($qq, q \in \{u, d, s\}$) using lattice QCD.**
 - (BO2) **Check, whether these potentials are sufficiently attractive to host bound states or resonances ($\rightarrow$ correspond to $\bar{b}\bar{b}qq$ tetraquarks) by using techniques from quantum mechanics and scattering theory.**
- (1) + (2) $\rightarrow$ **Born-Oppenheimer-Approximation:**
 - Developed in the context of molecular and solid state physics.
[M. Born, R. Oppenheimer, "Zur Quantentheorie der Molekeln," *Annalen der Physik* 389, Nr. 20, 1927]
 - Step (BO1) in the following not quantum mechanics, but (lattice) QCD.
 - Valid approximation for $m_q \ll m_b$ ($\bar{b}$ quarks almost at rest compared to light quarks).

step 1



step 2



BO1: $\bar{b}\bar{b}qq/B\bar{B}$ potentials (1)

- To determine $\bar{b}\bar{b}$ potentials $V_{qq,j_z,\mathcal{P},\mathcal{P}_x}(r)$, compute temporal correlation functions

$$\langle \Omega | \mathcal{O}_{BB,\Gamma}^\dagger(t) \mathcal{O}_{BB,\Gamma}(0) | \Omega \rangle \propto_{t \rightarrow \infty} e^{-V_{qq,j_z,\mathcal{P},\mathcal{P}_x}(r)t}$$

of operators

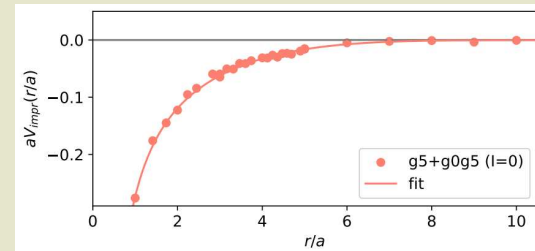
$$\mathcal{O}_{BB,\Gamma} = 2N_{BB}(\mathcal{C}\Gamma)_{AB}(\mathcal{C}\tilde{\Gamma})_{CD} \left(\bar{Q}_C^a(-\mathbf{r}/2) q_A^a(-\mathbf{r}/2) \right) \left(\bar{Q}_D^b(+\mathbf{r}/2) q_B^b(+\mathbf{r}/2) \right).$$

- Many different channels** (isospin/light flavor, angular momentum, parity).
 - Attractive as well as repulsive potentials.
 - Potentials with different asymptotic values (two heavy-light mesons $\in \{B, B^*, B_0^*, B_1^*\}$).
- The most attractive potential of a $B^{(*)}B^{(*)}$ meson pair has $(I, |j_z|, P, P_x) = (0, 0, +, -)$:

- $\psi^{(f)}\psi^{(f')} = ud - du$, $\Gamma \in \{(1 + \gamma_0)\gamma_5, (1 - \gamma_0)\gamma_5\}$.
- $\bar{Q}\bar{Q} = \bar{b}\bar{b}$, $\tilde{\Gamma} \in \{(1 + \gamma_0)\gamma_5, (1 + \gamma_0)\gamma_j\}$ (irrelevant).

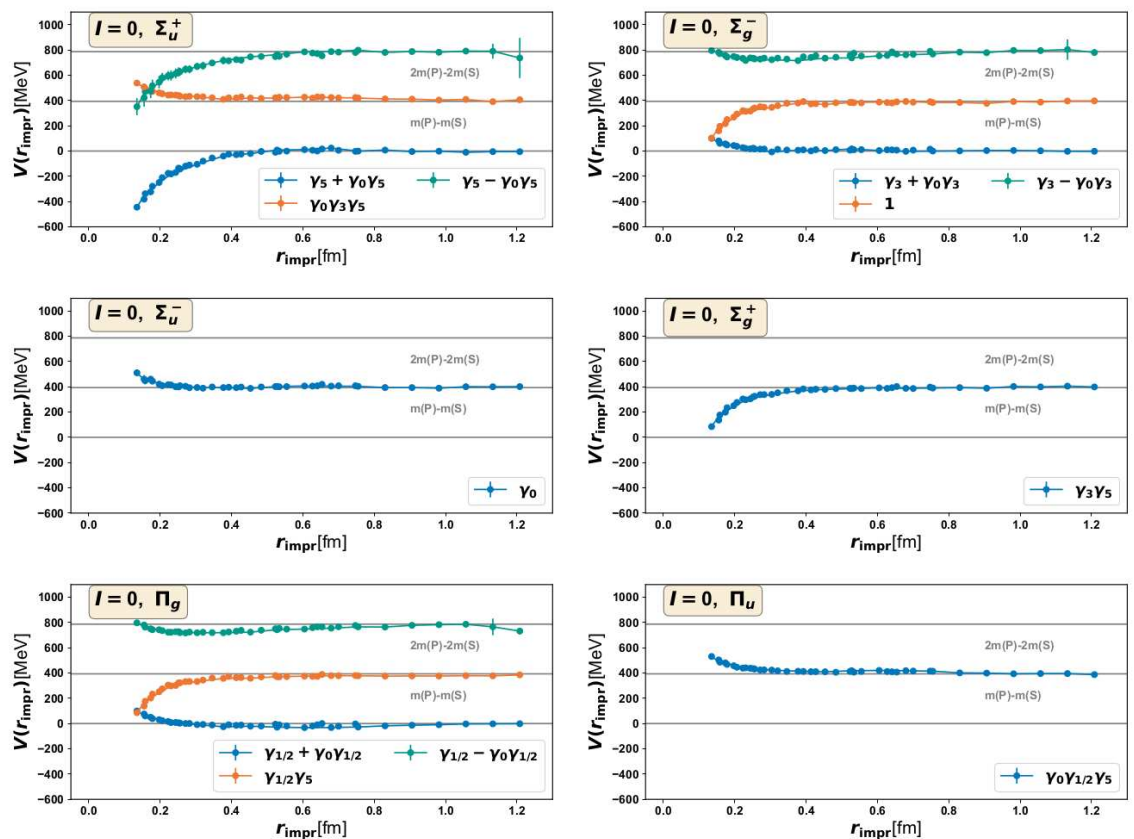
[P. Bicudo, K. Cichy, A. Peters, M.W., Phys. Rev. D **93**, 034501 (2016) [arXiv:1510.03441]]

[P. Bicudo, M. Marinkovic, L. Müller, M.W., PoS **LATTICE2024**, 124 (2024) [arXiv:2409.10786]]



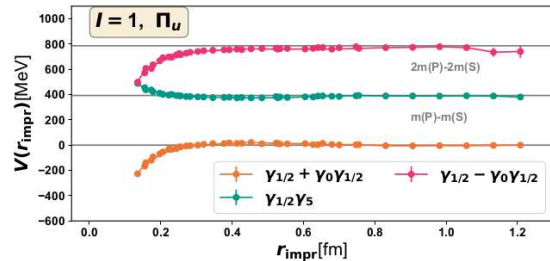
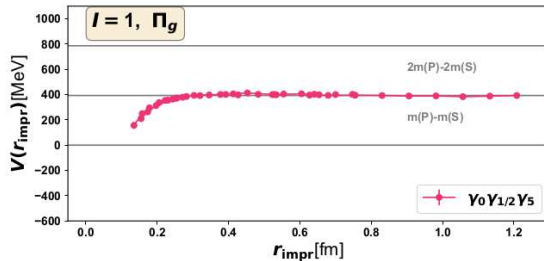
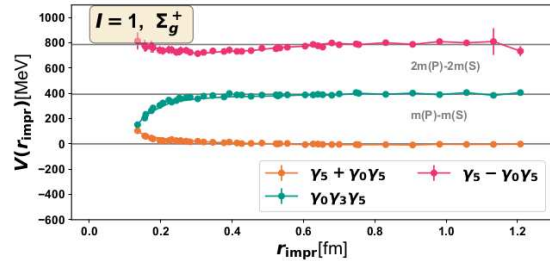
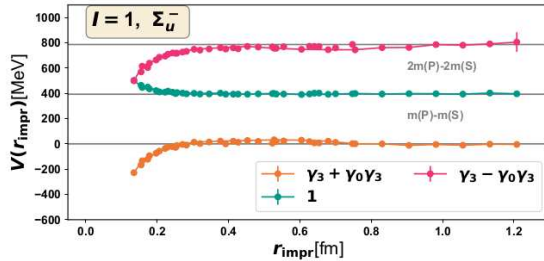
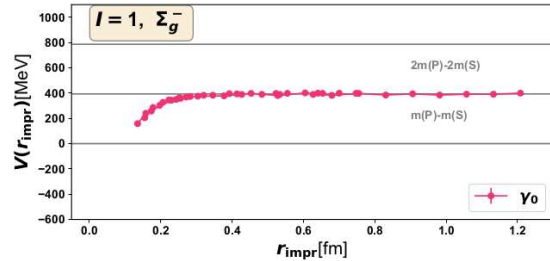
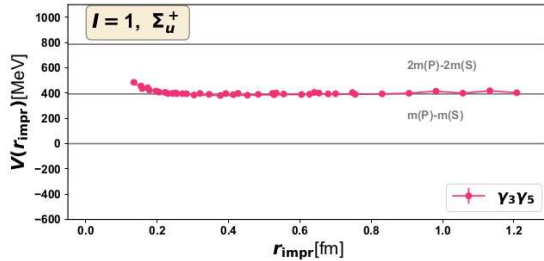
BO1: $\bar{b}bqq/B\bar{B}$ potentials (2)

[P. Bicudo, M. Marinkovic, L. Müller, M.W., PoS LATTICE2024, 124 (2024) [arXiv:2409.10786]]



BO1: $\bar{b}\bar{b}q\bar{q}/BB$ potentials (3)

[P. Bicudo, M. Marinkovic, L. Müller, M.W., PoS LATTICE2024, 124 (2024) [arXiv:2409.10786]]



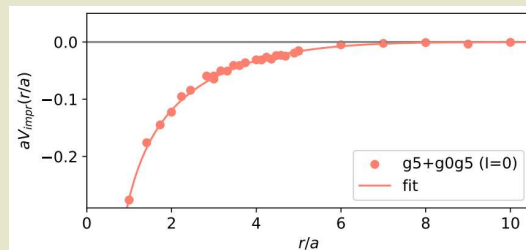
BO2: Stable $\bar{b}\bar{b}qq$ tetraquarks

- Solve the Schrödinger equation for the relative coordinate of the heavy quarks $\bar{b}\bar{b}$ using the previously computed $\bar{b}\bar{b}qq$ potentials,

$$\left(\frac{1}{m_b} \left(-\frac{d^2}{dr^2} + \frac{L(L+1)}{r^2} \right) + V_{qq,j_z,\mathcal{P},\mathcal{P}_x}(r) - 2m_B \right) R(r) = ER(r).$$

- Possibly existing bound states, i.e. $E < 0$, indicate QCD-stable $\bar{b}\bar{b}qq$ tetraquarks.
- There is a bound state for orbital angular momentum $L = 0$ of $\bar{b}\bar{b}$:
 - Binding energy $E = -90^{+43}_{-36}$ MeV with respect to the $B\bar{B}^*$ threshold.
 - Quantum numbers: $I(J^P) = 0(1^+)$.

[P. Bicudo, M.W., Phys. Rev. D 87, 114511 (2013) [arXiv:1209.6274]]



BO2: Further $\bar{b}\bar{b}qq$ results (1)

- Are there further QCD-stable $\bar{b}\bar{b}qq$ tetraquarks with other $I(J^P)$ and light flavor quantum numbers?
 - No, not for $qq = ud$ (both $I = 0, 1$), not for $qq = ss$.
[P. Bicudo, K. Cichy, A. Peters, B. Wagenbach, M.W., Phys. Rev. D **92**, 014507 (2015) [arXiv:1505.00613]]
 - $\bar{b}\bar{b}us$ was not investigated.
 - Strong evidence from full QCD computations that a QCD-stable $\bar{b}\bar{b}us$ tetraquark exists.
- Effect of heavy quark spins:
 - Expected to be $\mathcal{O}(m_{B^*} - m_B) = \mathcal{O}(45 \text{ MeV})$.
 - Previously ignored (potentials of static quarks are independent of the heavy spins).
 - In [P. Bicudo, J. Scheunert, M.W., Phys. Rev. D **95**, 034502 (2017) [arXiv:1612.02758]] included in a crude phenomenological way via a BB^* and a B^*B^* coupled channel Schrödinger equation with the experimental mass difference $m_{B^*} - m_B$ as input.
 - Binding energy reduced from around 90 MeV to 59 MeV.
 - Physical reason: the previously discussed attractive potential does not only correspond to a lighter BB^* pair, but has also a heavier B^*B^* contribution.

BO2: Further $\bar{b}\bar{b}qq$ results (2)

- Are there $\bar{b}\bar{b}qq$ tetraquark resonances?

- In

- [P. Bicudo, M. Cardoso, A. Peters, M. Pflaumer, M.W., Phys. Rev. D **96**, 054510 (2017) [arXiv:1704.02383]]

- resonances studied via standard scattering theory from quantum mechanics textbooks.

- Heavy quark spins ignored.

- Indication for $\bar{b}\bar{b}ud$ tetraquark resonance with $I(J^P) = 0(1^-)$ found, $E = 17^{+4}_{-4}$ MeV above the BB threshold, decay width $\Gamma = 112^{+90}_{-103}$ MeV.

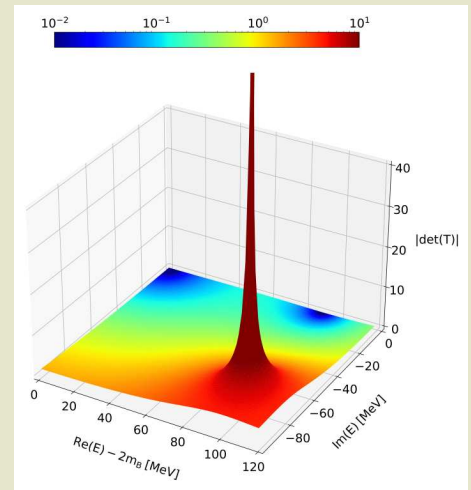
- In

- [J. Hoffmann, M.W., Phys. Rev. D **111**, no. 5, 054507 (2025), [arXiv:2412.06607]]

- heavy quark spins included.

- $\bar{b}\bar{b}ud$ resonance shifted upwards, slightly above the B^*B^* threshold.

- Physical reason: the relevant attractive potential does not only correspond to a lighter BB pair, but has also a heavier dominating B^*B^* contribution.



BO2: Further $\bar{b}\bar{b}qq$ results (3)

- Structure of the QCD-stable $\bar{b}\bar{b}ud$ tetraquark: meson-meson (BB) versus diquark-antidiquark (Dd).

– Use not just one but two operators,

$$\mathcal{O}_{BB,\Gamma} = 2N_{BB}(\mathcal{C}\Gamma)_{AB}(\mathcal{C}\tilde{\Gamma})_{CD}\left(\bar{Q}_C^a(-\mathbf{r}/2)\psi_A^{(f)a}(-\mathbf{r}/2)\right)\left(\bar{Q}_D^b(+\mathbf{r}/2)\psi_B^{(f)b}(+\mathbf{r}/2)\right)$$

$$\mathcal{O}_{Dd,\Gamma} = -N_{Dd}\epsilon^{abc}\left(\psi_A^{(f)b}(\mathbf{z})(\mathcal{C}\Gamma)_{AB}\psi_B^{(f')c}(\mathbf{z})\right)$$

$$\epsilon^{ade}\left(\bar{Q}_C^f(-\mathbf{r}/2)U^{fd}(-\mathbf{r}/2;\mathbf{z})(\mathcal{C}\tilde{\Gamma})_{CD}\bar{Q}_D^g(+\mathbf{r}/2)U^{ge}(+\mathbf{r}/2;\mathbf{z})\right),$$

compare the contribution of each operator to the $\bar{b}\bar{b}$ potential $V_{qq,jz,\mathcal{P},\mathcal{P}_x}(r)$.

[P. Bicudo, A. Peters, S. Velten, M.W., Phys. Rev. D **103**, 114506 (2021) [arXiv:2101.00723]]

- $r \lesssim 0.2$ fm: Clear diquark-antidiquark dominance.
- 0.5 fm $\lesssim r$: Essentially a meson-meson system.
- Integrate over t to estimate the composition of the tetraquark: % $BB \approx 60\%$, % $Dd \approx 40\%$.

