Stefano Frixione

A few thoughts on ISR at lepton colliders

(in 11 easy messages – look for $\#i$)

62^{nd} Bormio winter conference, 21/1/2026

Strategist:[noun] someone with a lot of skill and experience in planning, especially in military, political, or business matters

The European Strategy for particle physics

cannot be accused of being modest

- The tall guy is FCC-hh
- The not-that-tall-but-still-tall lad in front of it is FCC-ee

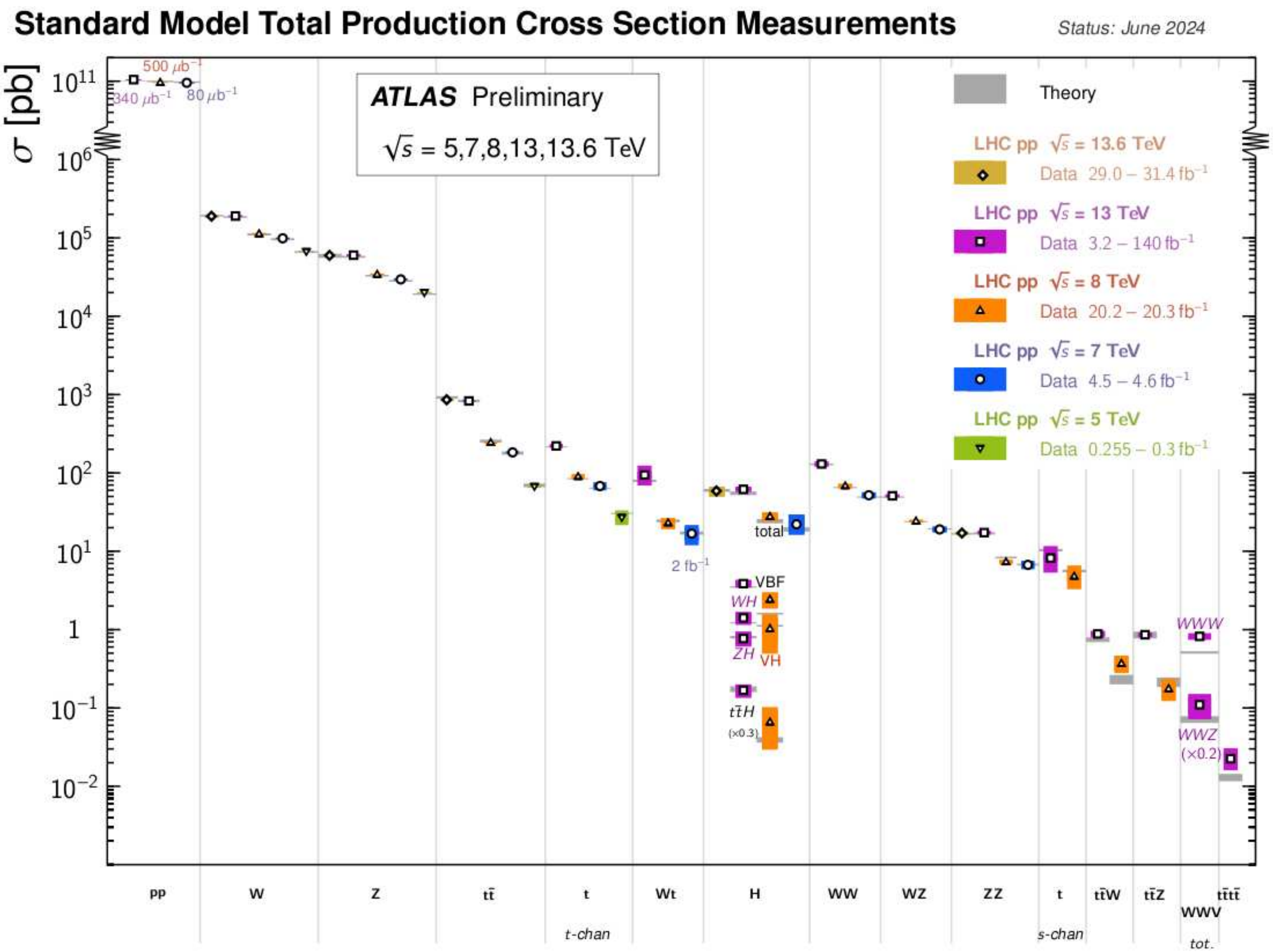
  (aka we'll-collect-the-same-stat-as-LEP-in-a-millisec-but-then-we'll-keep-going)
- The slightly-smaller-but-still-sizable chap in the back is the amount of theoretical work necessary to make any sense of it
- The afterthought in the foreground is a muon collider

A tad intimidated by the sheer scale of the project?

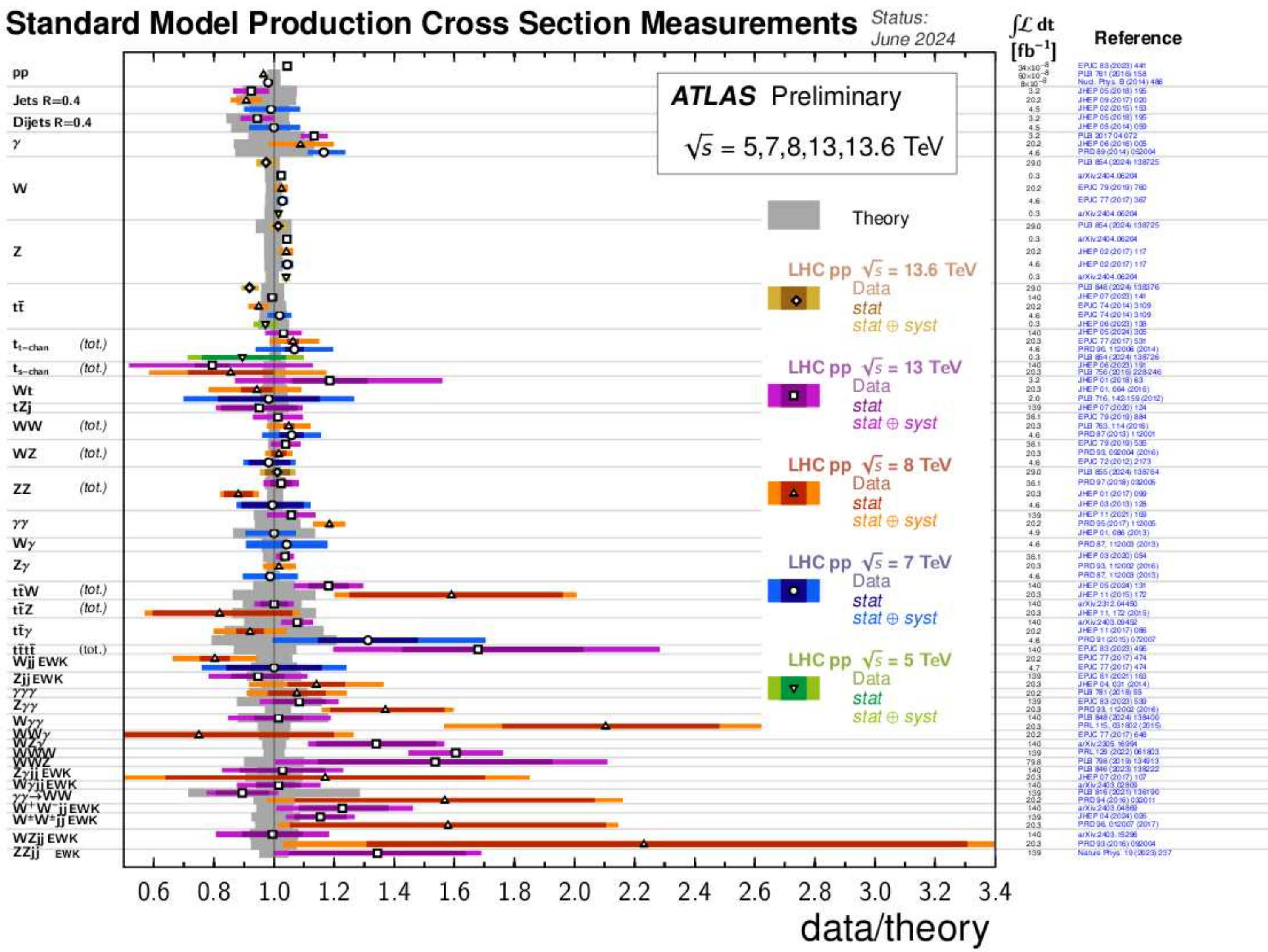
We need not to be, since:

a) we'll be long dead

b) we stand on the shoulders of giants $\longrightarrow$



The LHC has been/is/(inshallah)will be an astonishing success



The LHC has been/is/(inshallah)will be an astonishing success

Executive summary for the LHC:

Experiments agree, within (generally small) uncertainties, with the SM* across 12 orders of magnitude, and counting

Some of us didn't necessarily expect this: ⟶

* Predictions being the result of hard labour by a vast community

“I was shocked when SUSY particles were not discovered in the early days of the LHC” (M. Peskin in Symmetry, 12/1/21)*

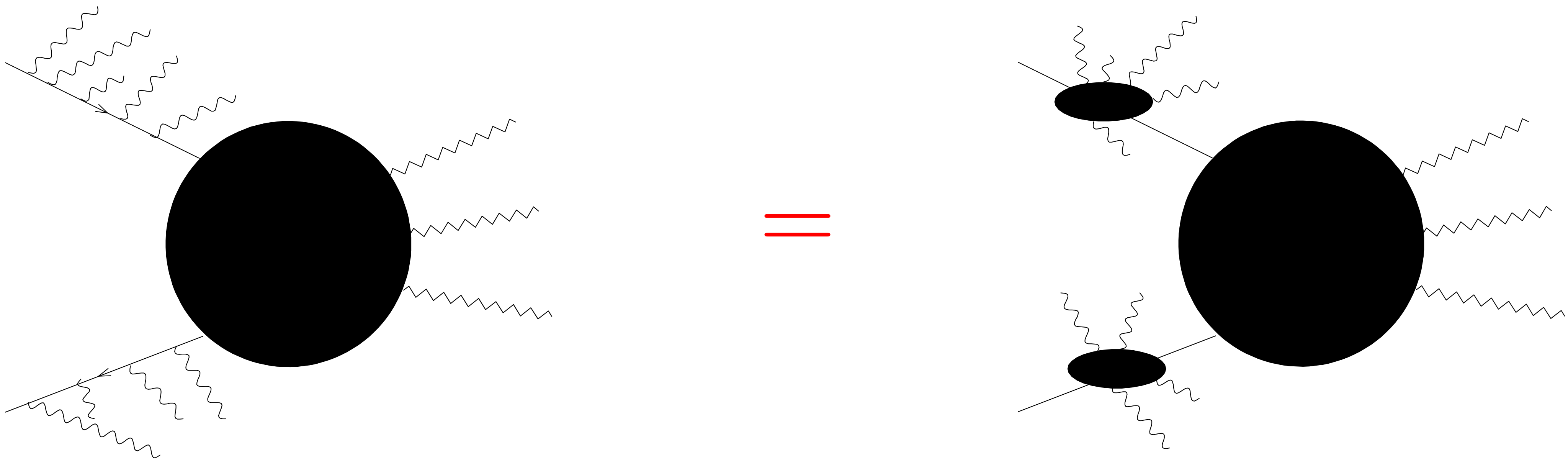
Part of the surprise stems from the fact that in ∼2007 we did not understand the SM (ie mostly QCD) as we now do

* The fact that a publication with title “Symmetry” has been issued on a date which is a palindrome does *NOT* necessarily imply that I've made this up

Perturbative QCD computations are extremely challenging – nowadays, NLO results are routine, but already NNLO ones require an immense amount of work, and NNNLO borders on the esoteric (N^{k}LO/LO$=\alpha_S^k$)

They are also unphysical, unless supplemented with non-perturbative quantities (the PDFs) through the factorisation theorem

Collinear factorisation



$$d\sigma = \text{PDF} \star \text{PDF} \star d\hat{\sigma}$$

$$d\sigma^{(H_1 H_2)}(P_1, P_2) = \sum_{ab} \int dz_1 dz_2 f_a^{(H_1)}(z_1, \mu^2) f_b^{(H_2)}(z_2, \mu^2) d\hat{\sigma}_{ab}(z_1 P_1, z_2 P_2, \mu$$

In simpler words:

- Factorisation is crucial
- We know what we're doing

Enters the future (~ 2050 or thereabouts):

Somewhere, someone will build an e^+e^- collider

(linear or circular; most likely FCCee)

(wearing a theory hat) That's an easy one, isn't it? We know how to calculate things from LEP, and most of the codes are still around (!)

Not quite: the (projected) immense precision of the experimental measurements will require comparable theoretical results, which the LEP-era ones are most definitely not

Hence, my first four take-home messages:

$\#1$:

EW is the new QCD.

Unfortunately, there are fundamental differences between the two theories, and having learned (the hard way) to carry out QCD computations will not *necessarily and/or completely* help with EW ones

$\#2$:

Forget E$_{\text{lectro}}$W$_{\text{eak}}$P$_{\text{seudo}}$O$_{\text{bservables}}$

(vehemently contested by some – too long to explain why I am right, but I am right nonetheless)

$\#3$:

Roughly speaking, mechanisms which are relevant to a ultra-high precision e^+e^- collider are so for a less-precise but more-energetic muon collider as well

Which must be kept in mind when considering the next item:

$\#4$:

Treating W and Z as partons is *leads to immense uncertainties*, unless the (muon) collider c.m. energy is in the ballpark of a 100 TeV

Barring the unthinkable, discoveries at e^+e^- machines are the outcomes of careful comparisons between theoretical predictions and experimental data

Therefore, let's dissect one such generic prediction

Consider a generic cross section, sufficiently inclusive:

$$\sigma = \alpha^b \sum_{n=0}^{\infty} \alpha^n \sum_{i=0}^{n} \sum_{j=0}^{n} \varsigma_{n,i,j} L^i \ell^j$$

This is symbolic, and only useful to expose the presence of:

$$\ell = \log \frac{Q^2}{\langle E_\gamma \rangle^2}\,, \qquad L = \log \frac{Q^2}{m^2}$$

Numerology: consider the production of $Z \to ll$ at:

- $\sqrt{Q^2} = m_Z$

$$\begin{aligned} L = 24.18 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} L = 0.06 \\ 0 \leq m_{ll} \leq m_Z\,, \quad \ell = 6.89 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} \ell = 0.017 \\ m_Z - 1\ \text{GeV} \leq m_{ll} \leq m_Z\,, \quad \ell = 10.60 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} \ell = 0.026 \end{aligned}$$

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Numerology: consider the production of $Z \to ll$ at:

- $\sqrt{Q^2} = 500$ GeV

$$\begin{aligned} L = 27.59 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} L = 0.069 \\ 0 \le m_{ll} \le m_Z\,, \quad \ell = 1.449 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} \ell = 0.0036 \\ m_Z - 1\ \mathrm{GeV} \le m_{ll} \le m_Z\,, \quad \ell = 1.453 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} \ell = 0.0036 \end{aligned}$$

It takes a lot of brute force (i.e. fixed-order results to some $\mathcal{O}(\alpha^n)$) to overcome the enhancements due to L and ℓ.

It is always convenient to first improve by means of factorisation formulae:

$$d\sigma(L,\ell) = \mathcal{K}_{soft}(\ell;L)\beta(L)d\mu \quad (1)$$

$$= \mathcal{K}_{coll}(L;\ell) \otimes d\hat{\sigma}(\ell) \quad (2)$$

Use of:

(1) YFS (resummation of ℓ)

(2) collinear factorisation (resummation of L)

Common features: $\mathcal{K}$ is an *all-order* universal factor; β and $d\hat{\sigma}$ are process-specific and computed order by order

(still brute force, but to a lesser extent)

YFS

Aim: soft resummation for:

$$\left\{ e^+(p_1) + e^-(p_2) \longrightarrow X(p_X) + \sum_{i=0}^{n} \gamma(k_n) \right\}_{n=0}^{\infty}$$

Achieved with:

$$\begin{aligned} d\sigma(L,\ell) &= \mathcal{K}_{soft}(\ell;L)\beta(L)d\mu \\ &= e^{Y(p_1,p_2,p_X)} \sum_{n=0}^{\infty} \beta_n \left(\mathcal{R}p_1, \mathcal{R}p_2, \mathcal{R}p_X; \{k_i\}_{i=0}^{n}\right) d\mu_{X+n\gamma} \end{aligned}$$

This is symbolic, and stands for both the EEX and CEEX approaches [hep-ph/0006359 Jadach, Ward, Was] that build upon the original YFS work [Ann.Phys.13(61)379]

EEX: exclusive (in the photons) exponentiation, matrix element level

CEEX: coherent exclusive (in the photons) exponentiation, amplitude level, including interference

YFS

Aim: soft resummation for:

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Achieved with:

$$d\sigma(L,\ell) \;=\; e^{Y(p_1,p_2,p_X)} \sum_{n=0}^{\infty} \beta_n \left(\mathcal{R}p_1, \mathcal{R}p_2, \mathcal{R}p_X; \{k_i\}_{i=0}^{n}\right) d\mu_{X+n\gamma}$$

- Y essentially universal (process dependence only through kinematics); resums ℓ
- The soft-finite β_n are process-specific, and are constructed by means of local subtractions involving matrix elements and eikonals (i.e. *not* BN)

$$\beta_n = \alpha^b \sum_{i=0}^{n} \alpha^i \sum_{j=0}^{i} c_{n,i,j} L^j$$

- For a given n, matrix elements have different multiplicities, hence the need for the kinematic mapping $\mathcal{R}$

Collinear factorisation

Aim: collinear resummation for:

$$\left\{ k(p_k) + l(p_l) \longrightarrow X(p_X) + \sum_{i=0}^{n} a_i(k_n) \right\}_{n=0}^{\infty} \qquad a_i = e^{\pm}, \gamma \ldots$$

with initial-state particles stemming from beams:

$$(k,l) = (e^+, e^-)\,, \quad (k,l) = (e^+, \gamma)\,, \quad (k,l) = (\gamma, e^-)\,, \quad (k,l) = (\gamma, \gamma)\,, \ldots$$

Master formula:

$$\begin{aligned} d\sigma(L,\ell) &= \mathcal{K}_{coll}(L;\ell) \otimes d\hat{\sigma}(\ell) \\ \longrightarrow \quad d\sigma_{kl} &= \sum_{ij} \int dz_+ dz_- \, \Gamma_{i/k}(z_+, \mu^2, m^2) \, \Gamma_{j/l}(z_-, \mu^2, m^2) \\ & \qquad \times d\hat{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2; p_X, \{k_i\}_{i=0}^{n}) \end{aligned}$$

- $\Gamma_{\alpha/\beta}$ universal (the PDF); resums L
- The collinear-finite $d\hat{\sigma}_{ij}$ are process-specific, and are the standard short-distance matrix elements, constructed order by order (*with* BN). May or may not include resummation of other large logs (including ℓ)

YFS vs collinear factorisation

Both are systematically improvable in perturbation theory:
in YFS the β_n's (fixed-order), in collinear factorisation both the PDFs (logarithmic accuracy) and the $d\hat{\sigma}$'s (fixed-order, resummation)

+ YFS: very little room for systematics. Exceptions are the kinematic mapping $\mathcal{R}$, and the quark masses (when the quarks are radiators). Renormalisation schemes??

− Collinear factorisation: systematic variations much larger. At the LL (used in phenomenology so far) a rigorous definition of uncertainties is impossible (parameters are arbitrary), and comparisons with YFS are largely fine tuned

− YFS: the computations of β_n are not standard (EEX) and highly non-trivial (CEEX)

+ Collinear factorisation: the computations of $d\hat{\sigma}_{ij}$ are standard

$\#5$:

YFS is naturally suited to describing threshold production of narrow objects (such as the Z); collinear factorisation is appropriate for anything else

(but one can consider threshold processes as well)

Collinear factorisation has many analogies with its hadronic counterpart: what has been learned at the LHC will not be wasted

#6:

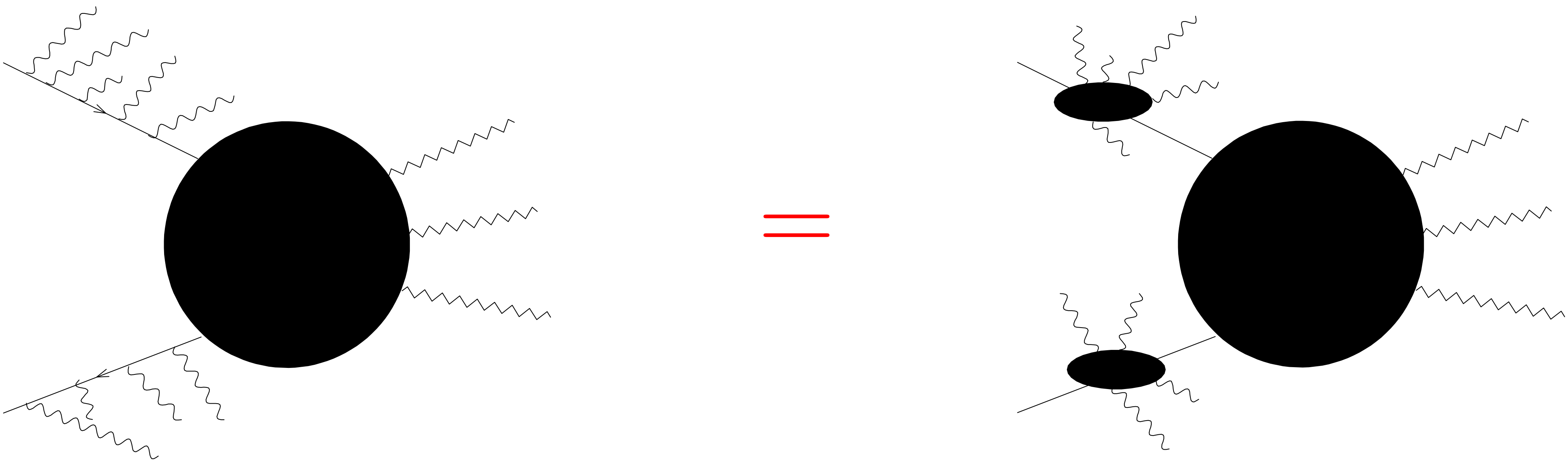
If the message was lost in the technicalities, it is this: the factorisation theorem I am speaking about has *the same* functional form as its QCD counterpart (for all practical purposes, the incoming leptons behave as hadrons)

The crucial difference: the PDFs can *entirely* be computed perturbatively

Hence, let's stick to

COLLINEAR FACTORISATION

Collinear factorisation



$$d\sigma = \text{PDF} \star \text{PDF} \star d\hat{\sigma}$$

$$d\sigma^{(e^+e^-)}(P_1, P_2) = \sum_{ab} \int dz_1 dz_2 f_a^{(e^+)}(z_1, \mu^2) f_b^{(e^-)}(z_2, \mu^2) d\hat{\sigma}_{ab}(z_1 P_1, z_2 P_2, \mu^2$$

All physics simulations based on collinear factorisation done so far are based on a LL-accurate picture

This is not tenable at high energies/high statistics:

- accuracy is insufficient (see e.g. W^+W^- production)
- systematics not well defined

Step 0 was to upgrade PDFs from LL to NLL accuracy: increase of precision, and meaningful systematics, in particular factorisation-scheme dependence

z-space LO+LL PDFs $(\alpha \log(Q^2/m^2))^k$:

$\sim$ 1992

- $0 \le k \le \infty$ for $z \simeq 1$ (Gribov, Lipatov)
- $0 \le k \le 3$ for $z < 1$ (Skrzypek, Jadach; Cacciari, Deandrea, Montagna, Nicrosini; Skrzypek)
- matching between these two regimes
- for e^-

z-space NLO+NLL PDFs $(\alpha \log(Q^2/m^2))^k + \alpha\,(\alpha \log(Q^2/m^2))^{k-1}$:

$\longrightarrow$ 1909.03886, 1911.12040, 2105.06688, 2207.03265 (Bertone, Cacciari, Frixione, Stagnitto, Zaro, Zhao)

- $0 \le k \le \infty$ for $z \simeq 1$
- $0 \le k \le 3$ for $z < 1$ $\iff$ $\mathcal{O}(\alpha^3)$
- matching between these two regimes
- for e^+, e^-, γ, and light quarks
- both numerical and analytical
- factorisation schemes: $\overline{\mathrm{MS}}$ and Δ (that has DIS-like features)

Bear in mind that PDFs are fully defined only after adopting a definite *factorisation scheme*, which is the choice of the finite terms associated with the subtraction of the collinear poles

- ♦ 1911.12040 $\longrightarrow$ $\overline{\rm MS}$
- ♦ 2105.06688 $\longrightarrow$ a DIS-like scheme (called Δ)

At variance with the QCD case, there is also an interesting *renormalisation-scheme* dependence of QED PDFs

Asymptotic $\overline{\mathrm{MS}}$ solution

Non-singlet $\equiv$ singlet; photon is more complicated

$$\Gamma_{\mathrm{NLL}}(z,\mu^2) \stackrel{z\to 1}{\longrightarrow} \frac{e^{-\gamma_{\mathrm{E}}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)}\, \xi_1 (1-z)^{-1+\xi_1}$$
$$\times \left\{ 1 + \frac{\alpha(\mu_0)}{\pi} \left[\left(L_0 - 1\right) \left(A(\xi_1) + \frac{3}{4} \right) - 2B(\xi_1) + \frac{7}{4} \right.\right.$$
$$\left.\left. + \left(L_0 - 1 - 2A(\xi_1) \right) \log(1-z) - \log^2(1-z) \right] \right\}$$

where $L_0 = \log \mu_0^2/m^2$, and:

$$\begin{aligned} A(\kappa) &= -\gamma_{\mathrm{E}} - \psi_0(\kappa) \\ B(\kappa) &= \frac{1}{2}\gamma_{\mathrm{E}}^2 + \frac{\pi^2}{12} + \gamma_{\mathrm{E}}\,\psi_0(\kappa) + \frac{1}{2}\,\psi_0(\kappa)^2 - \frac{1}{2}\,\psi_1(\kappa) \end{aligned}$$

with:

$$
\begin{aligned}
\xi_1 &= 2t - \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1-e^{-2\pi b_0 t}\right)\left(\frac{20}{9}n_F + \frac{4\pi b_1}{b_0}\right) \\
&= 2t + \mathcal{O}(\alpha t) = \eta_0 + \ldots \\
\hat{\xi}_1 &= \frac{3}{2}\,t + \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1-e^{-2\pi b_0 t}\right)\left(\lambda_1 - \frac{3\pi b_1}{b_0}\right) \\
&= \frac{3}{2}\,t + \mathcal{O}(\alpha t) = \lambda_0\eta_0 + \ldots \\
\lambda_1 &= \frac{3}{8} - \frac{\pi^2}{2} + 6\zeta_3 - \frac{n_F}{18}(3+4\pi^2)
\end{aligned}
$$

and:

$$
\begin{aligned}
t &= \frac{1}{2\pi b_0}\log\frac{\alpha(\mu)}{\alpha(\mu_0)} \\
&= \frac{\alpha(\mu)}{2\pi}\,L - \frac{\alpha^2(\mu)}{4\pi}\left(b_0 L^2 - \frac{2b_1}{b_0}L\right) + \mathcal{O}(\alpha^3)\,, \qquad L = \log\frac{\mu^2}{\mu_0^2}\,.
\end{aligned}
$$

Asymptotic Δ solution

Non-singlet $\equiv$ singlet; photon is trivial

$$\Gamma_{\rm NLL}(z,\mu^2) \quad \stackrel{z\to 1}{\longrightarrow} \quad \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)}\, \xi_1 (1-z)^{-1+\xi_1} \times \left[\left(1+\frac{3\alpha(\mu_0)}{4\pi}L_0\right)\sum_{p=0}^{\infty}\mathcal{S}_{1,p}(z) - \frac{\alpha(\mu_0)}{\pi}L_0 \sum_{p=0}^{\infty}\mathcal{S}_{2,p}(z)\right]$$

The $\mathcal{S}_{i,p}(z)$ functions are increasingly suppressed at $z \to 1$ with growing p. The dominant behaviour is:

$$\Gamma_{\rm NLL}(z,\mu^2) \quad \stackrel{z\to 1}{\longrightarrow} \quad \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)}\, \xi_1 (1-z)^{-1+\xi_1} \times \left[\frac{\alpha(\mu)}{\alpha(\mu_0)} + \frac{\alpha(\mu)}{\pi}L_0\left(A(\xi_1)+\log(1-z)+\frac{3}{4}\right)\right]$$

- A vastly different logarithmic behaviour w.r.t. the $\overline{\rm MS}$ case
  However, $\Gamma_{\rm NLL}^{(\overline{\rm MS})} - \Gamma_{\rm NLL}^{(\Delta)} = \mathcal{O}(\alpha^2)$

Key facts

- Both $\overline{\mathrm{MS}}$ and Δ results feature an integrable singularity at $z \to 1$, basically identical to the LL one

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  (to some unknown accuracy)

- Owing to the integrable singularity, it is essential to have large-z analytical results: the PDFs convoluted with cross sections are obtained by matching the small- and intermediate-z numerical solution with the large-z analytical one

$\#7$:

Use NLL PDFs and the Δ scheme rather than LL ones and $\overline{\mathrm{MS}}$

Δ is so useful owing to everything being perturbatively computable:

its hadronic counterpart has been all but forgotten

On top of increased precision, for sensible phenomenology we need:

[2207.03265; Bertone, Cacciari, Frixione, Stagnitto, Zaro, Zhao]

- evolution with all fermion families (leptons and quarks), including their respective mass thresholds
- renormalisation schemes other than $\overline{\mathrm{MS}}$: $\alpha(m_Z)$ and G_μ
- assess implications by studying realistic observables in physical processes

Sample results for:

$$
\begin{aligned}
e^+e^- &\longrightarrow q\bar{q} \\
e^+e^- &\longrightarrow t\bar{t} \\
e^+e^- &\longrightarrow W^+W^-
\end{aligned}
$$

with $q\bar{q}$ production (massless quarks) restricted to ISR QED radiation.
The other two are in the SM

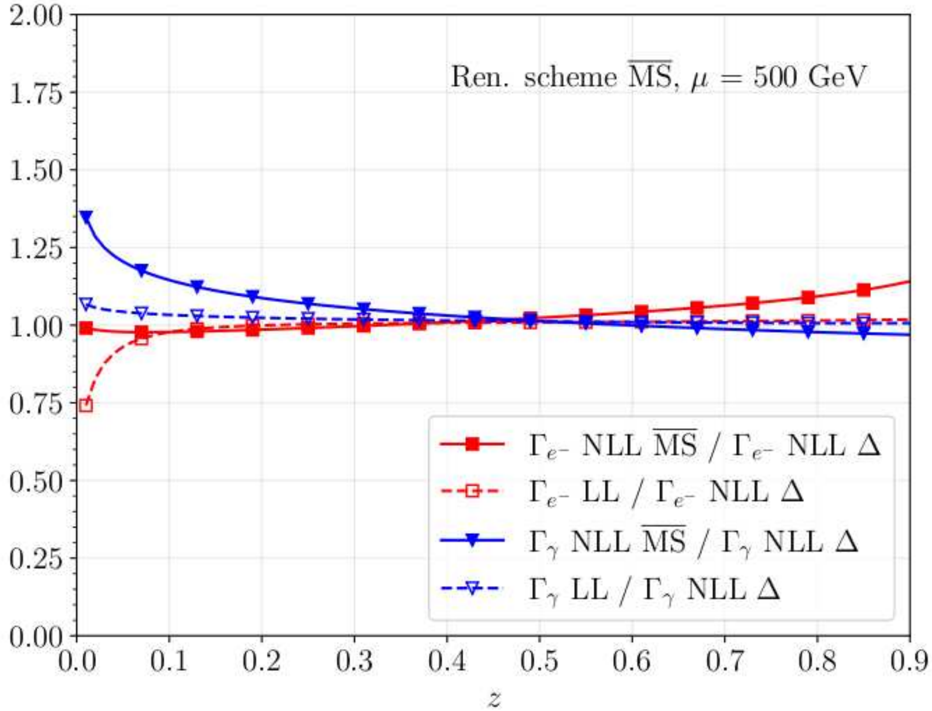
NLO accuracy, automated generation with MG5_aMC@NLO
(this version is now public, v3.5.0) [2108.10261; Frixione, Mattelaer, Zaro, Zhao]

What is plotted:

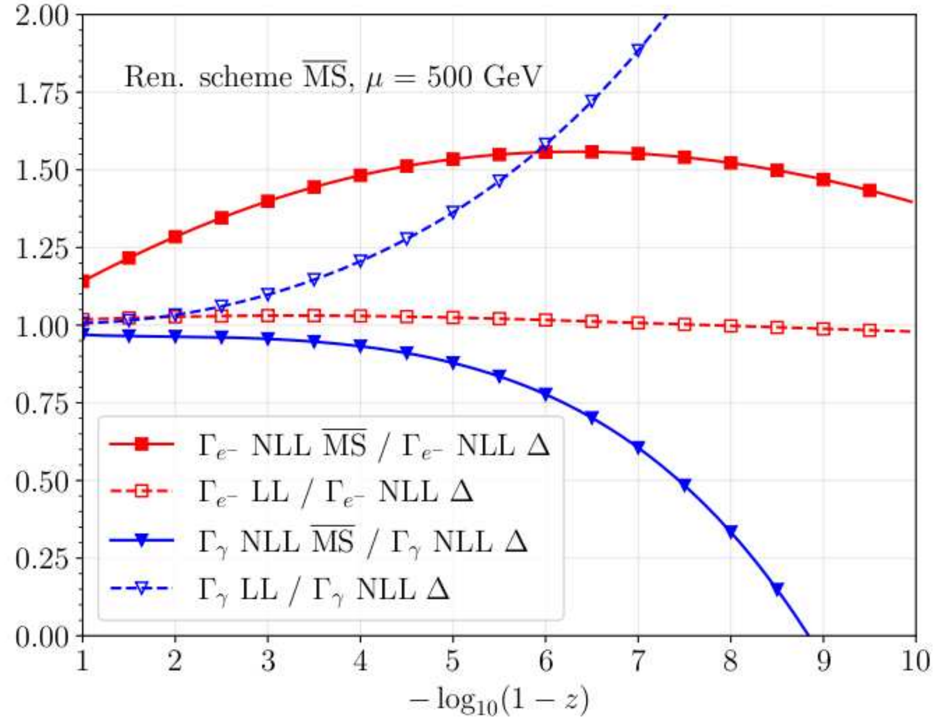
$$
\sigma(\tau_{min}) = \int d\sigma\, \Theta\left(\tau_{min} \le \frac{M^2_{p\bar{p}}}{s}\right), \qquad p = q\,, t\,, W^+
$$

$\tau_{min} \sim 1$ is sensitive to soft emissions (not resummed)

Dependence of PDFs on factorisation scheme



$z < 1$

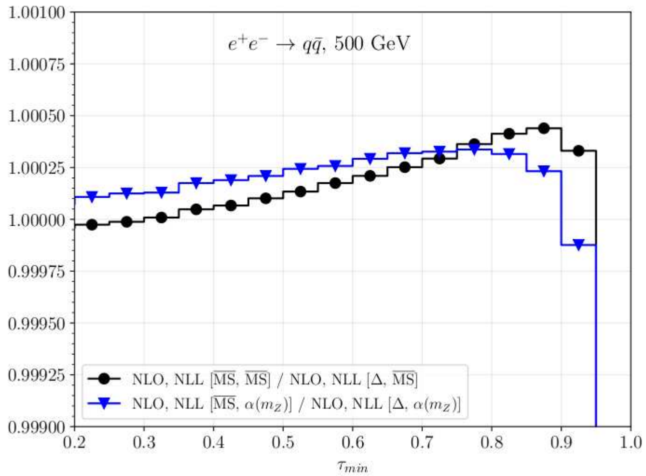


$z \simeq 1$

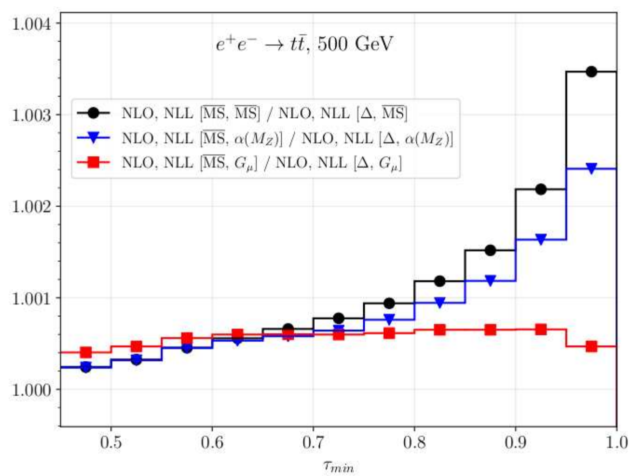
Very large dependence at the NLL at $z \to 1$ ($\mathcal{O}(1)$); this is particularly significant *(but unphysical!)* since the electron has an integrable divergence there

Electron at NLL in the Delta scheme close to the LL result (differences of $\mathcal{O}(5\%)$)

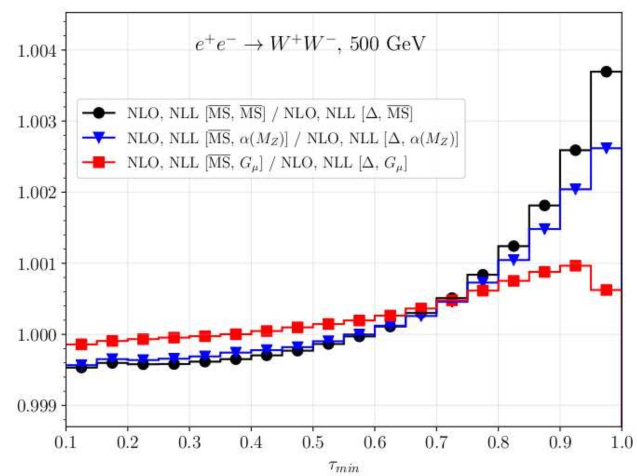
Dependence of observables on factorisation scheme



$q\bar{q}$



$t\bar{t}$



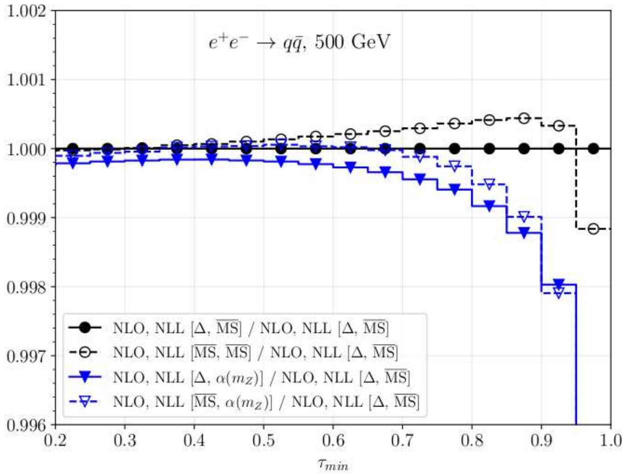
W^+W^-

$\mathcal{O}(1)$ differences for PDFs down to $\mathcal{O}(10^{-4} - 10^{-3})$ for observables

In the $\overline{\rm MS}$ scheme, huge cancellations between PDFs and short-distance cross sections

Behaviour qualitatively similar for different renormalisation schemes

Factorisation vs renormalisation scheme dependence



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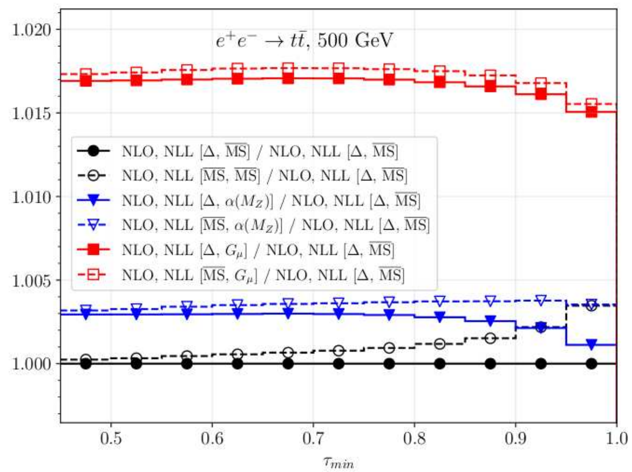

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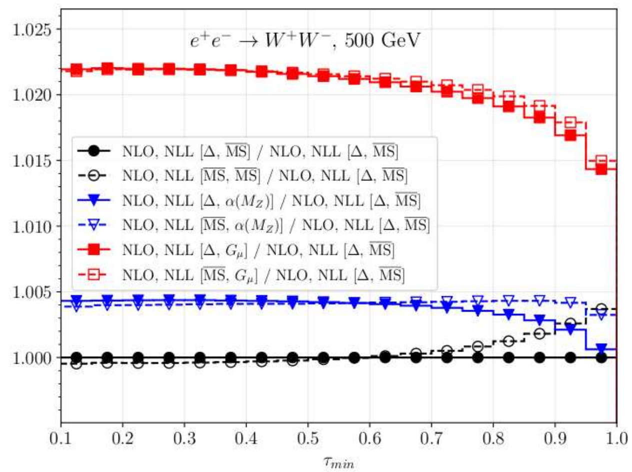
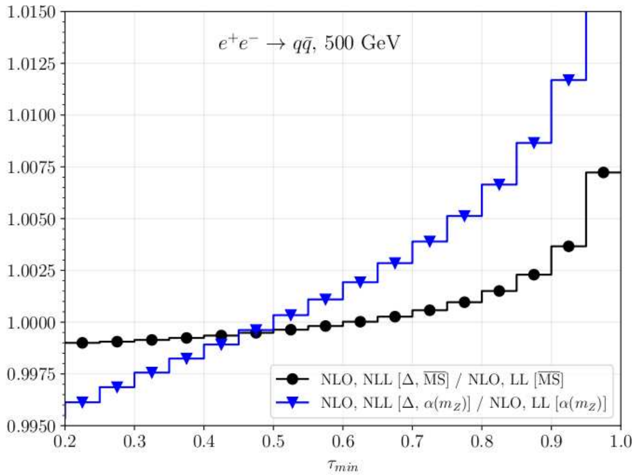


W^+W^-

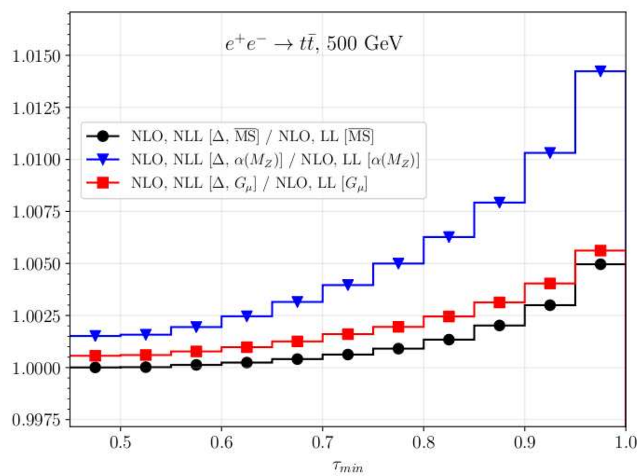
Renormalisation-scheme dependence much larger than factorisation-scheme dependence, with process-dependent pattern

Depending on the precision, renormalisation scheme is an informed choice; factorisation scheme always induces a systematic

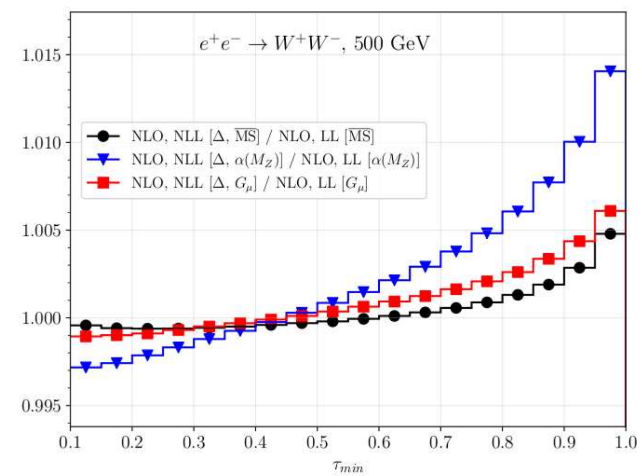
NLL vs LL



$q\bar{q}$



$t\bar{t}$

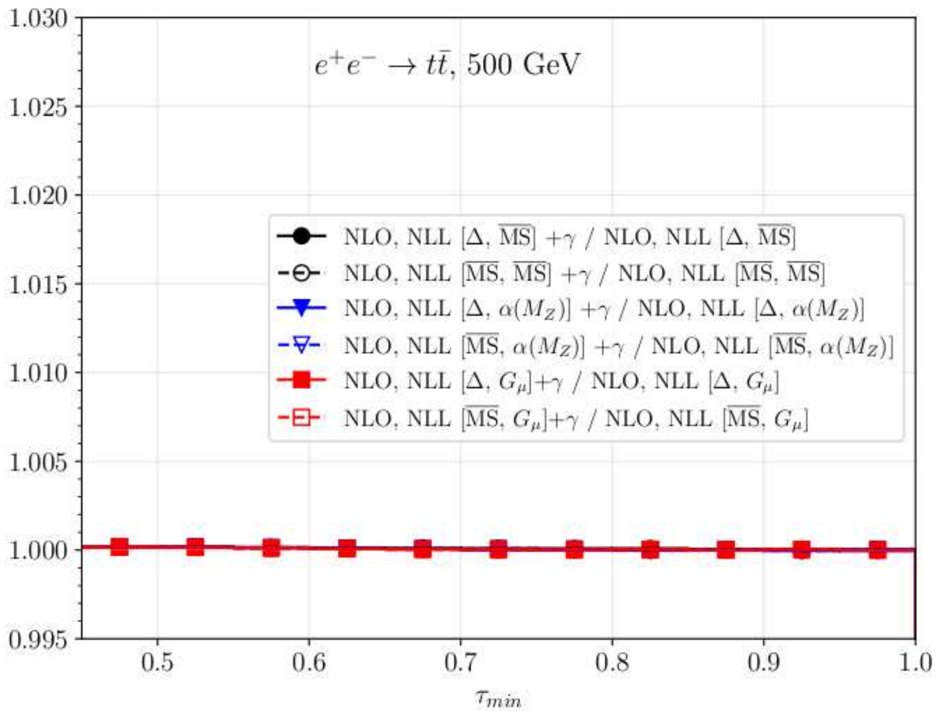


W^+W^-

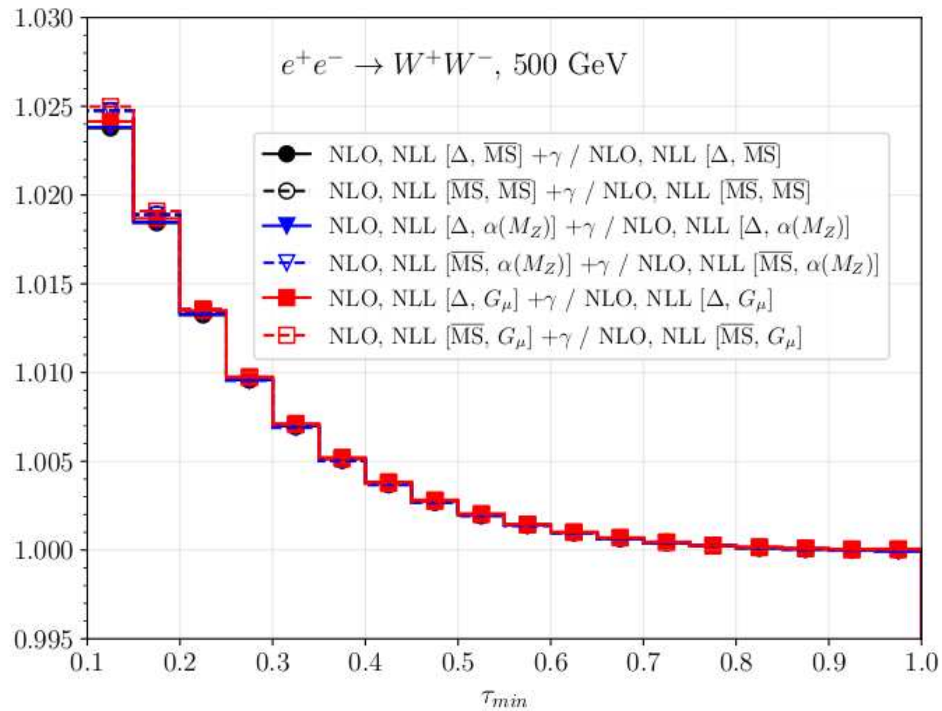
Effects are non trivial

Pattern dependent on the process (and on the observable) as well as on the renormalisation scheme

Impact of $\gamma\gamma$ channel



$t\bar{t}$



W^+W^-

Essentially independent of factorisation and renormalisation schemes: a genuine physical effect

Utterly negligible for $t\bar{t}$, significant for W^+W^- – process dependence is not surprising

Thus:

- The inclusion of NLL contributions into the electron PDF has an impact of $\mathcal{O}(1\%)$ (precise figures are observable and renormalisation-scheme dependent)
- This estimate does not include the effects of the photon PDF
- The comparison between $\overline{\rm MS}$- and Δ-based results shows differences compatible with non-zero $\mathcal{O}(\alpha^2)$ effects, as expected
  (but: these are potentially *large in the soft region*)
- Renormalisation-scheme dependence is of $\mathcal{O}(0.5\%)$

$\#8$: If the target is a $10^{-\rm some\ large\ number}$ relative precision, these effects must *all* be taken into account

The power of automation

$\sqrt{s}$ [GeV]	$\sigma(e^+e^- \to q\bar{q})$ [pb]	$\sigma(e^+e^- \to W^+W^-)$ [pb]	$\sigma(e^+e^- \to ZH)$ [pb]	$\sigma(W^+W^- \to H)$ [pb]	$\sigma(e^+e^- \to t\bar{t})$ [pb]
20.0	898.8	-	-	-	-
30.0	434.6	-	-	-	-
40.0	259.9	-	-	-	-
50.0	182.1	-	-	-	-
60.0	153.0	-	-	-	-
70.0	177.7	-	-	-	-
80.0	423.9	-	-	-	-
88.0	3891.0	-	-	-	-
91.2	29250.0	-	-	-	-
94.0	8953.0	-	-	-	-
125.0	417.9	-	-	-	-
157.5	177.4	-	-	-	-
162.5	162.0	-	-	-	-
165.0	155.2	8.773	-	0.00021	-
217.0	-	17.63	0.04278	0.004497	-
240.0	-	16.62	0.1998	0.005859	-
350.0	-	11.57	0.1306	0.024613	0.3771
360.0	-	11.22	0.1236	0.027064	0.5534

Cross-sections have been computed with MADGRAPH5_AMC@NLOv̇3.5.0 [1, 6], exploiting the recent developments for lepton colliders [9, 3]. In particular, ISR partonic densities with NLL-accurate evolution [7, 2, 3] have been employed, using the so-called Δ factorisation scheme [8]. All cross-sections include NLO EW and QCD corrections (the latter only when relevant), with the exception of $e^+e^- \to q\bar{q}$ where only QCD corrections are computed. NLO EW corrections are computed in the G_μ scheme; all fermions, with the exception of the top quark, are considered massless. Contributions from photons in the initial state are included whenever NLO EW corrections are computed. It is worth to note that the only processes where initial-state photons contribute at the LO are $e^+e^- \to t\bar{t}$ and $e^+e^- \to W^+W^-$. The following parameters have been employed:

$$m_t = 173.33\,\text{GeV}\,,\ \mathrm{m_W} = 80.419\,\text{GeV}\,,\ \mathrm{m_Z} = 91.189\,\text{GeV}\,,\ \mathrm{m_H} = 125\,\text{GeV}\,,\ \mathrm{G}_\mu = 1.166\times10^{-5}\,\text{GeV}^{-2}\,. \quad (1)$$

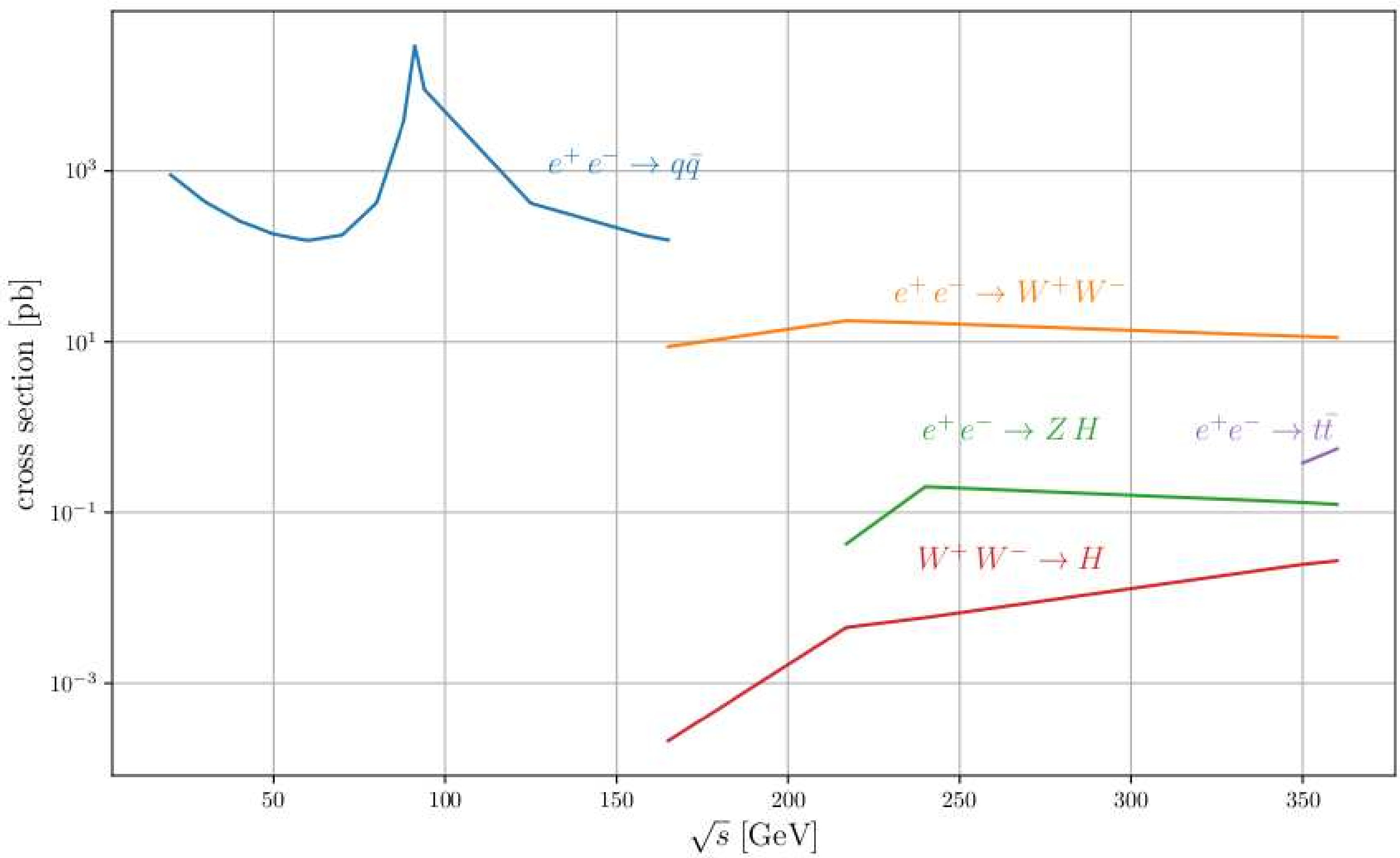
The cross section for Higgs production in VBF has been obtained as the difference of the cross sections for the processes $e^+e^- \to H\nu_e\bar{\nu}_e$ and $e^+e^- \to H\nu_\mu\bar{\nu}_\mu$, both computed at NLO EW accuracy in the complex mass scheme [4, 5]. For these cross sections, the following non-zero widths have been employed:

$$\Gamma_t = 1.3776\,\text{GeV}\,,\quad \Gamma_\mathrm{W} = 2.093\,\text{GeV}\,,\quad \Gamma_\mathrm{Z} = 2.499\,\text{GeV}\,, \quad (2)$$

MG5_aMC@NLO, EW(+QCD) NLO accurate results, NLL PDFs

A few days of work (Selvaggi, Zaro)

The power of automation



MG5_aMC@NLO, EW(+QCD) NLO accurate results, NLL PDFs

A few days of work (Selvaggi, Zaro)

$\#9$:

Automation has allowed an exponential growth of data-theory comparisons at the LHC (and freed a few PhD slaves in the process).

There is no reason why its success cannot be replicated in e^+e^- physics

Are we done?

Not quite

- What was done at the NLL gives one a blueprint to go to NNLL, if need be. Most of the ingredients are available from QCD, but one still has to figure out the $z \to 1$ behaviour analytically

- In an orthogonal direction, one must achieve an exclusive generation, at the desired logarithmic accuracy

Exclusive means the ability to retain the information on the dof's of the particles stemming from the (ISR) branchings that do not enter the hard process

- Well established within YFS; not so much within collinear factorisation

- We cannot blindly apply MC@NLO or Powheg: hadron and lepton PDFs have dramatically different behaviours

- Besides, there is currently no NLL-accurate ISR hadronic shower

$\#10$:

Sooner or later, progress will rely almost solely on the availability of matched predictions – it has happened in hadronic collisions, and it will happen in e^+e^- collisions as well

Finally...

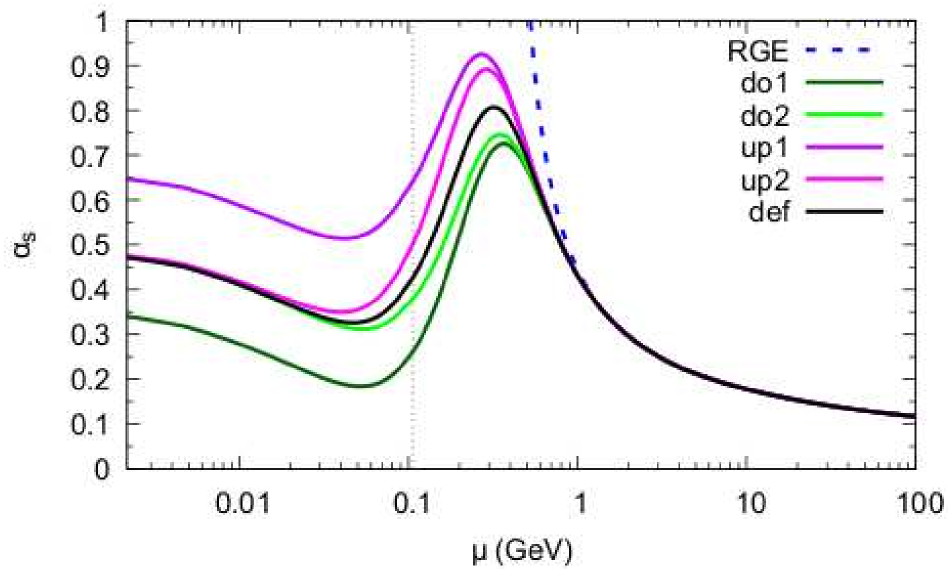
At a certain level of precision, the impact of the *strongly-interacting* partonic content of the incoming leptons cannot be ignored

In YFS, the corresponding contributions enter the β_n terms; in collinear factorisation, they entail the presence of quarks and gluon PDFs

- I'm not aware of attempts to address this issue in YFS
- In collinear factorisation there are now two different approaches, applied so far to the PDFs of the muon (the case of the electron is conceptually identical)

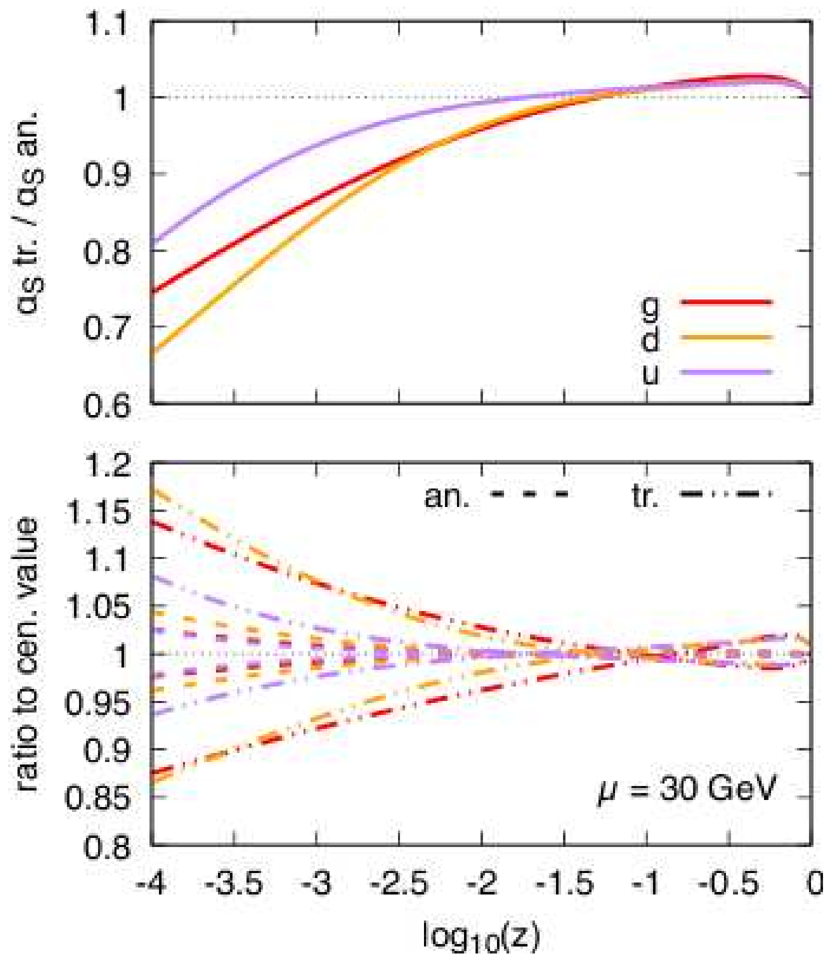
The quark and gluon PDFs force one to consider α_s in the infrared

- 2103.09844 (Han, Ma, Xie) and 2303.16964 (Garosi, Marzocca, Trifinopoulos) bypass the problem by setting $\alpha_s(\mu) = 0$ for $\mu < Q_0$, with $Q_o = \mathcal{O}(0.5\ \mathrm{GeV})$
  ("truncated" approach)
- 2309.07516 (SF, Stagnitto) adopts a parametrisation of α_s in the infrared motivated by dispersion relations (Webber; Dokshitzer, Marchesini, Webber)
  ("analytical" approach)



In the case of the muon PDFs, there are significant differences between the truncated and the analytical approaches

I expect these to be potentially even larger for electron PDFs



PDFs ratios and their uncertainties

In the case of the muon PDFs, there are significant differences between the truncated and the analytical approaches

I expect these to be potentially even larger for electron PDFs

$\sigma(p_T^{cut} = 10$ GeV) [pb]	an.	tr.
$\mathcal{O}(\alpha_S^2)$	$18.33^{+1.30\%}_{-1.25\%}$	$15.00^{+10.23\%}_{-10.99\%}$
γ-ind.	$8.24^{+0.68\%}_{-0.91\%}$	$7.56^{+3.71\%}_{-3.75\%}$
total	$26.58^{+1.11\%}_{-1.15\%}$	$22.57^{+8.04\%}_{-8.56\%}$

Table 4: Total dijet rates for $p_T^{cut} = 10$ GeV, in pb.

$\sigma(p_T^{cut} = 100$ GeV) [fb]	an.	tr.
$\mathcal{O}(\alpha_S^2)$	$41.38^{+0.03\%}_{-0.03\%}$	$41.17^{+0.46\%}_{-0.85\%}$
γ-ind.	$90.03^{+0.01\%}_{-0.02\%}$	$89.67^{+0.24\%}_{-0.32\%}$
total	$136.91^{+0.00\%}_{-0.00\%}$	$136.35^{+0.28\%}_{-0.48\%}$

Table 5: Total dijet rates for $p_T^{cut} = 100$ GeV, in fb.

Dijet cross sections at 10 TeV

In the case of the muon PDFs, there are significant differences between the truncated and the analytical approaches

I expect these to be potentially even larger for electron PDFs

- There are several issues with the truncated approach, and they all boil down to the *logarithmically-divergent* sensitivity to an arbitrary cutoff

But in any case, here's the final message:

$\#11$:

He who waits long enough will see colours in leptons

Conclusions

If you are a PhD student sort of risk-averse (e.g. no wingsuit flights), by taking into account progress in medicine and with some luck, you'll be able to see the next hadron collider (maybe even to work on it: especially since there will be no money for your pension)

For the rest of us, it will be a lepton collider. The good news: there is plenty to do, and so far chatGPT does not seem to have a clue