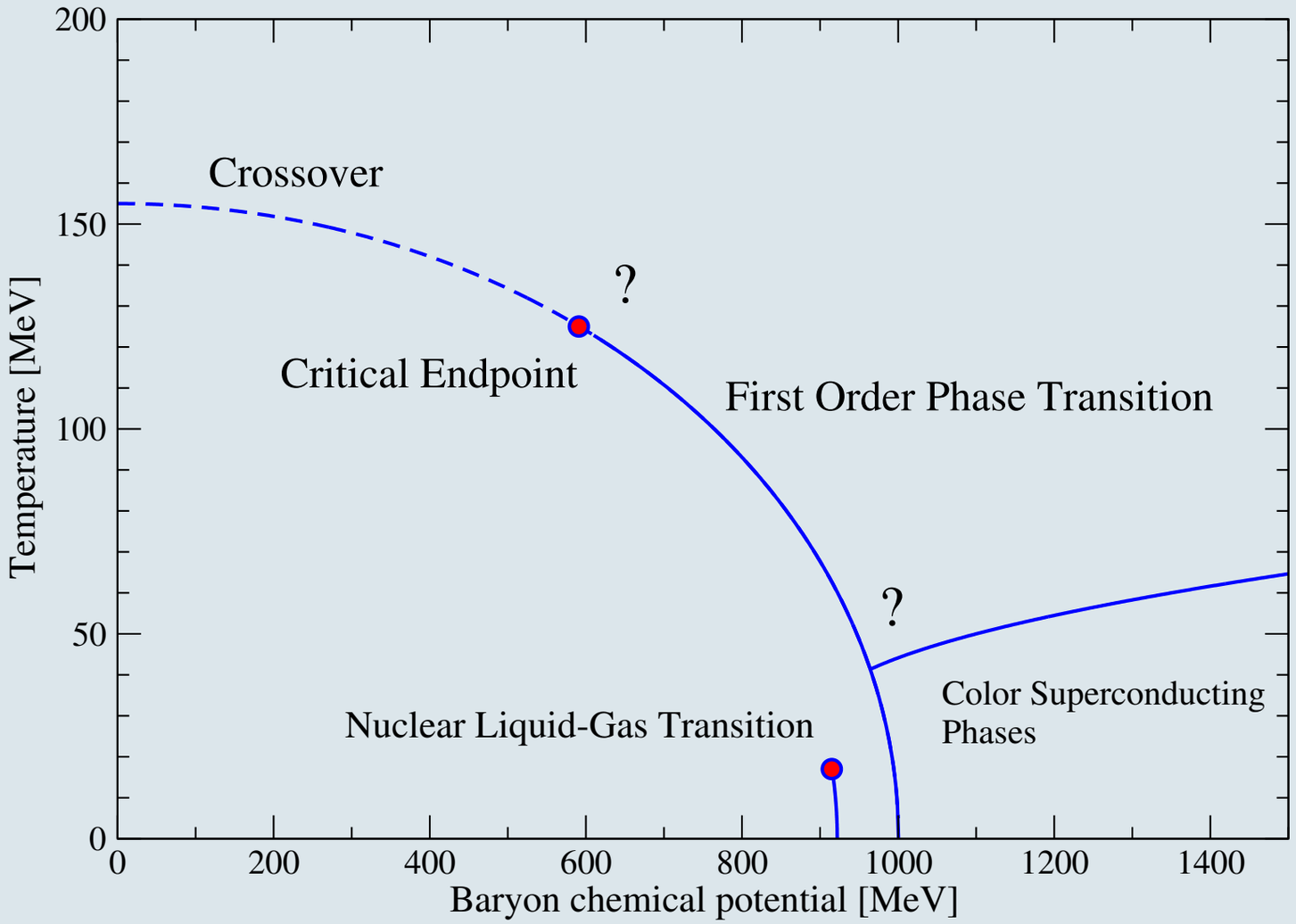
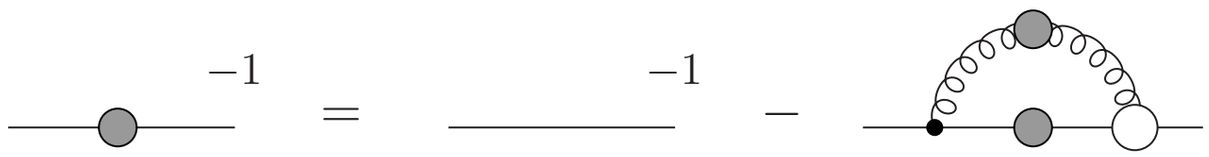


Columbia plot, critical point and bound states in dense QCD

Overview

1. Introduction: dynamical mass generation



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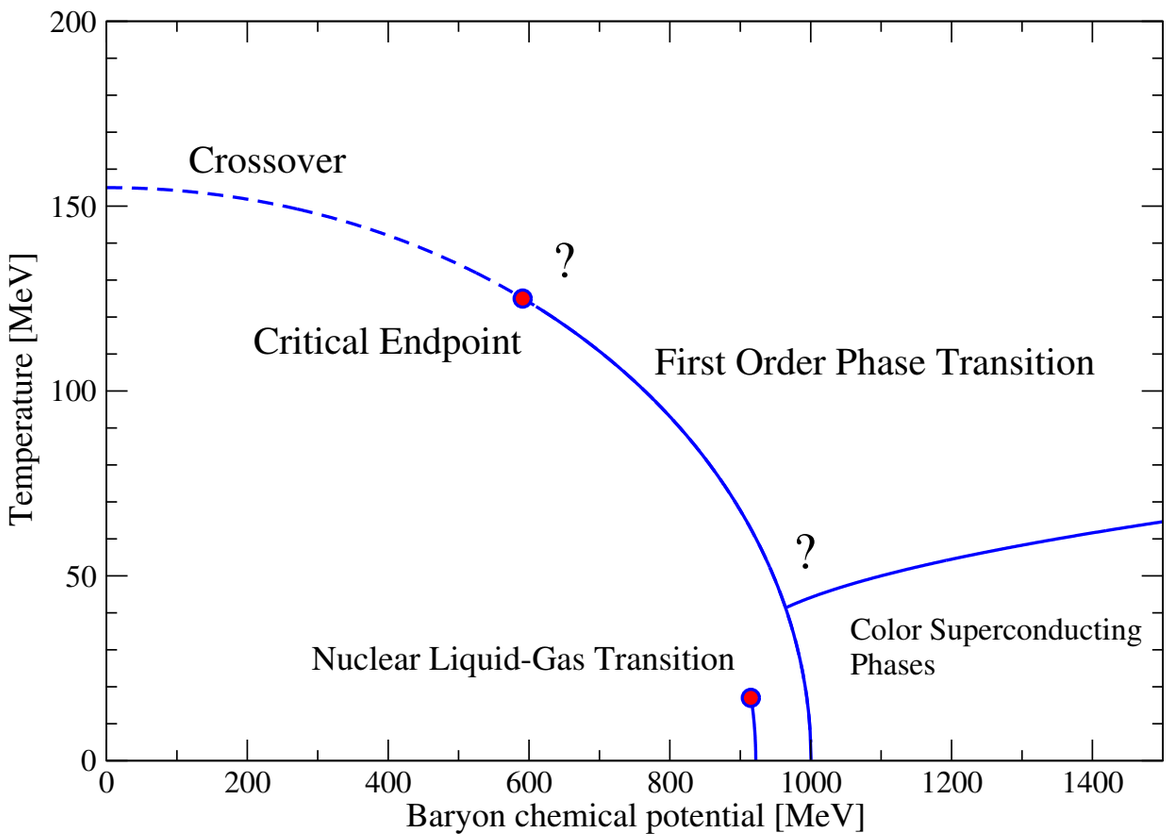
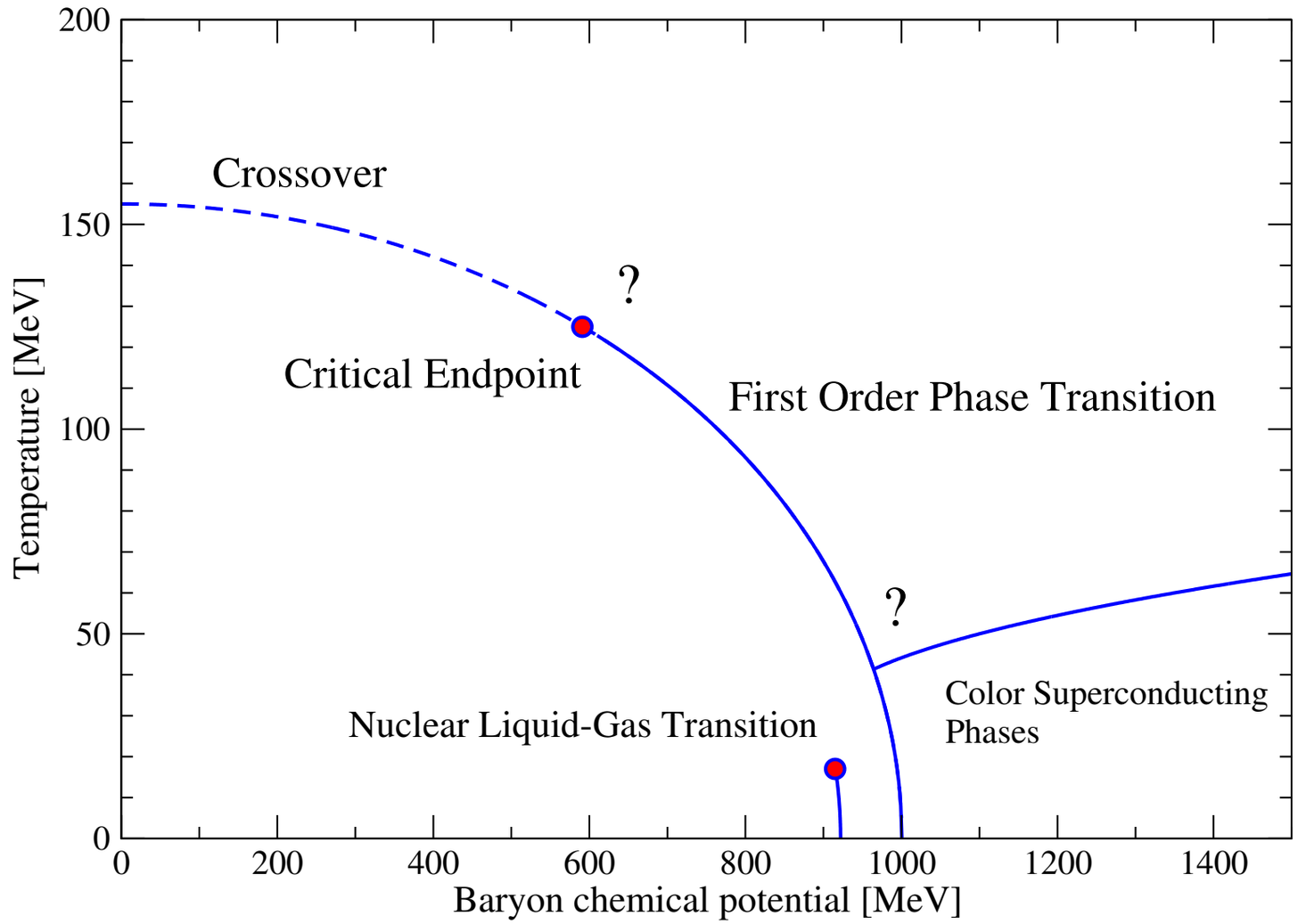


Phase diagram of quark matter: QCD



Many interesting open questions:

- Existence and location of critical point ?
- Details of phase transitions ??
- Consequences for early universe and physics of neutron stars

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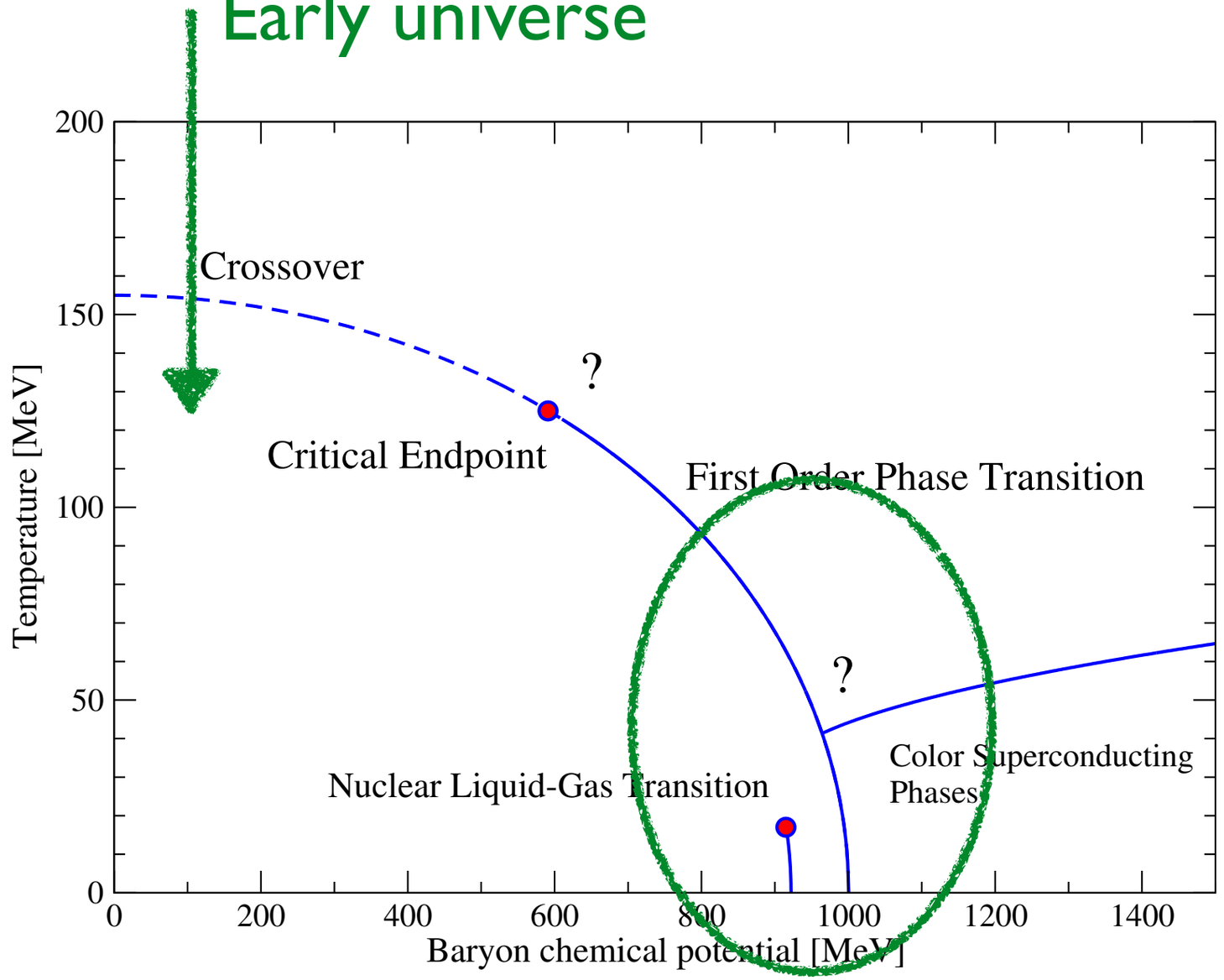

Neutron star (mergers)

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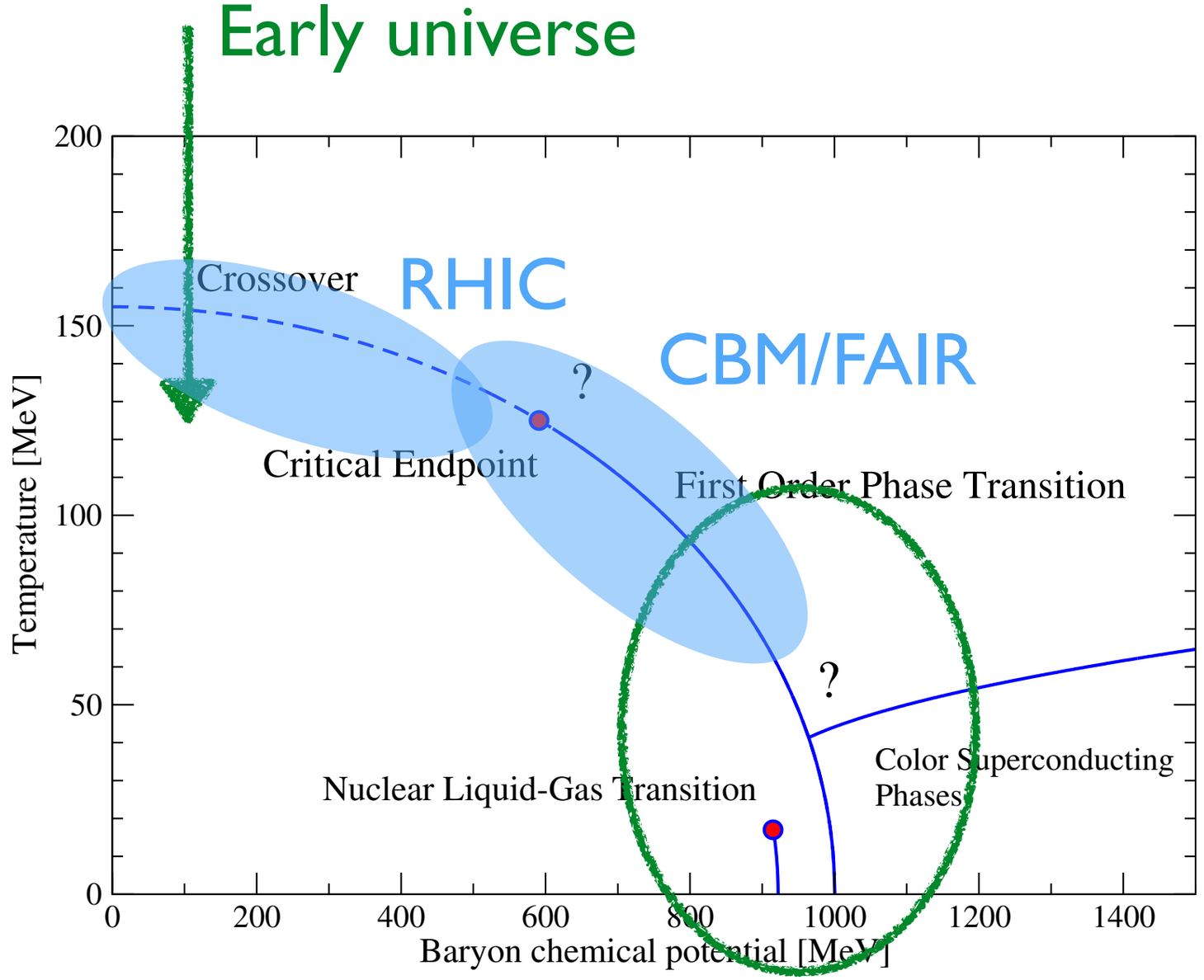

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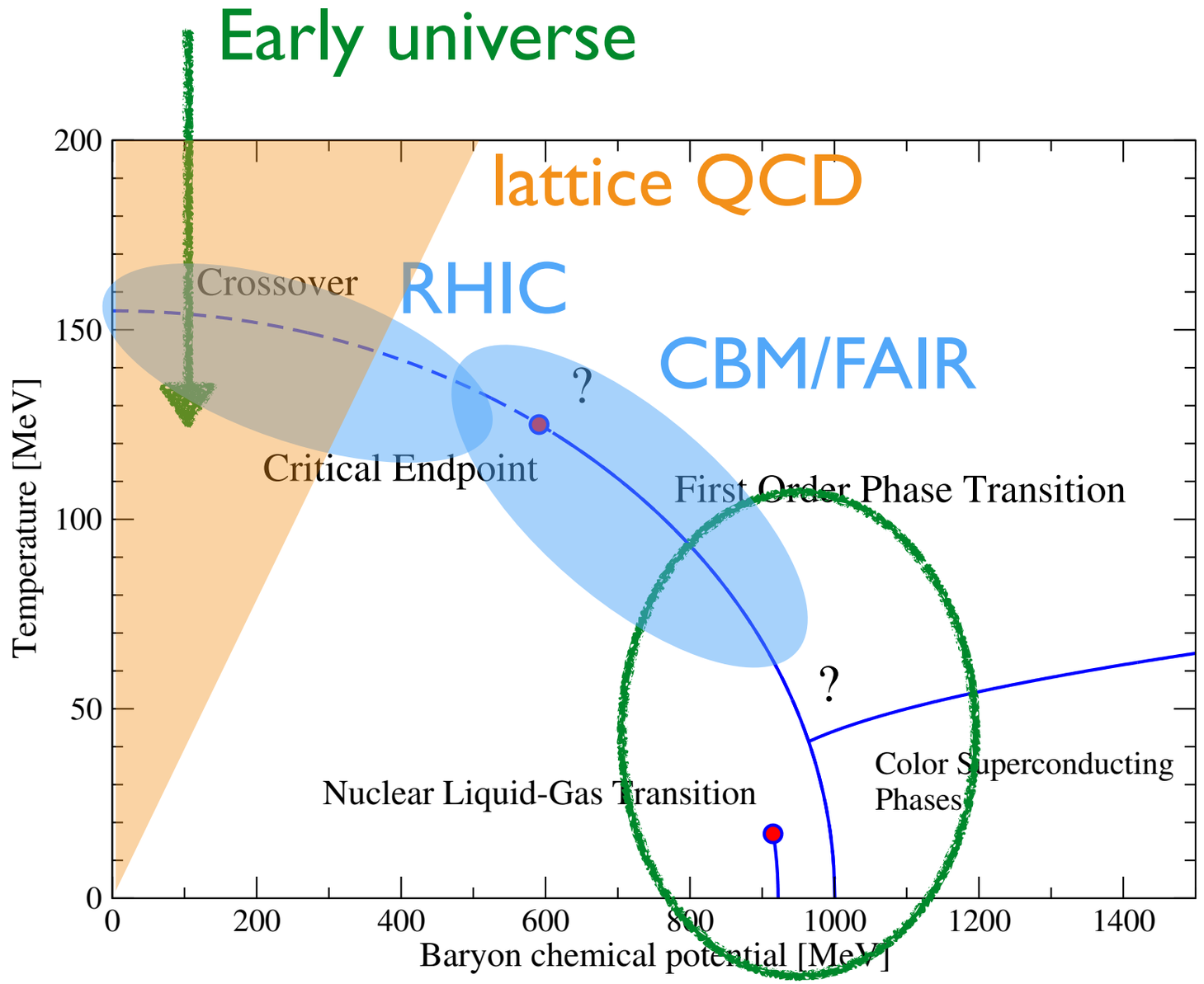

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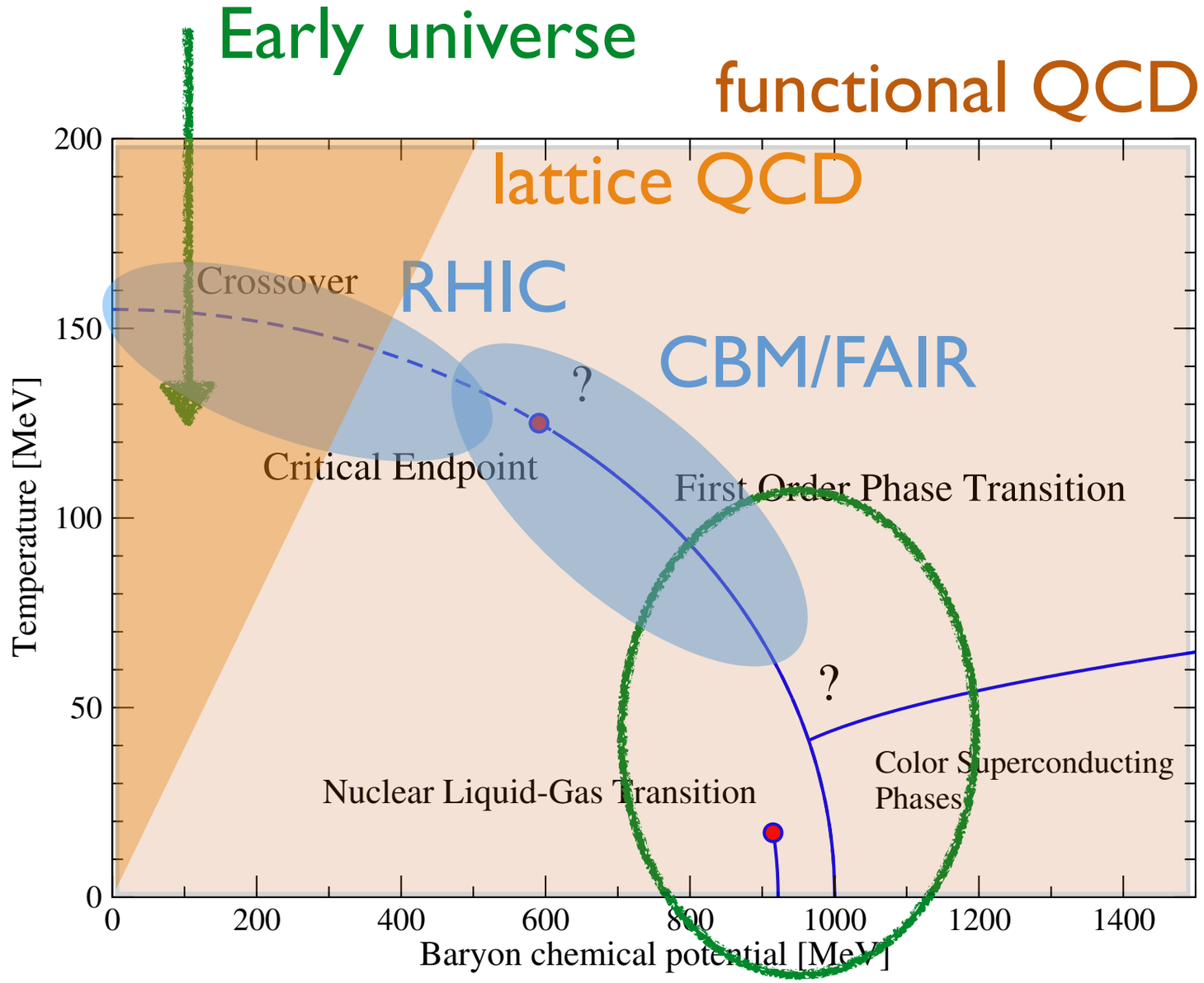
Phase diagram of quark matter: QCD



Neutron star (mergers)

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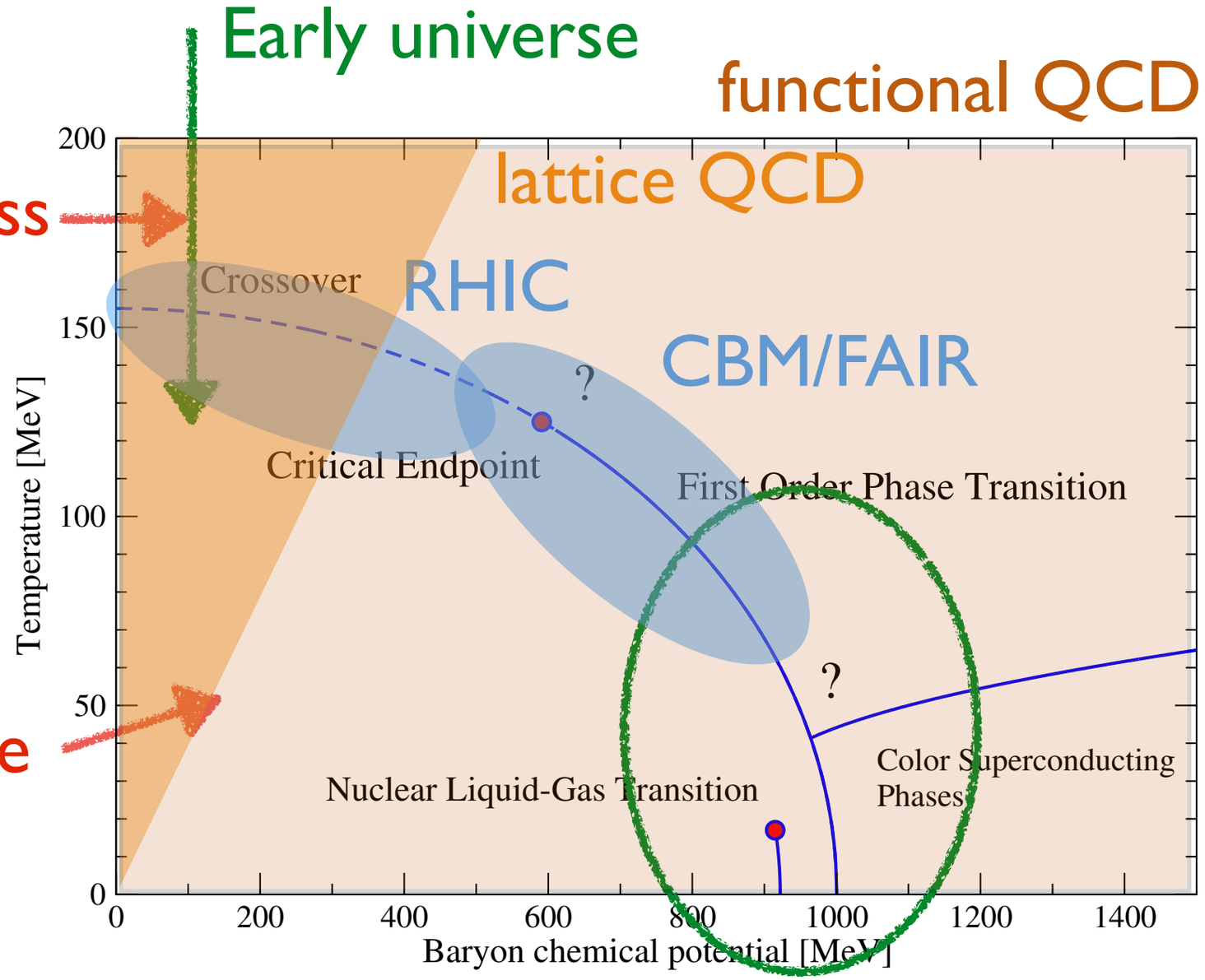
Phase diagram of quark matter: QCD

Quarks (almost) massless

Quarks massive



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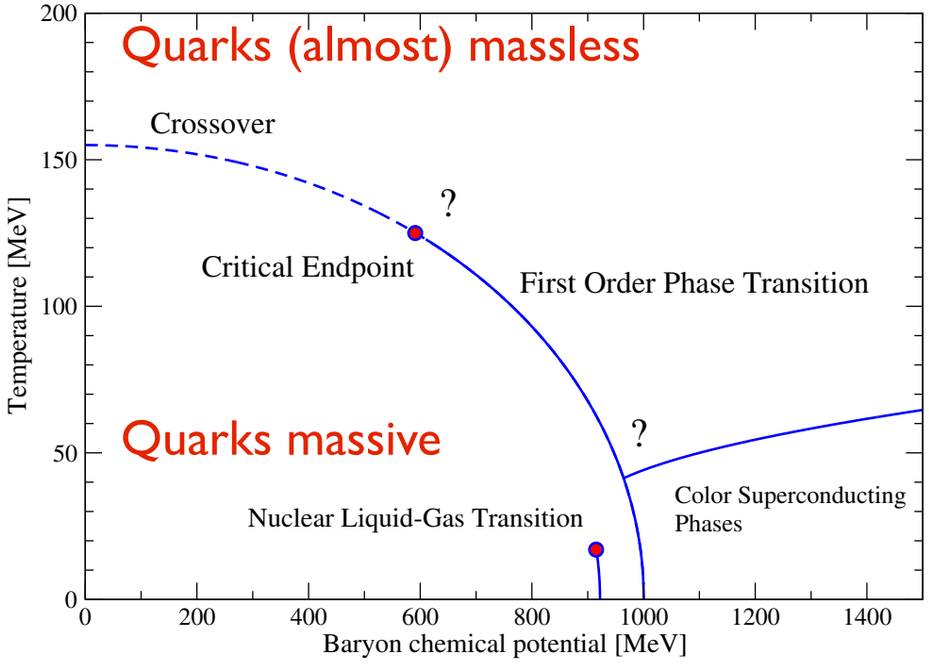
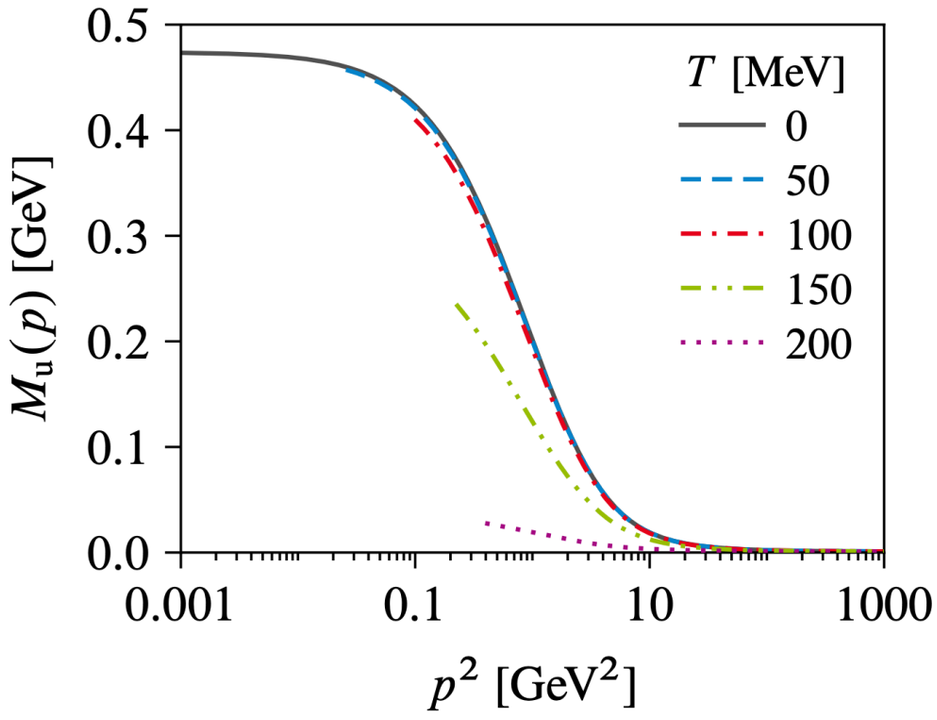
Columbia plot

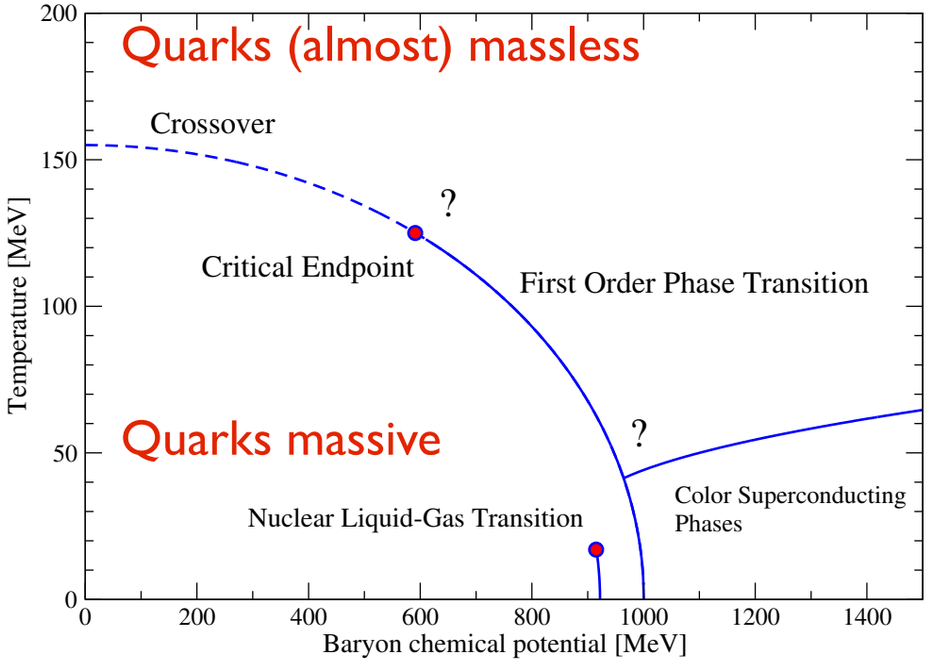


$$m_{EW} \to M(p^2)$$

$$S^{-1}(p) = \frac{(i\not{p} + M(p^2))}{Z_f(p^2)}$$



Columbia plot



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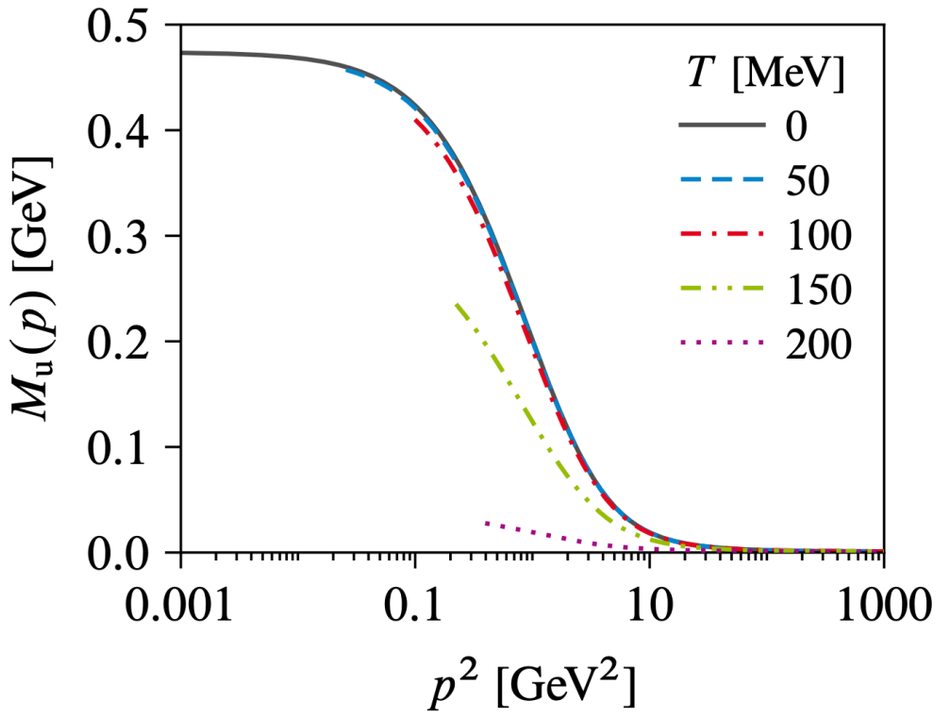
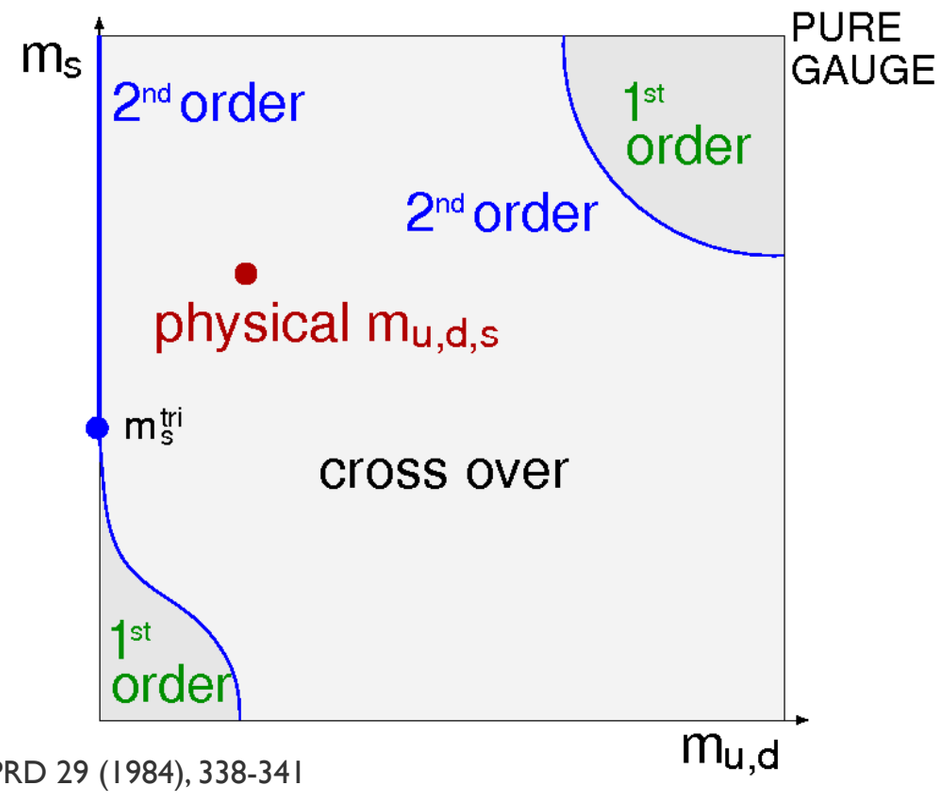
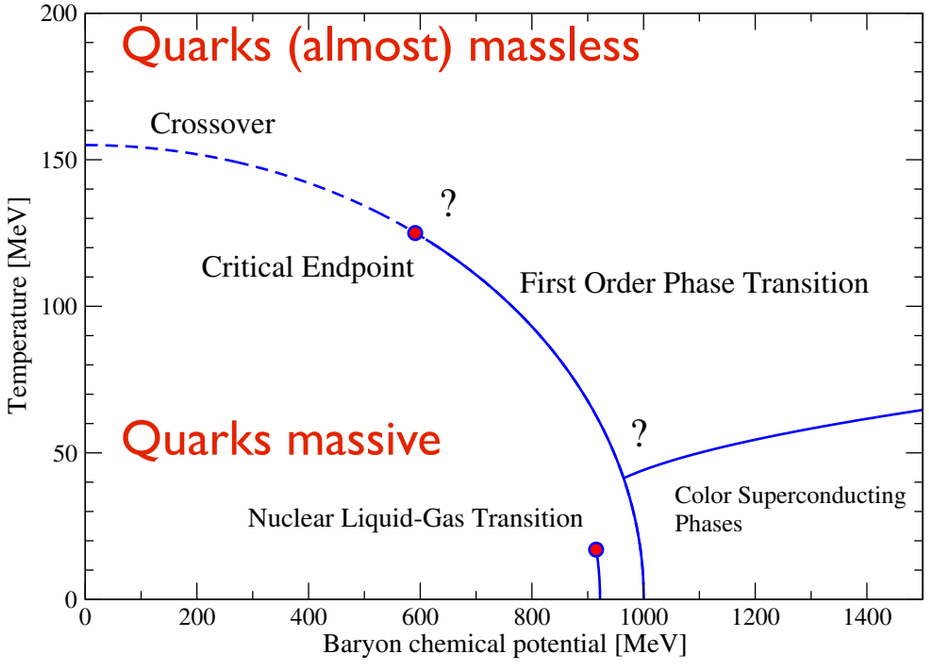




Pisarski and Wilczek, PRD 29 (1984), 338-341
Brown et a. PRL 65 (1990) 2491

Columbia plot



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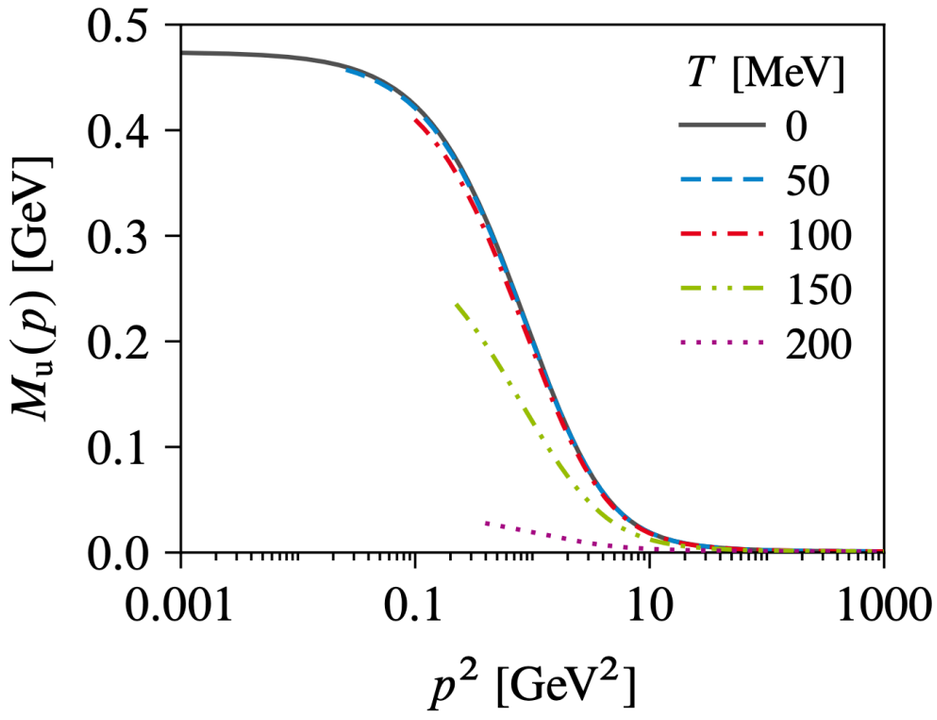
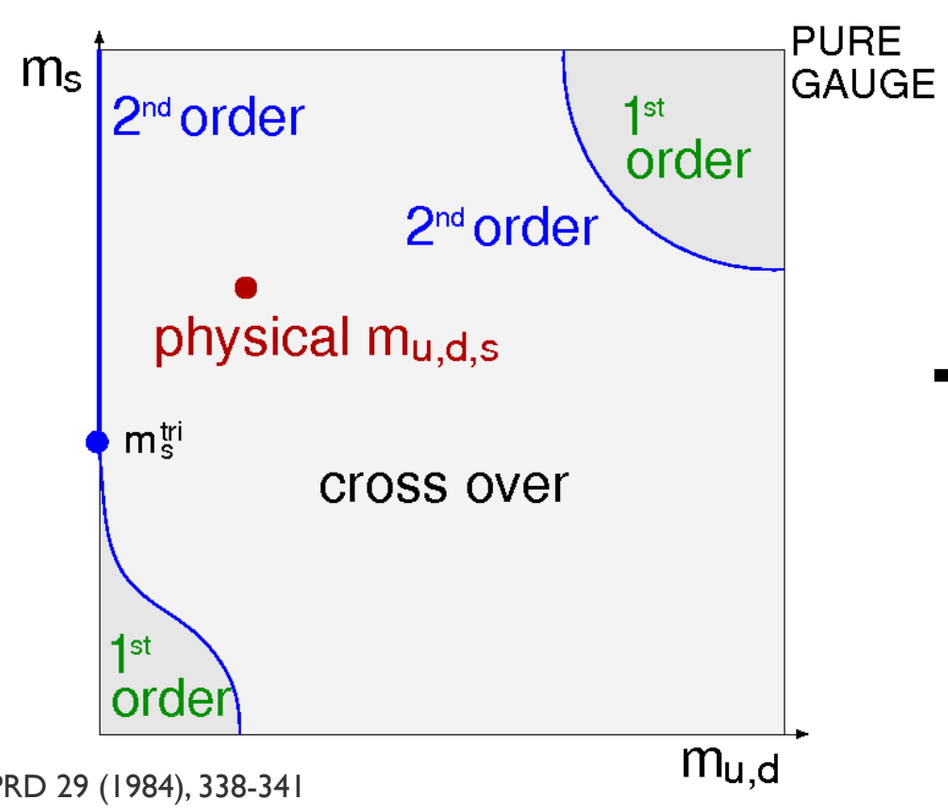
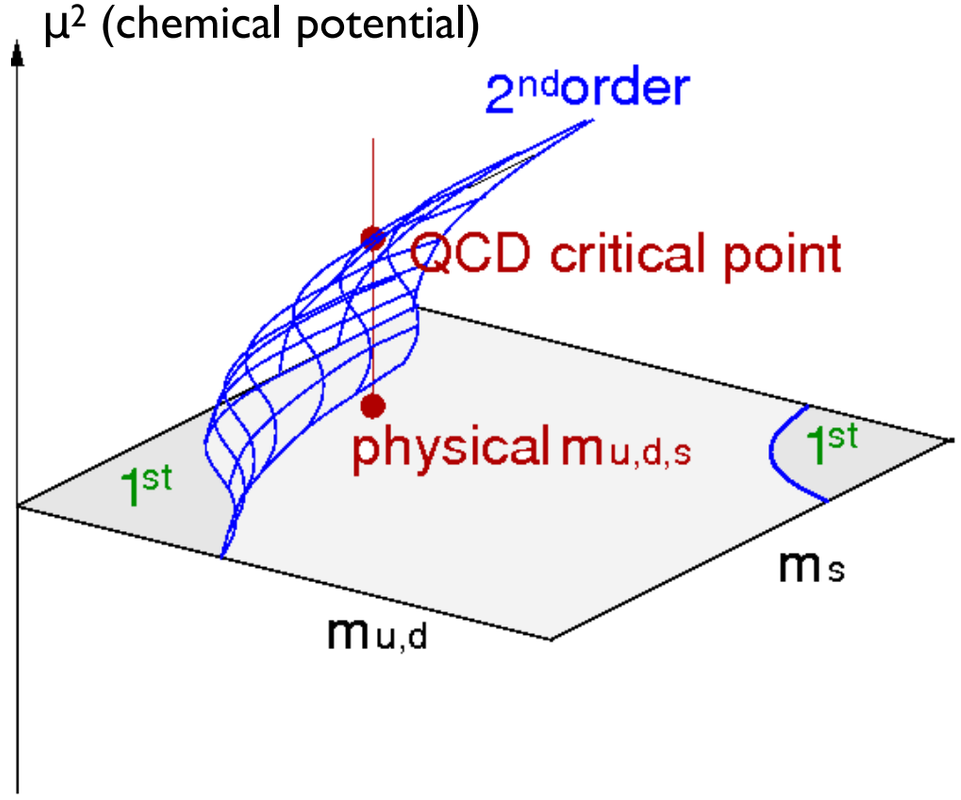
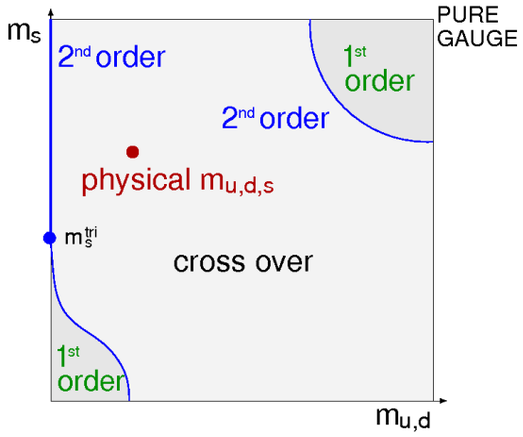




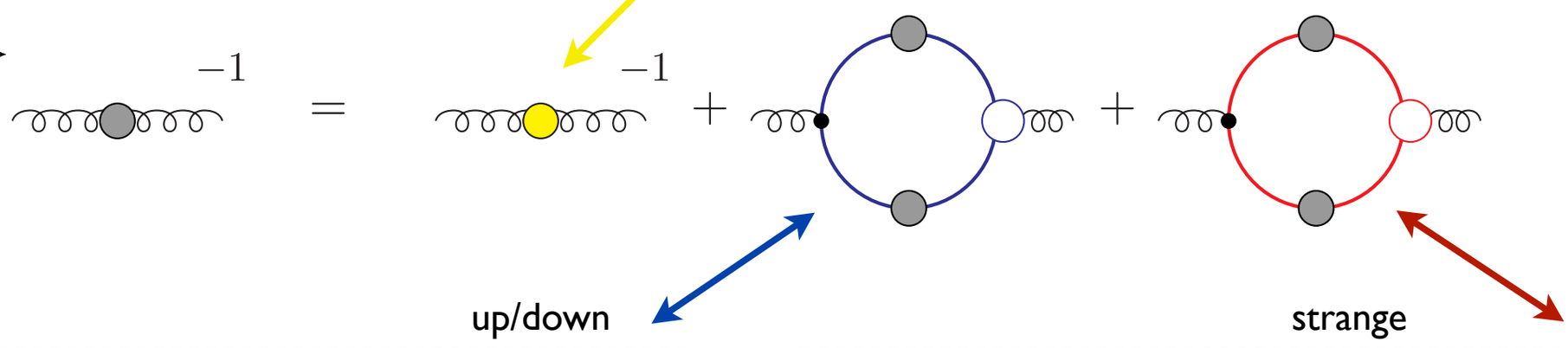


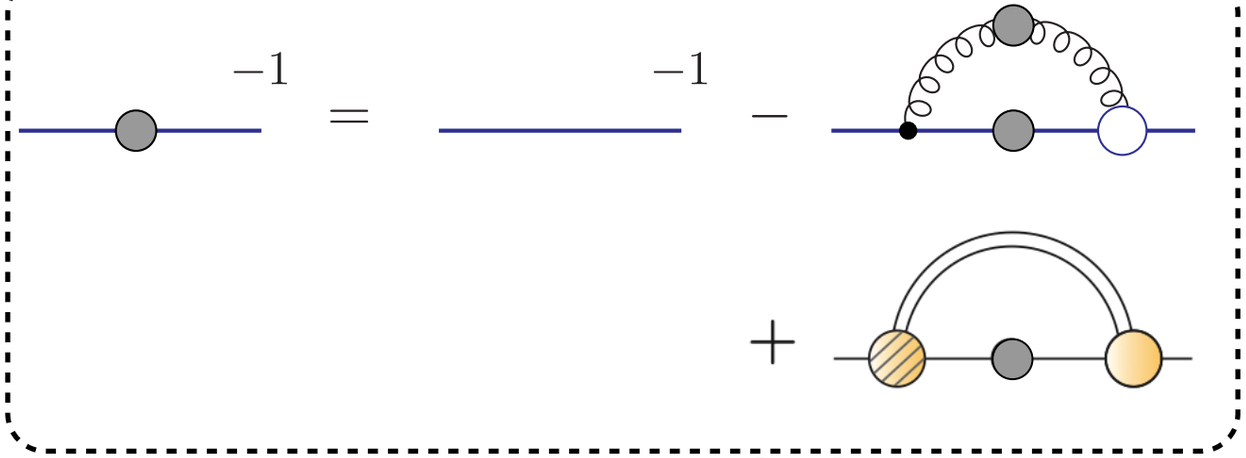
Pisarski and Wilczek, PRD 29 (1984), 338-341
Brown et a. PRL 65 (1990) 2491

N_f=2+1-QCD with functional methods (DSEs)

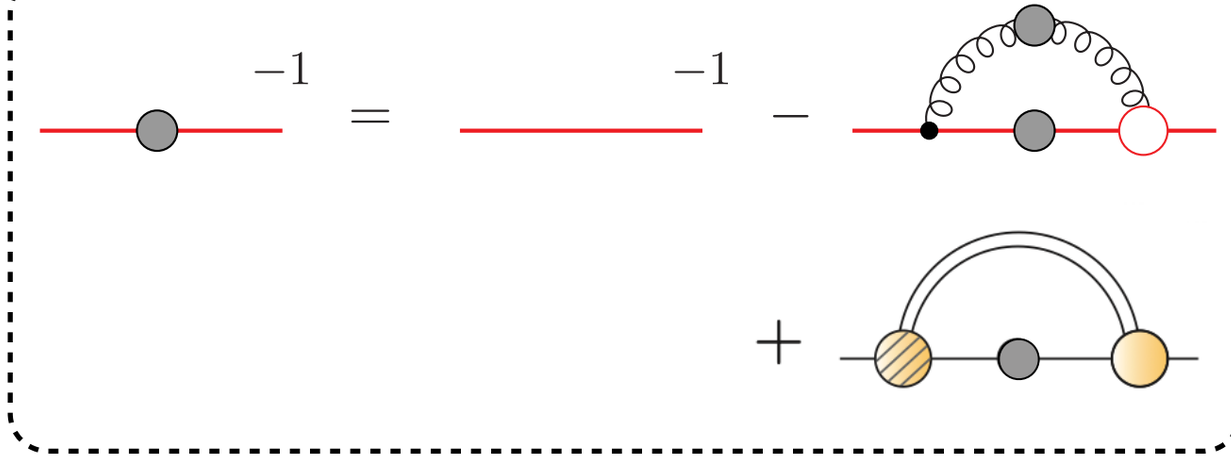


Yang-Mills theory





light mesons

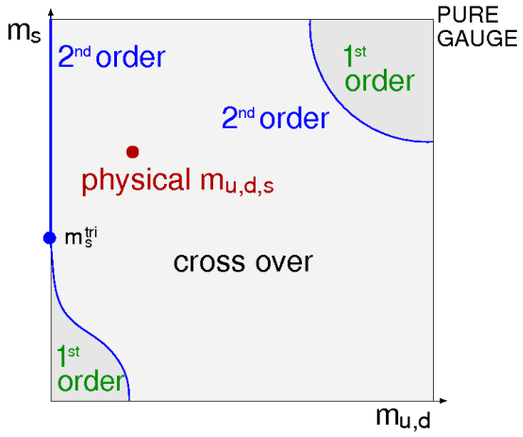


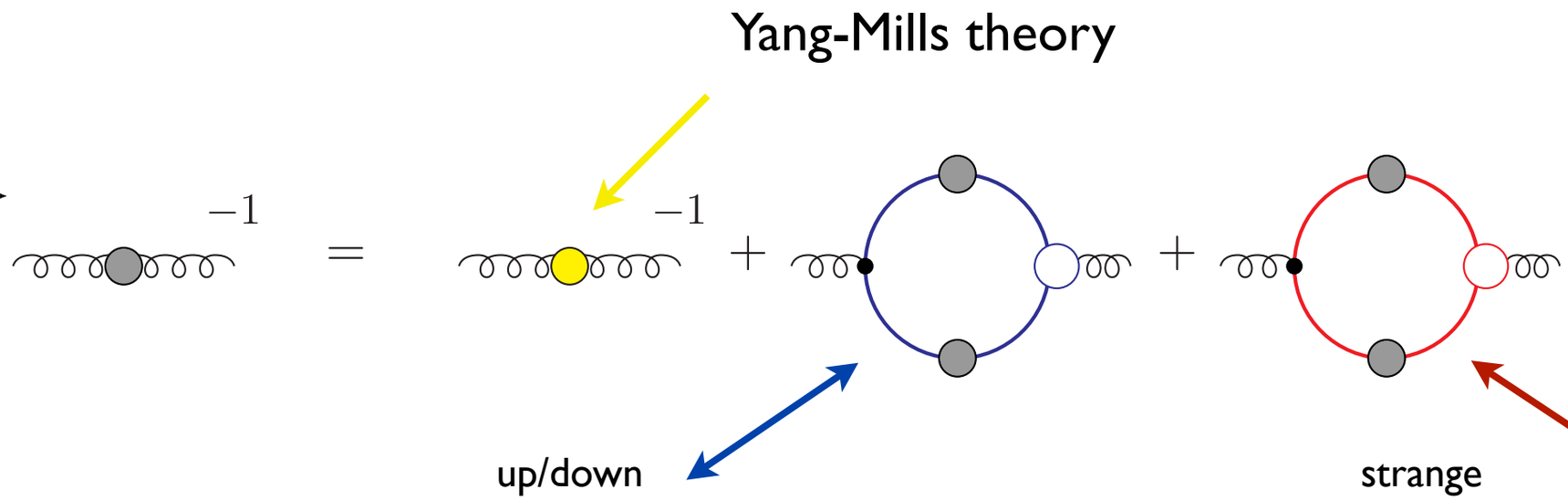
strange mesons

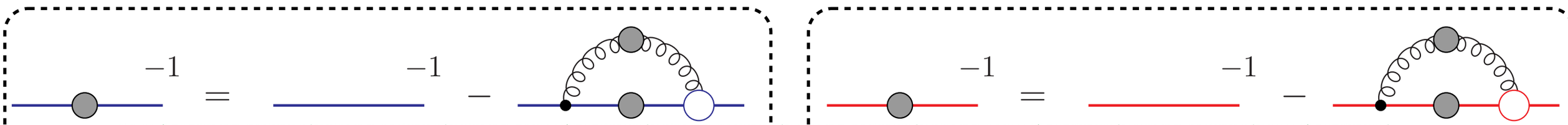
$$S^{-1}(p) = \frac{(i\not{p} + M(p^2))}{Z_f(p^2)}$$

CF, Luecker, Welzbacher, PRD 90 (2014) 034022
Gunkel, CF, PRD 104 (2021) [2106.08356]

N_f=2+1-QCD with functional methods (DSEs)





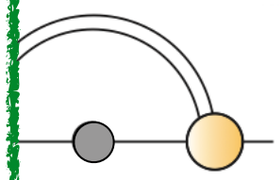


FRG: Fu, Rennecke, Pawlowski, PRD 101 (2020)

FRG/DSE: Gao, Pawlowski, PLB 820 (2021)

DSE: Gunkel, CF, PRD 104 (2021)

quality of truncations on same level Pawlowski, CF, in prep.



mesons

$S^{-1}(p) =$

CF, Luecker, Welzbacher, PRD 90 (2014) 034022

Gunkel, CF, PRD 104 (2021) [2106.08356]

Overview

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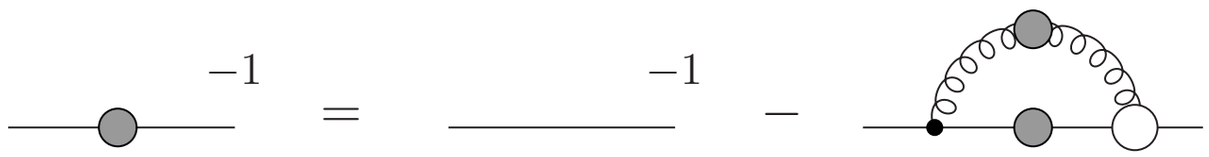

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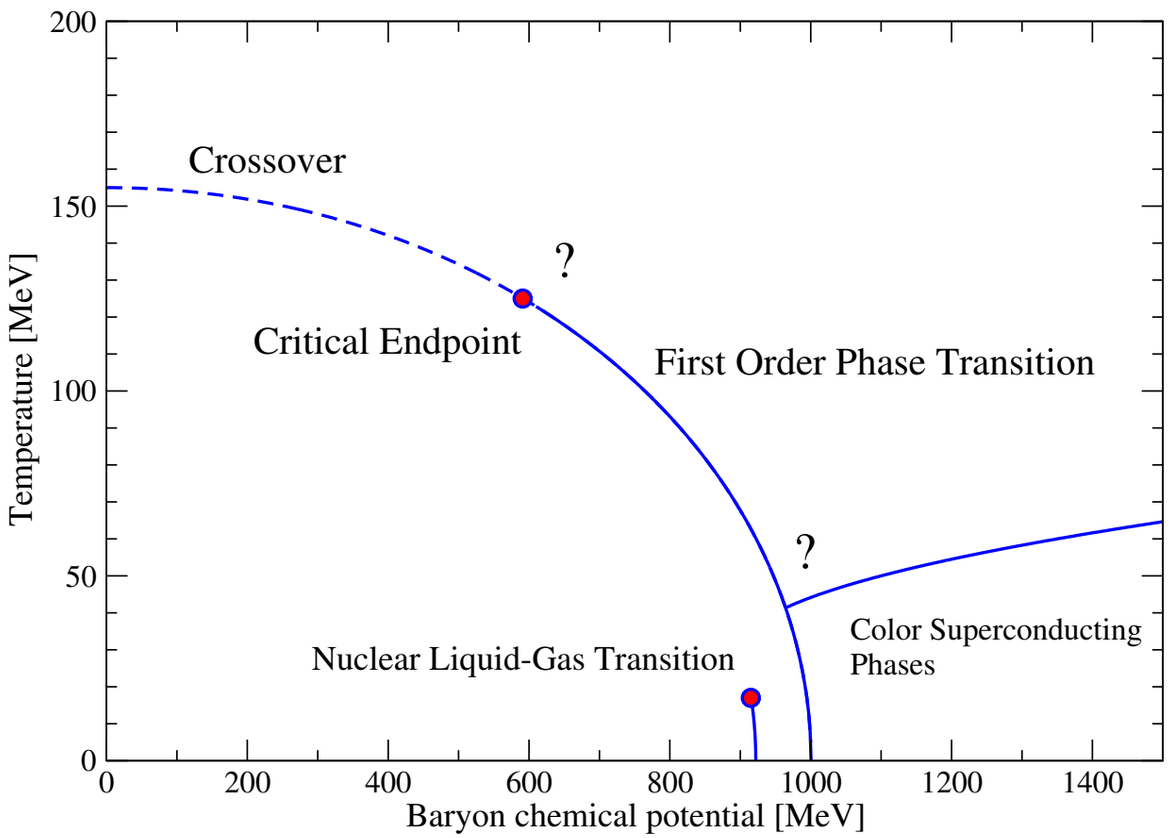
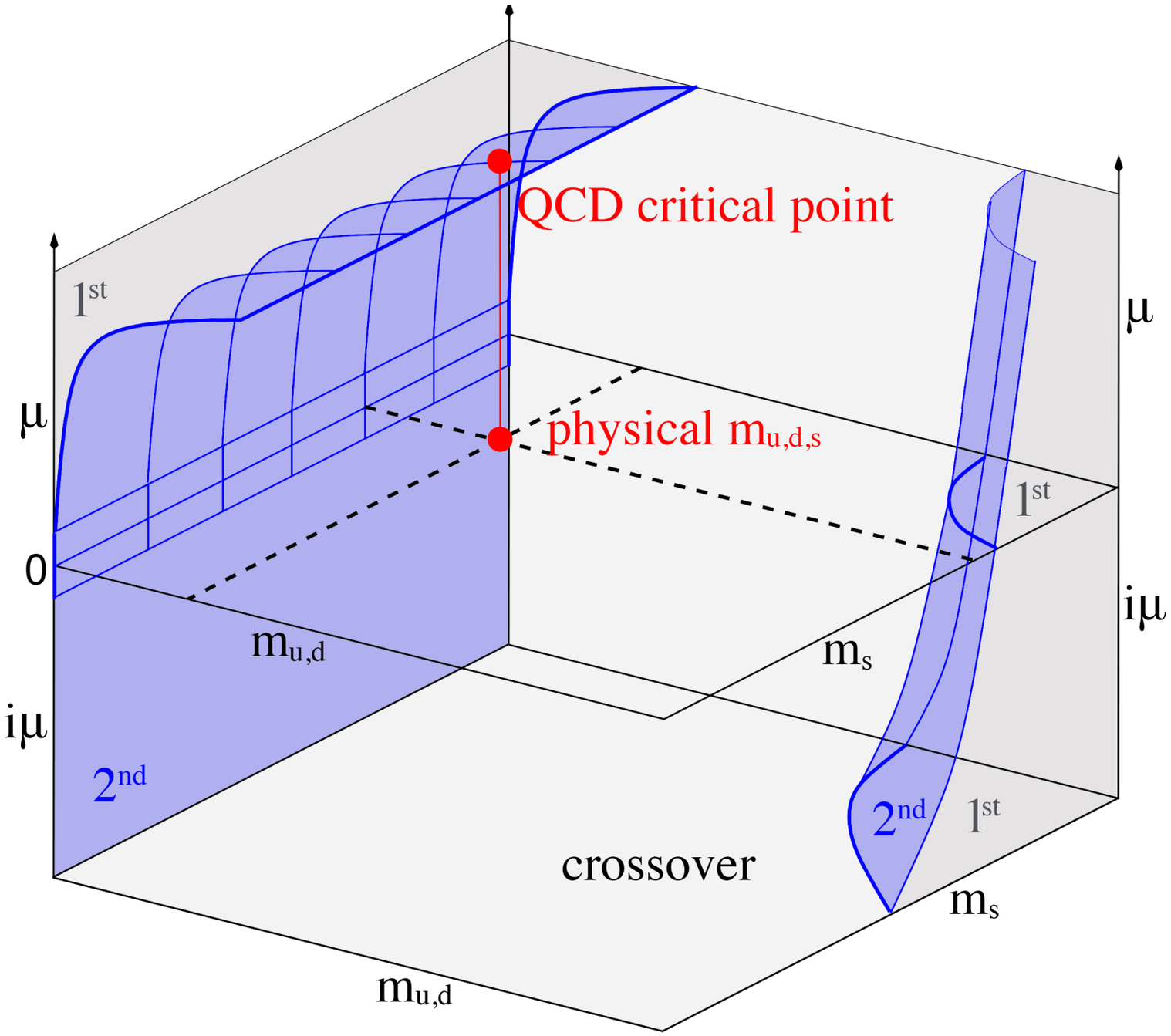
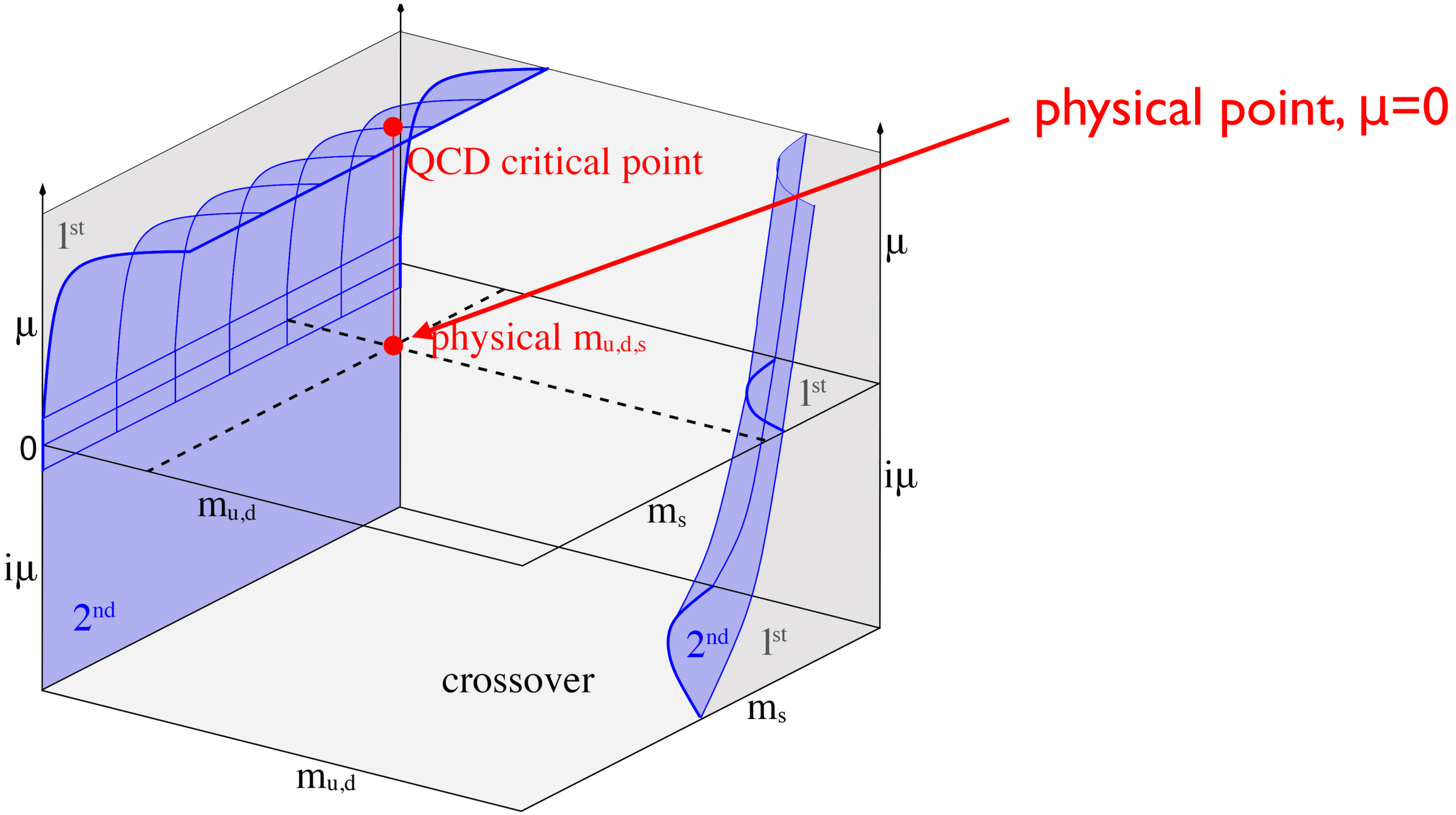


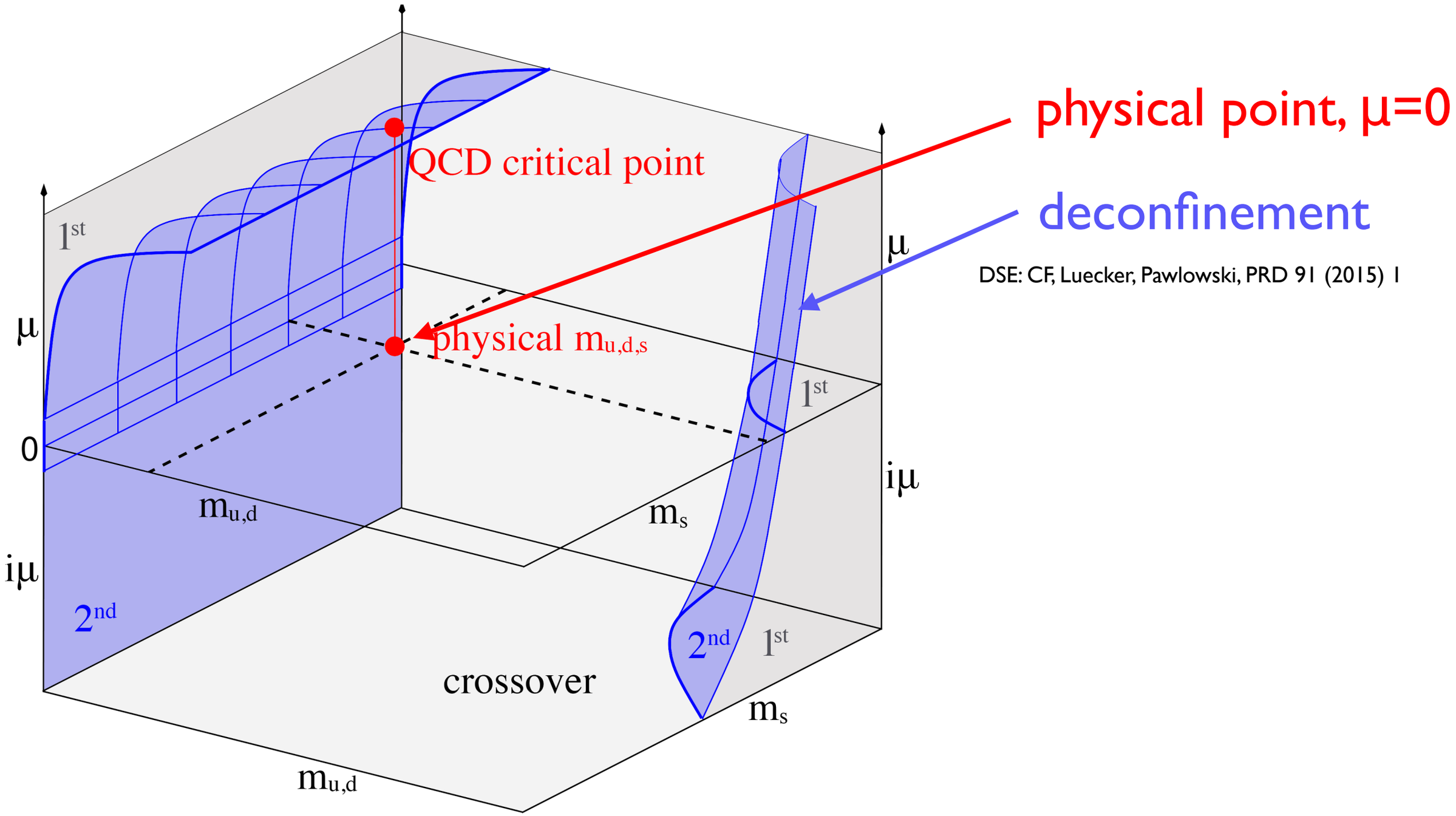
Roadmap


QCD critical point
physical $m_{u,d,s}$
1st
μ
0
$m_{u,d}$
m_s
iμ
2nd
crossover
1st
μ
1st
iμ
2nd
1st
m_s
$m_{u,d}$

Roadmap


QCD critical point
physical point, μ=0
physical $m_{u,d,s}$
1st
μ
0
$m_{u,d}$
m_s
1st
μ
iμ
2nd
2nd
1st
crossover
m_s
$m_{u,d}$

Roadmap

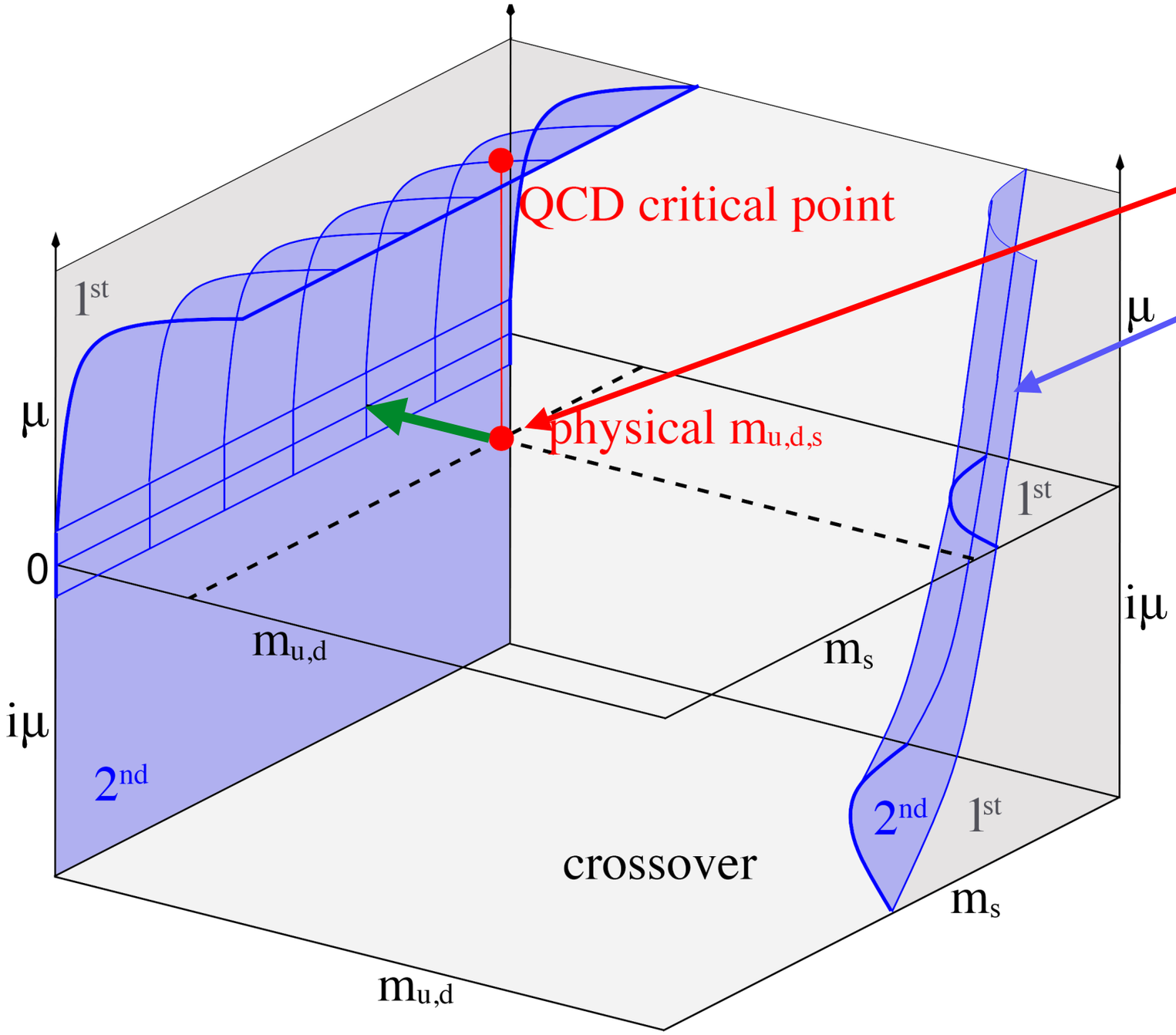


physical point, μ=0

deconfinement

DSE: CF, Luecker, Pawlowski, PRD 91 (2015) 1

Roadmap



physical point, μ=0

deconfinement

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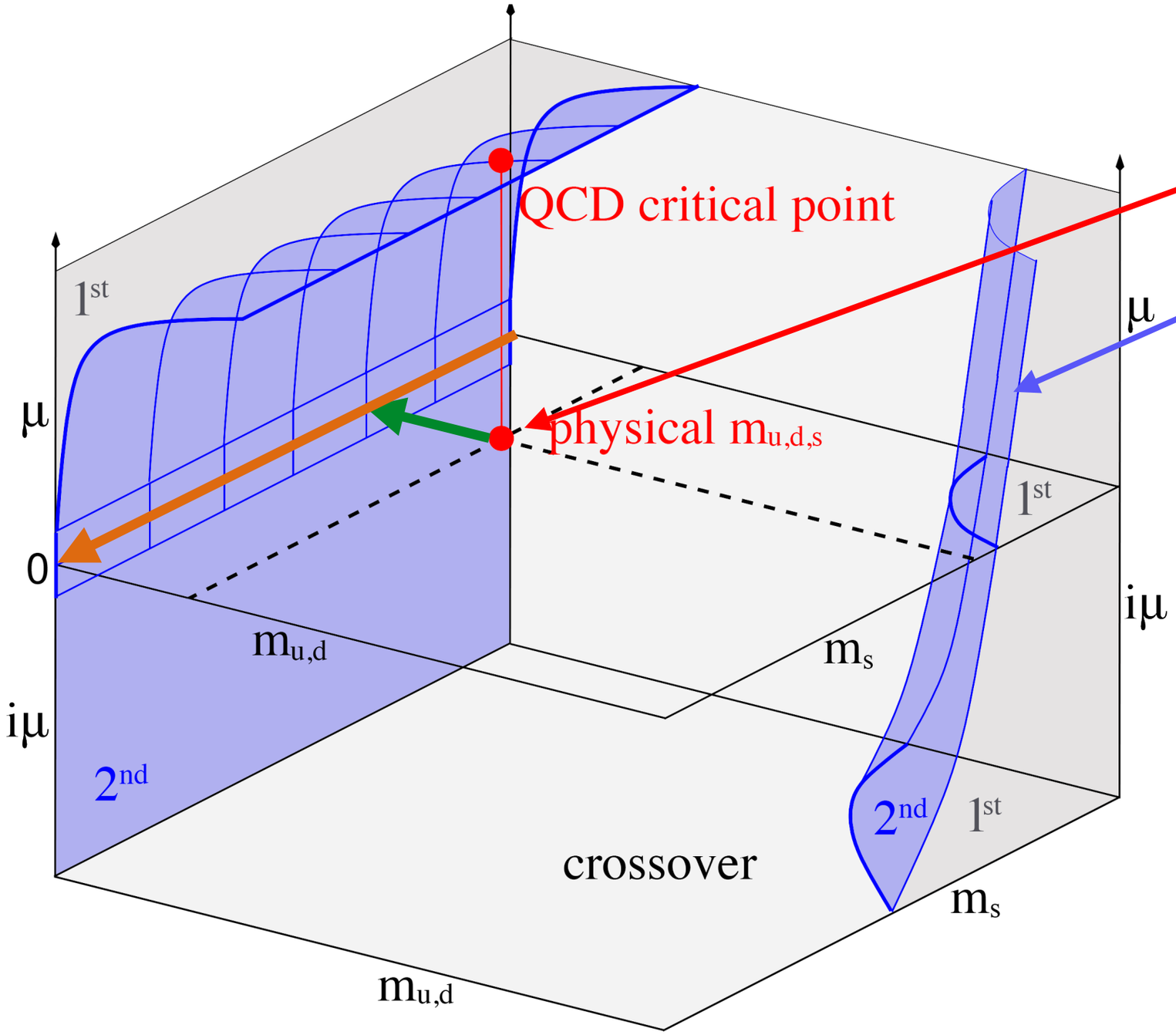
towards chiral limit

FRG: Braun et al, PRD 102 (2020) 5, 056010
FRG/DSE: Gao and Pawlowski PRD 105(2022) 094020
DSE: Bernhardt and CF, PRD 108 (2023) 114018

Roadmap



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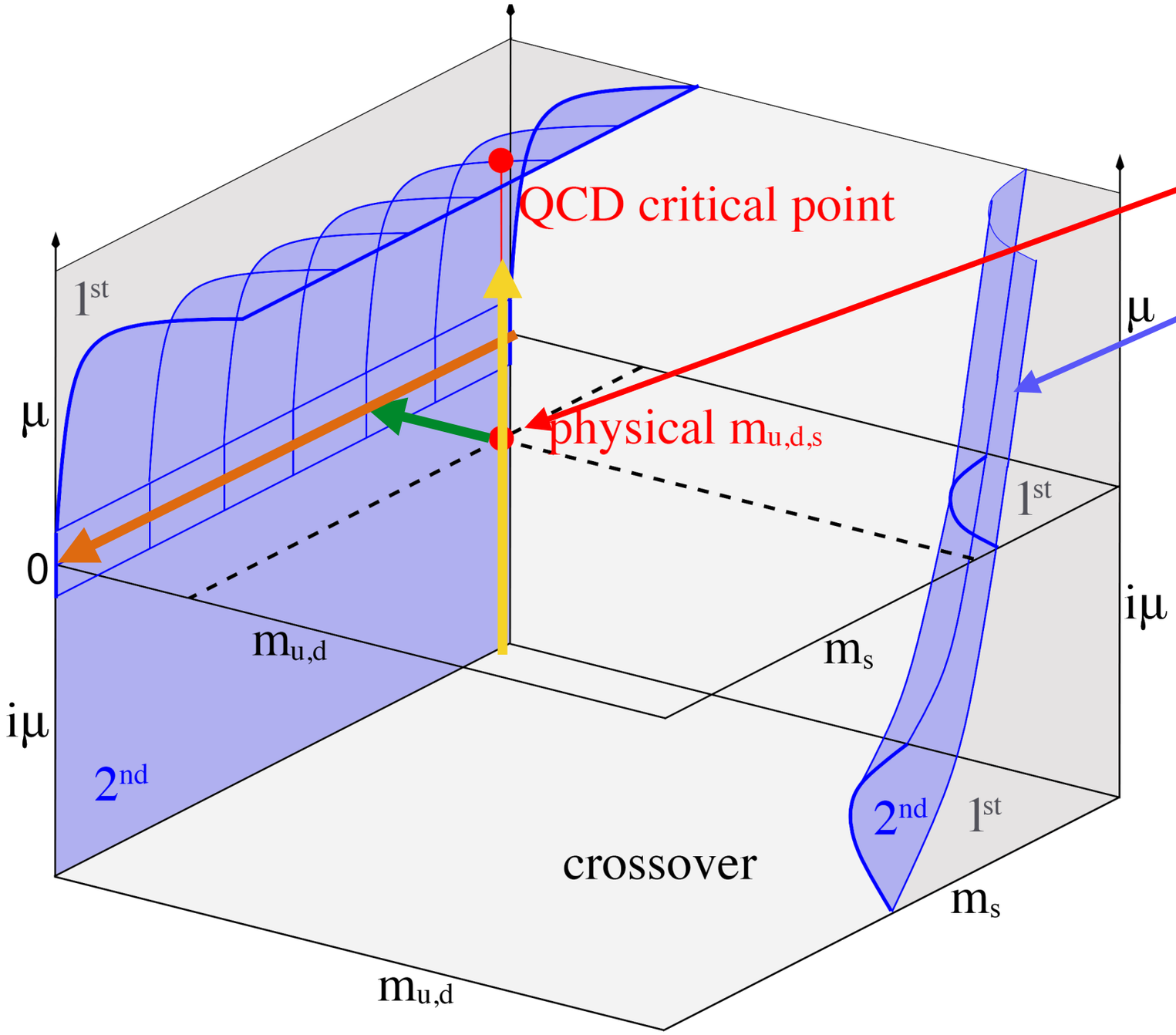

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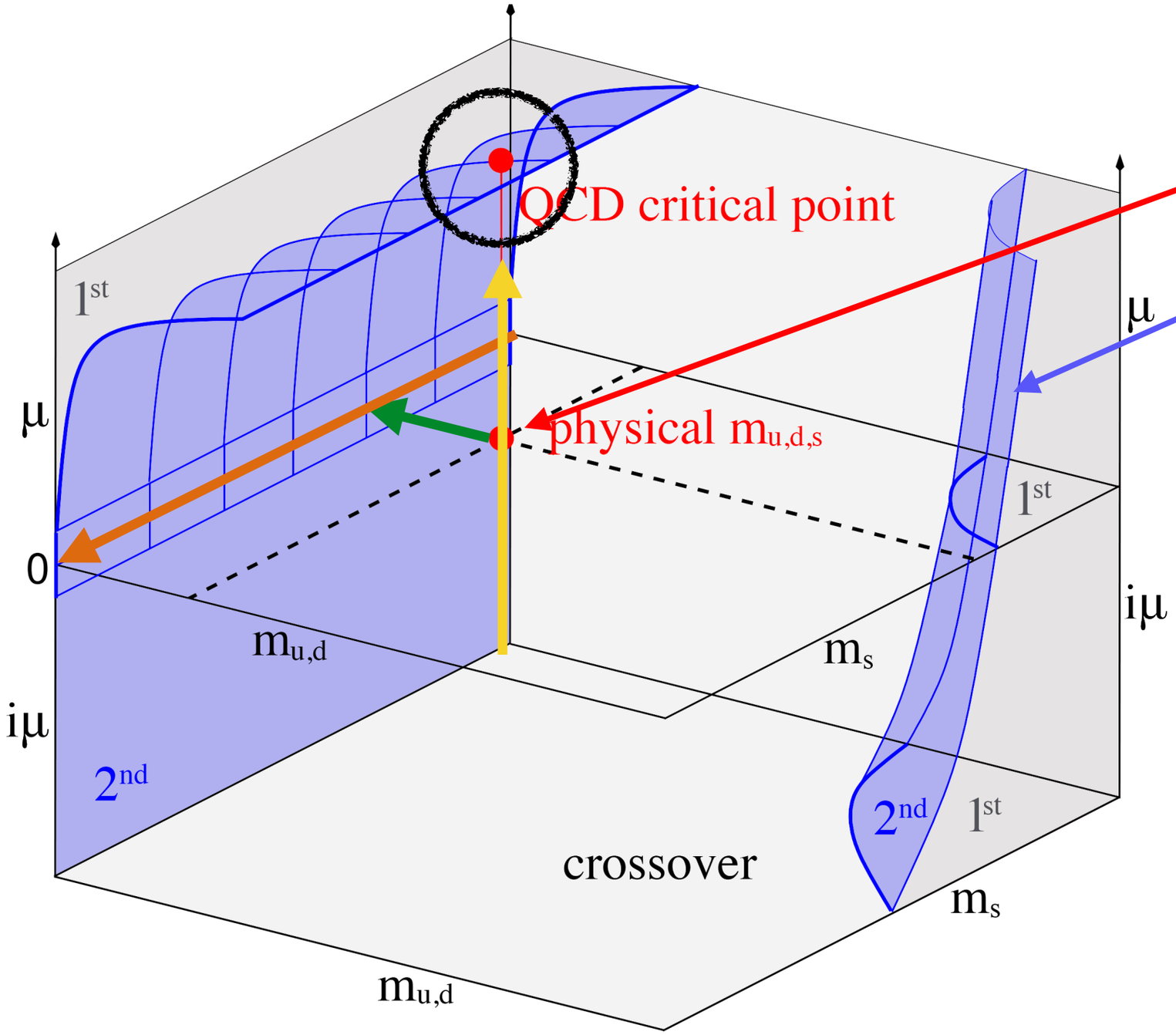
chiral limit

DSE: Bernhardt and CF, PRD 108 (2023) 114018

imaginary μ

FRG (Nf=2): Braun et. al. PRL 106 (2011)
DSE: Bernhardt, CF, EPJA 59 (2023) 8, 181

Roadmap



physical point, μ=0

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DSE: CF, Luecker, Pawlowski, PRD 91 (2015) 1

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imaginary μ

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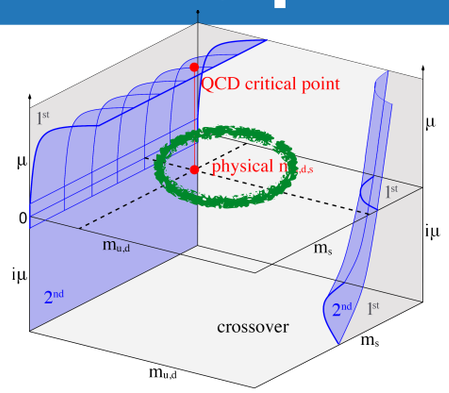
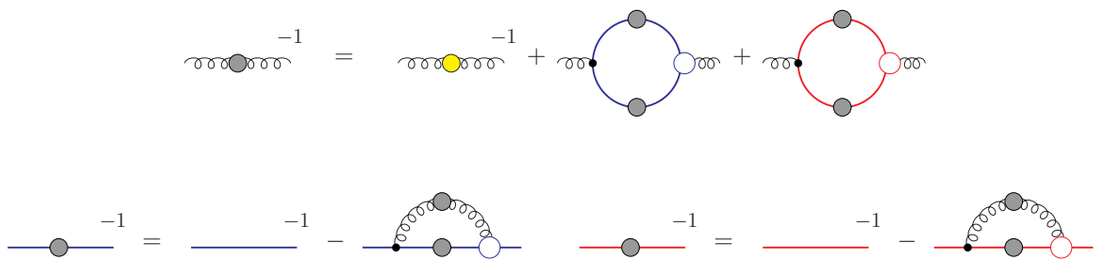
location of CEP

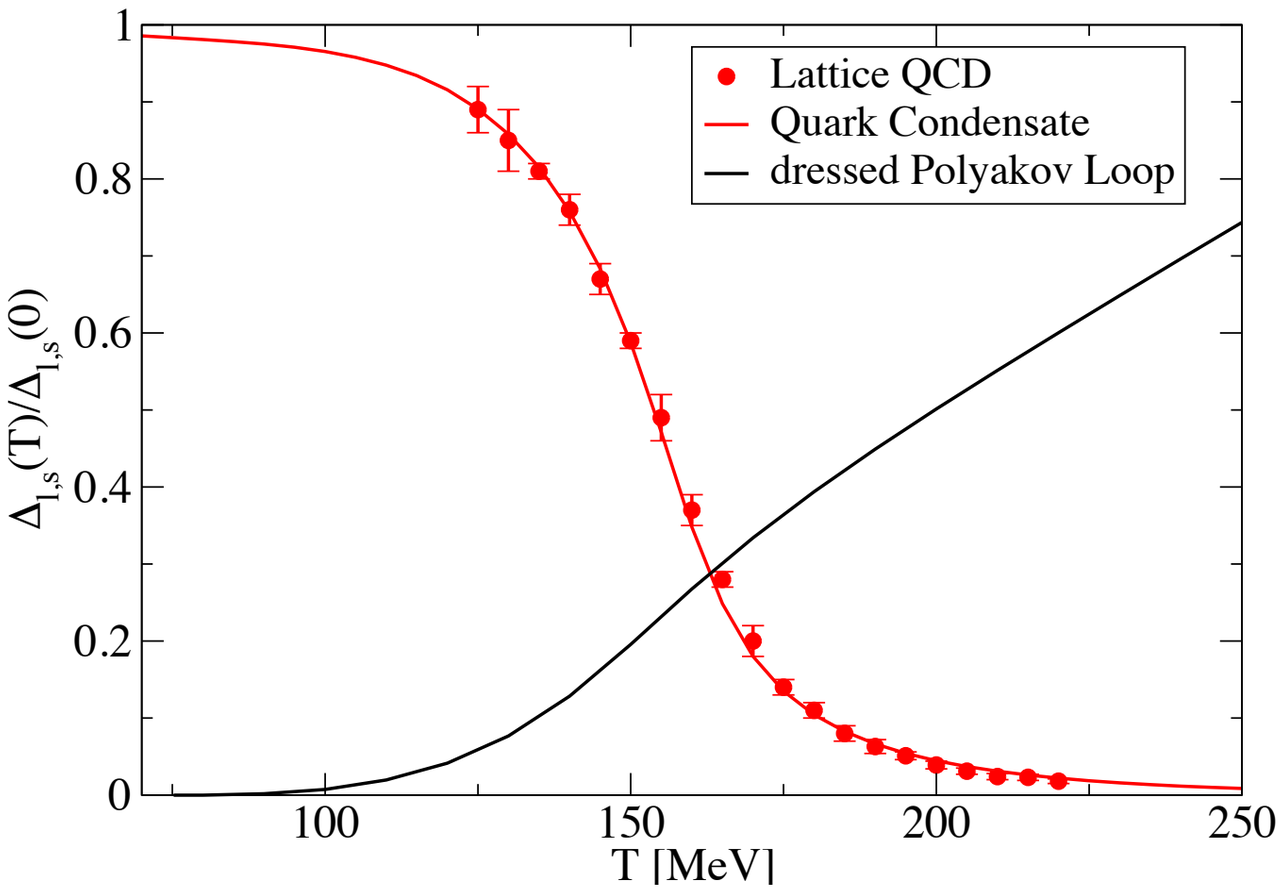
FRG: Fu, Rennecke, Pawlowski, PRD 101 (2020)
FRG/DSE: Gao, Pawlowski, PLB 820 (2021)
DSE: Gunkel, CF, PRD 104 (2021)

N_f=2+1, μ=0, physical point



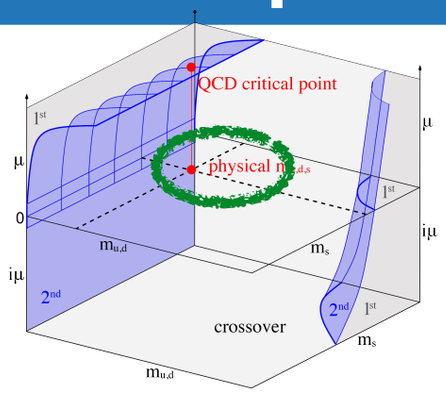


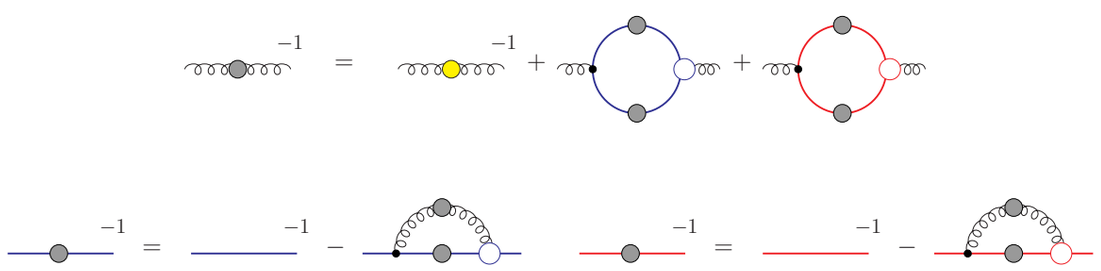
$$\Delta_{u,d,s} \sim Tr(S_{u,d,s})$$



Lattice: Borsanyi *et al.* [Wuppertal-Budapest], JHEP 1009(2010) 073
DSE: CF, Luecker, PLB 718 (2013) 1036,
CF, Luecker, Welzbacher, PRD 90 (2014) 034022

N_f=2+1, μ=0, physical point





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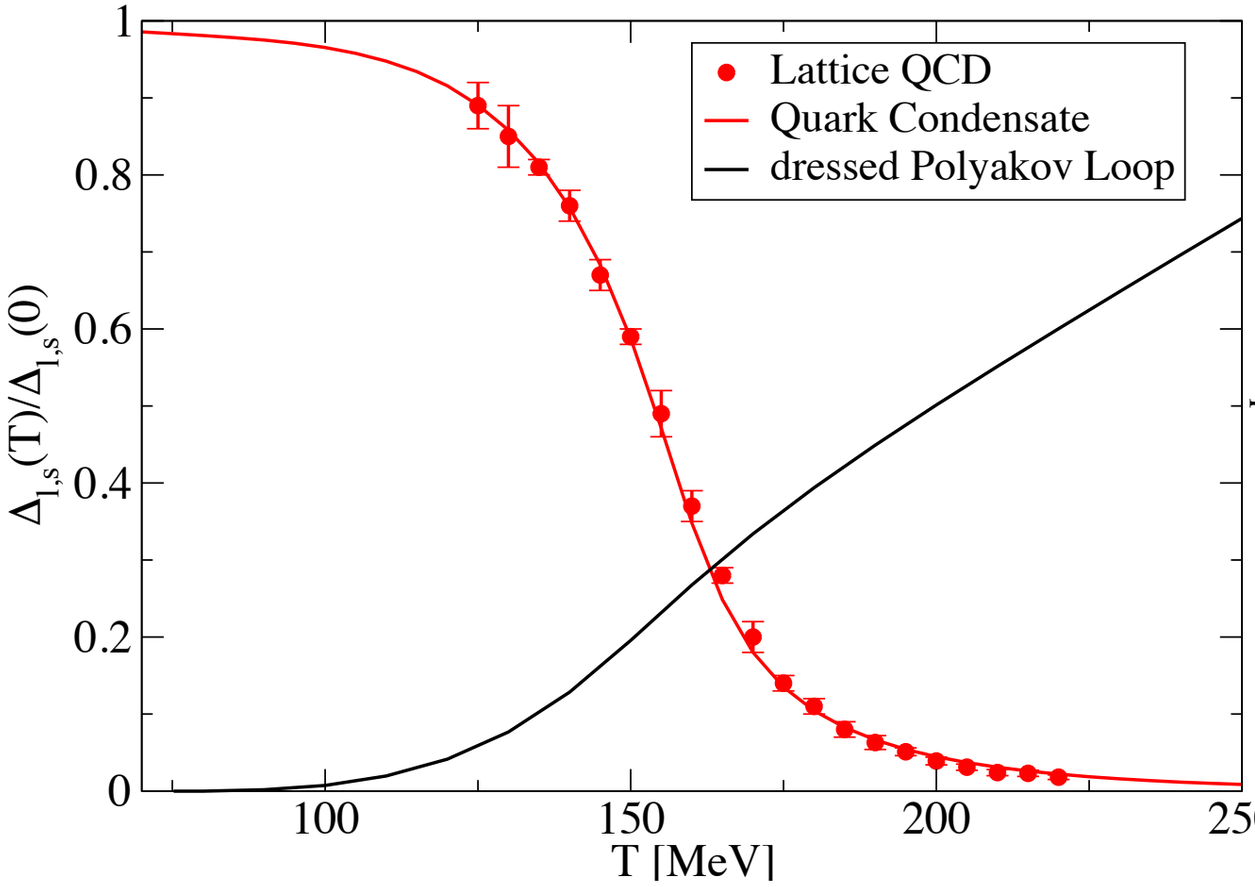

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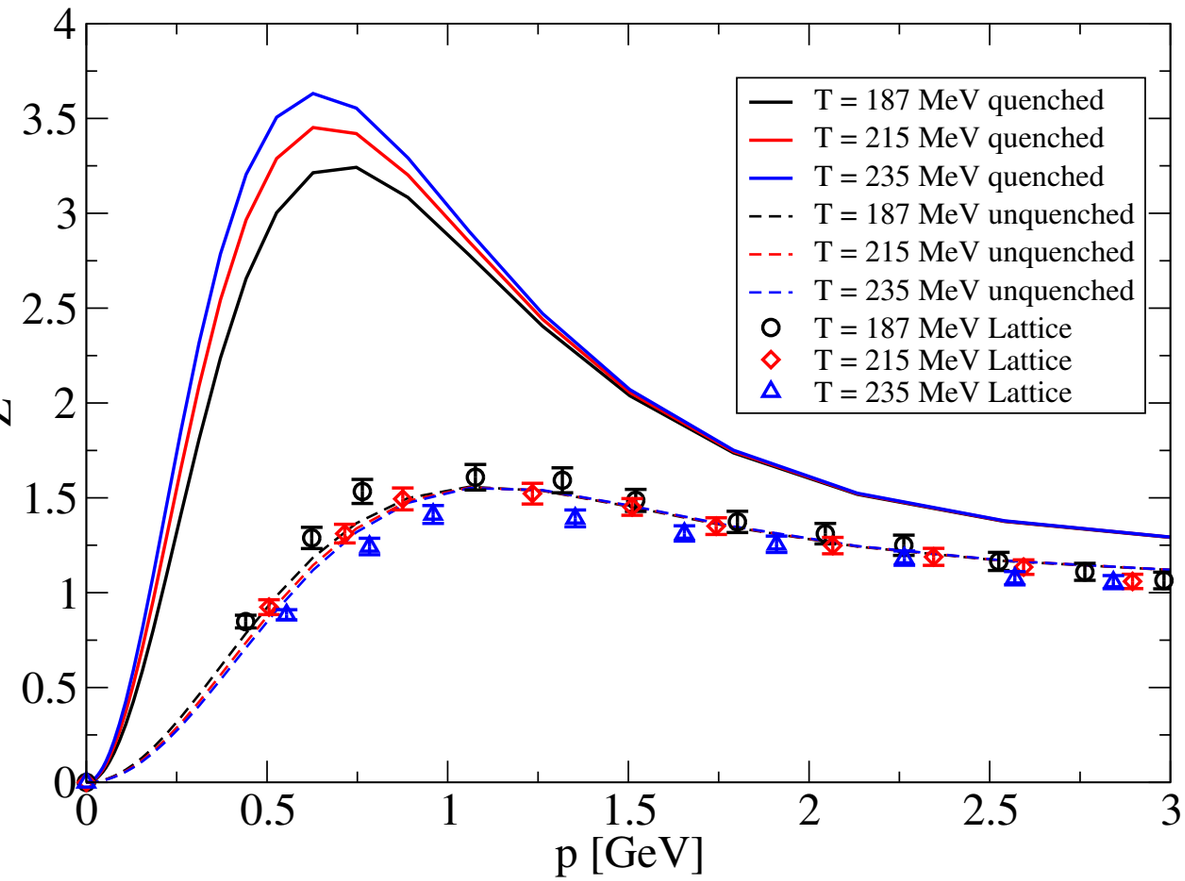
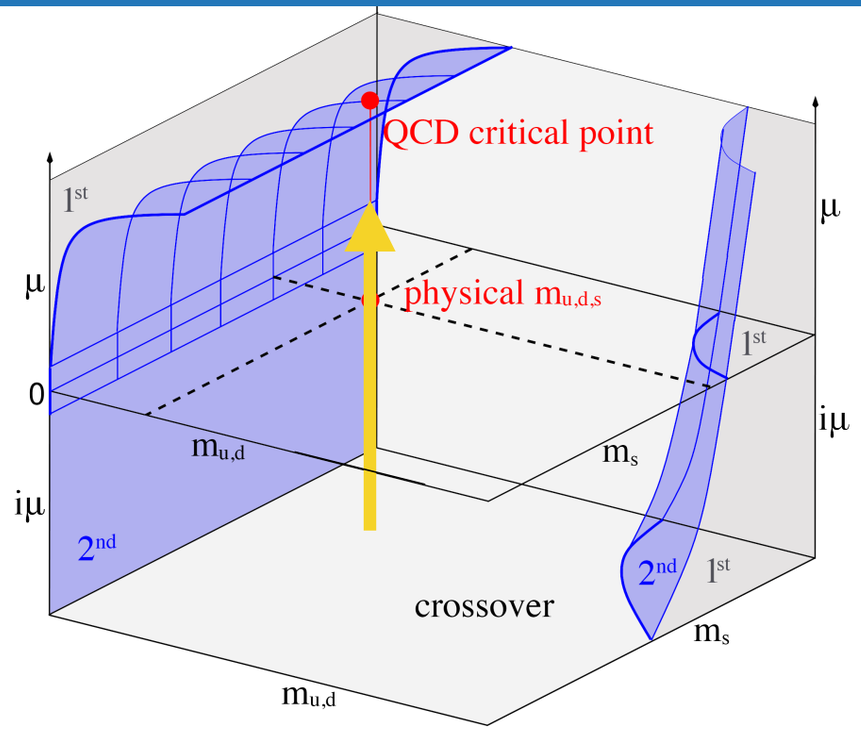
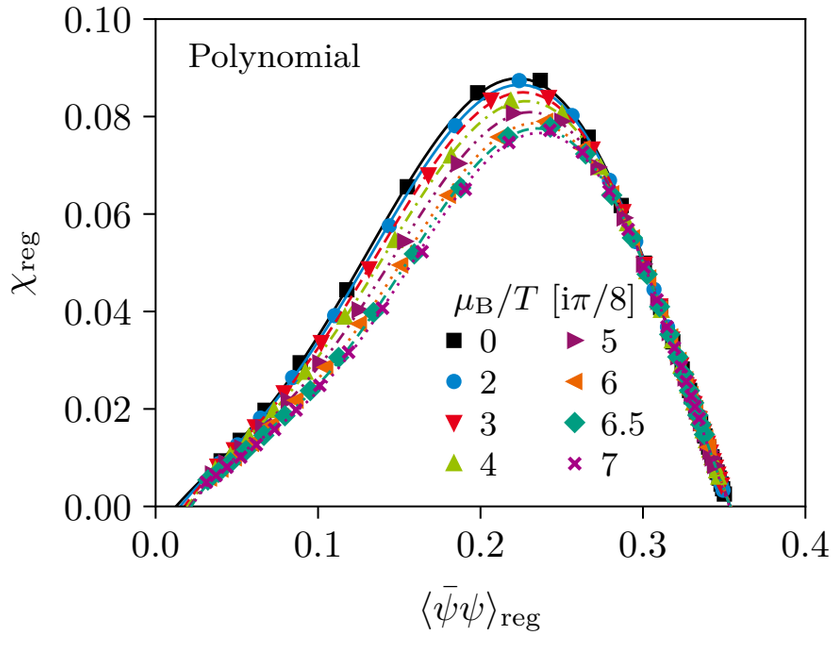


Lattice: Aouane, et al.PRD D87 (2013), [arXiv:1212.1102]
DSE: CF, Luecker, PLB 718 (2013) 1036, [arXiv:1206.5191]
CF, Luecker, Welzbacher, PRD 90 (2014) 034022

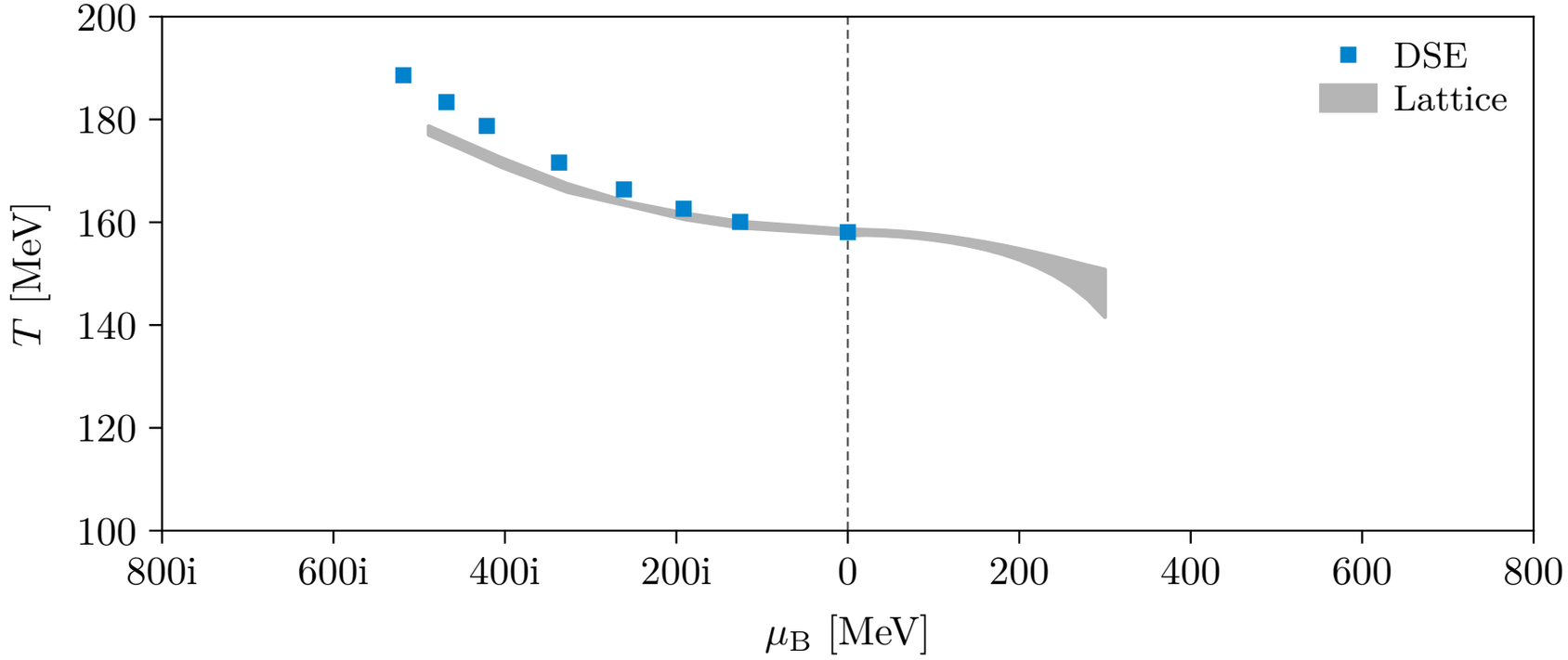
- quantitative agreement: DSE prediction verified by lattice
  FRG: similar results Fu, Rennecke, Pawlowski, PRD 101 (2020)

Extrapolation from imaginary chemical potential





$$\chi(T) = \frac{\partial\langle\bar{\psi}\psi\rangle(T)}{\partial m_u}$$



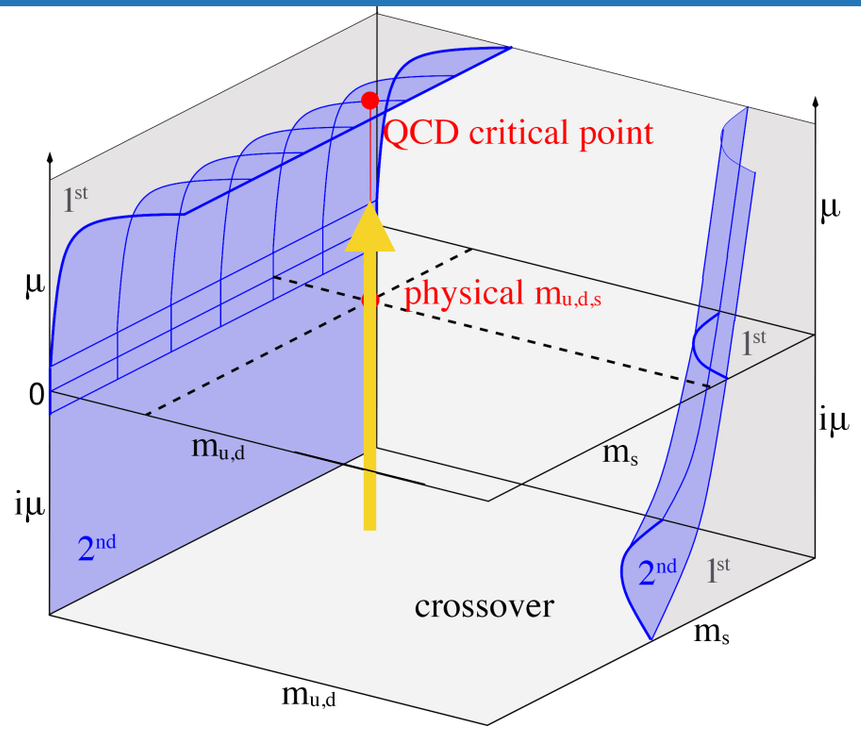
Lattice: Borsanyi et al. PRL 125 052001 (2020)
DSE: Bernhardt, CF, EPJA 59 (2023) 8, 181

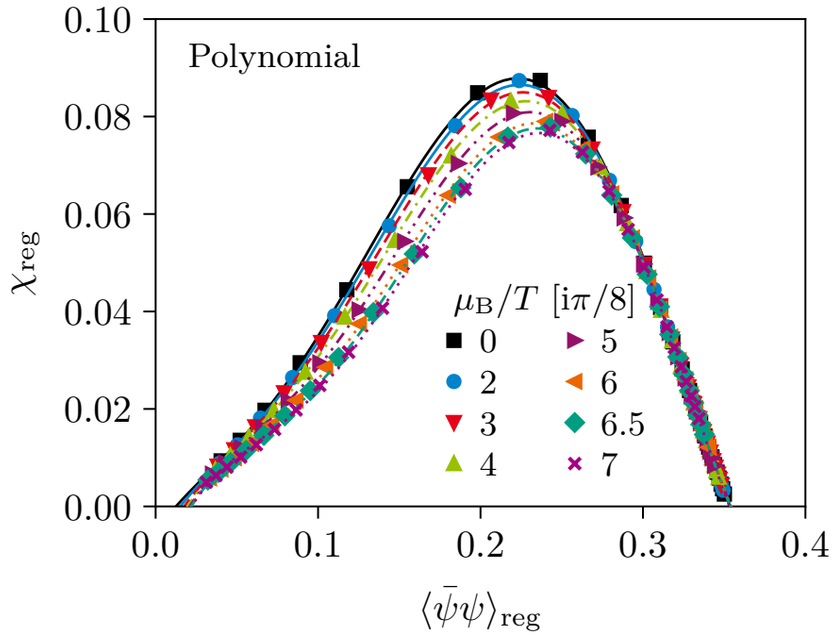
see also FRG (Nf=2): Braun et. al. PRL 106 (2011)

$$\frac{T_c(\mu_B)}{T_c} = 1 - \kappa_2\left(\frac{\mu_B}{T_c}\right)^2 - \kappa_4\left(\frac{\mu_B}{T_c}\right)^4$$

$$\kappa_2^{\rm poly} = 0.0196\,, \quad \kappa_4^{\rm poly} = 0.00015\,,$$

Extrapolation from imaginary chemical potential





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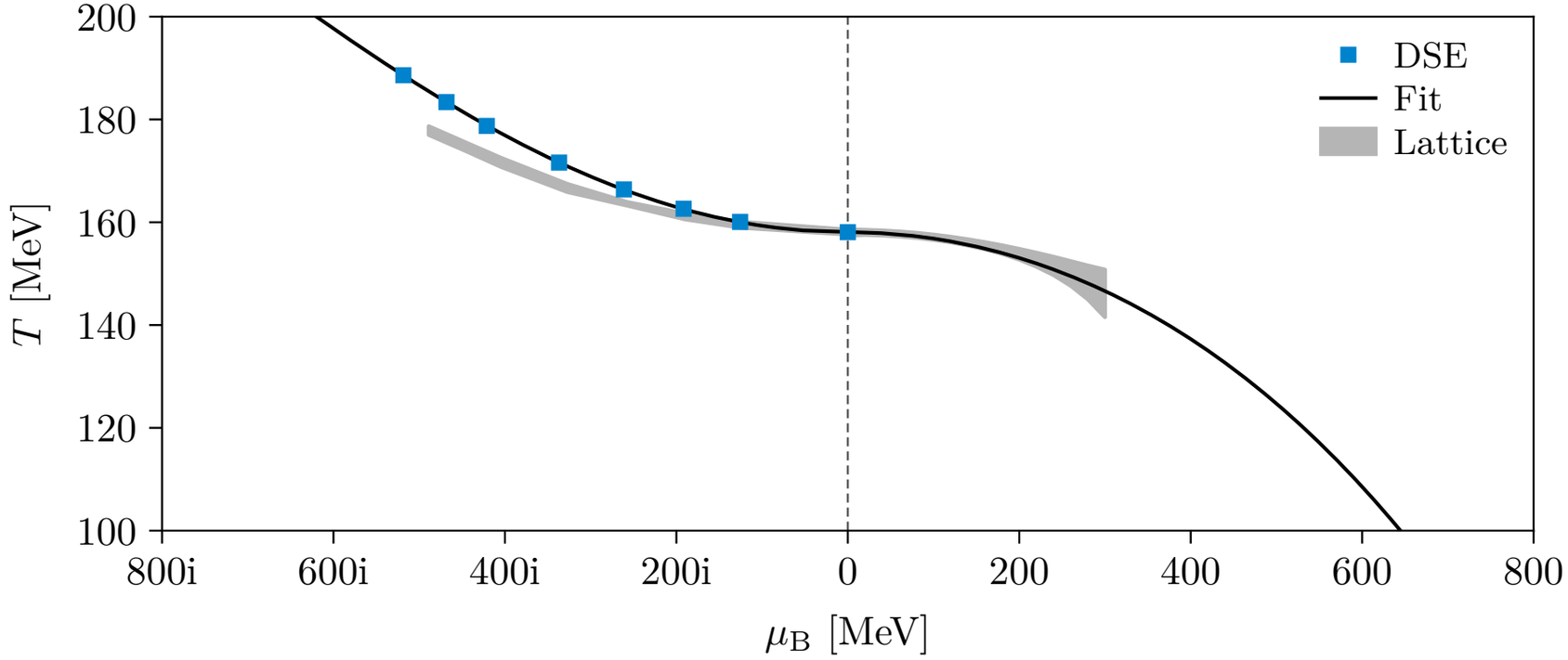

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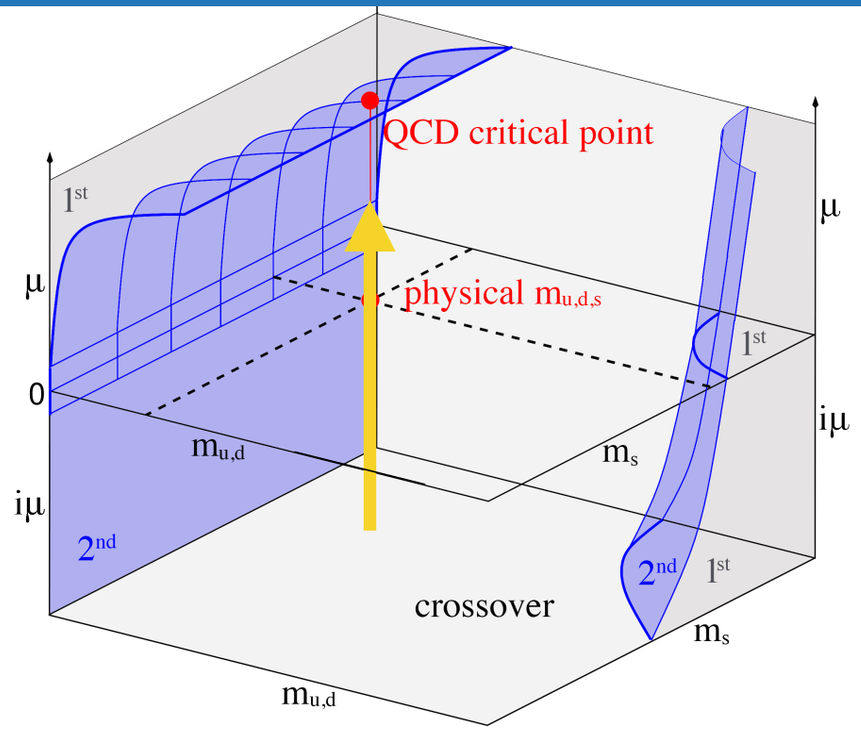

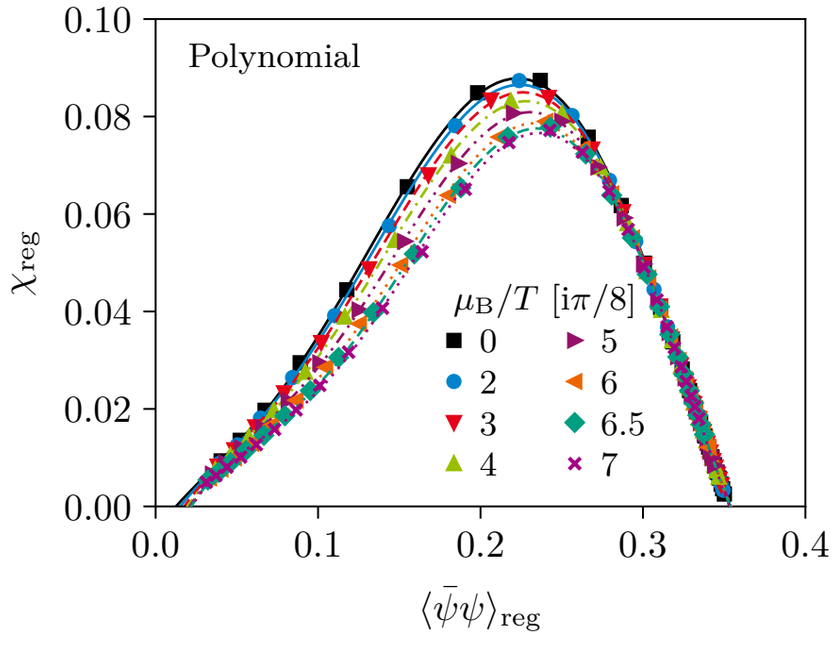

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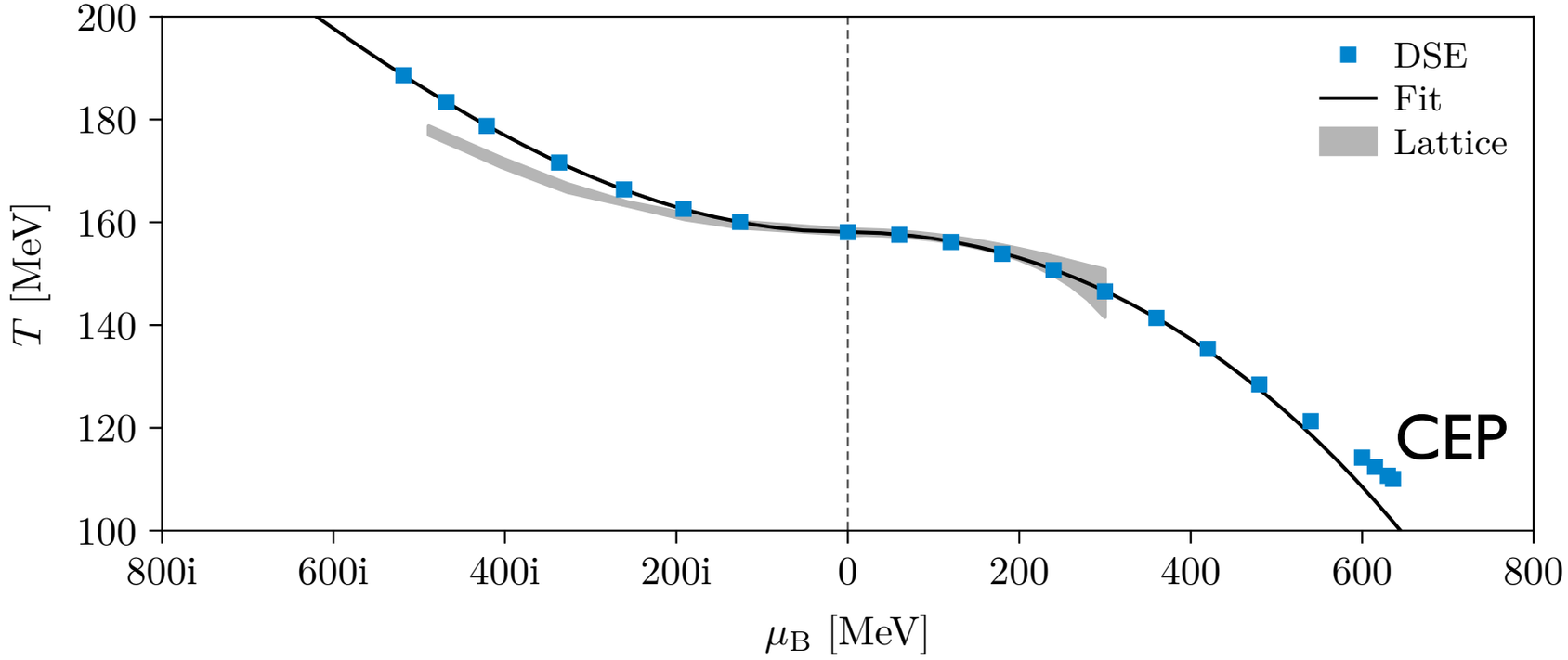

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- Extrapolation works very well!

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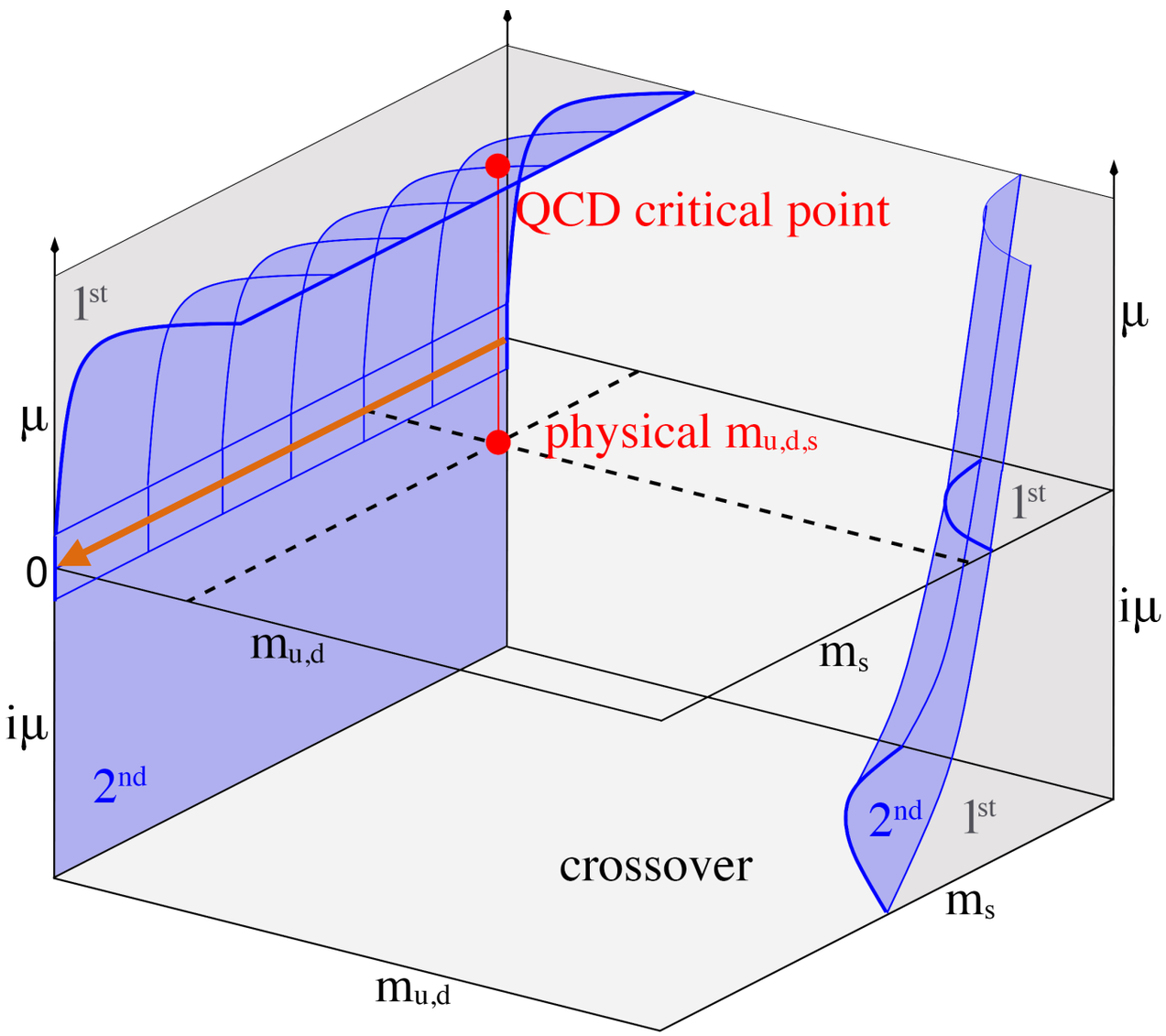
Columbia plot - chiral critical surface



no first order region found

DSE: Bernhardt and CF, PRD 108 (2023) 114018

Lattice: Cuteri, Philipsen and Sciarra, JHEP 11 (2021), 141
Dini, et al, PRD 105 (2022) no.3, 034510

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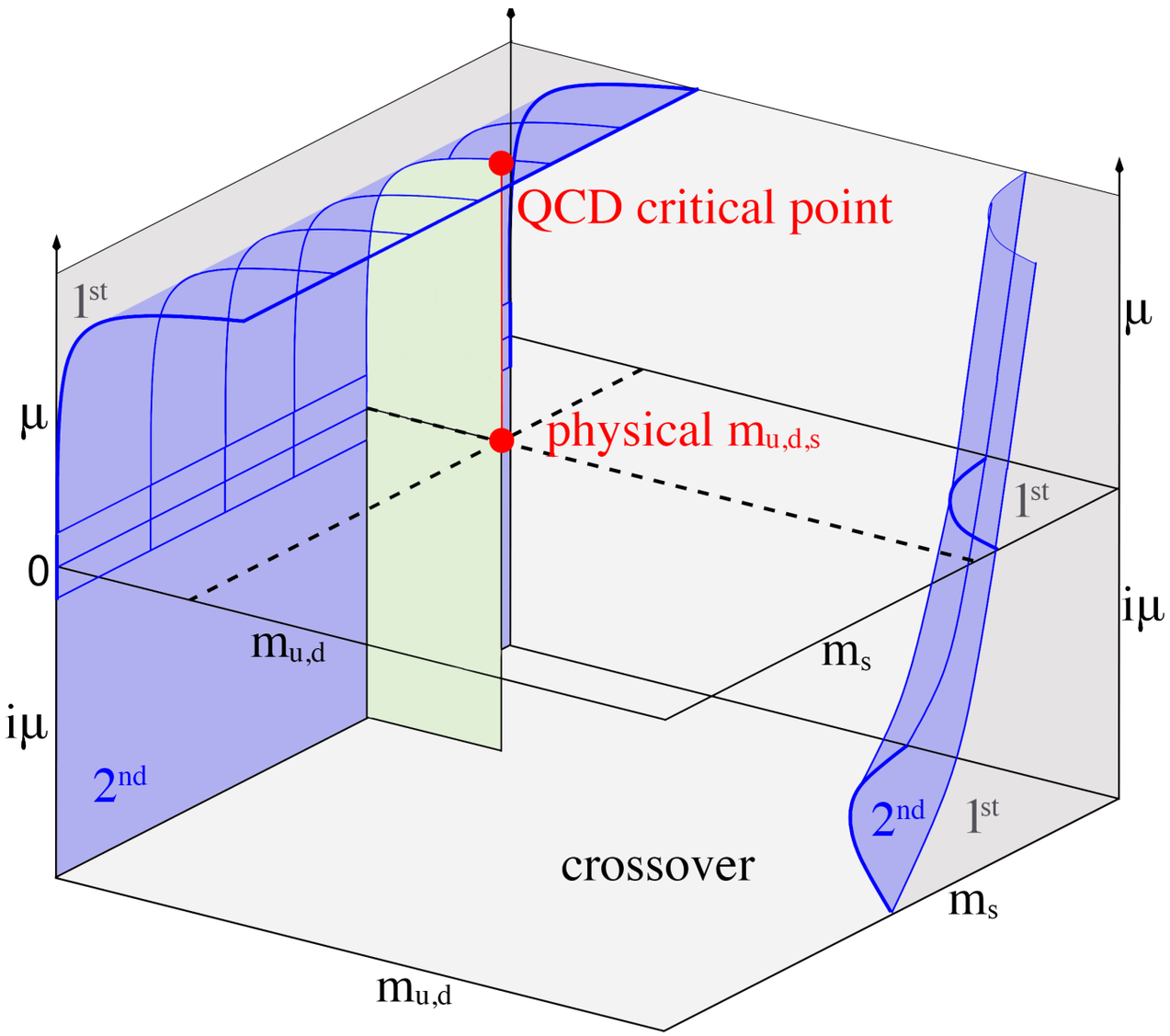

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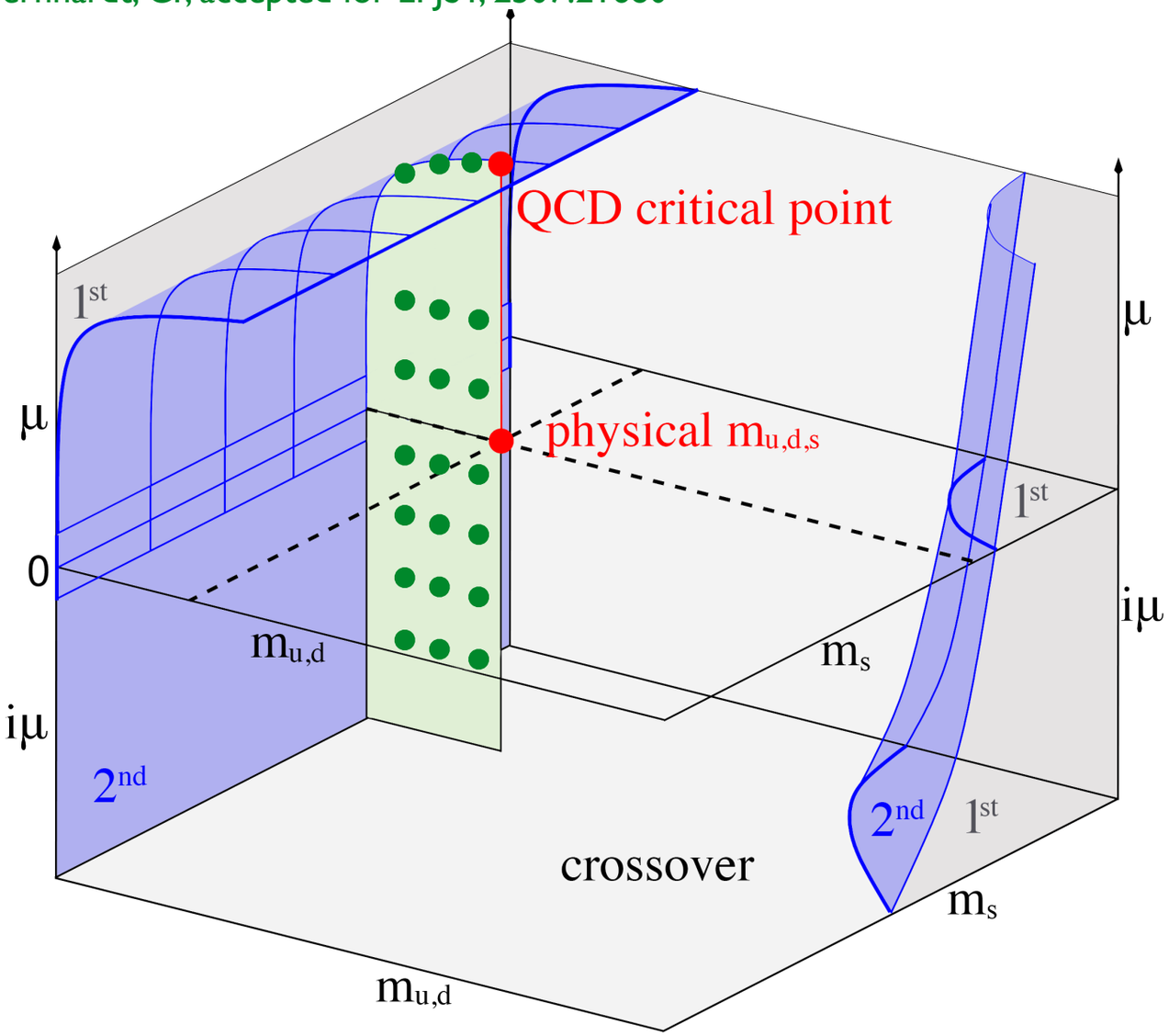

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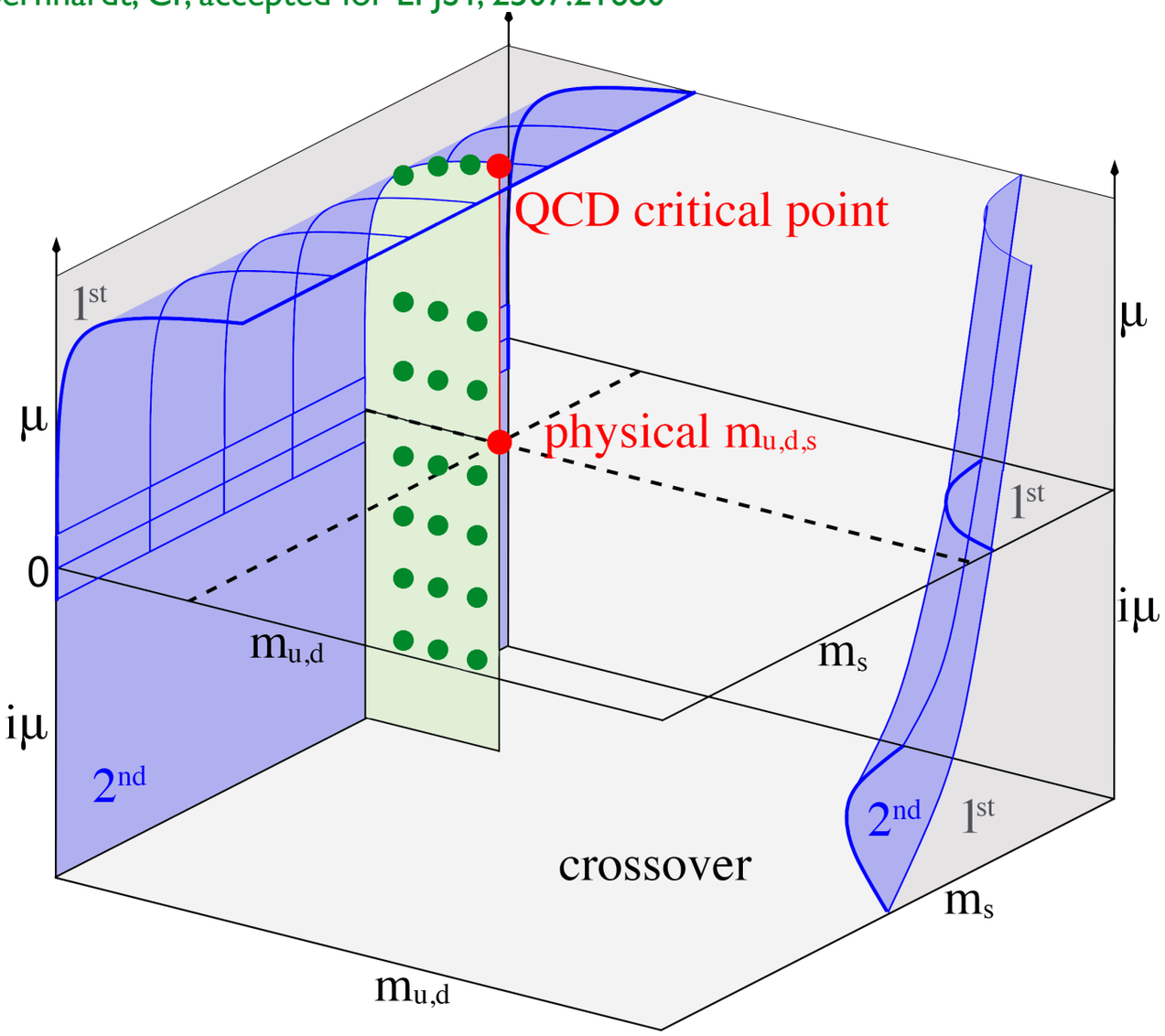

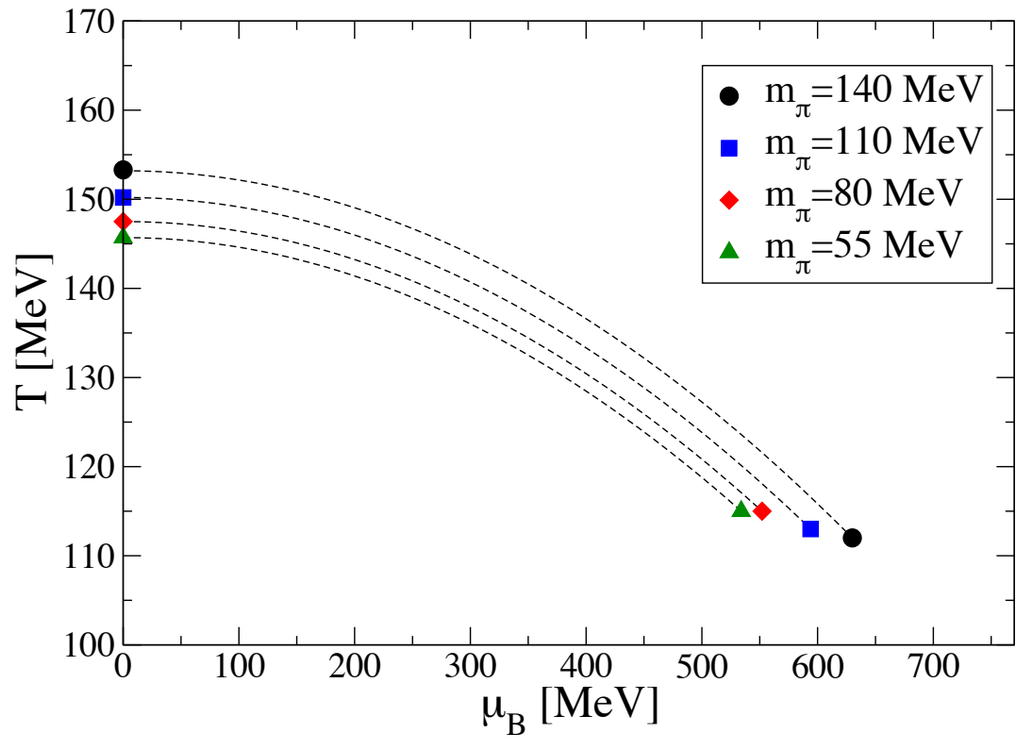

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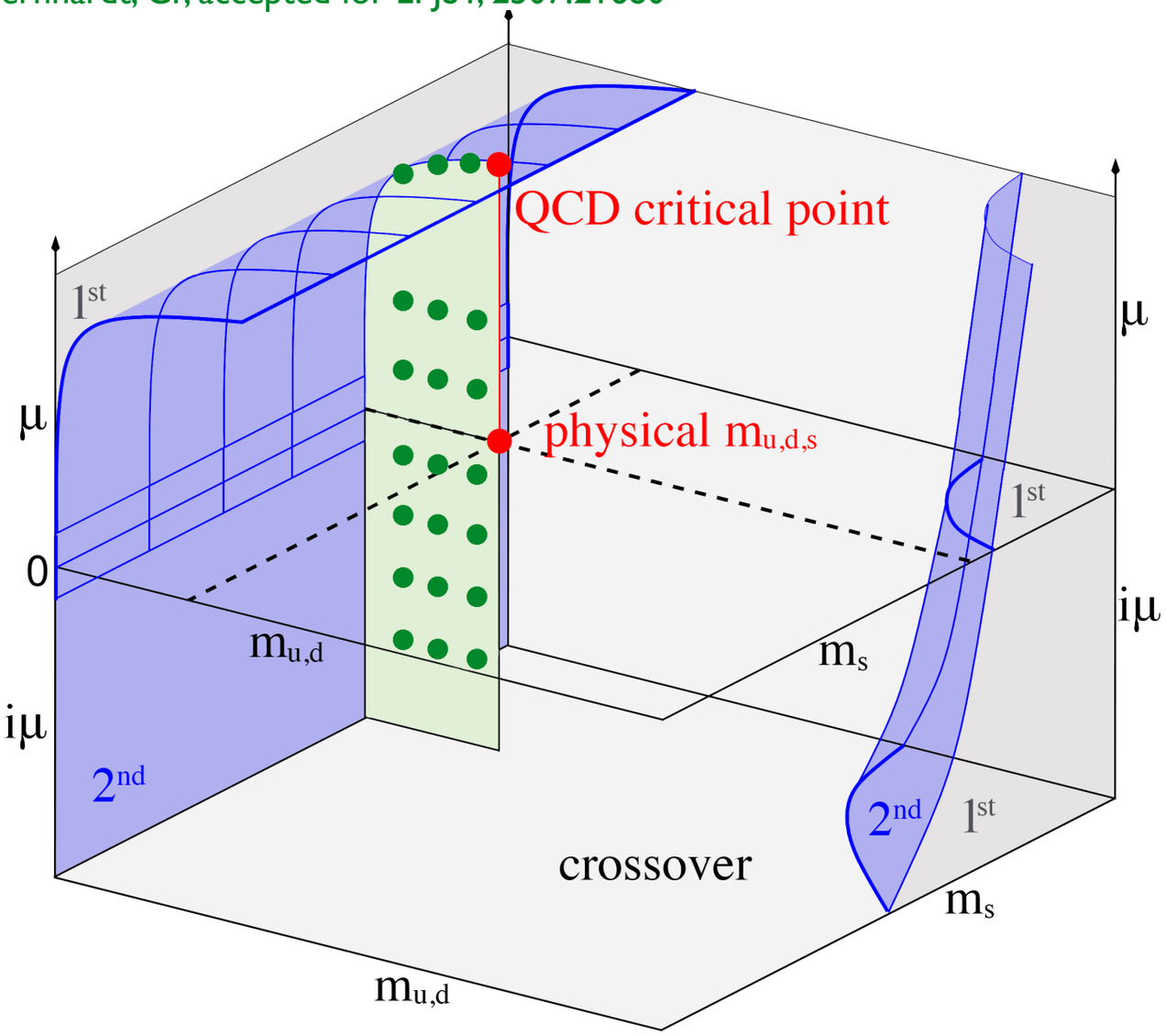

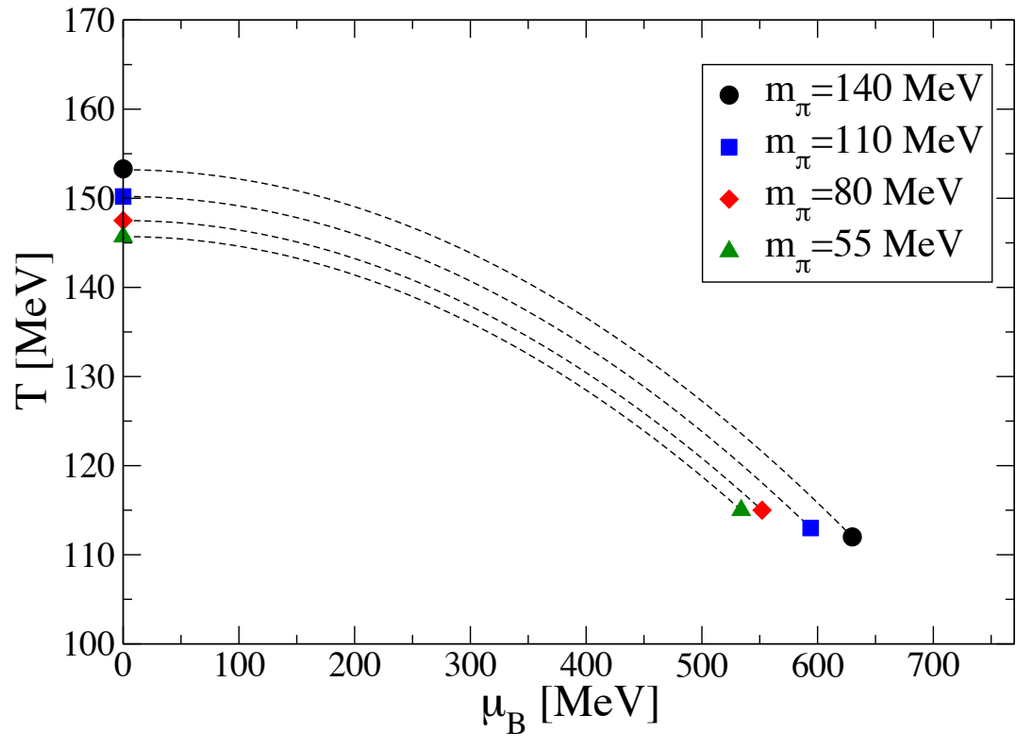

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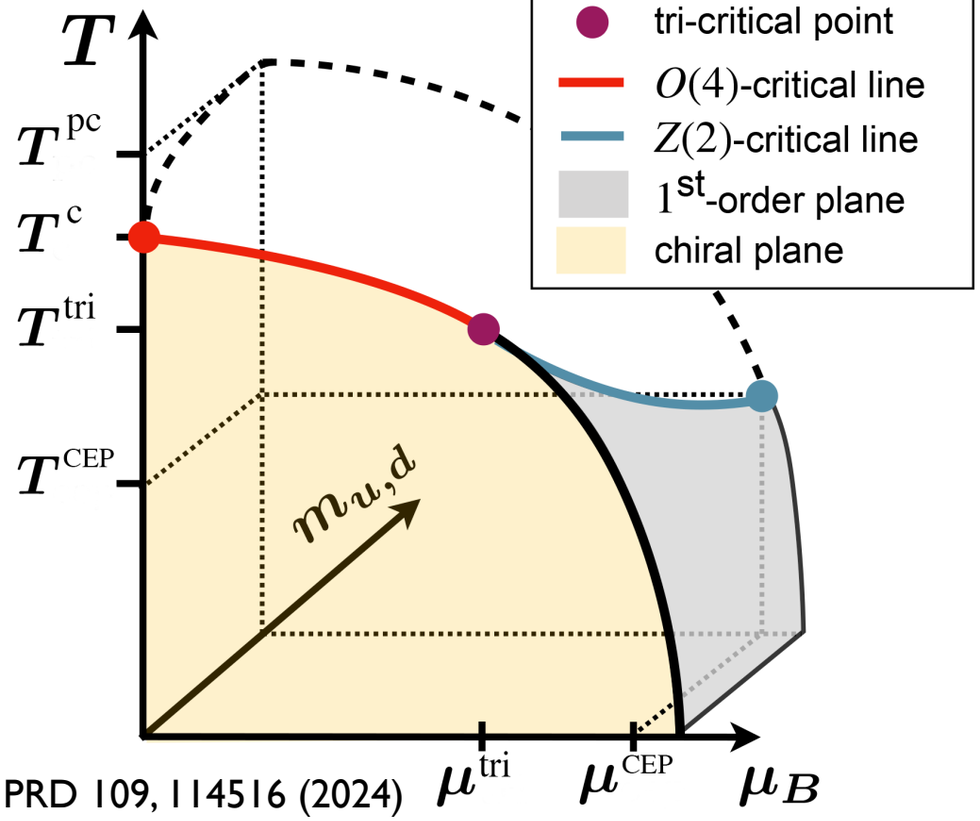
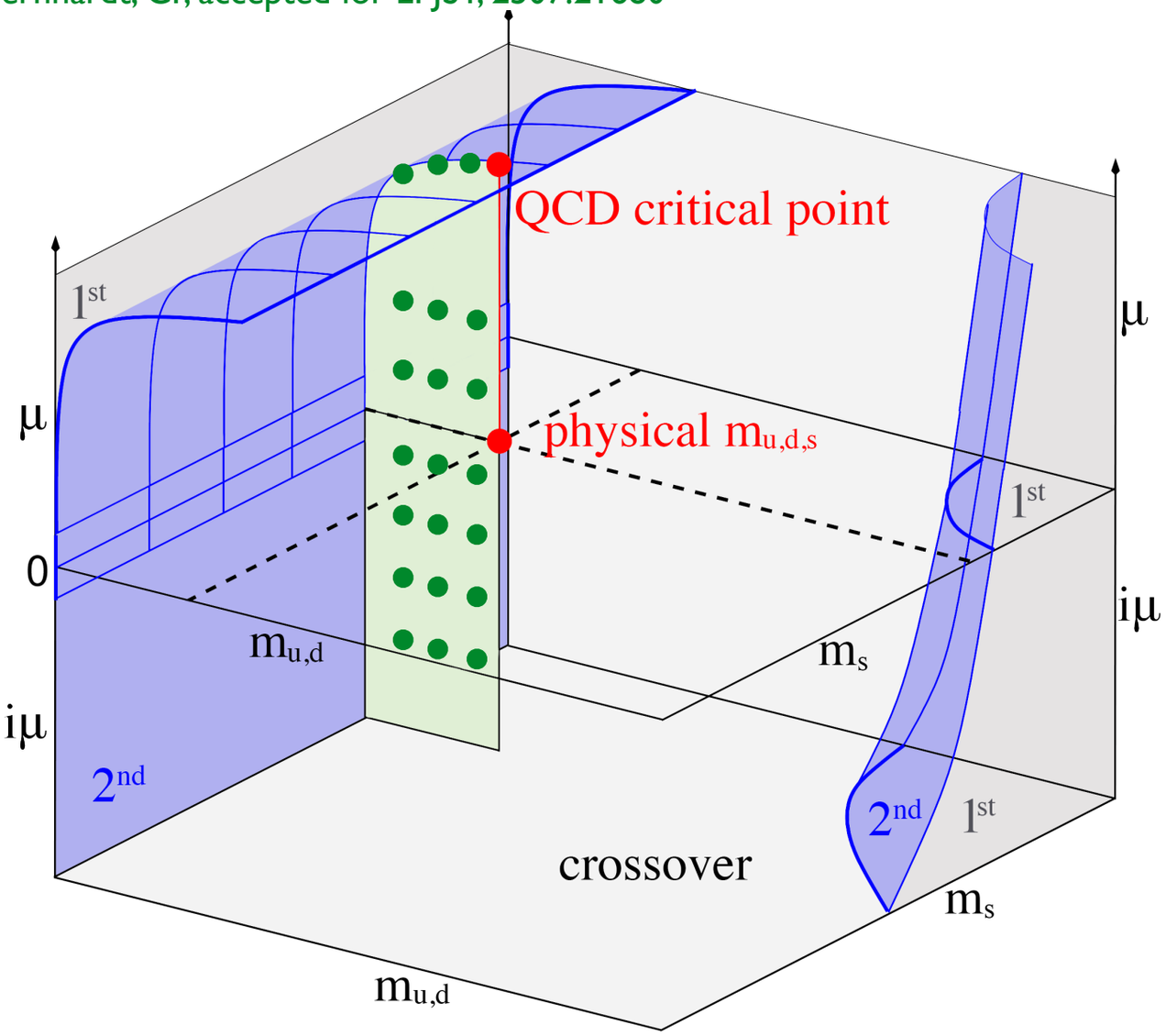
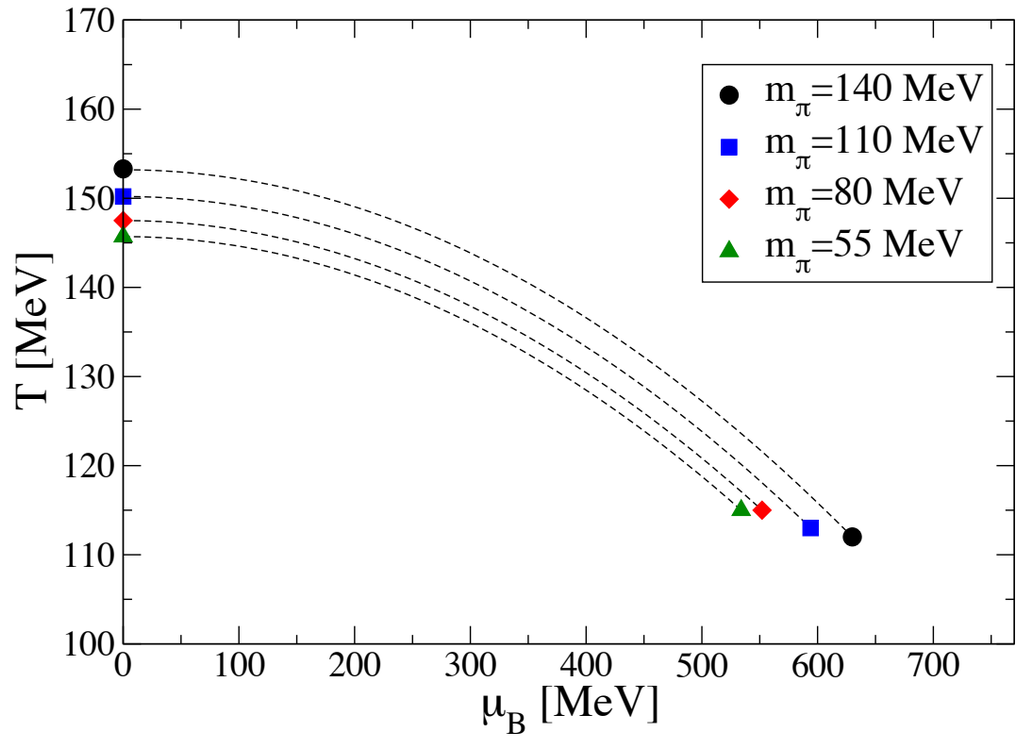


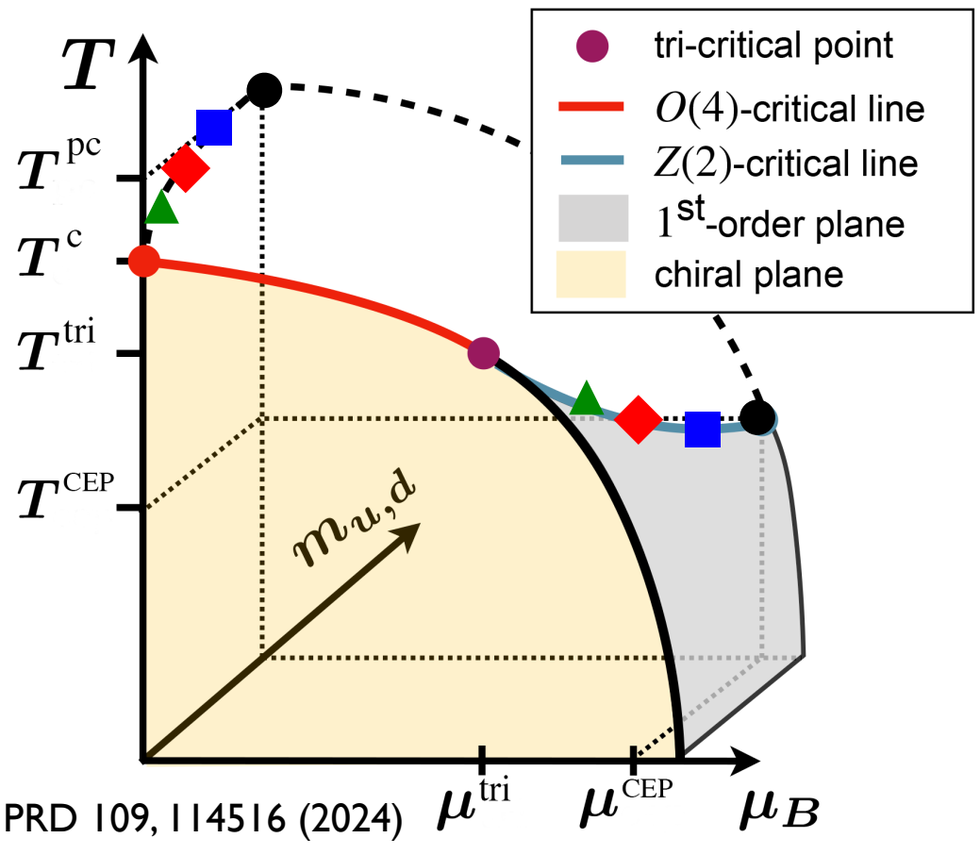
Ding, et al. PRD 109, 114516 (2024)

Columbia plot - chiral critical surface

Bernhardt, CF, accepted for EPJST, 2507.21680







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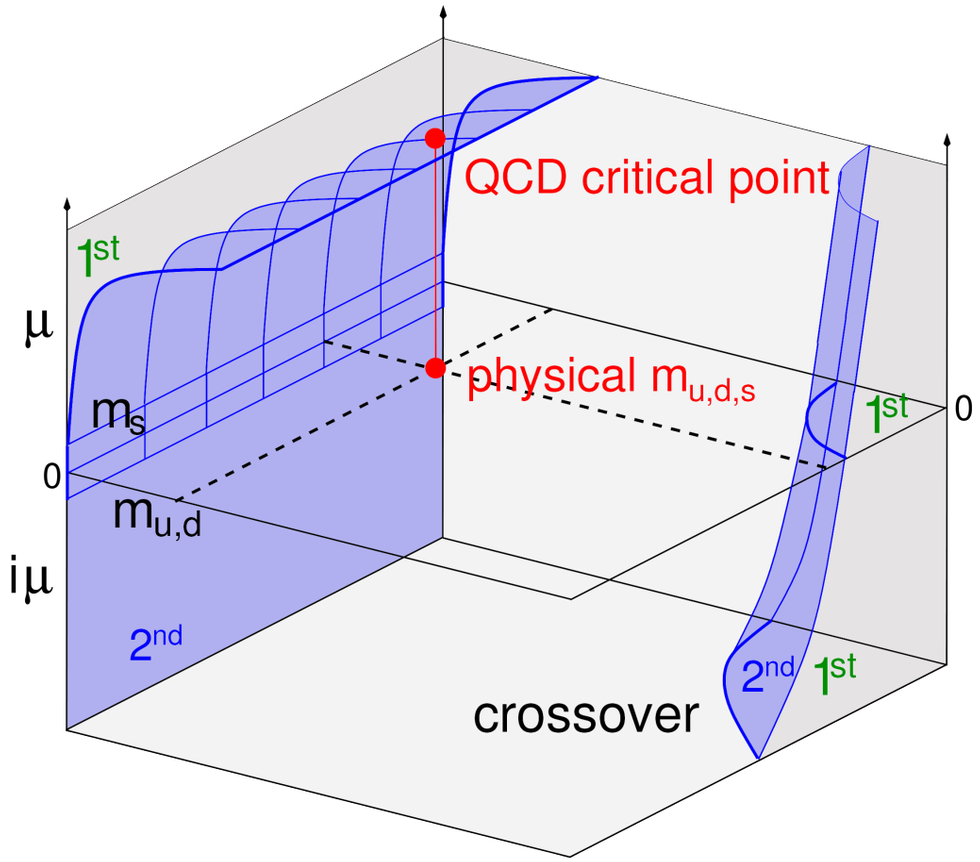
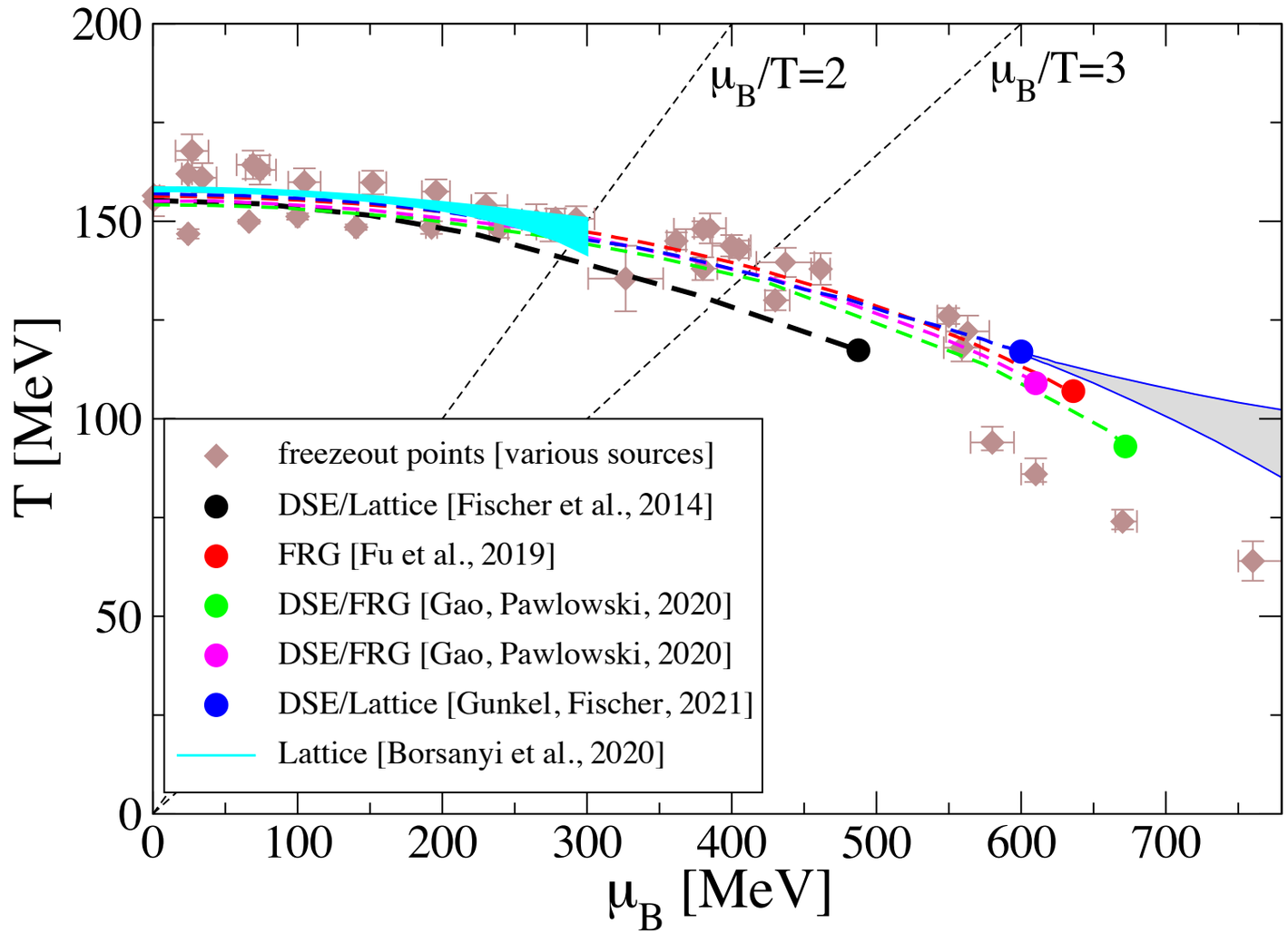
Location of CEP





- how stable is this result ??
  - crosscheck DSE-FRG ✔

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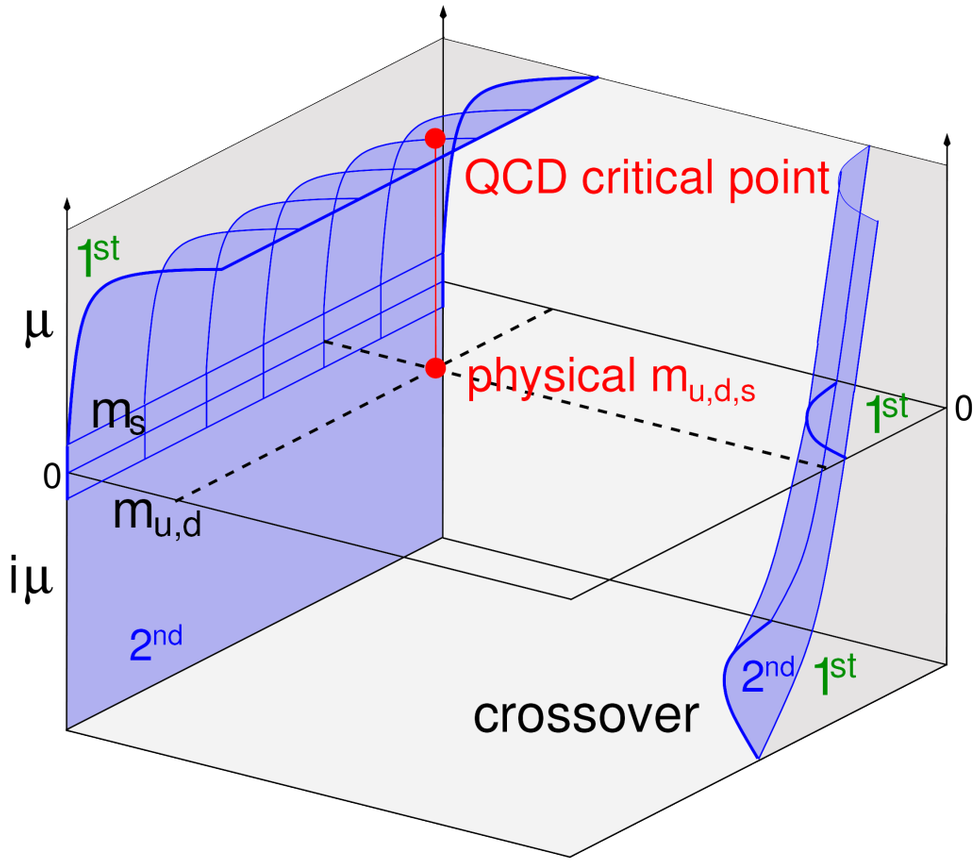

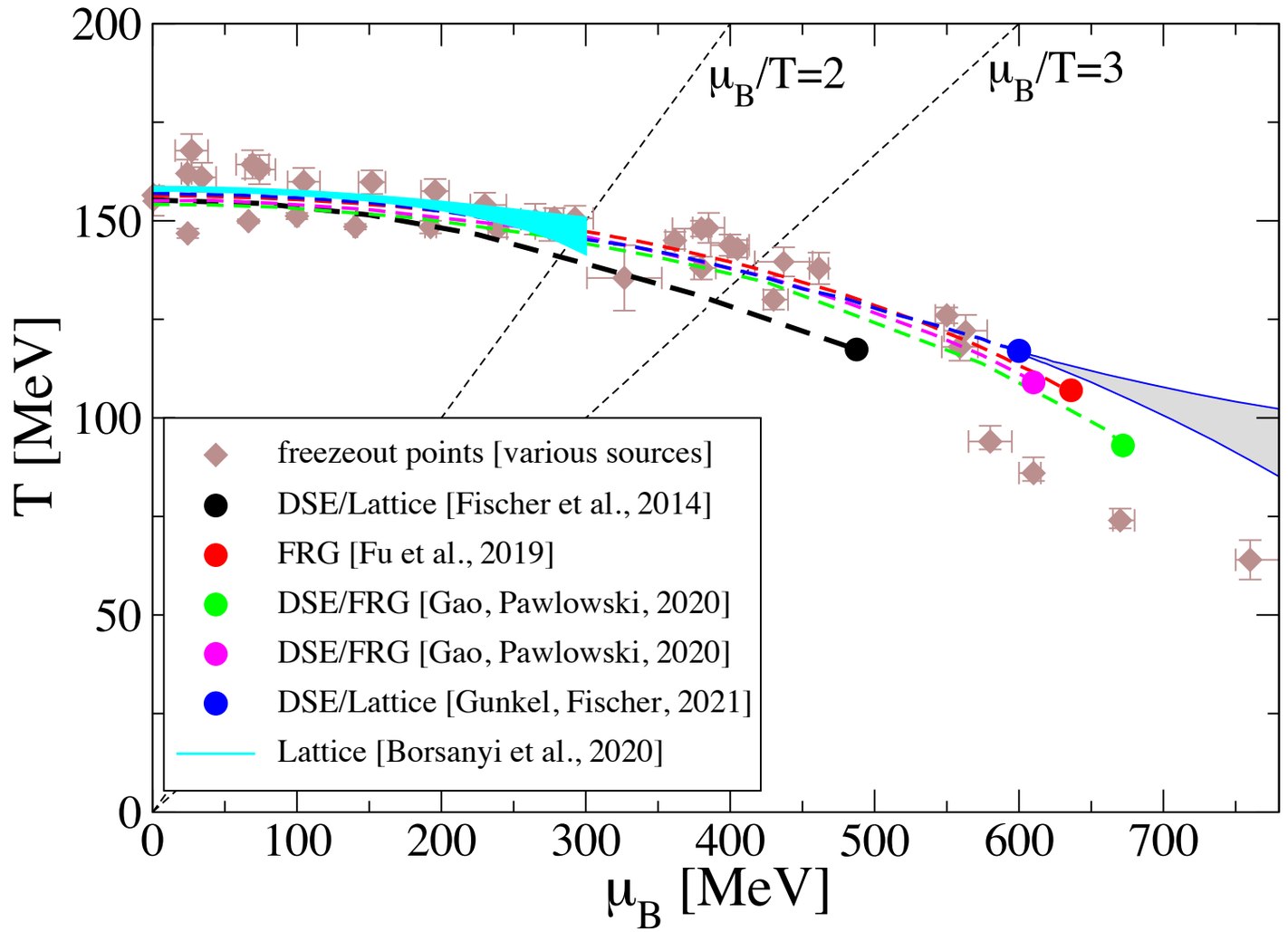

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CF, Luecker, Welzbacher, PRD 90 (2014) 034022

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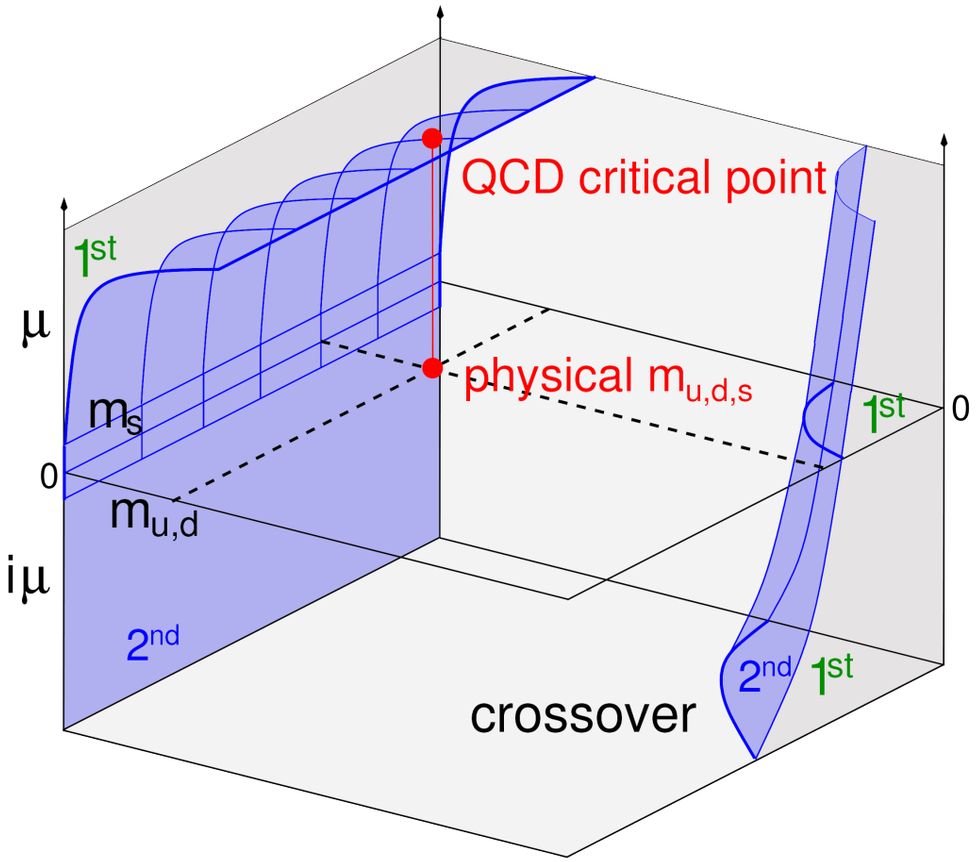

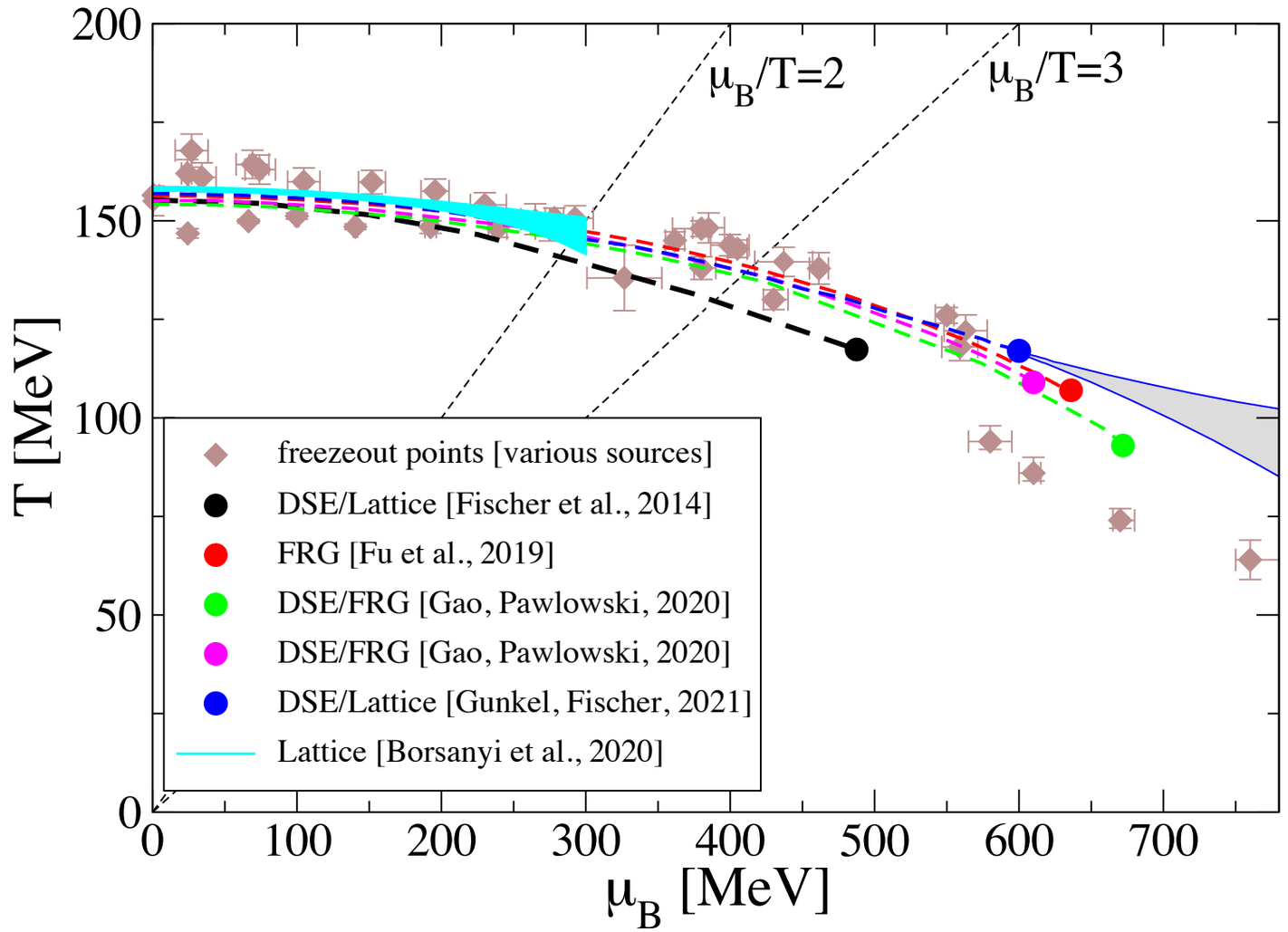

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  - baryon effects ✔ Eichmann, CF, Welzbacher, PRD93 (2016); Braun et al. PRD 101 (2020)

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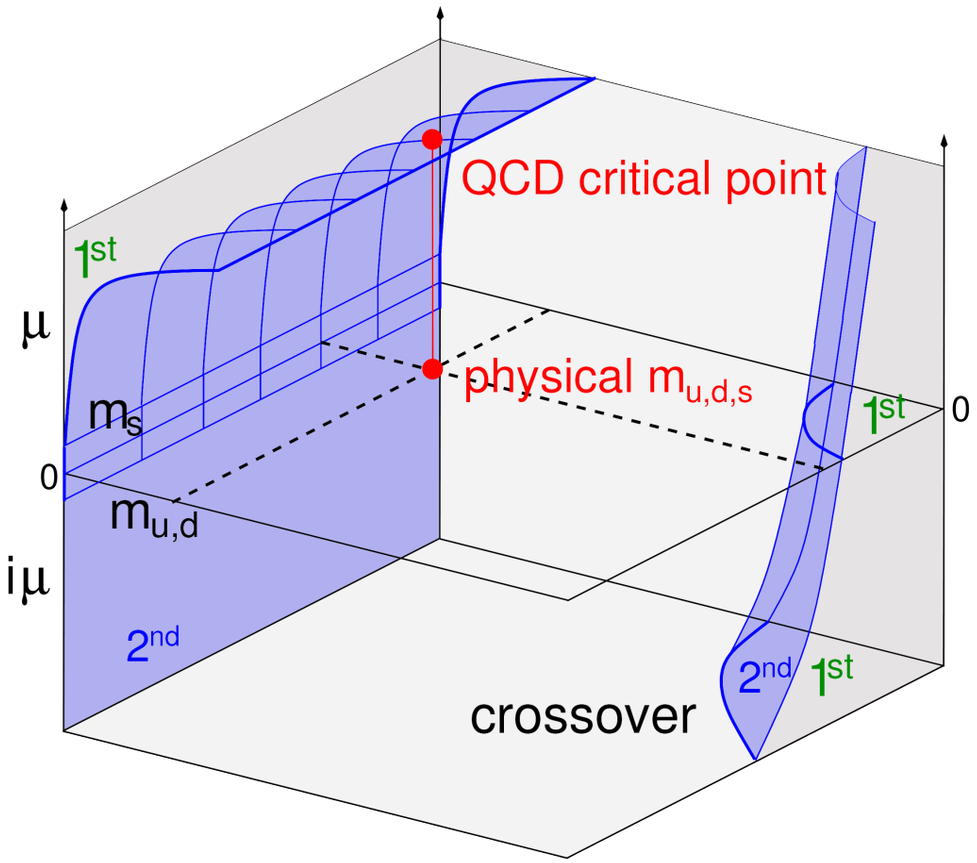

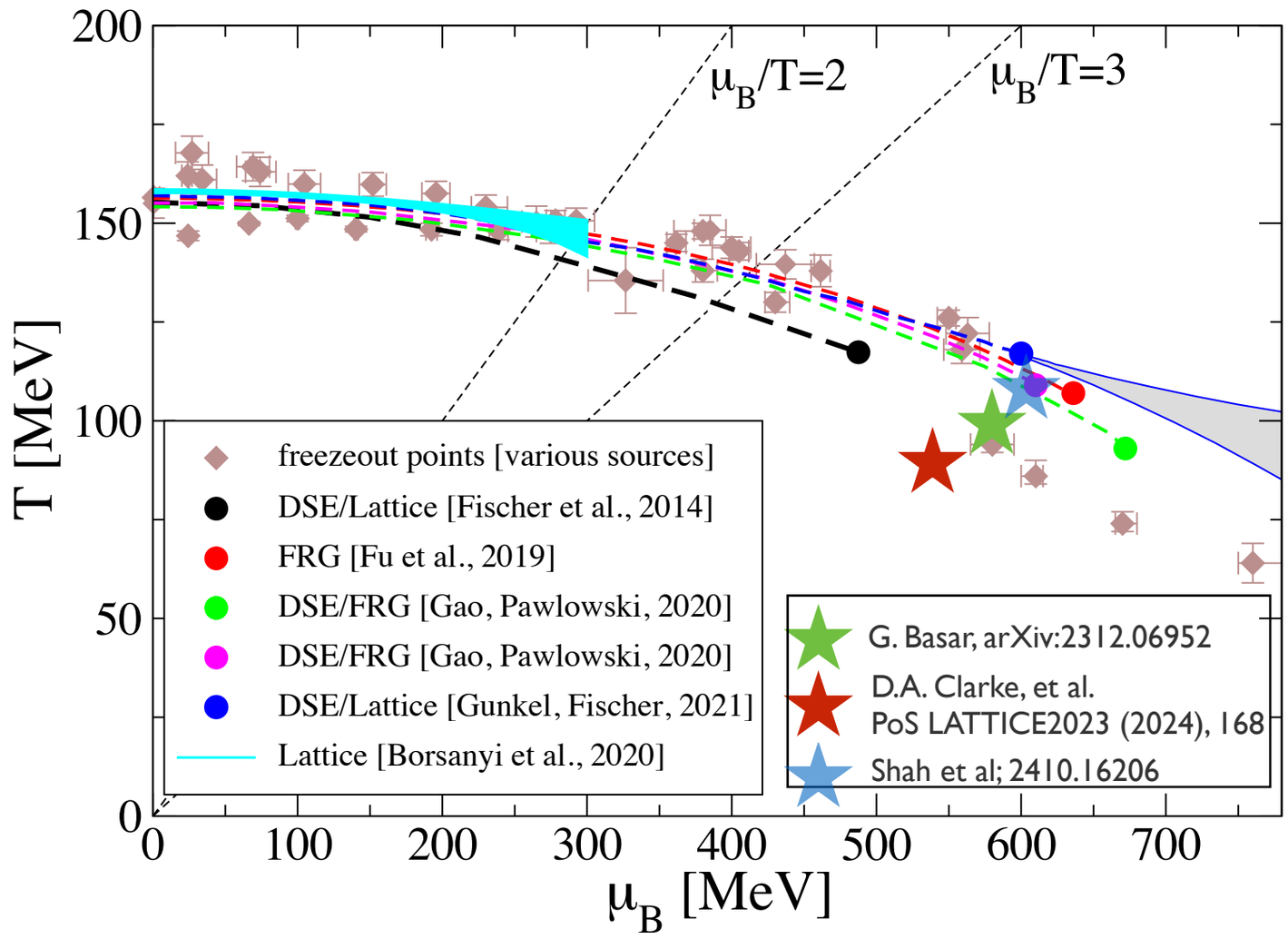

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  - 'cross-check' with lattice (✔)

CF, Luecker, Welzbacher, PRD 90 (2014) 034022

Eichmann, CF, Welzbacher, PRD93 (2016)
Braun et al. PRD 101 (2020)

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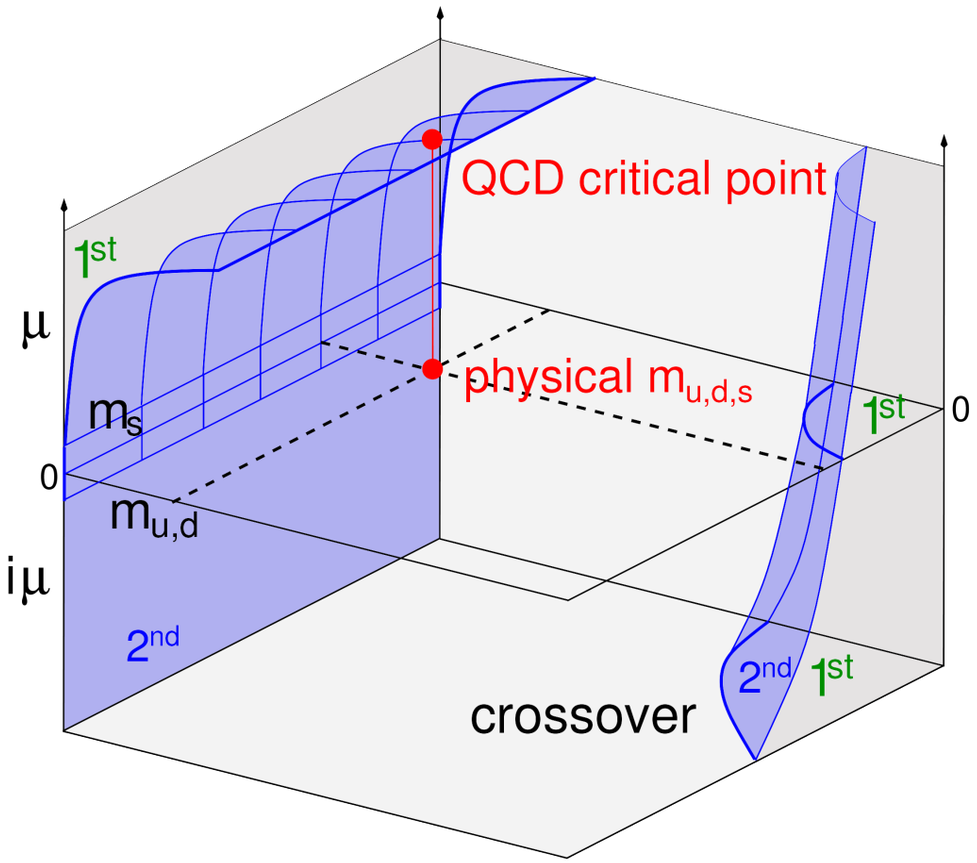

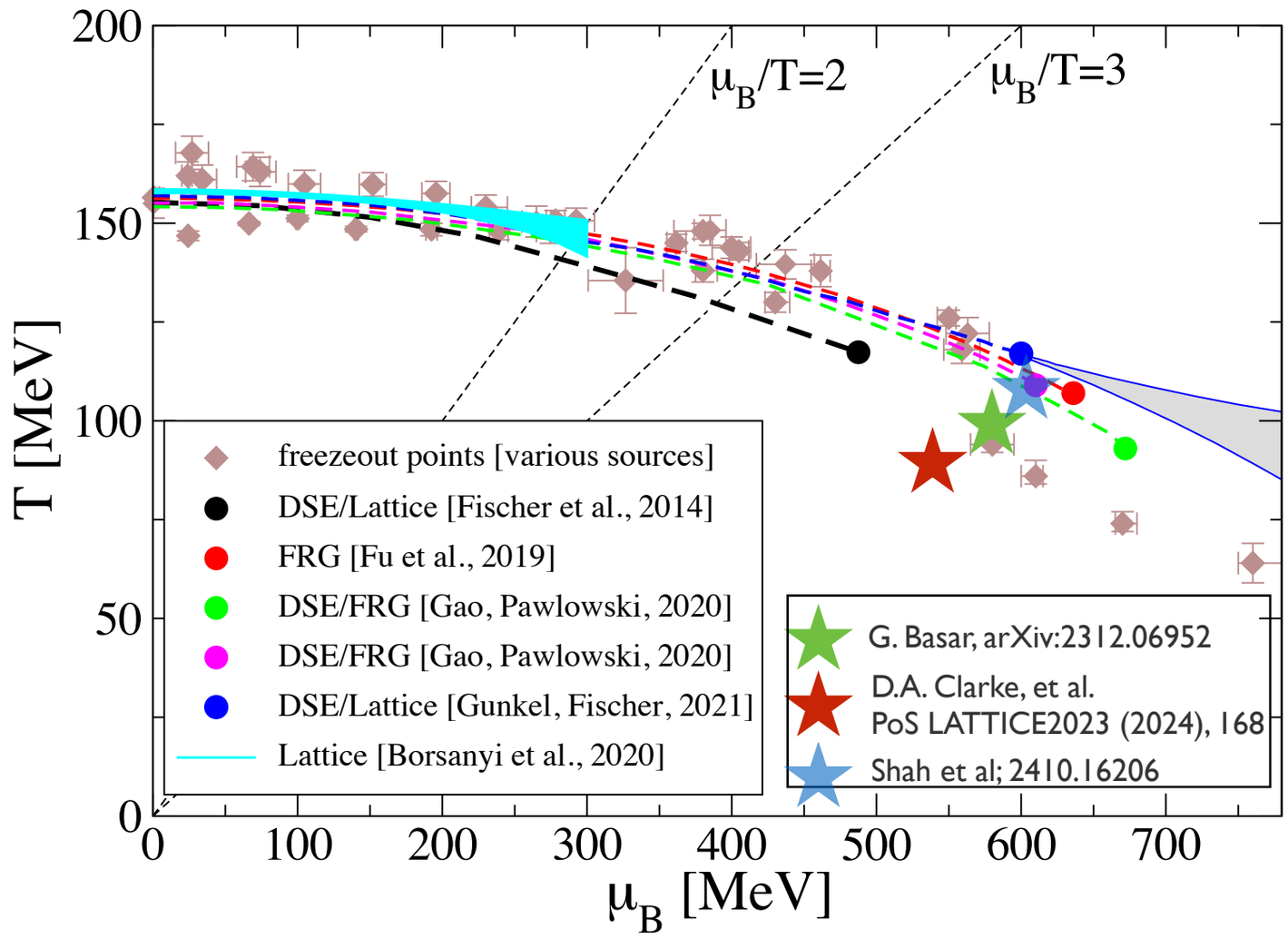

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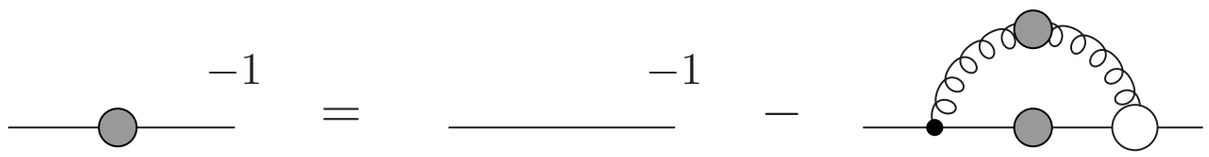
Braun et al. PRD 101 (2020)

T. F. Motta, J. Bernhardt, M. Buballa and CF, PRD 108 (2023)
T. F. Motta, J. Bernhardt, M. Buballa and CF, acc. for PRD, [2411.02285]

Overview

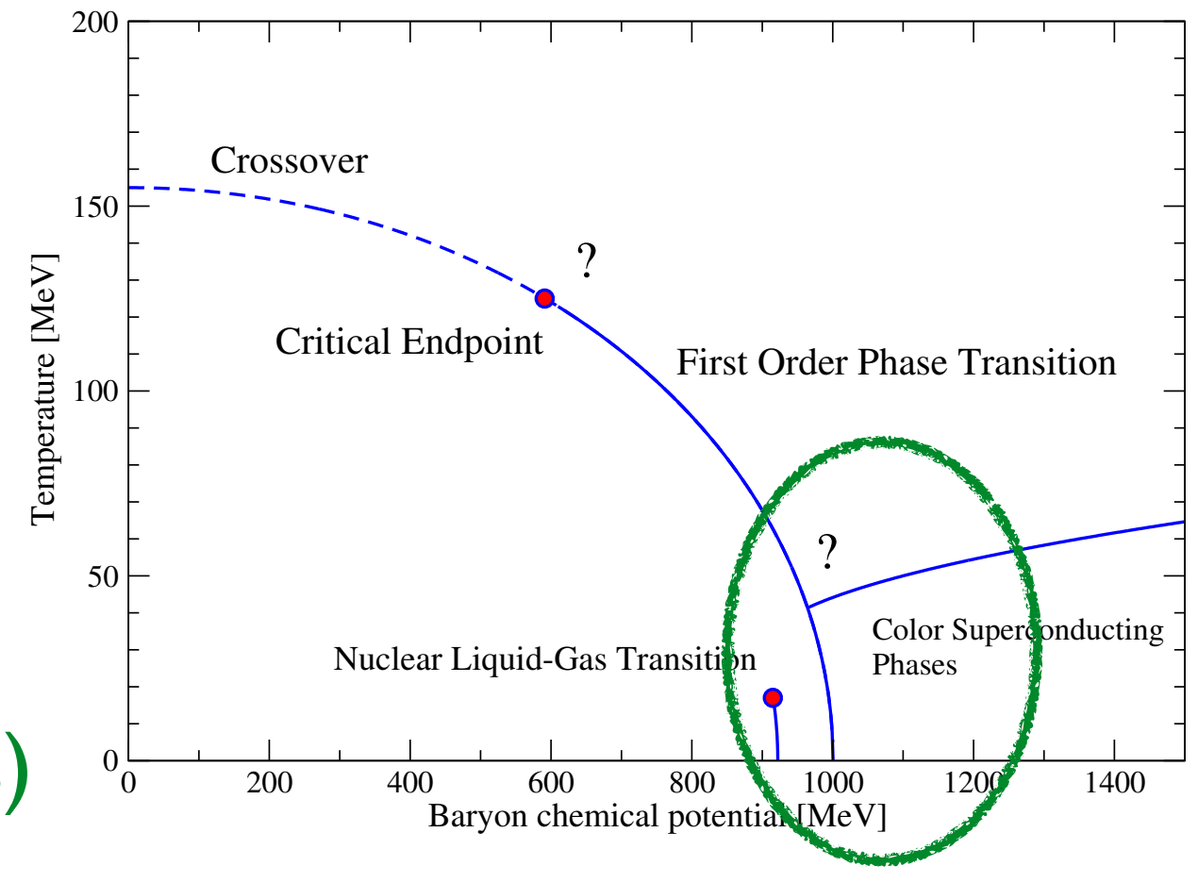
1.Introduction: dynamical mass generation



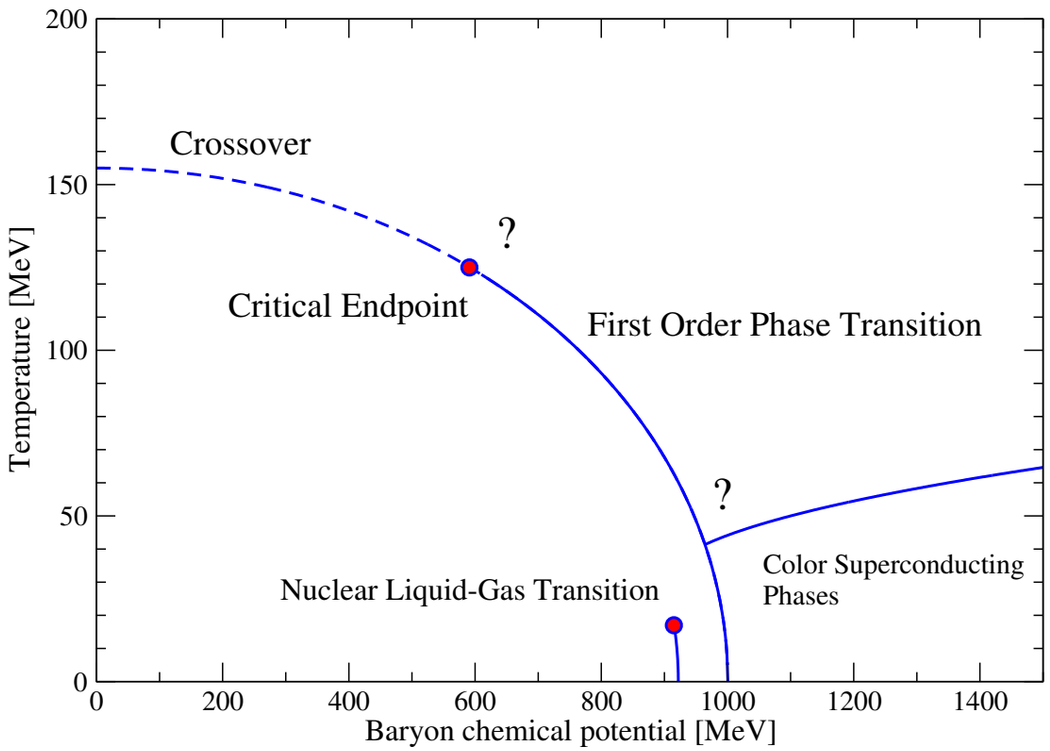
2.Large T, small μ: the quest for the critical end point

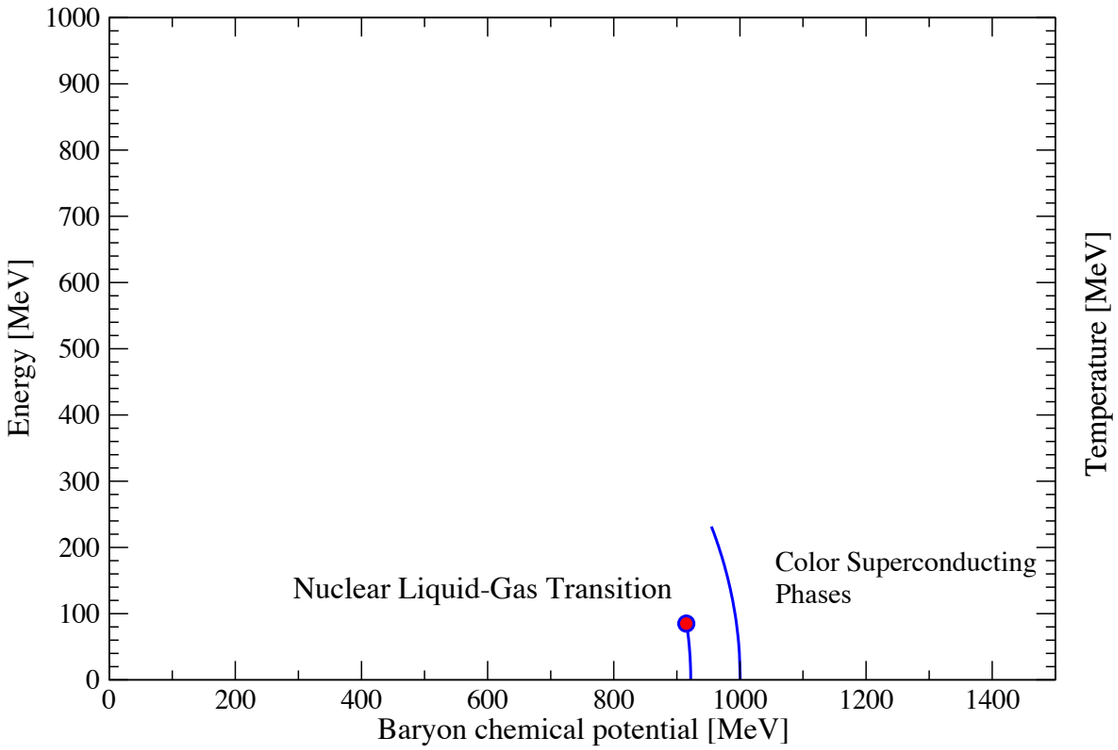
3.Small T, large μ: the quest for the equation of state

Neutron star (mergers)



Phase transitions at zero T: what to expect…





"silver blaze": no excitations until

T. D. Cohen, PRL 91 , 222001 (2003)

$$\mu_B \leq m_N^*$$

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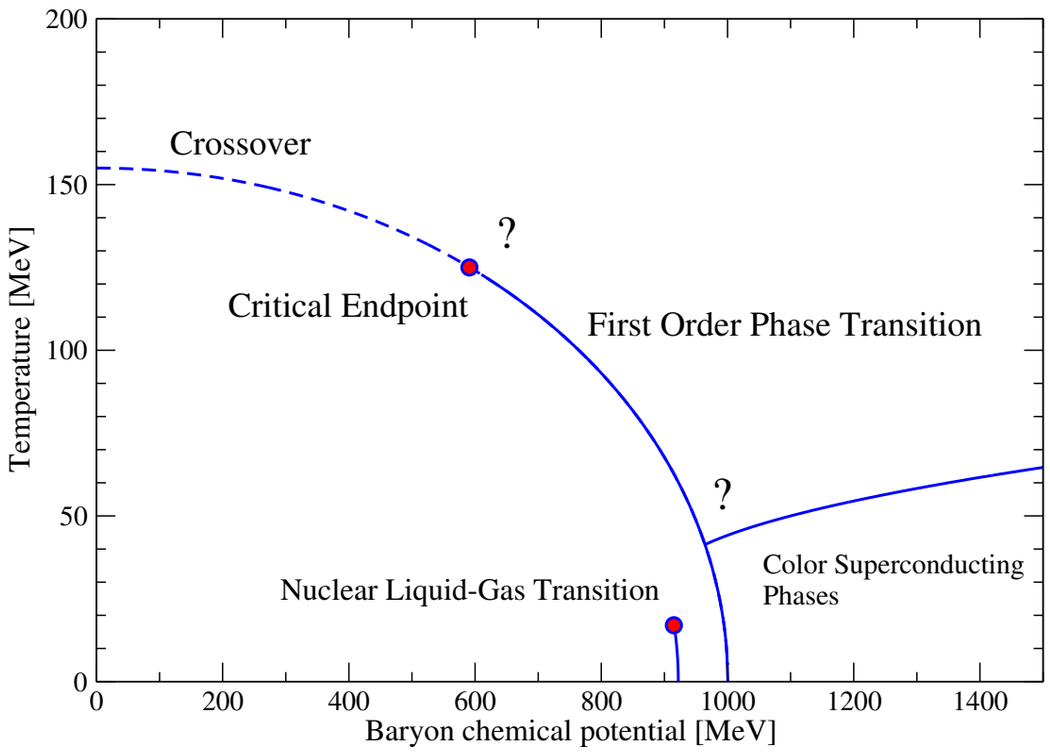

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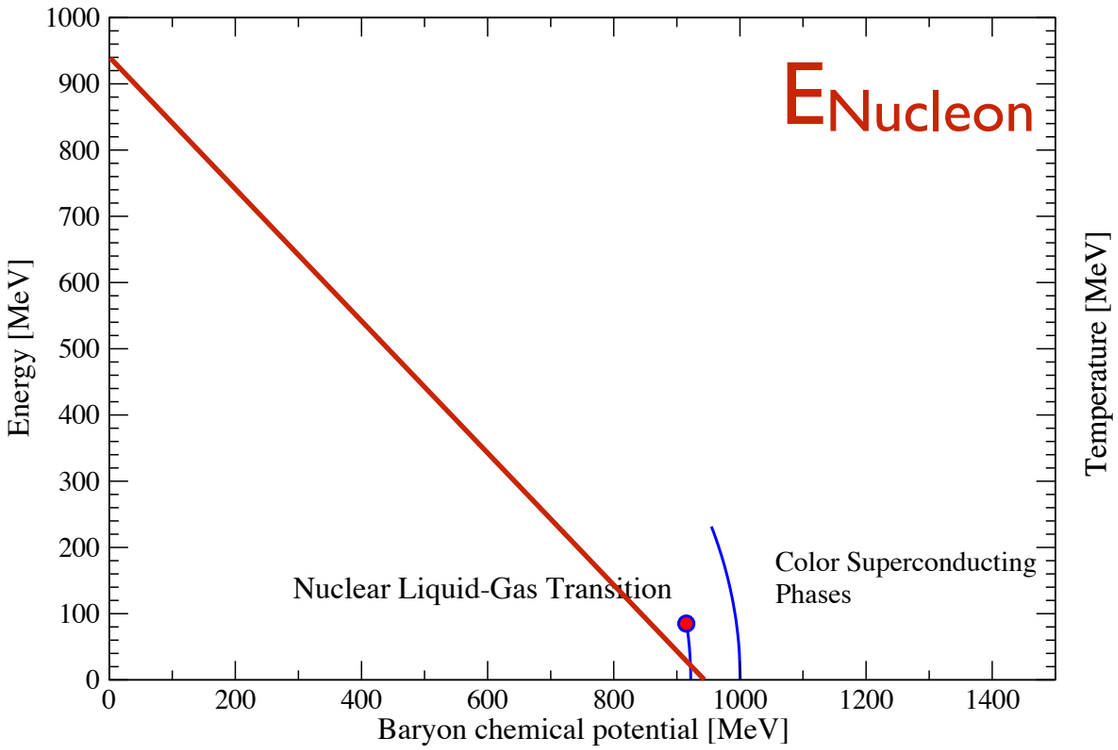

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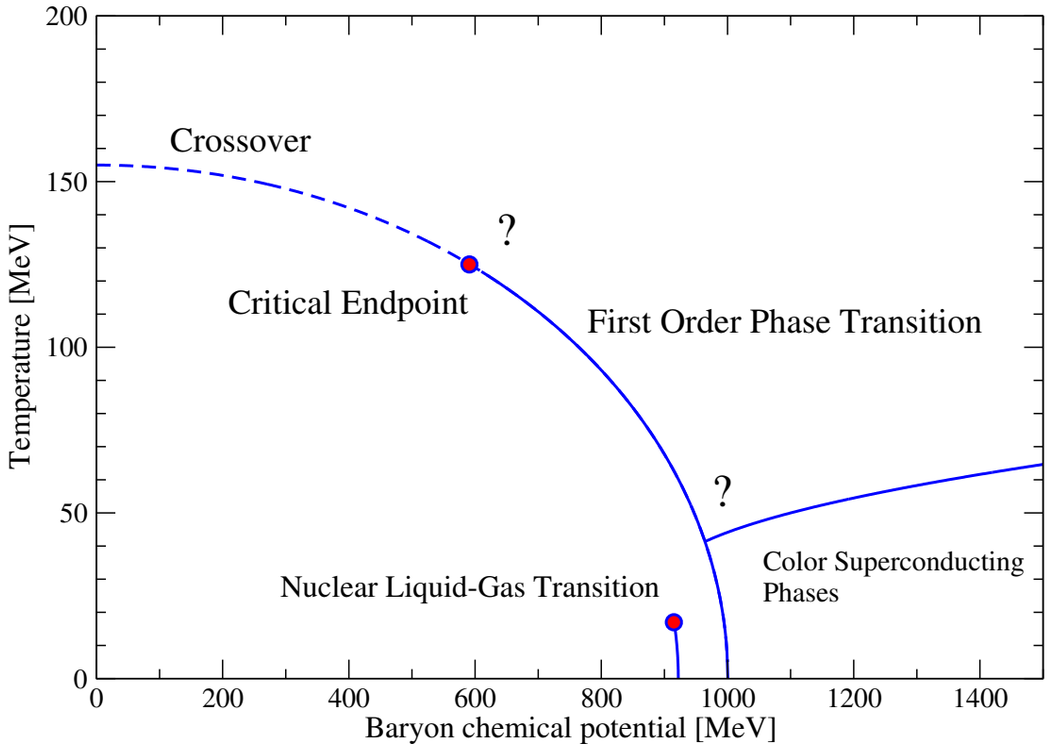

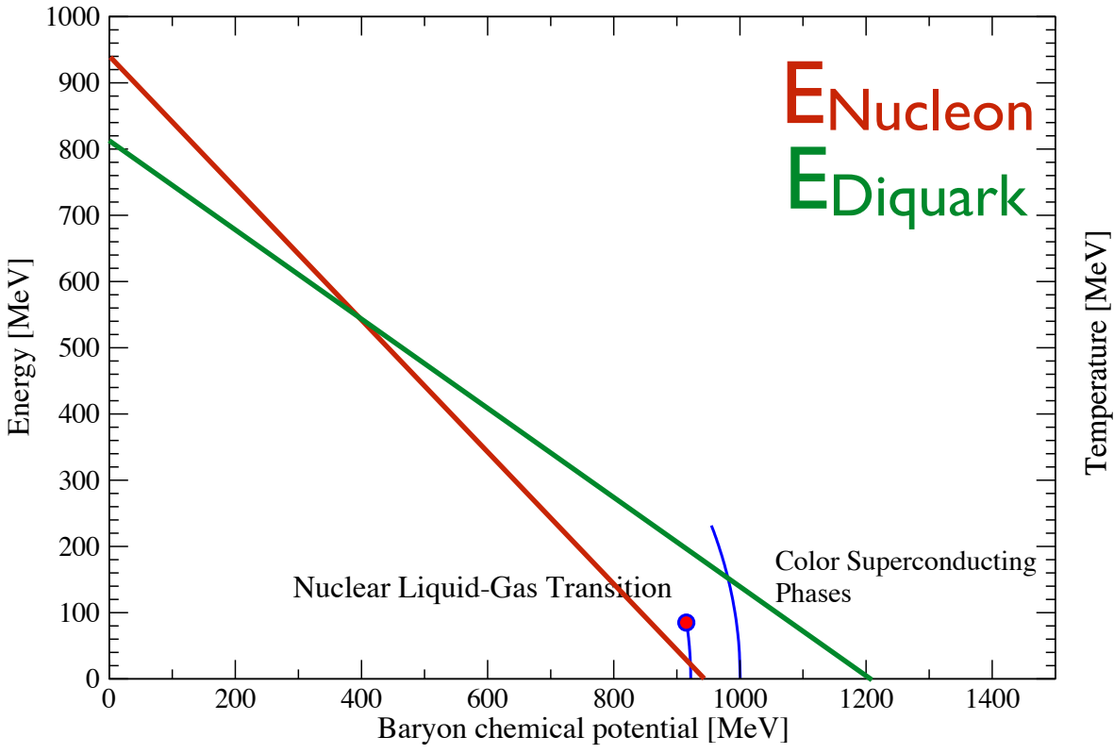

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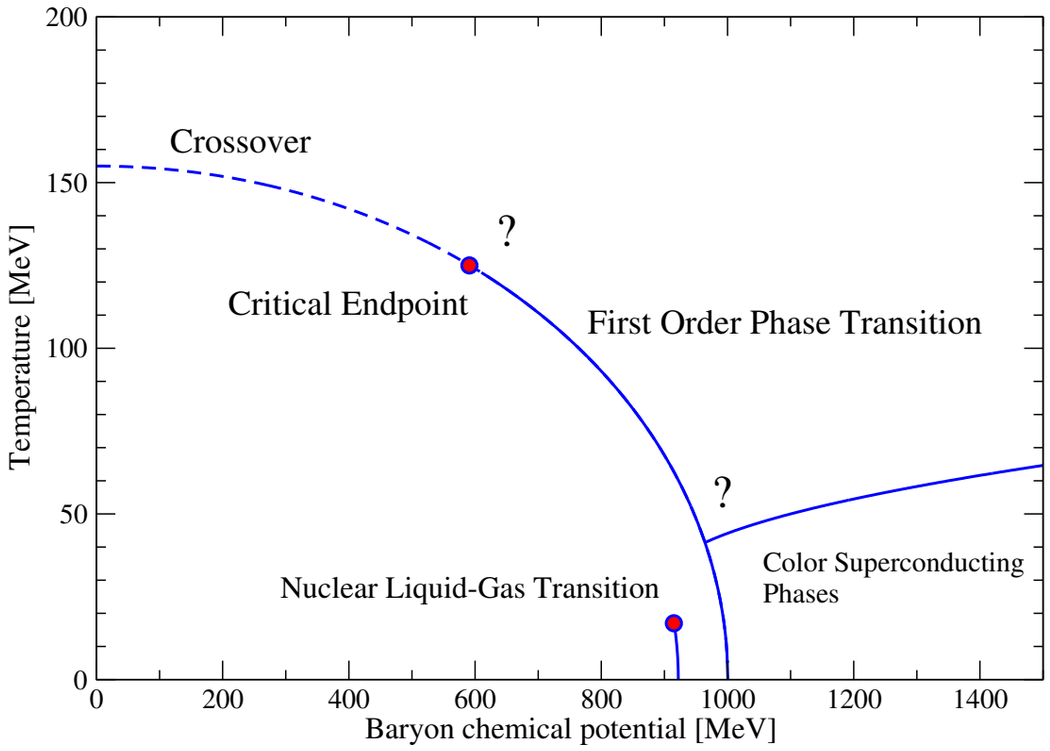

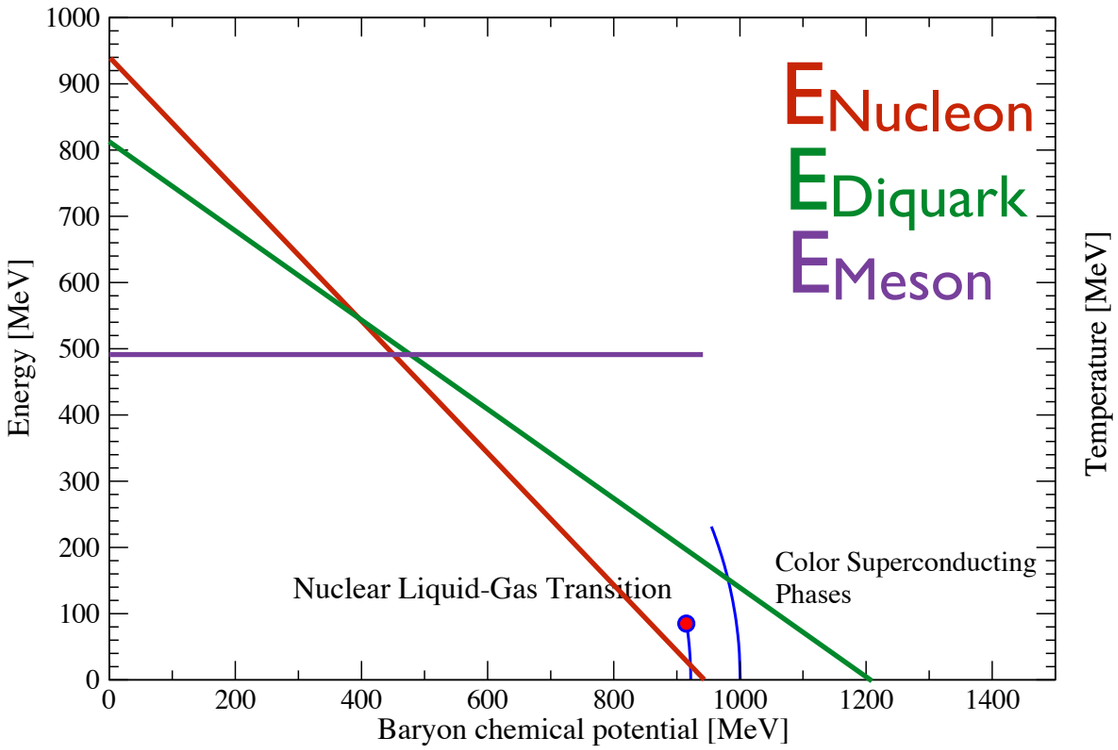

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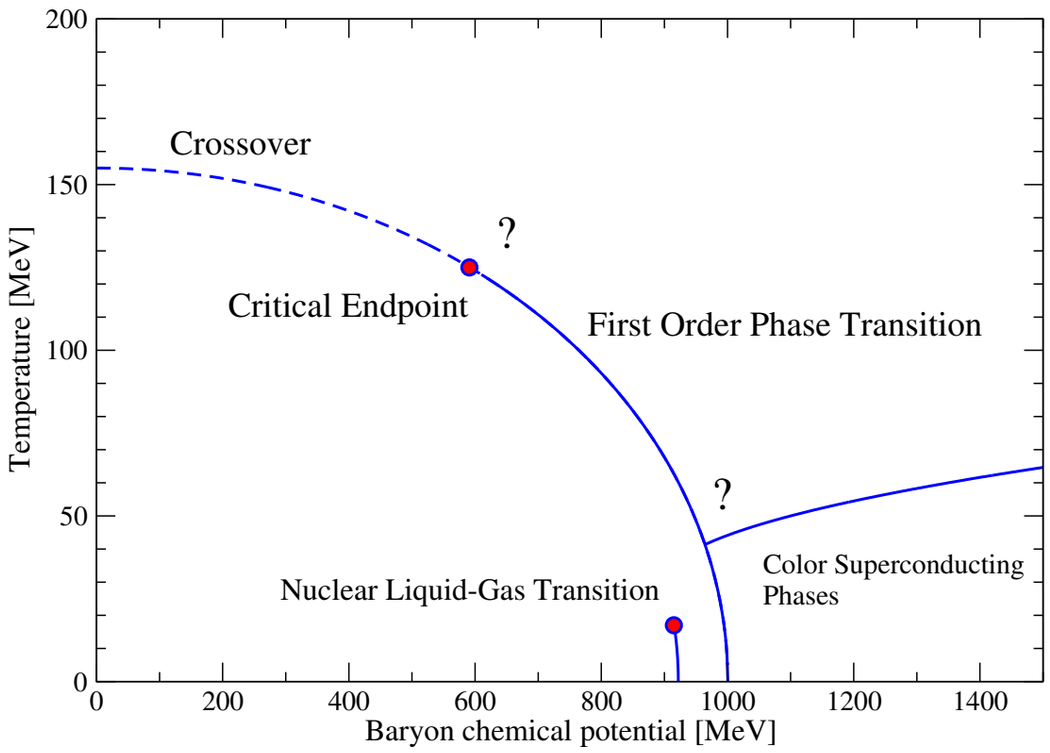

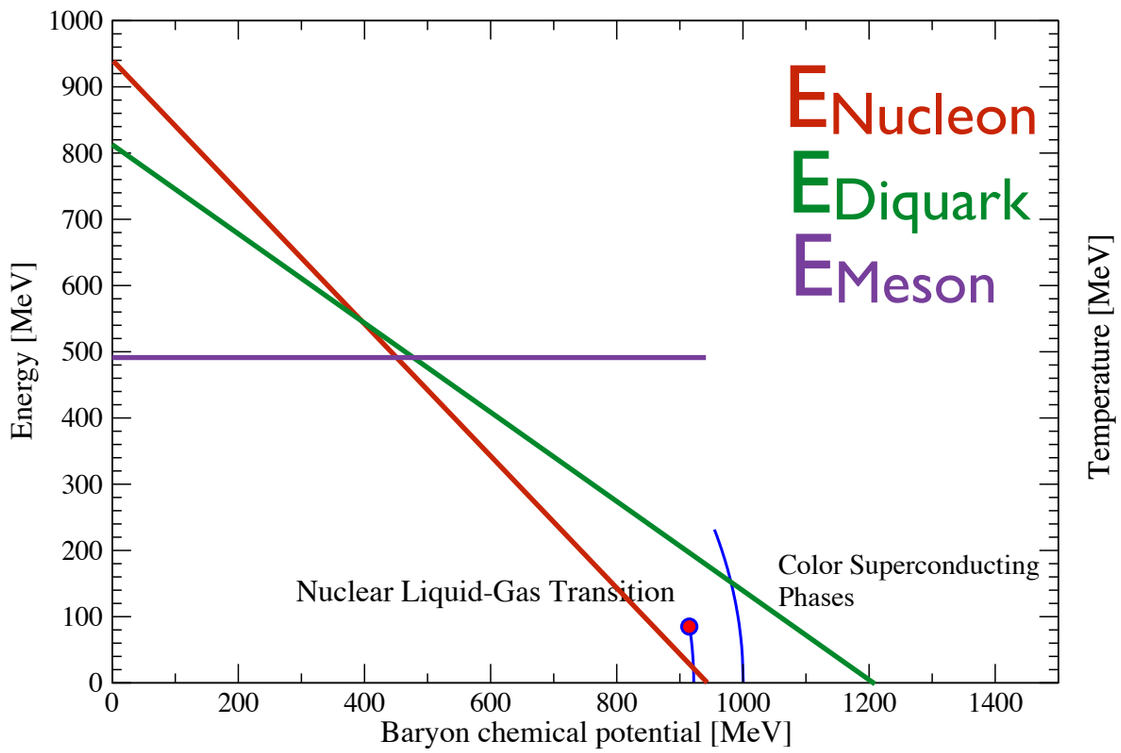

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  Gunkel, CF, EPJ A 57 (2021) no. 4, 147

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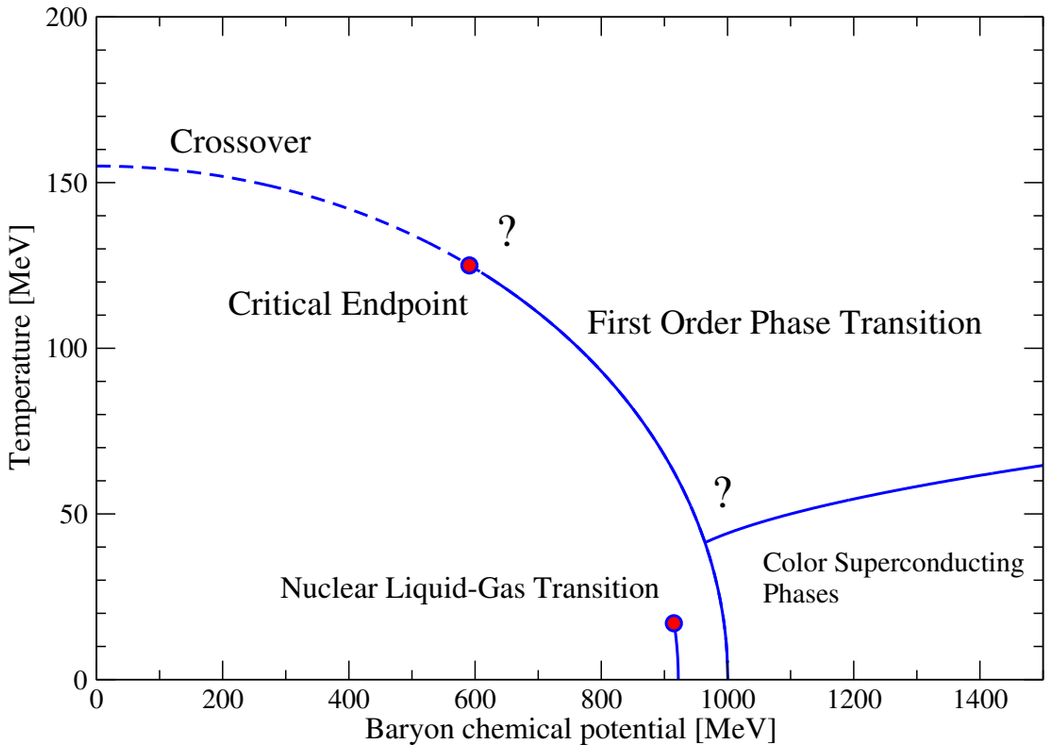

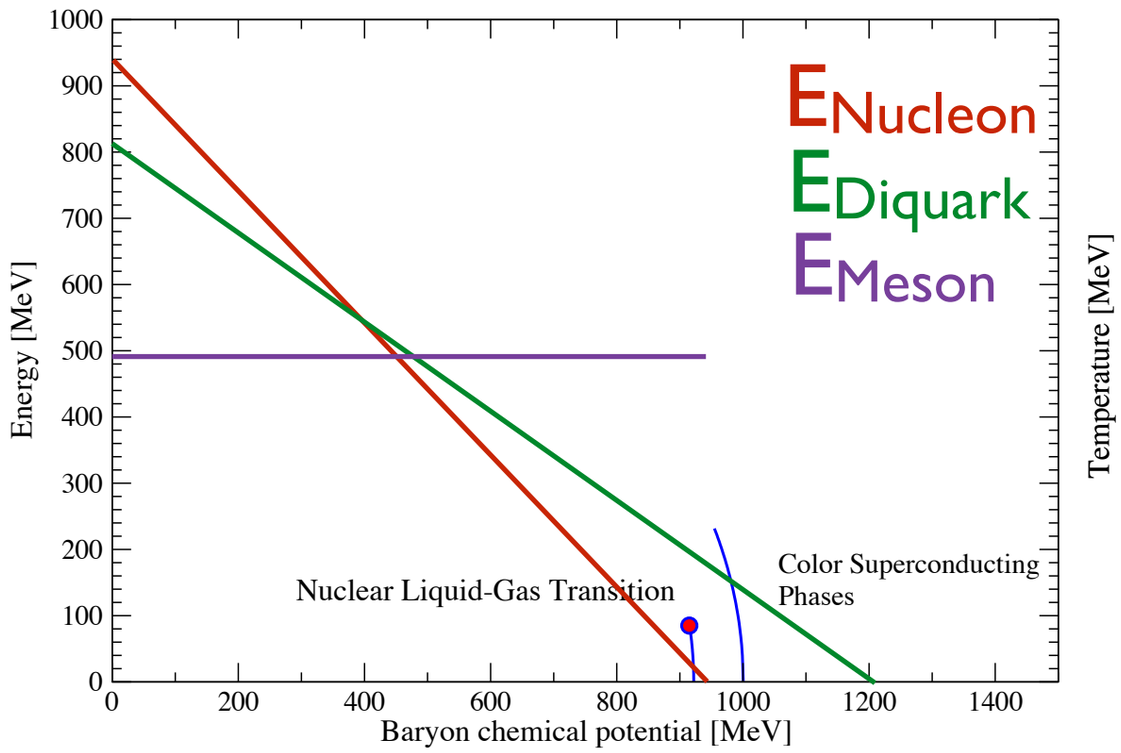

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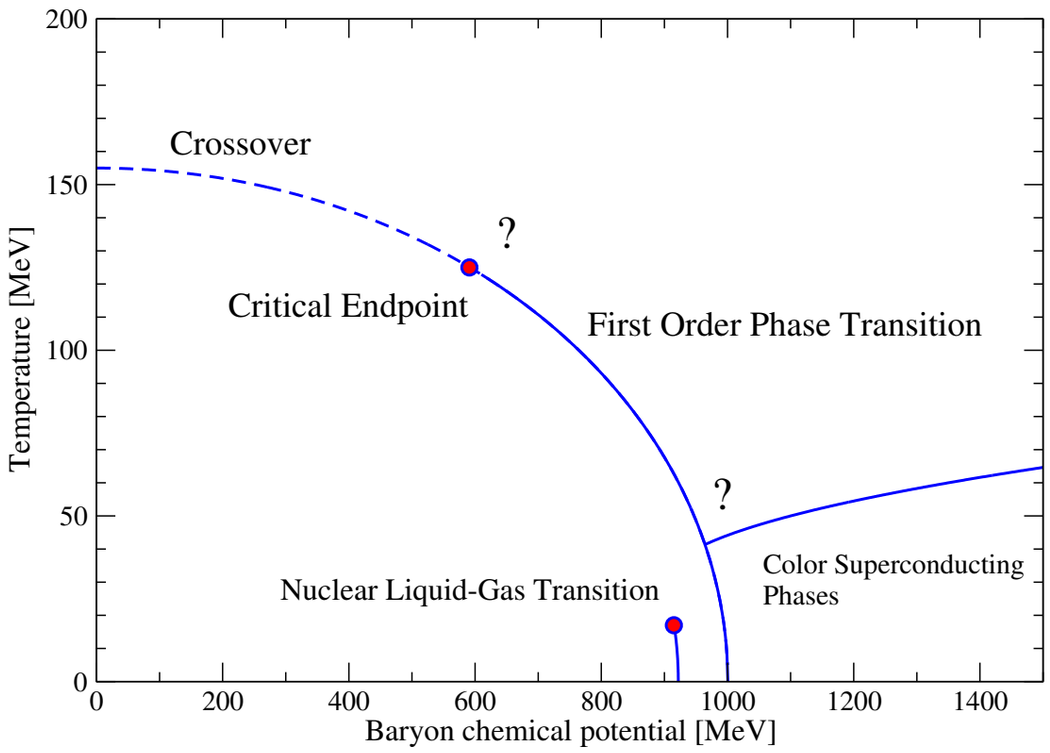
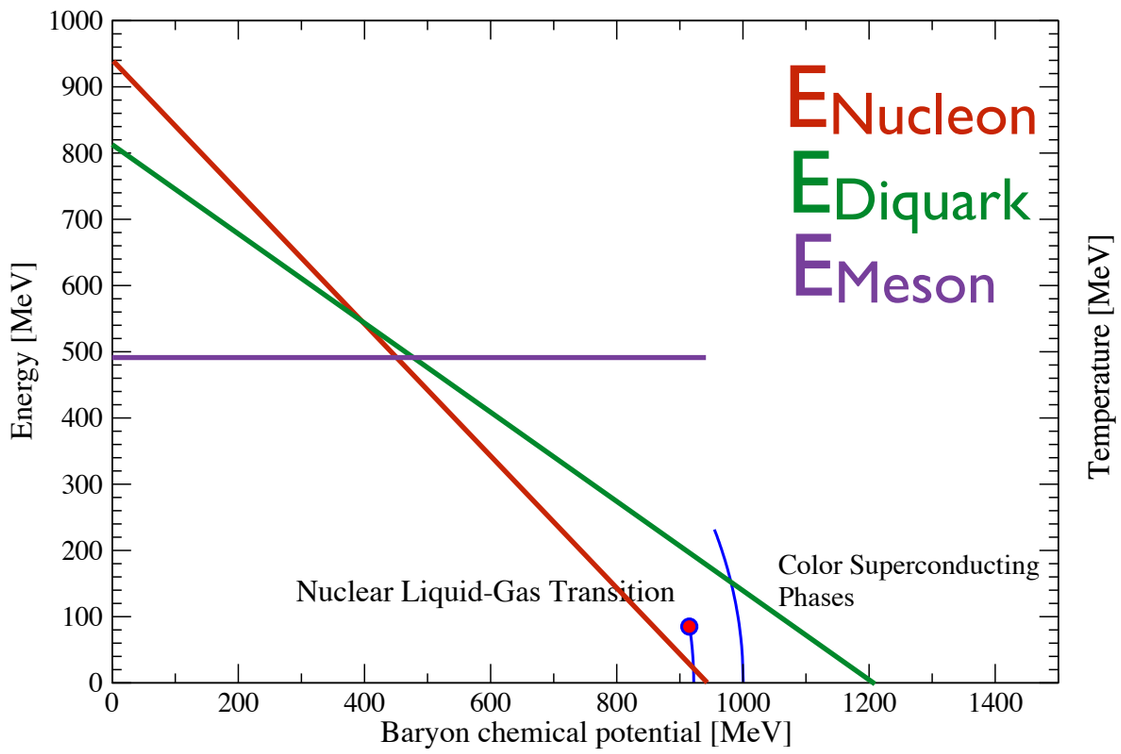
- Meson ✔ Gunkel, CF, Isserstedt, EPJ A 55 (2019) no.9, 169; Gunkel, CF, EPJ A 57 (2021) no. 4, 147
- Diquarks — Müller, Buballa, Wambach, arXiv:1603.02865; Graetz, CF work in progress

Phase transitions at zero T: what to expect…





"silver blaze": no excitations until

T. D. Cohen, PRL 91 , 222001 (2003)

$$\mu_B \leq m_N^*$$

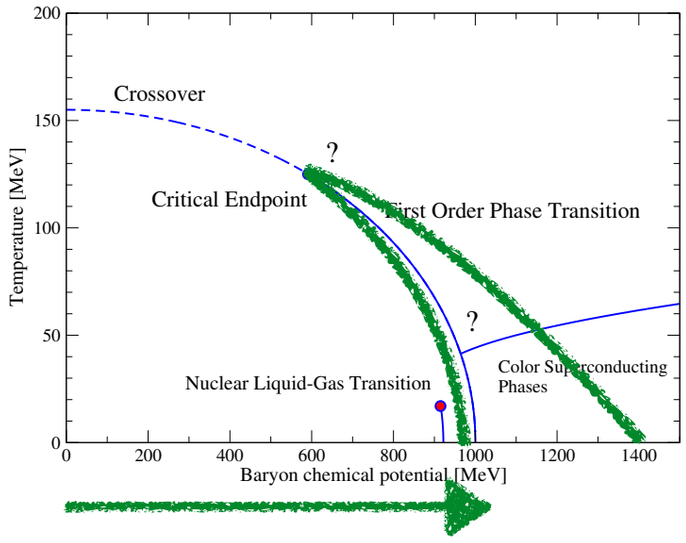
$$E_{\rm Nucleon} = m_N - \mu_B$$

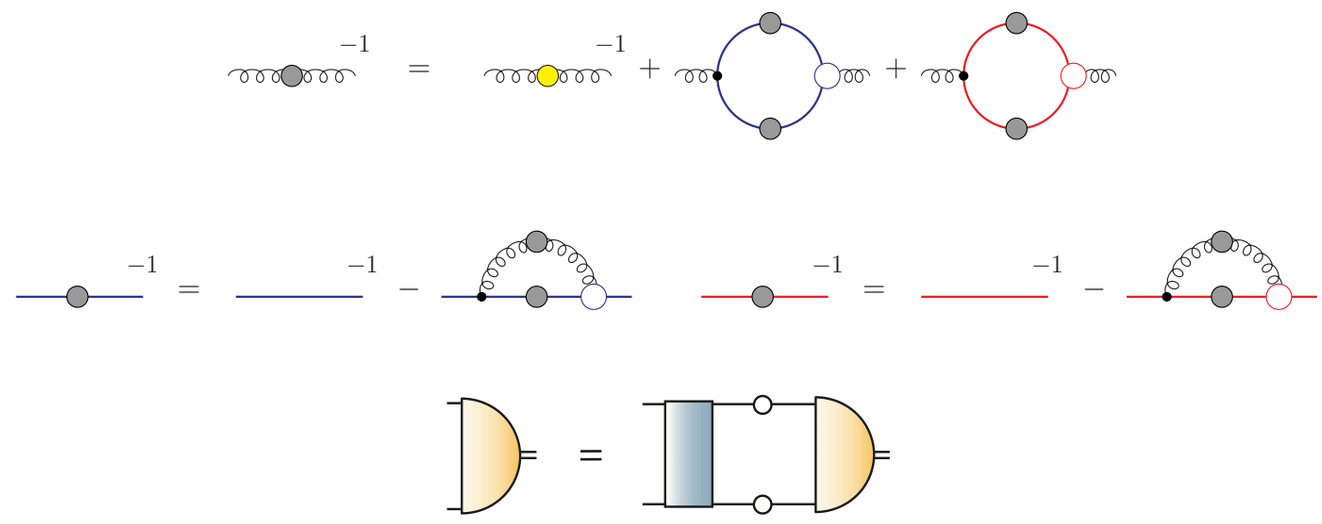
$$E_{\rm Diquark} = m_D - 2/3\,\mu_B$$

$$E_{\rm Meson} = m_M$$

- Meson ✔ Gunkel, CF, Isserstedt, EPJ A 55 (2019) no.9, 169; Gunkel, CF, EPJ A 57 (2021) no. 4, 147
- Diquarks — Müller, Buballa, Wambach, arXiv:1603.02865; Graetz, CF work in progress
- Baryons — Eichmann, CF, Welzbacher, PRD93 (2016); Fukushima, et al., PRD. D 110 (2024) no.7, 076022; Gao at al, arXiv:2504.00539; Graetz, CF work in progress

Meson properties at finite chemical potential

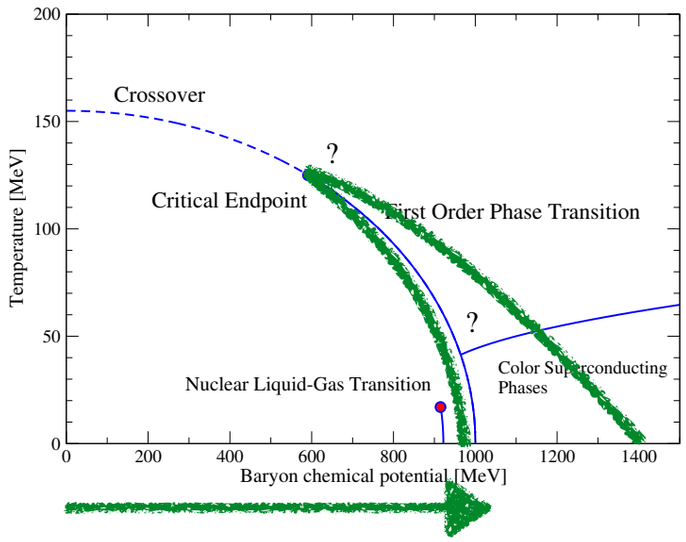


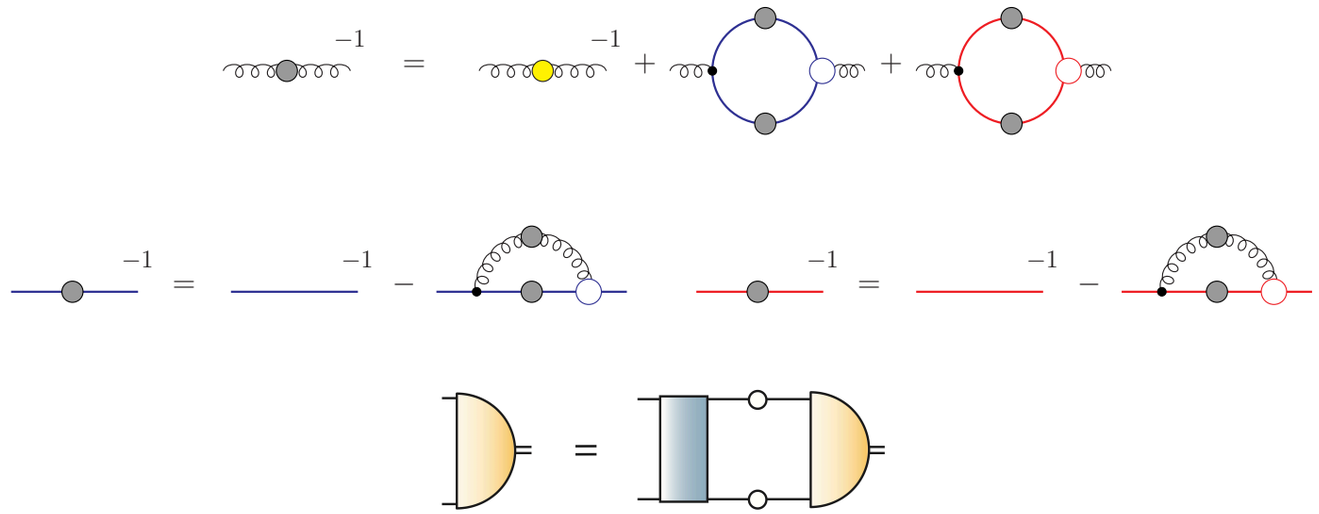


- Quarks/meson wave functions do change !

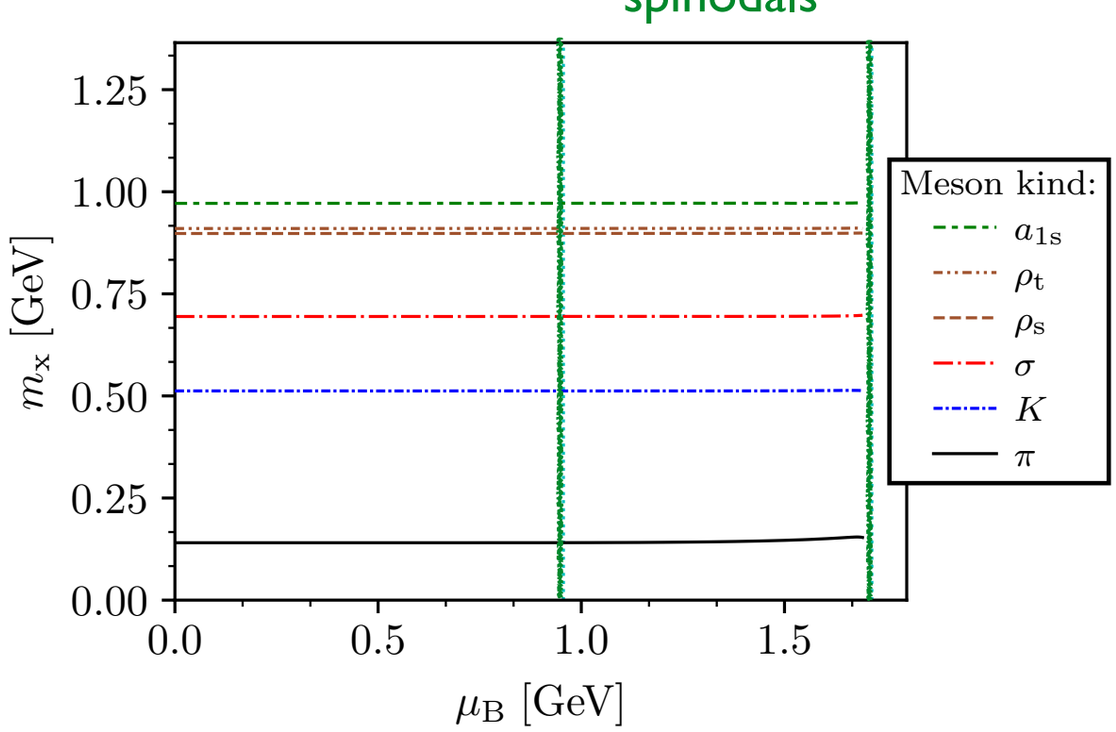
Gunkel, CF, Isserstedt, EPJ A 55 (2019) no.9, 169
Gunkel, CF, EPJ A 57 (2021) no. 4, 147

Meson properties at finite chemical potential

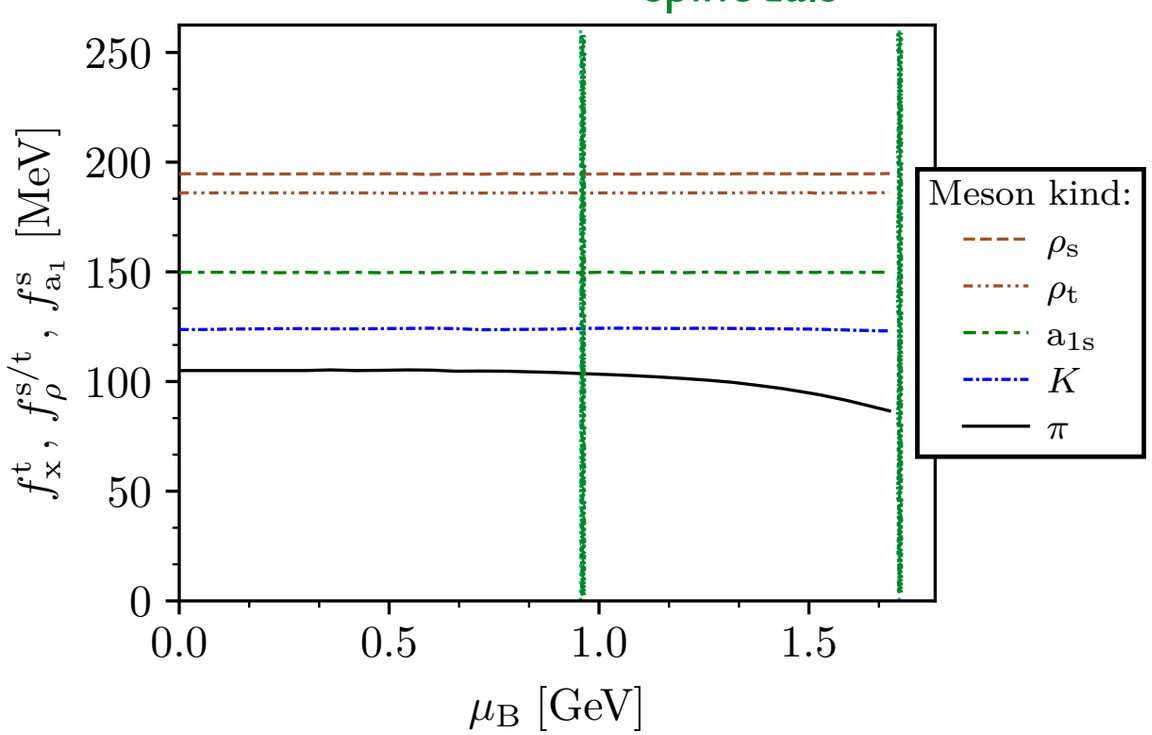




spinodals



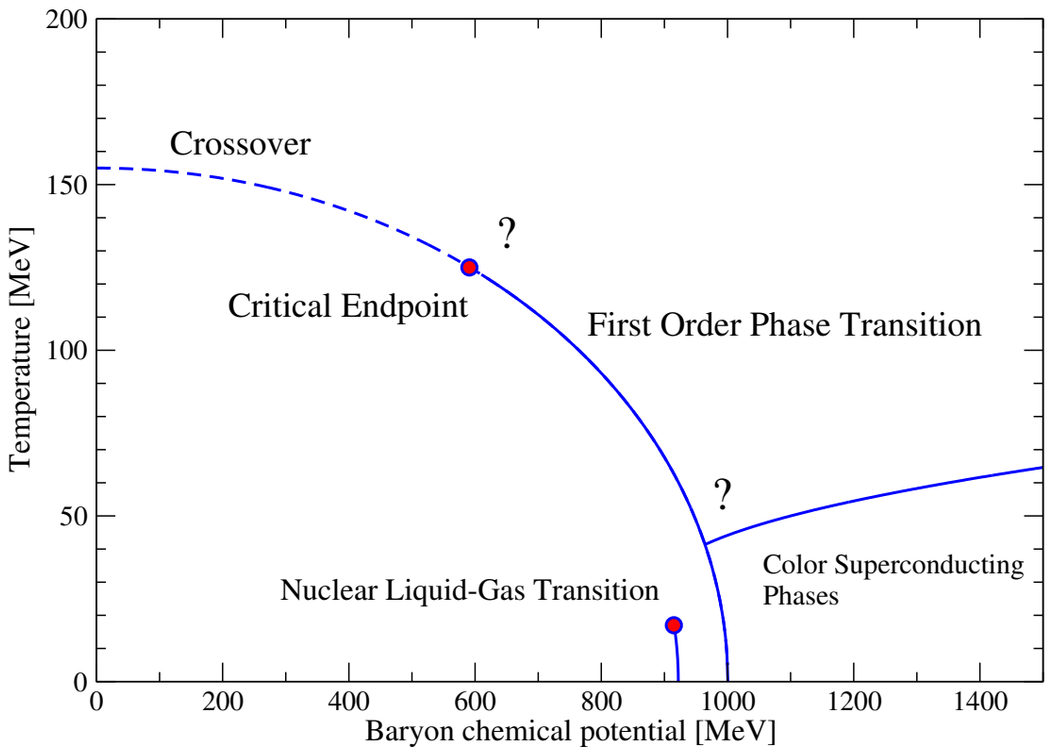
spinodals



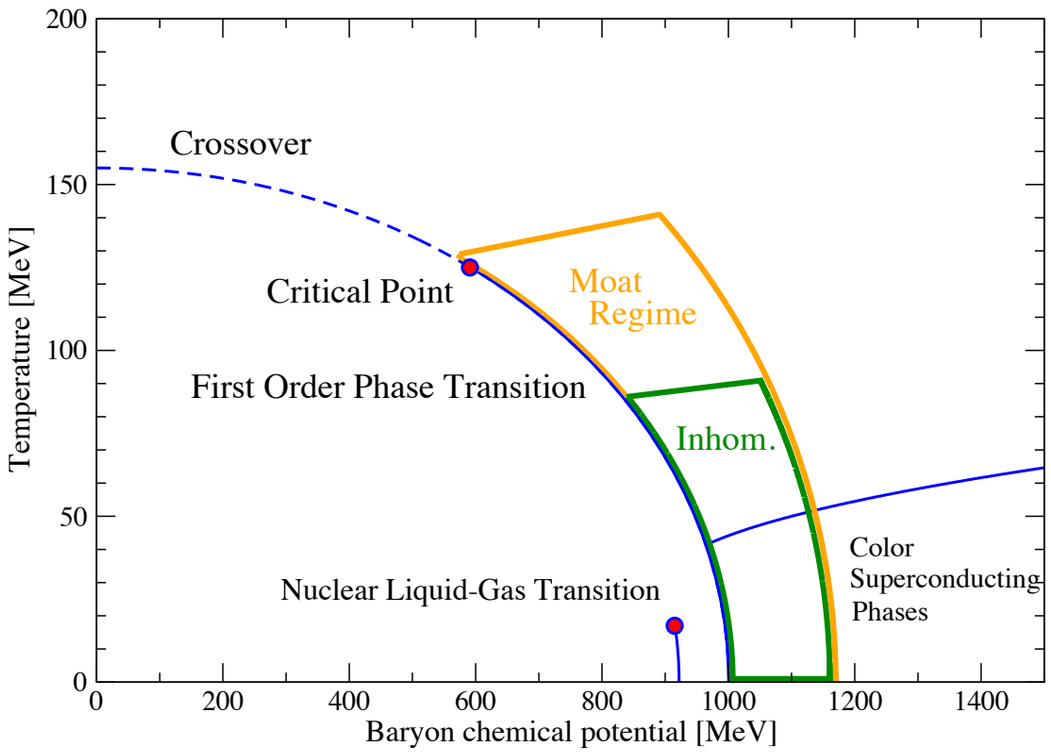
- Quarks/meson wave functions do change !
- But: Silver blaze satisfied

Gunkel, CF, Isserstedt, EPJ A 55 (2019) no.9, 169
Gunkel, CF, EPJ A 57 (2021) no. 4, 147

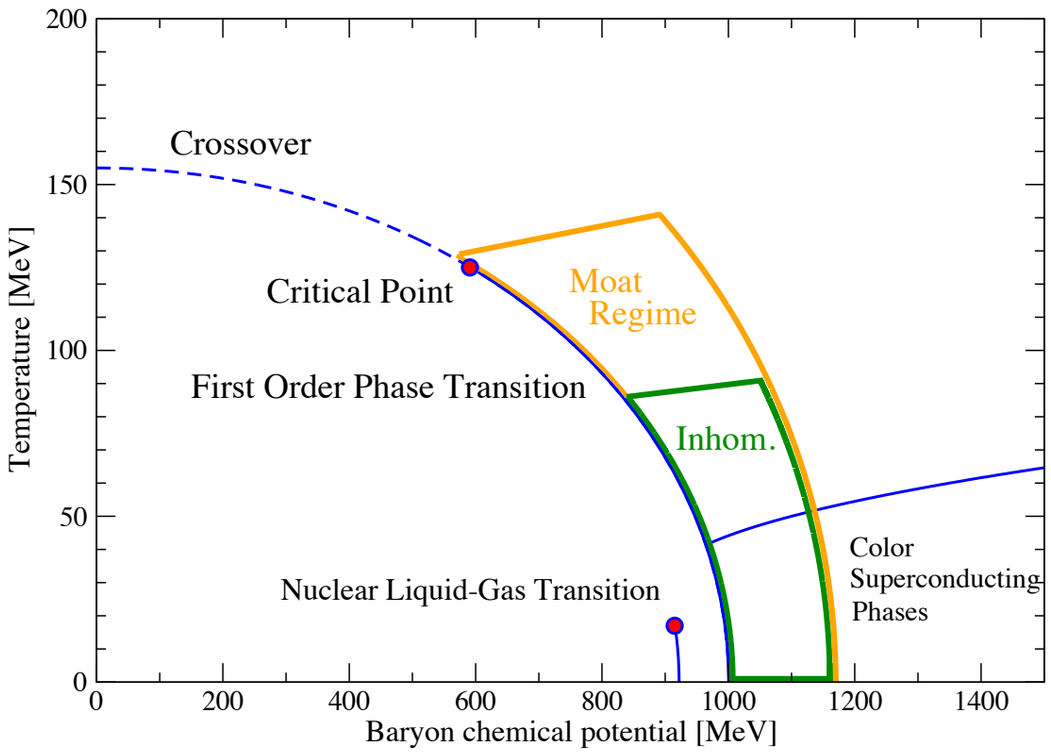
Phase transitions at non-zero T: what to expect…



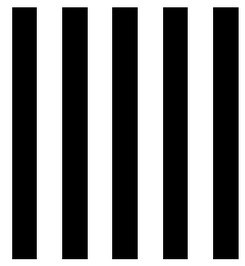
Phase transitions at non-zero T: what to expect…



Phase transitions at non-zero T: what to expect…

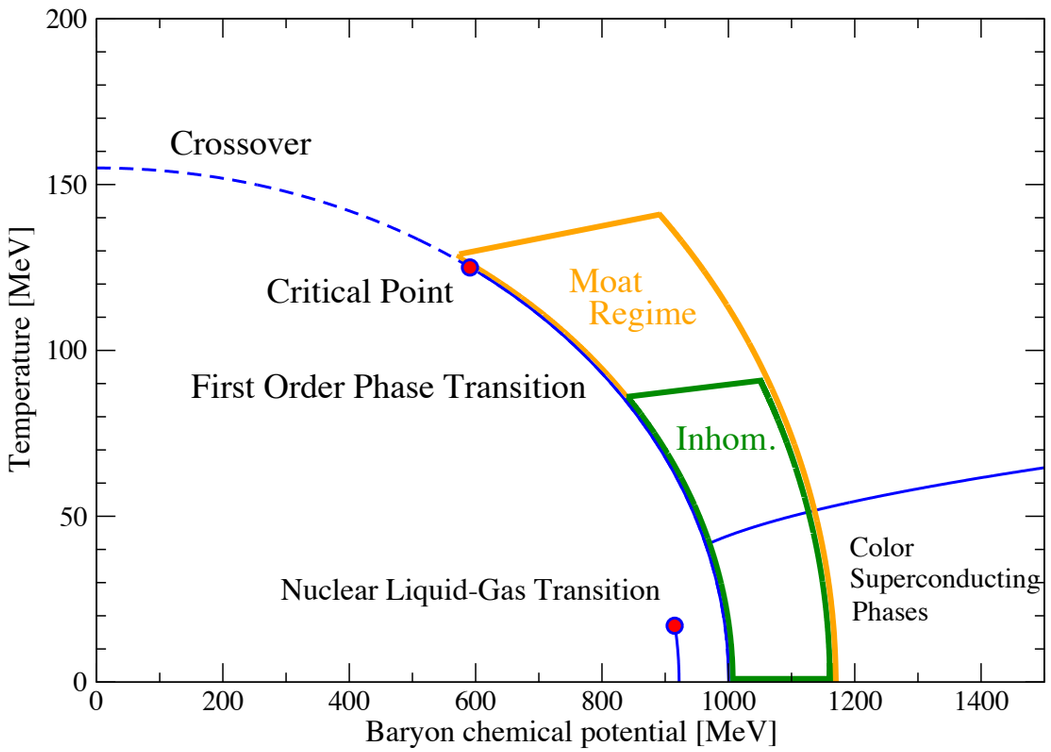


$\langle \bar{\Psi}\Psi \rangle$

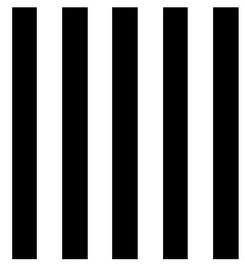


$\langle \bar{\Psi}\Psi \rangle(x)$

Phase transitions at non-zero T: what to expect…



$\langle \bar{\Psi}\Psi \rangle$



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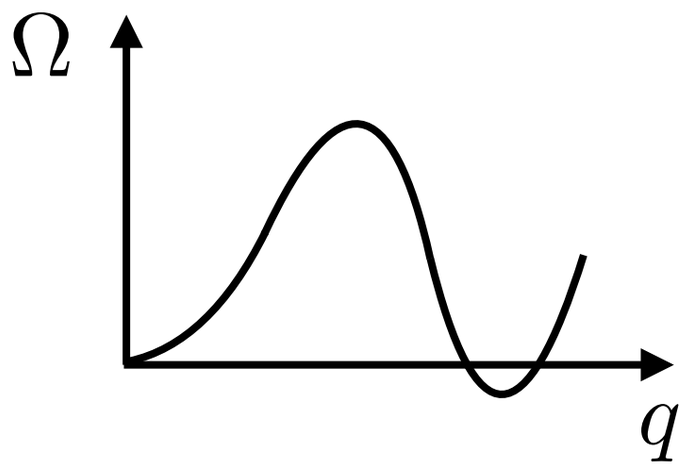
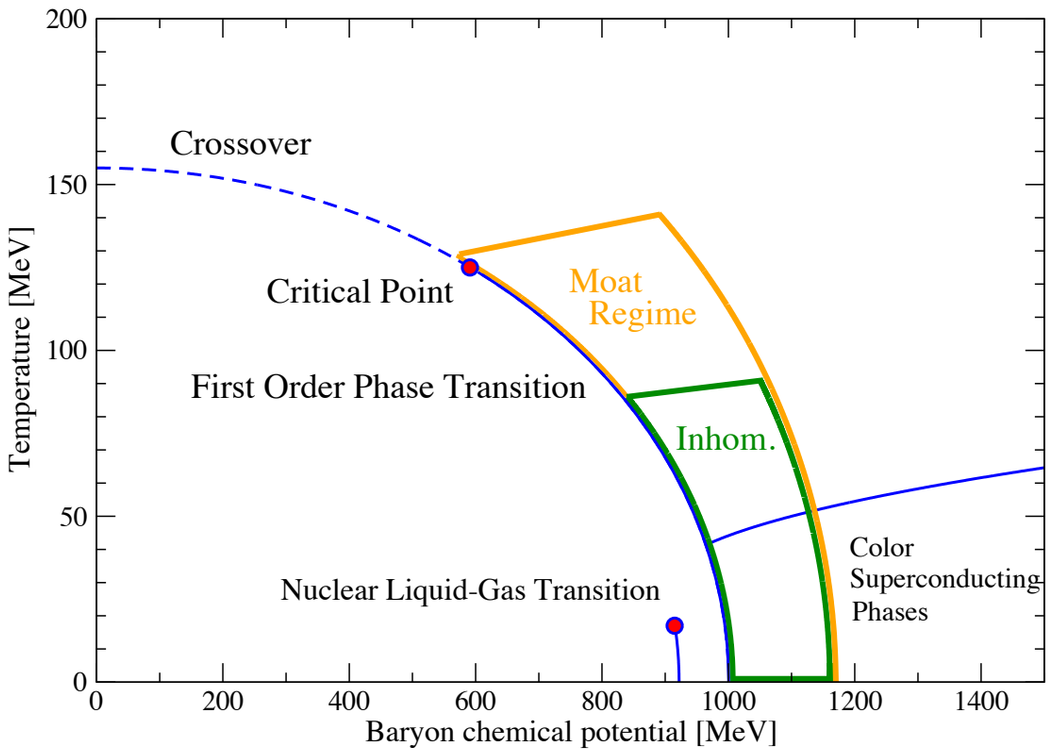


inhom. phase

Buballa and Carignano, PPNP 81 (2015), 39-96

Phase transitions at non-zero T: what to expect…



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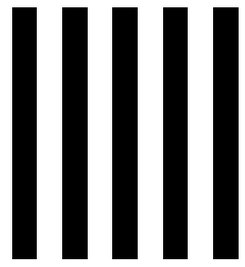
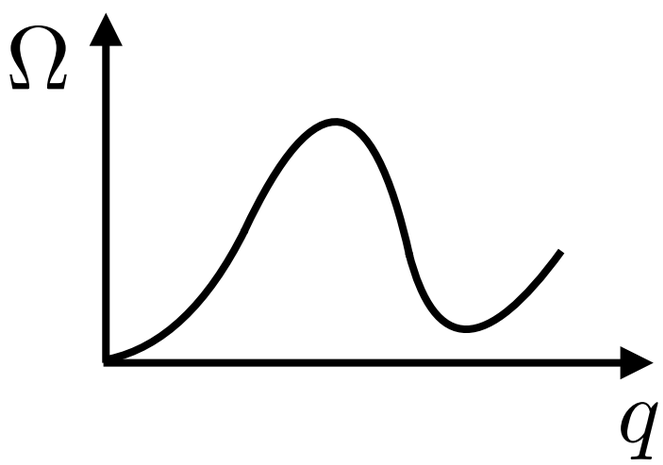
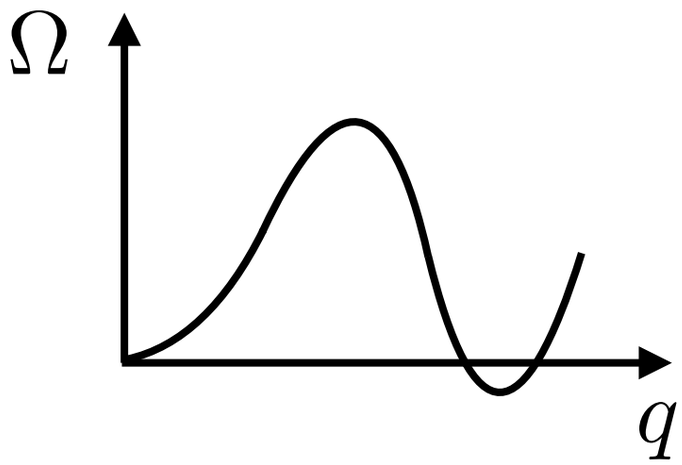


$\langle \bar{\Psi}\Psi \rangle(x)$



moat regime

Fu, Pawlowski, Rennecke, PRD 101, (2020) 5 054032
Fu, Pawlowski, Pisarski, Rennecke, Wen, Yin, arXiv:2412.15949



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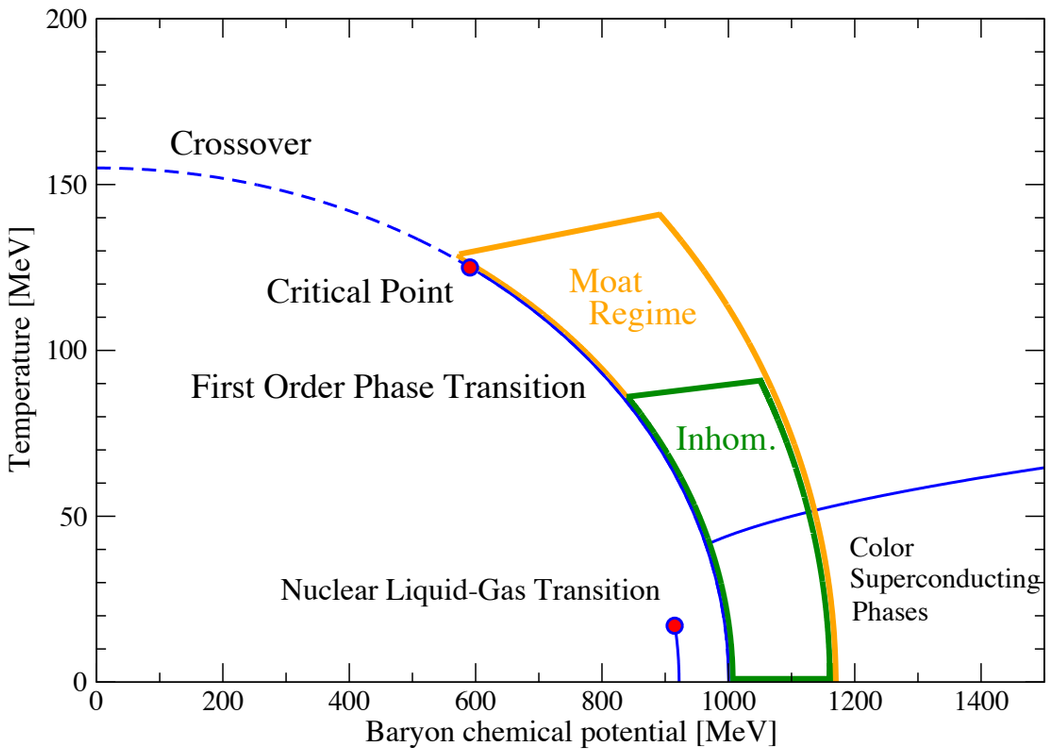
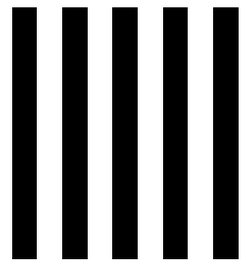
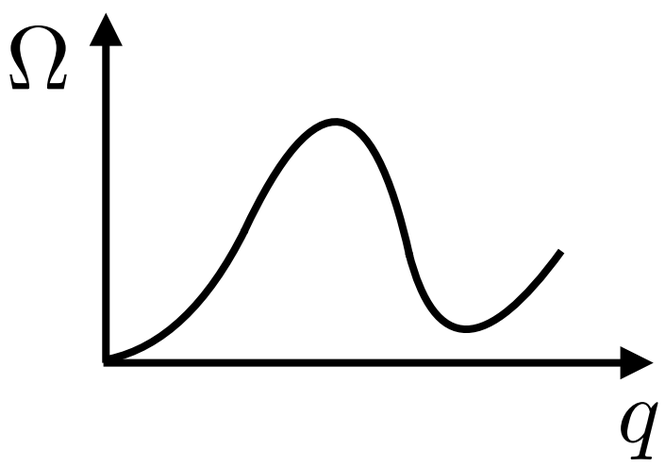


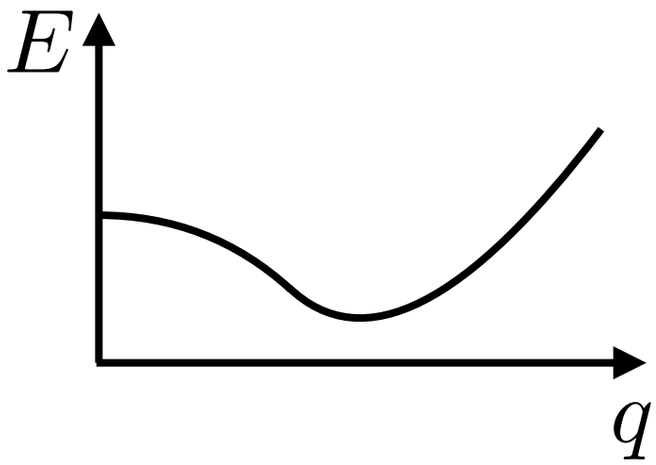
$\langle \bar{\Psi}\Psi \rangle$



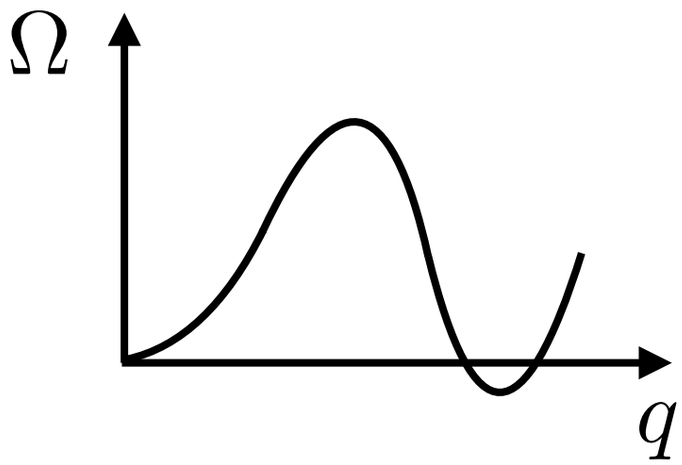
$\langle \bar{\Psi}\Psi \rangle(x)$



moat regime



nontrivial
dispersion relation



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Fu, Pawlowski, Rennecke, PRD 101, (2020) 5 054032
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Pisarski, Rennecke, PRL 127 (2021), 15 152302

Buballa and Carignano, PPNP 81 (2015), 39-96

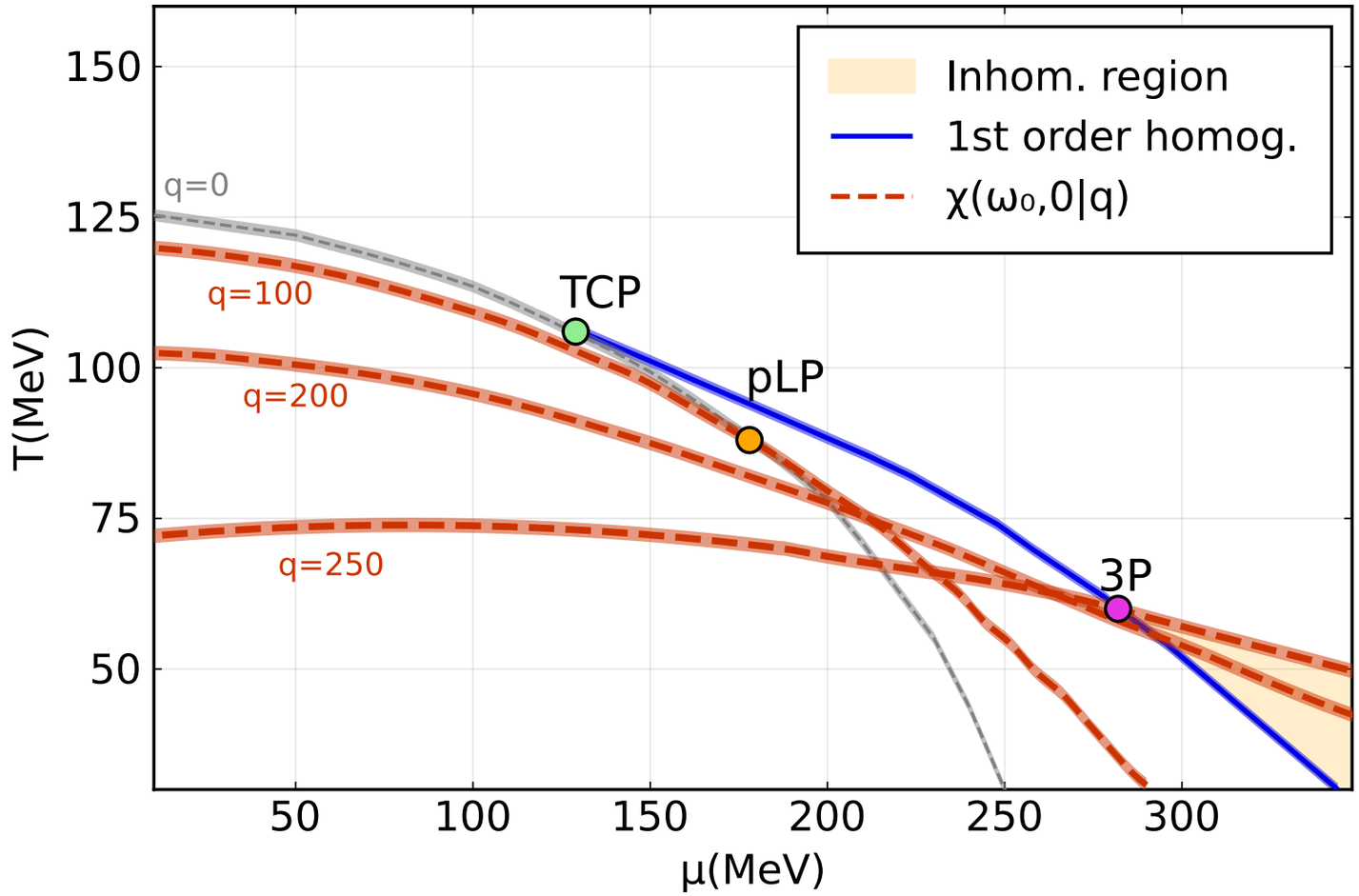
Inhomogeneous phases (simple truncation)

$$\xi = e^{i\gamma_5\tau_3 q\cdot x}\psi$$

$$\langle\bar{\psi}(x)\psi(x)\rangle = \cos(2q\cdot x)\,\langle\bar{\xi}\xi\rangle$$
$$\langle\bar{\psi}(x)i\gamma_5\tau_3\psi(x)\rangle = \sin(2q\cdot x)\,\langle\bar{\xi}\xi\rangle$$

stability analysis

$$\chi(q) = \frac{\partial M}{\partial m} < 0$$



- unambiguous identification of inhomogeneous region
- locations of CEP and 3P are far apart!

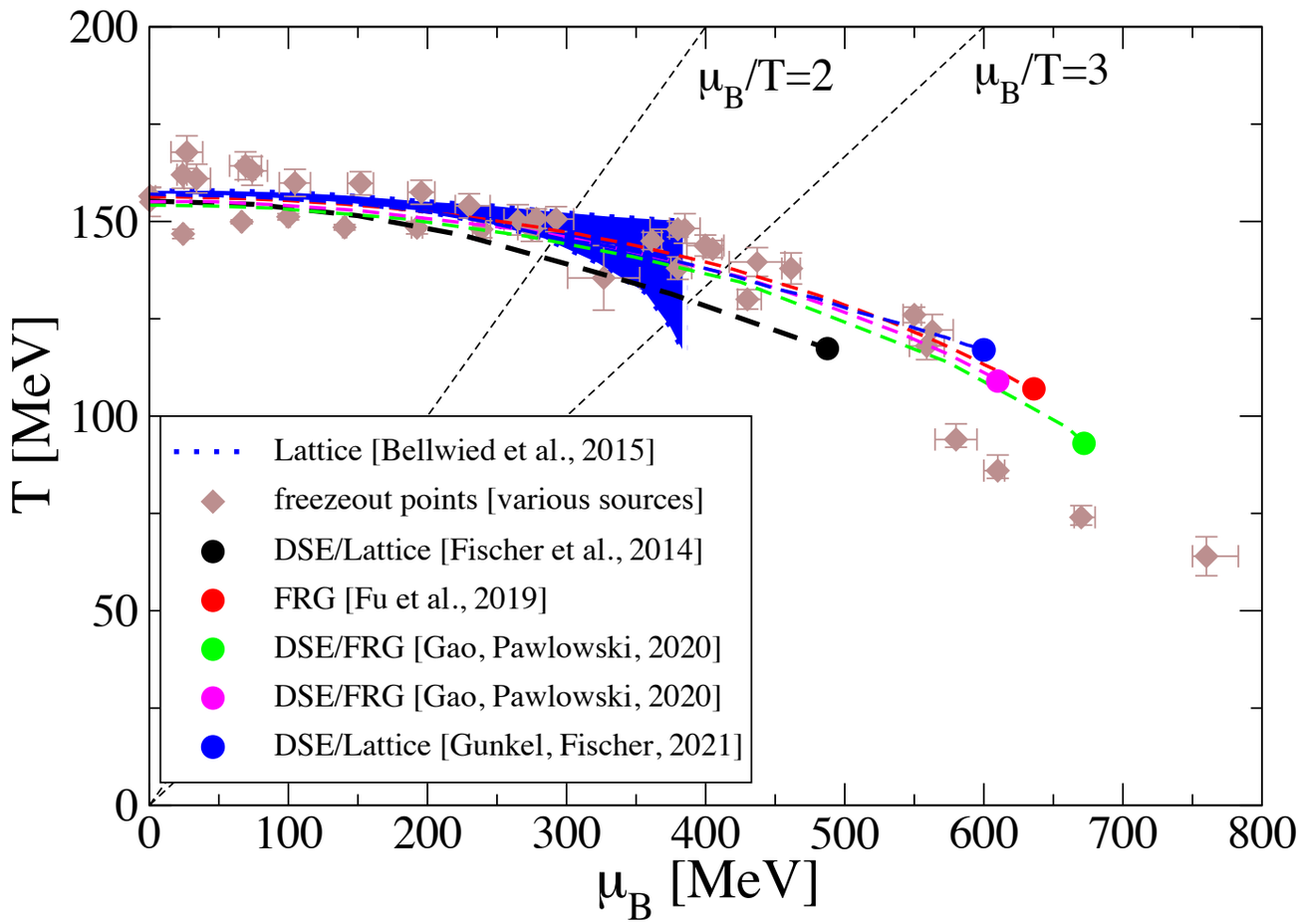
T. F. Motta, J. Bernhardt, M. Buballa and CF, PRD 108 (2023), PRD 110 (2024), PRD 111 (2025)

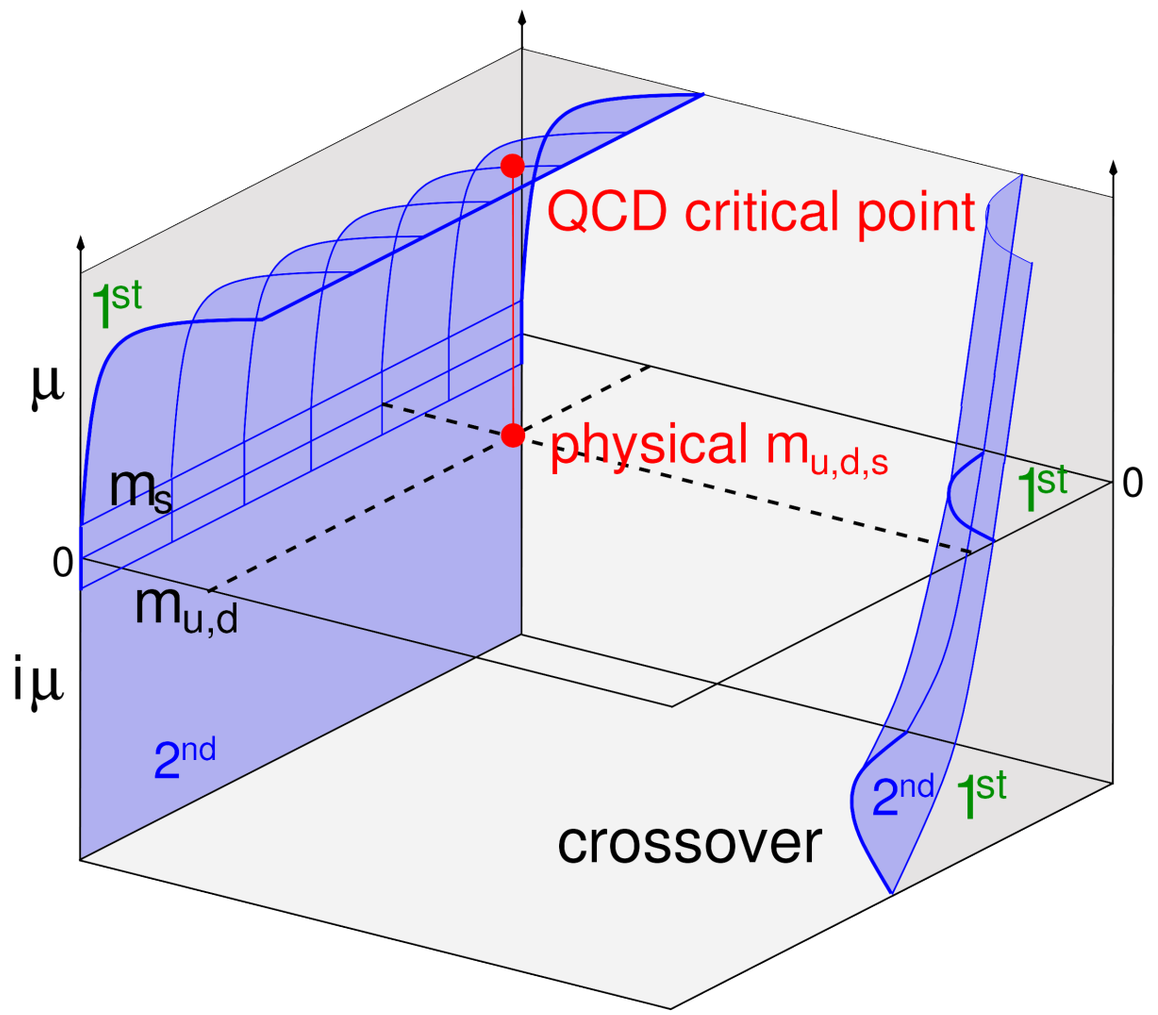
Summary: QCD with functional methods

Main goals:

- **one** framework for all areas of hadron physics:
  mesons, baryons, 'exotic states', form factors,
  hadronic contributions to precision observables (g-2)
- **same** framework for QCD phase diagram

Main results:





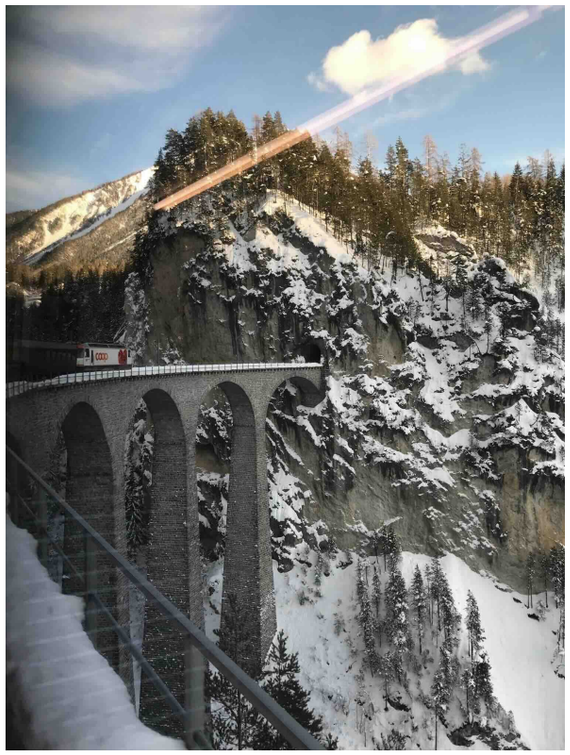
Summary:

2012

2013

2018

2019

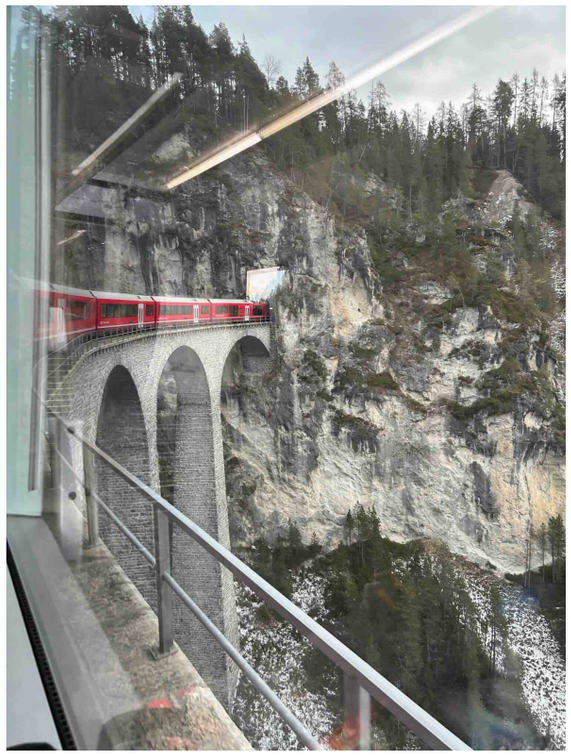


2023

2024

2025

2026

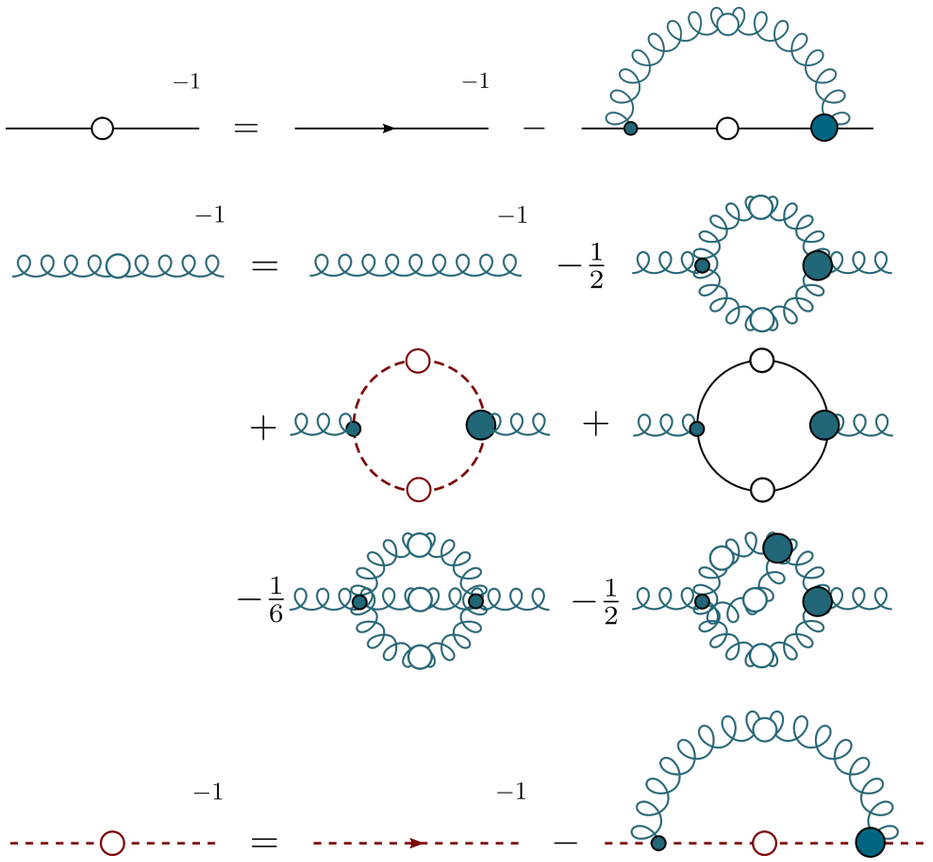


Bernina-Express:
Landwasser Viadukt

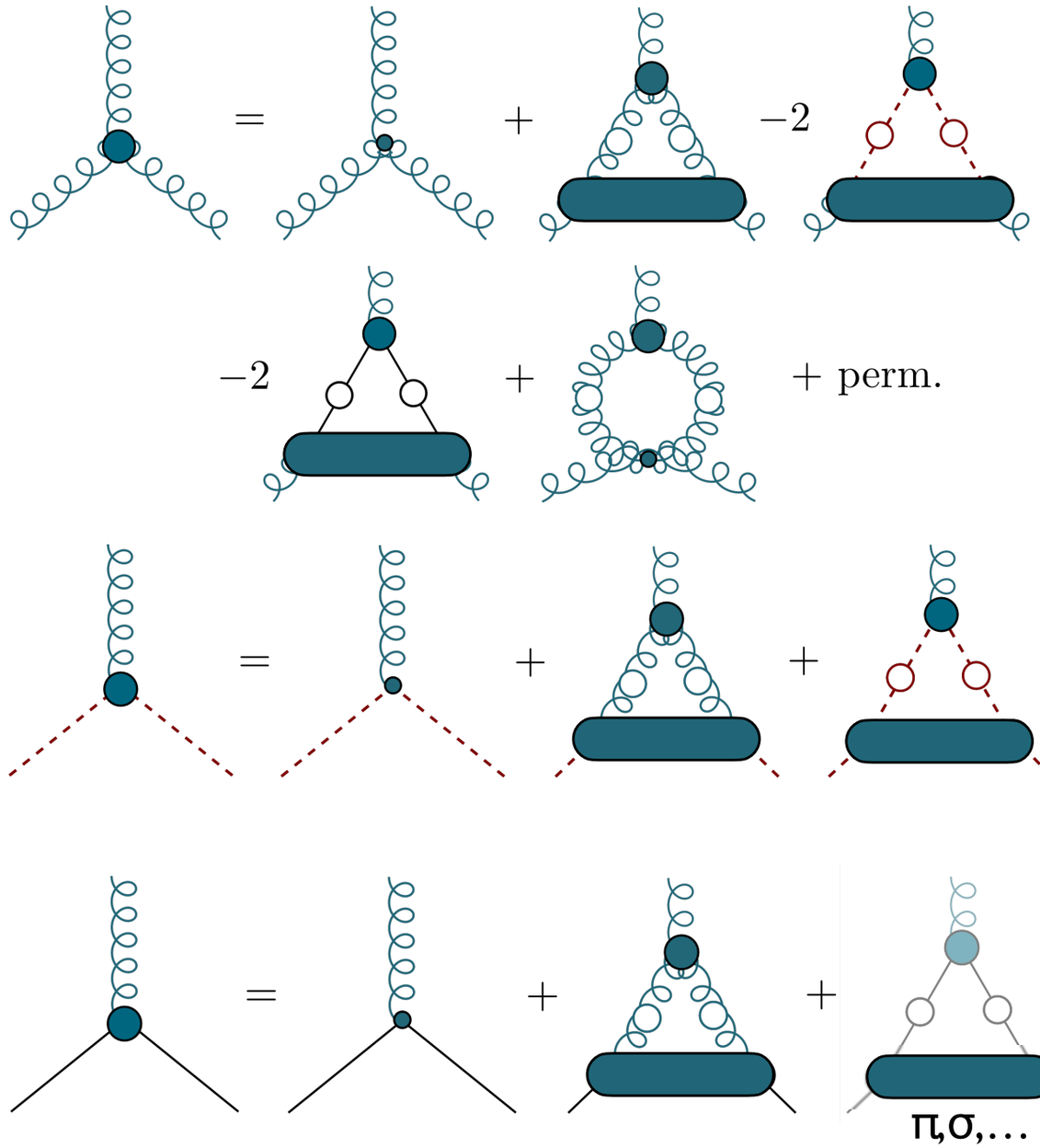
Backup

QCD with functional methods (DSE)

propagators



vertices



Review: Eichmann, Sanchis-Alepuz, Williams, Alkofer, CF, PPNP 91, 1-100 [1606.09602]

CF, Alkofer, PRD67 (2003) 094020
Williams, CF, Heupel, PRD93 (2016) 034026
Huber, PRD 101 (2020) 114009

QCD with functional methods (DSE)

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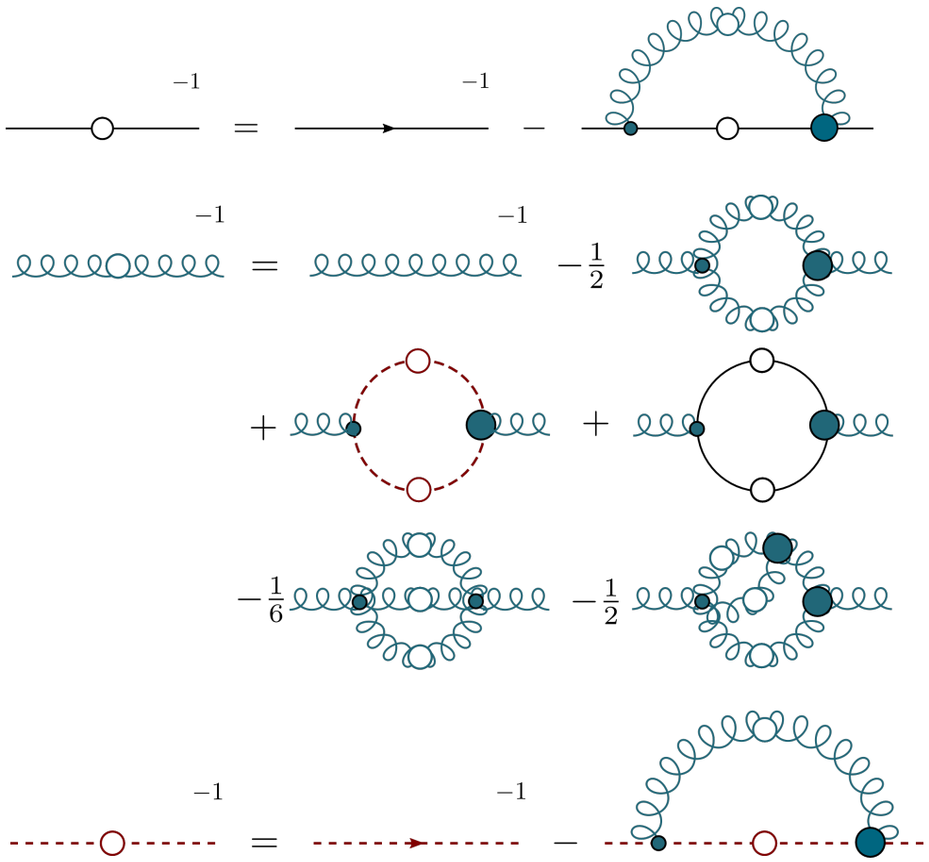

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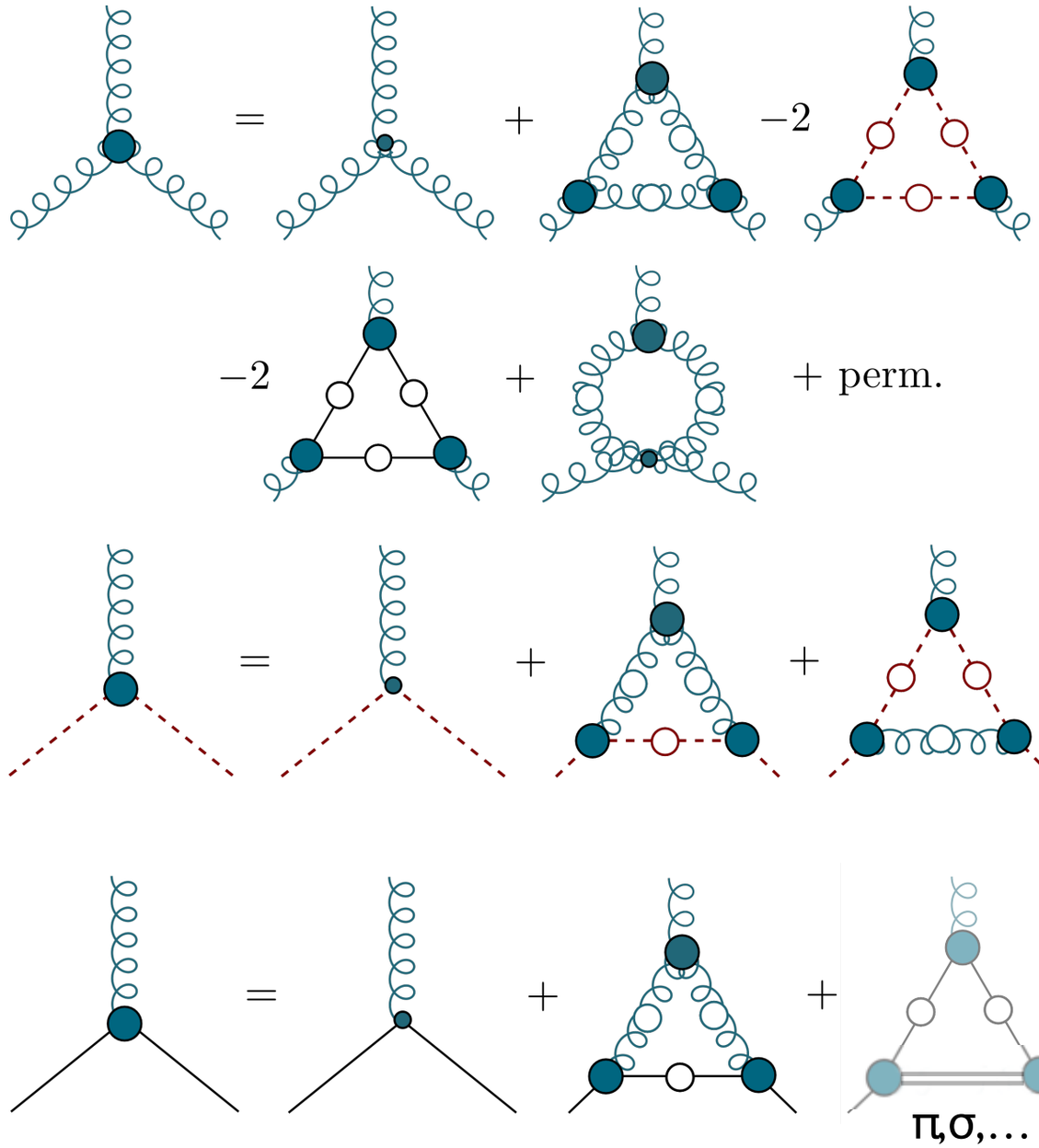

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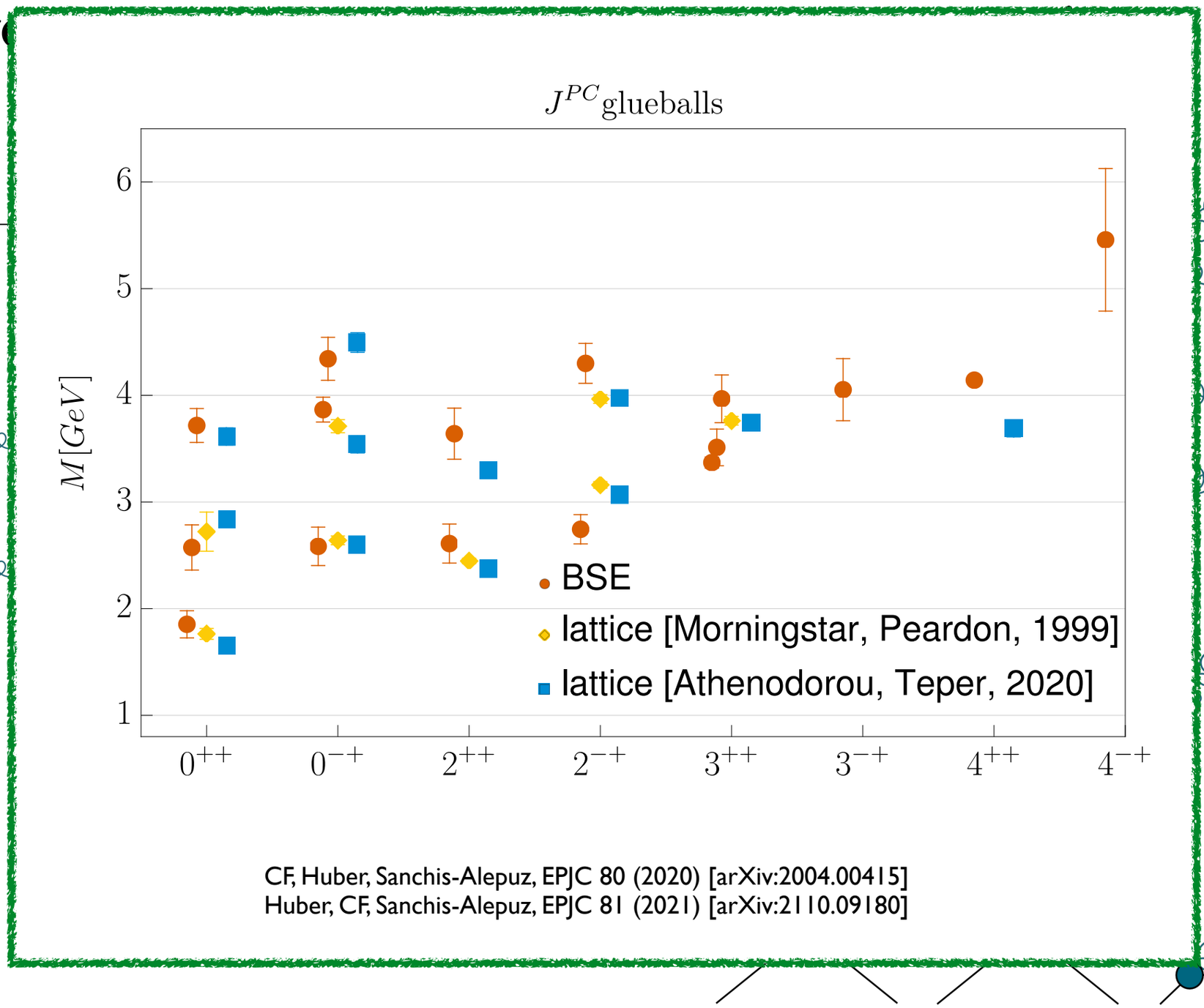
QCD with functional methods (DSE)



CF, Huber, Sanchis-Alepuz, EPJC 80 (2020) [arXiv:2004.00415]
Huber, CF, Sanchis-Alepuz, EPJC 81 (2021) [arXiv:2110.09180]

-2

+ perm.

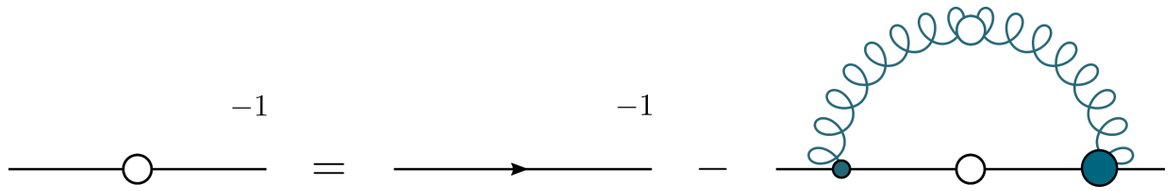
$\pi,\sigma,\ldots$

Review: Eichmann, Sanchis-Alepuz, Williams, Alkofer, CF, PPNP 91, 1-100 [1606.09602]

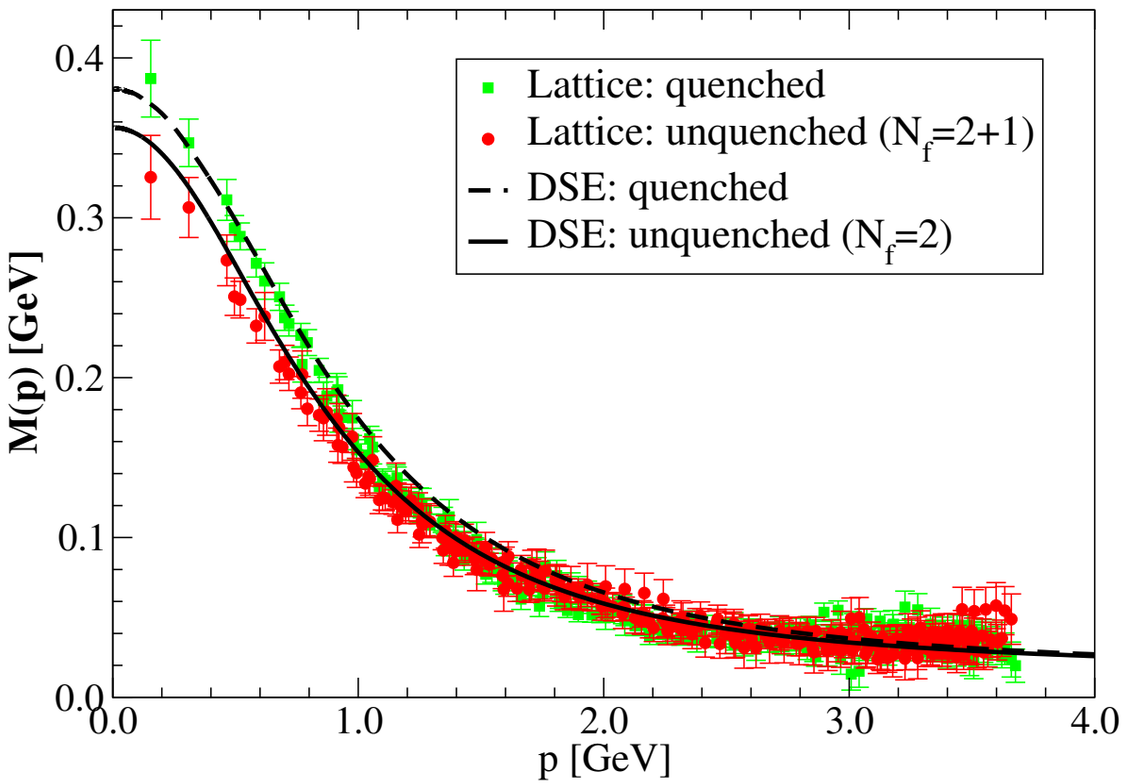
CF, Alkofer, PRD67 (2003) 094020
Williams, CF, Heupel, PRD93 (2016) 034026
Huber, PRD 101 (2020) 114009

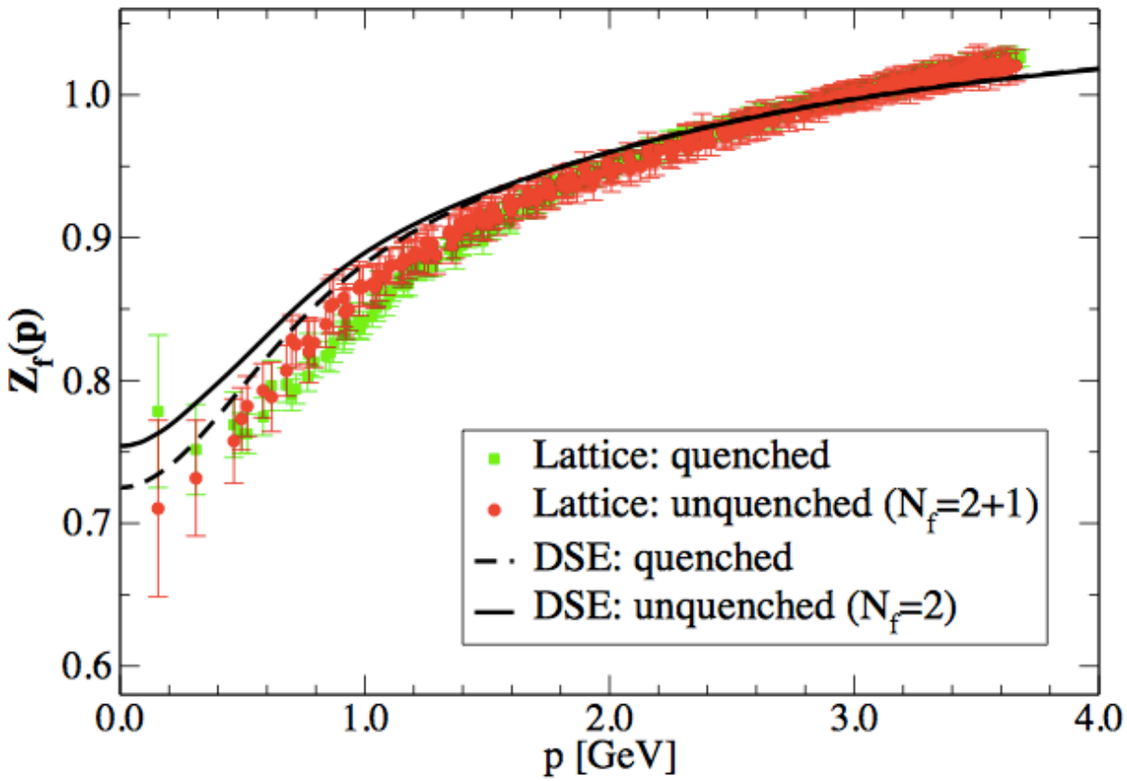
Dynamical mass generation

$$S^{-1}(p) = \frac{(i\not{p} + M(p^2))}{Z_f(p^2)}$$



DSE: CF, Nickel, Williams, EPJ C 60 (2009) 47
Lattice: P. O. Bowman, et al PRD 71 (2005) 054507





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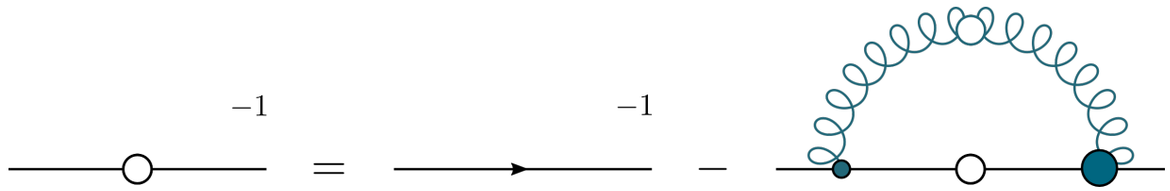

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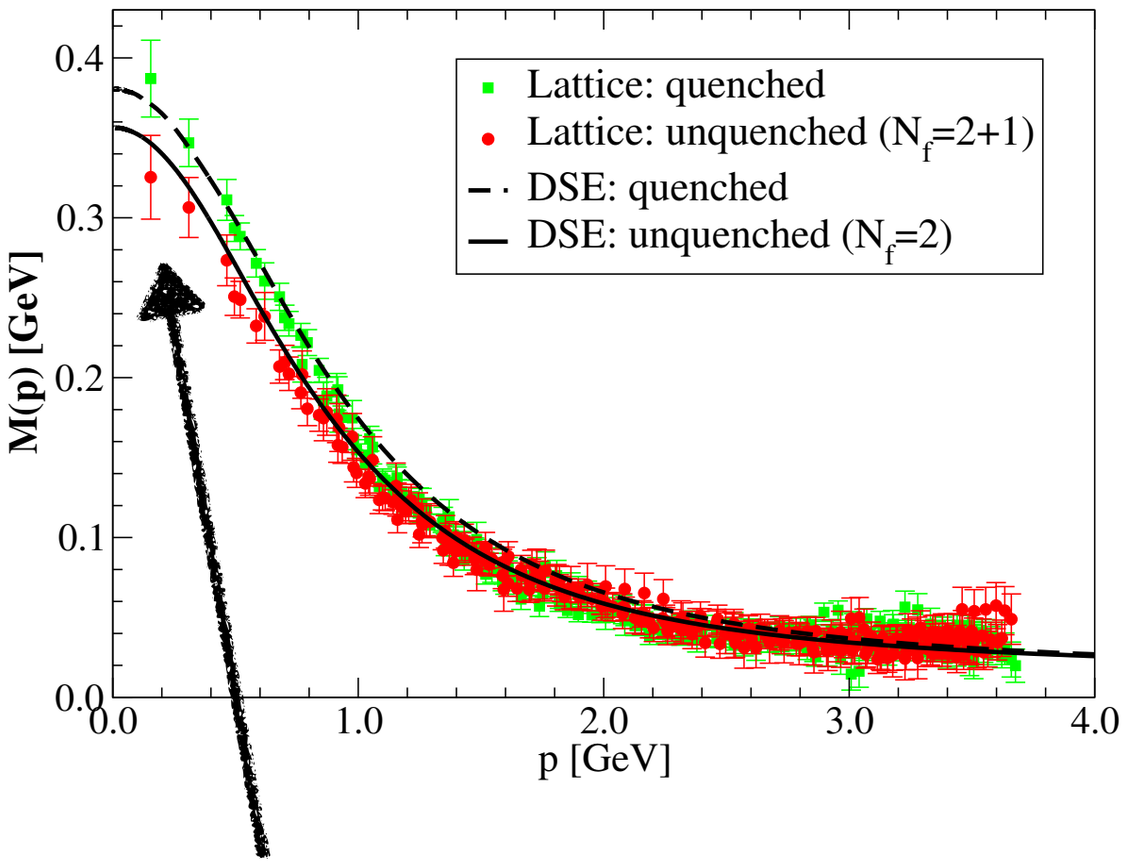

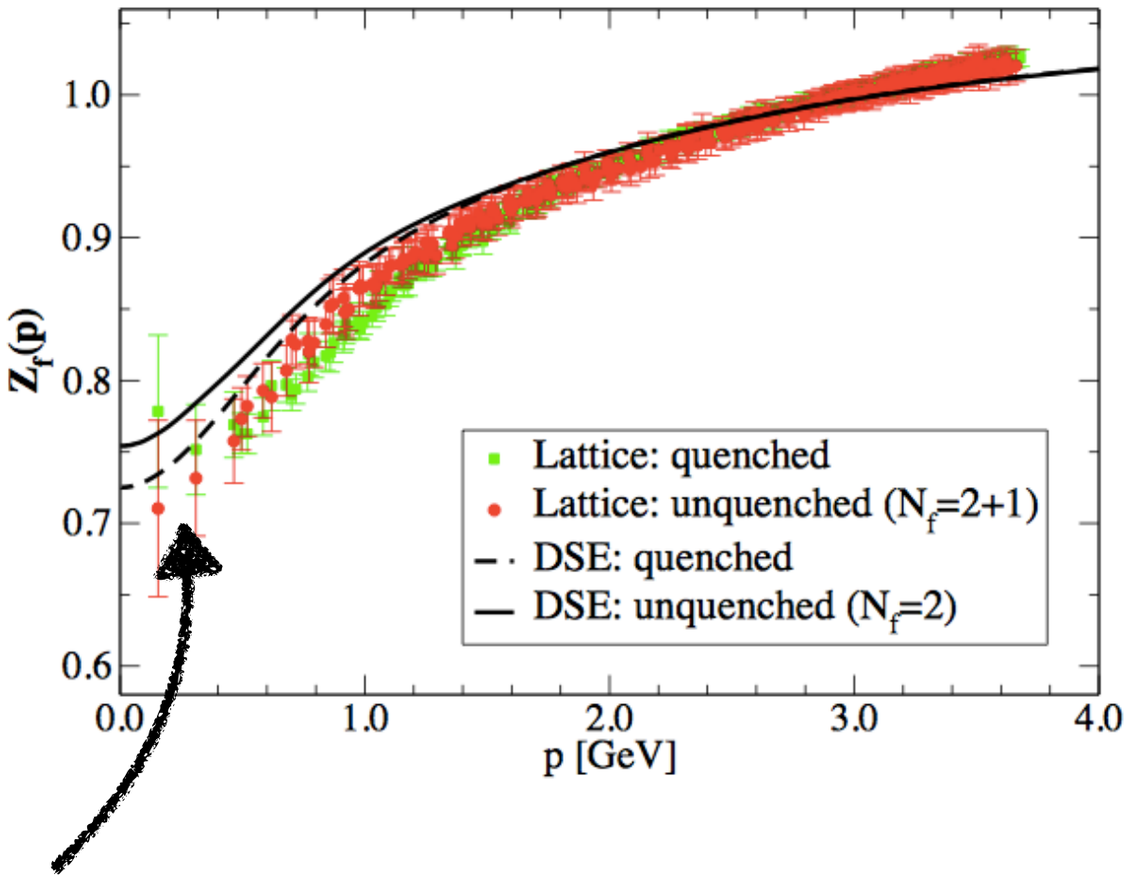

‘constituent quark’: large mass - very composite

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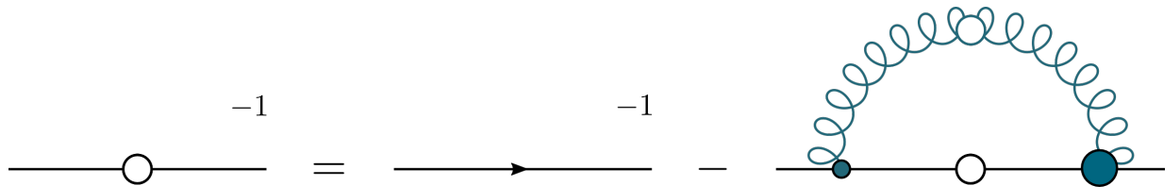
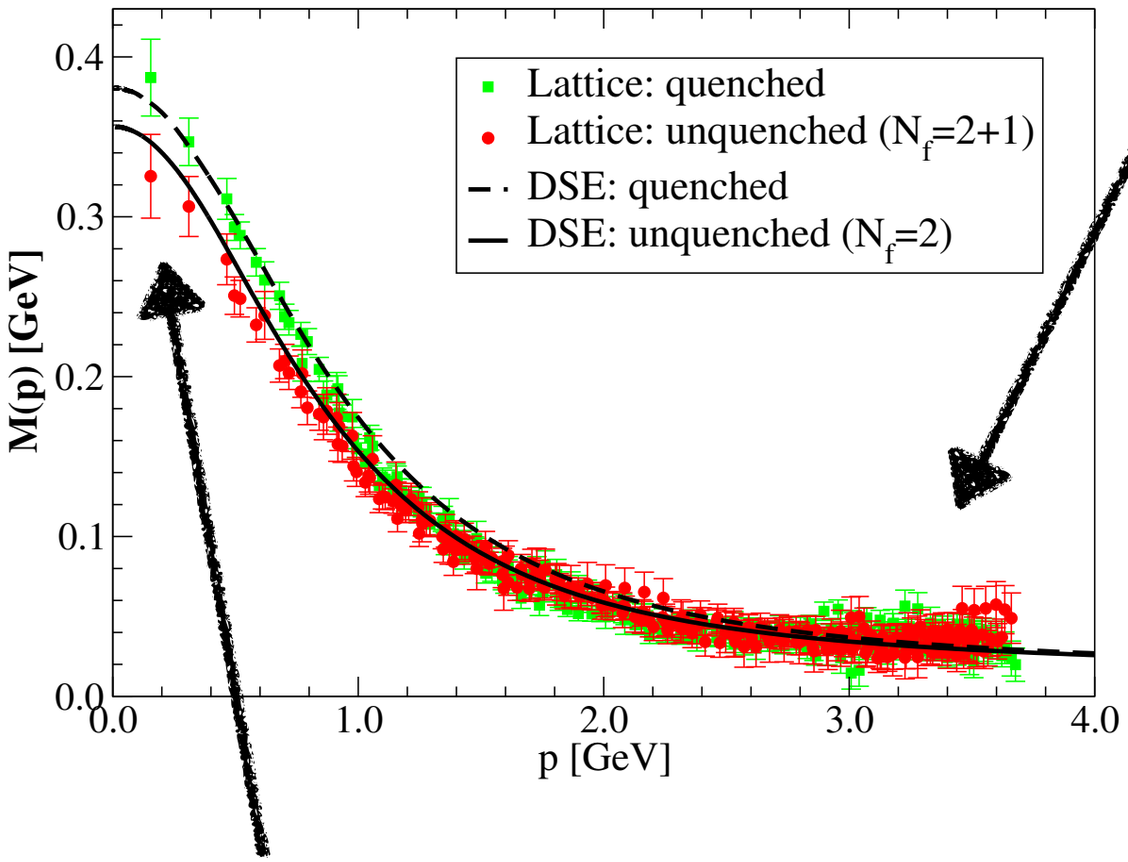
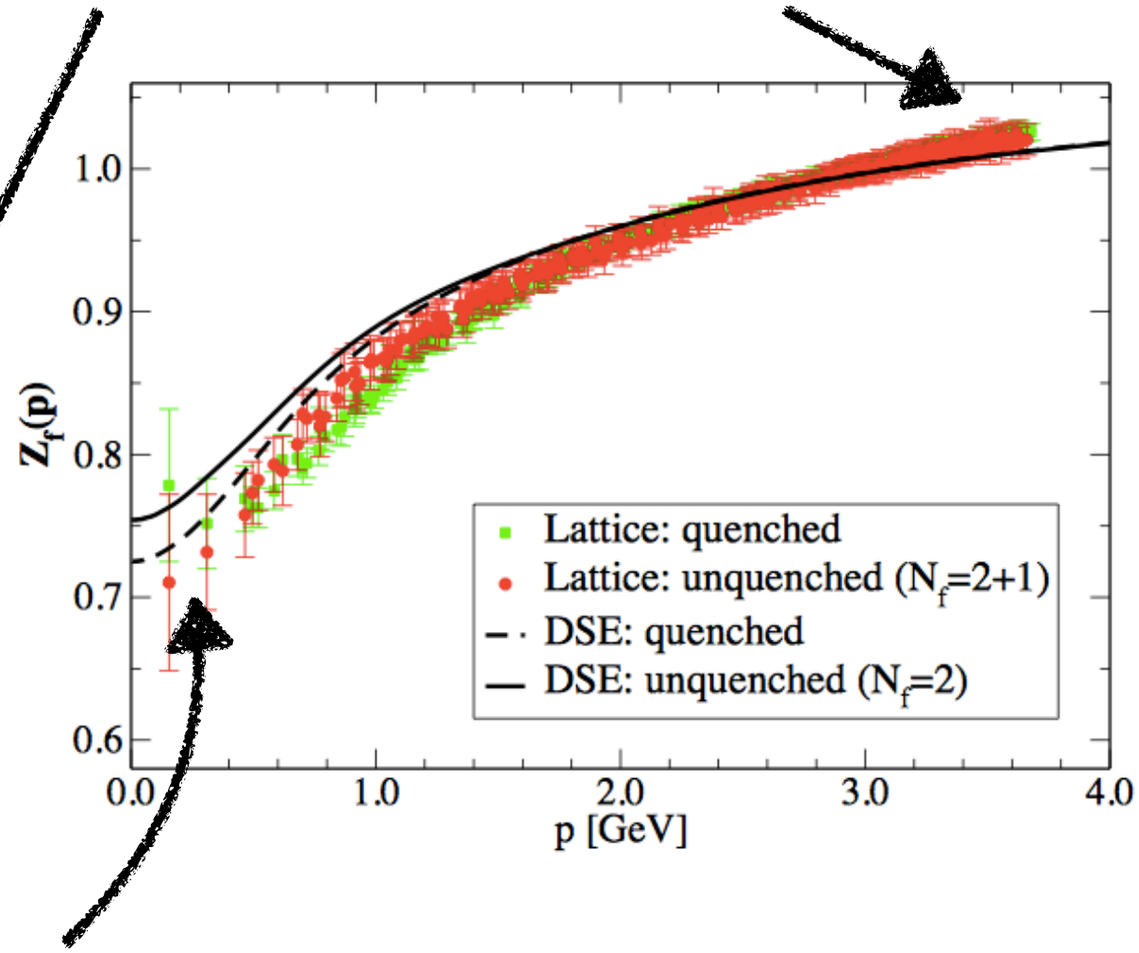


DSE: CF, Nickel, Williams, EPJ C 60 (2009) 47
Lattice: P. O. Bowman, et al PRD 71 (2005) 054507

‘current quark’: small mass; non-composite

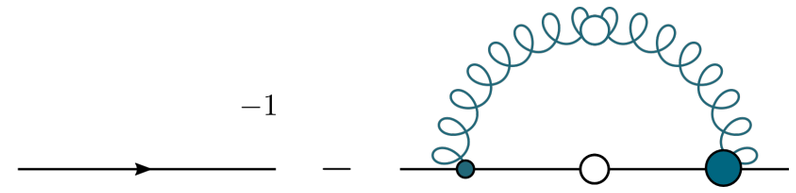




‘constituent quark’: large mass - very composite

Dynamical mass generation

$$S^{-1}(p) = \frac{(i\not{p} + M(p^2))}{}$$



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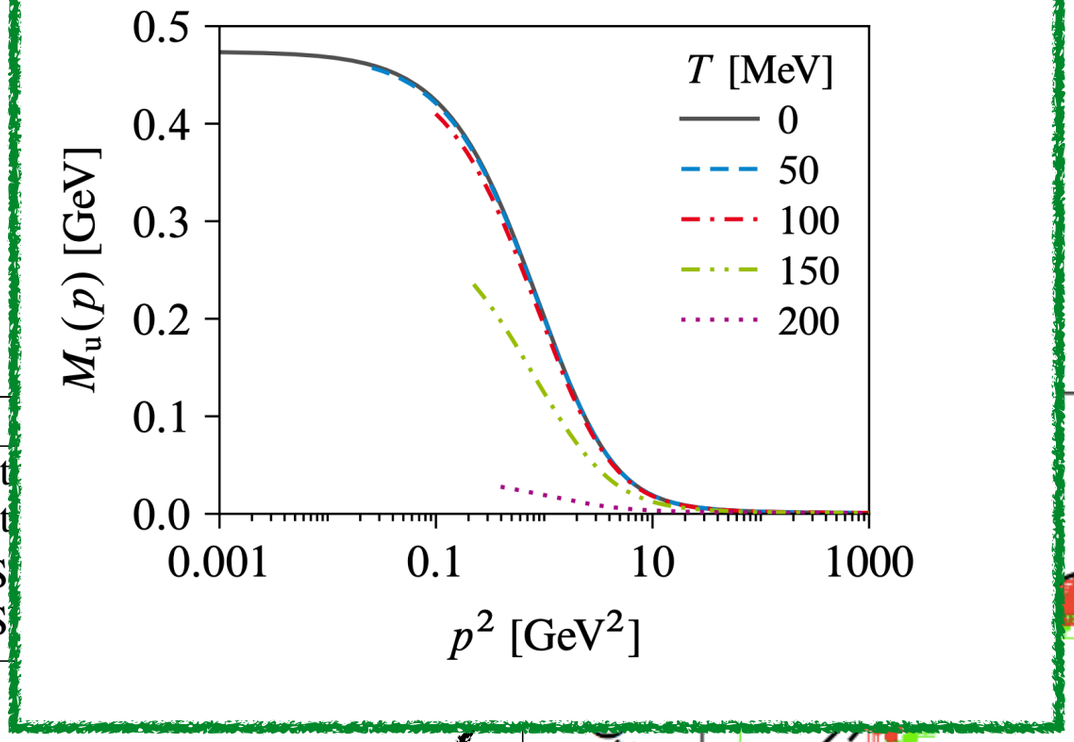
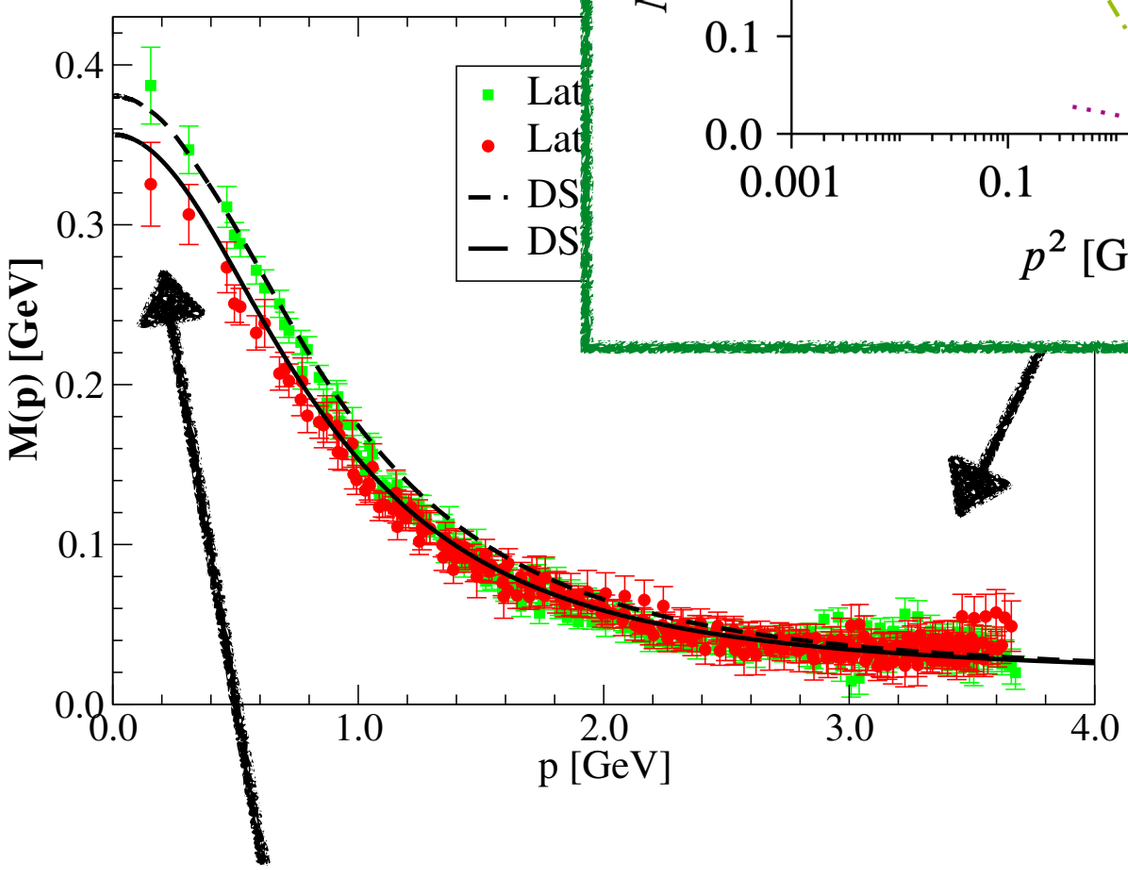
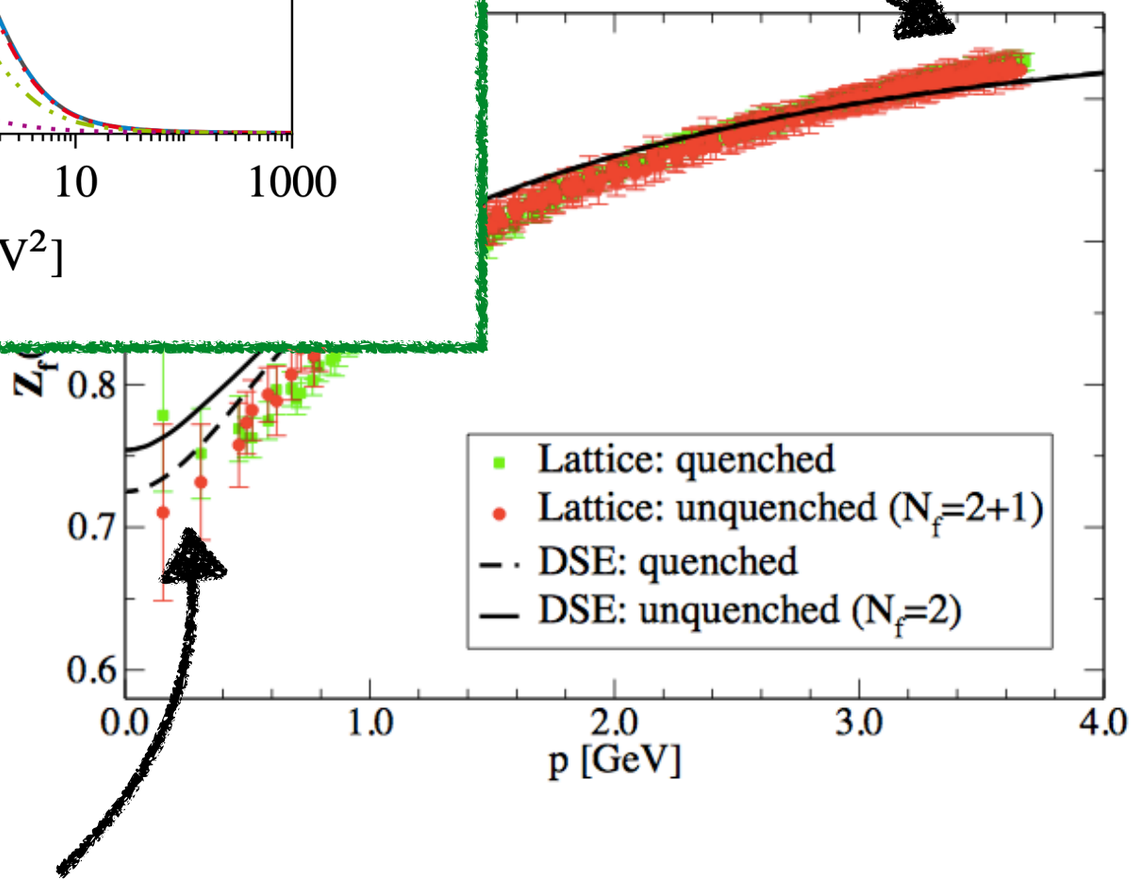


mass; non-composite





‘constituent quark’: large mass - very composite