GLOBAL FITS OF PARTON DISTRIBUTION FUNCTIONS (I)

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17th September 2025

MPA summer school 2025

Goal of the lectures

➡ Give an overview on our understanding on the structure of the proton: from Feynman parton model to modern QCD picture

➡ Introduce basic concepts and techniques behind PDF global fits

➡ Wealth of ingredients involved from low to high energy: non-perturbative effects, perturbative QCD, experimental measurements, statistical and mathematical problems, higher order calculations, phenomenology tools, machine learning.

➡ Discuss PDF-related phenomenology at the LHC (mostly), EIC and beyond

➡ Discuss current frontiers and challenges

Disclaimer: these lectures are far from providing a complete picture of the topic. You can find complementary information in excellent lectures on PDFs from W. Giele, G. Salam, A. Martin, P. Nadolsky, S. Forte, D. Stump, W. Melnitchouk, D. Stump, A. Guffanti, J. Rojo … at recent graduate schools

References

- G. Ridolfi "Notes on deep-inelastic scattering and the Parton model"
- Ellis, Stirling, Webber "QCD and collider physics"
- Dissertori, Knowles, Schmelling "Quantum Chromo Dynamics"
- Ubiali, arXiv:2404.08508
- Snowmass 2021 review "Proton structure at the precision frontier" arXiv:2203.13923
- Kovarik, Nadolsky, Soper arXiv:1905.06957
- Gao, Harland-Lang, Rojo arXiv:1709.04922
- Accardi, et al. arXiv:1603.08906
- PDG review
  https://pdg.lbl.gov/2023/reviews/rpp2023-rev-structure-functions.pdf

List of references complemented by specific references during the lectures

Outline

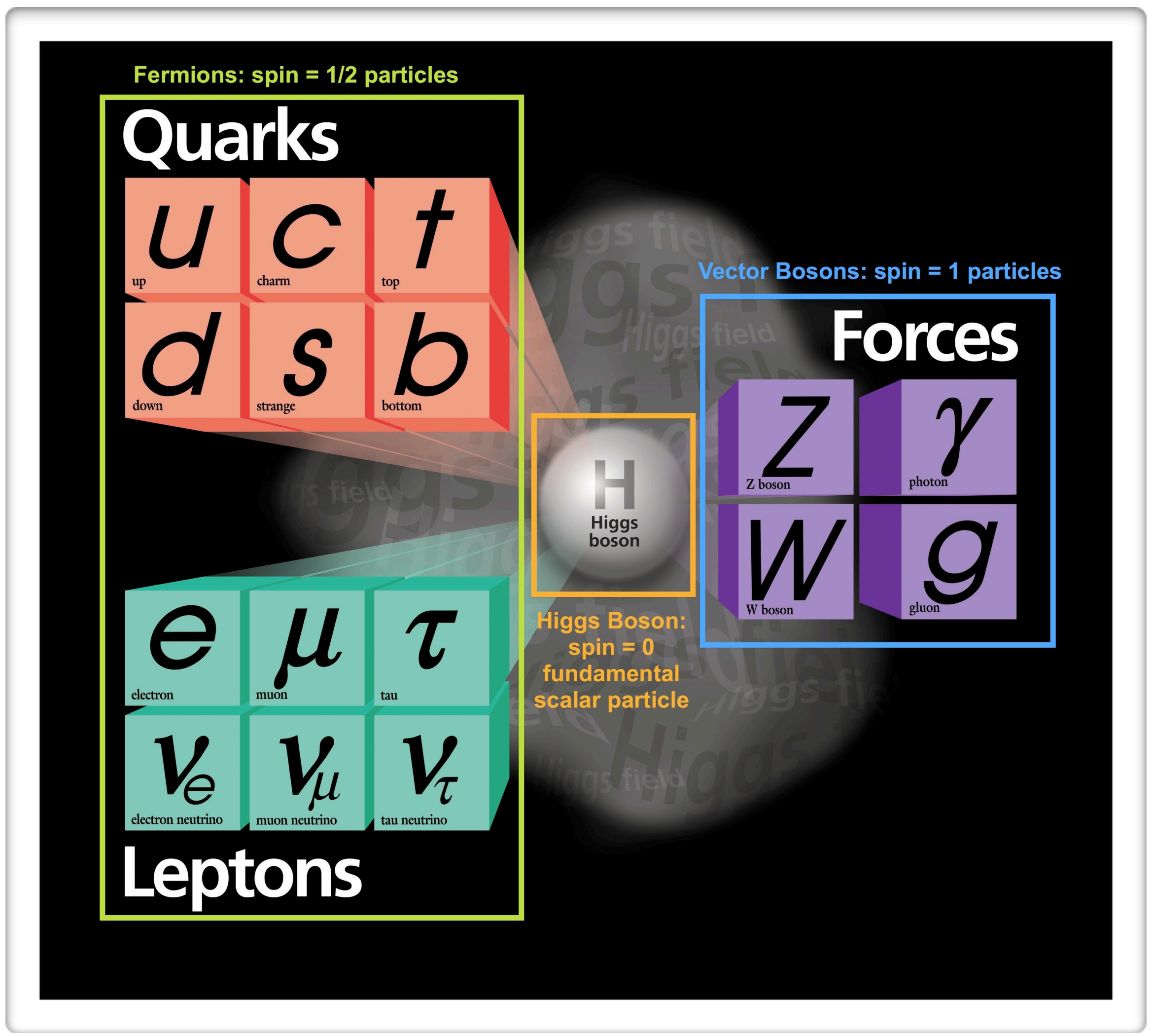
- **First lecture (Today)**
- Second & third lectures (tomorrow)
    - Ingredients of a PDF global fits
    - Experimental input
    - Methodological aspects
    - Theoretical aspects
    - New frontiers and challenges

- Motivation:
  the high energy big picture
- Parton Model and QCD
- Collinear Factorisation

Motivation:
the high-energy big picture

Standard Model of particle physics

- Standard Model (SM) of particle physics one of the greatest triumph of Quantum Field Theories in the past century
- SM remarkably successful theory: no convincing deviations so far from its predictions, but necessarily incomplete



$$\boxed{SU(3)_c \times SU(2)_L \times U(1)_Y}$$

Spontaneous EW Symmetry breaking

Quantum Chromo Dynamics

$$SU(3)_c \times U(1)_{\mathrm{e.m.}}$$

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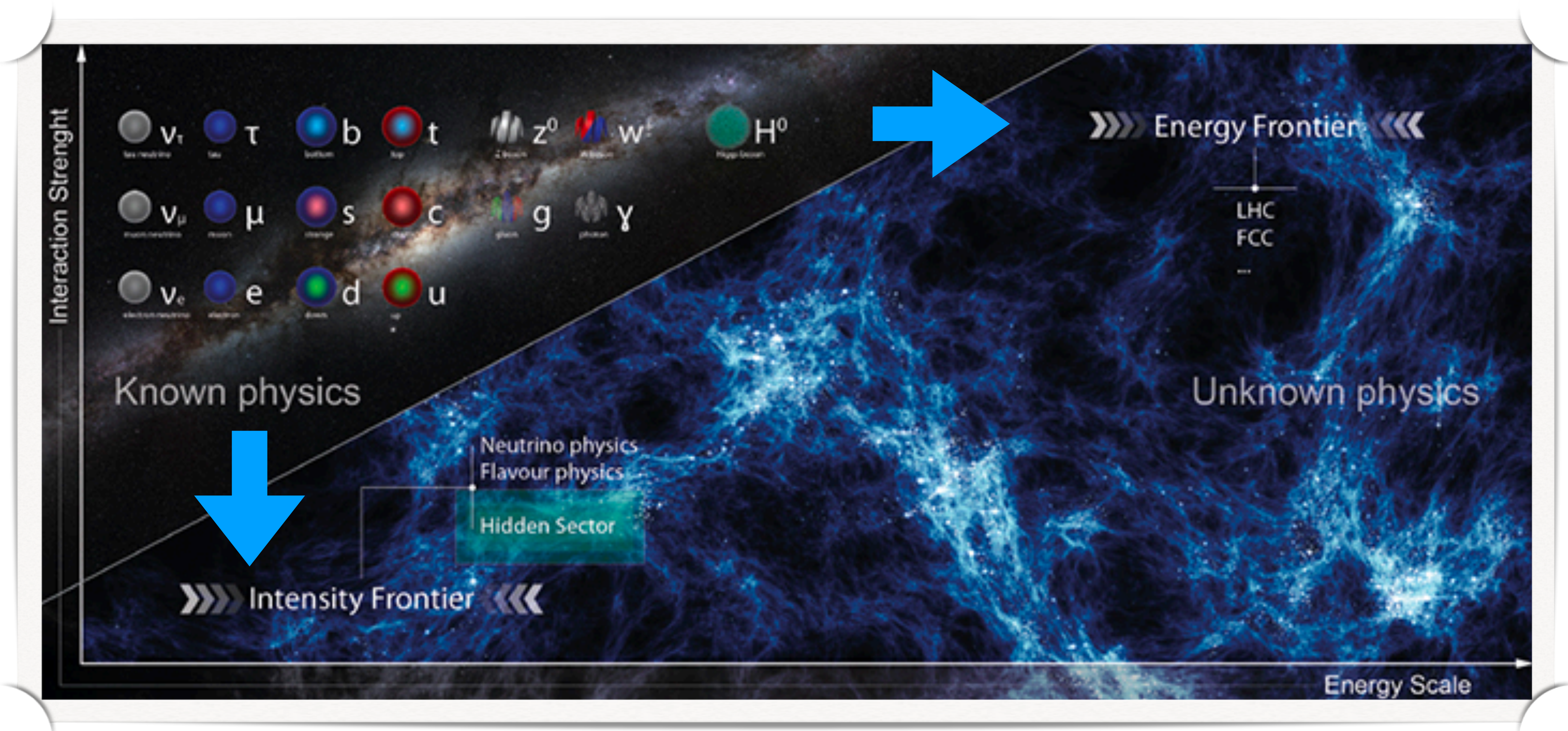


Collider events: a theory view

- Collinear factorisation: the LHC master formula
- Divide et impera!

$$\sigma^{pp\to ab} = \sum_{i,j=-n_f}^{n_f} \int dz_1\, dz_2\, f_i(z_1,\mu_F)\, f_j(z_2,\mu_F)\, \hat{\sigma}^{ij\to ab}(z_1 z_2 S, \alpha_s(\mu_R), \mu_F) + \mathcal{O}\left(\frac{\Lambda^n}{S^n}\right)$$

PDFs: universal functions that parametrise the proton structure

Partonic cross section computed in perturbative QCD + EW corrections

Parton showers, hadronisation

Soft stuff: higher twists, multiple parton interaction, underlying event

PDF uncertainties

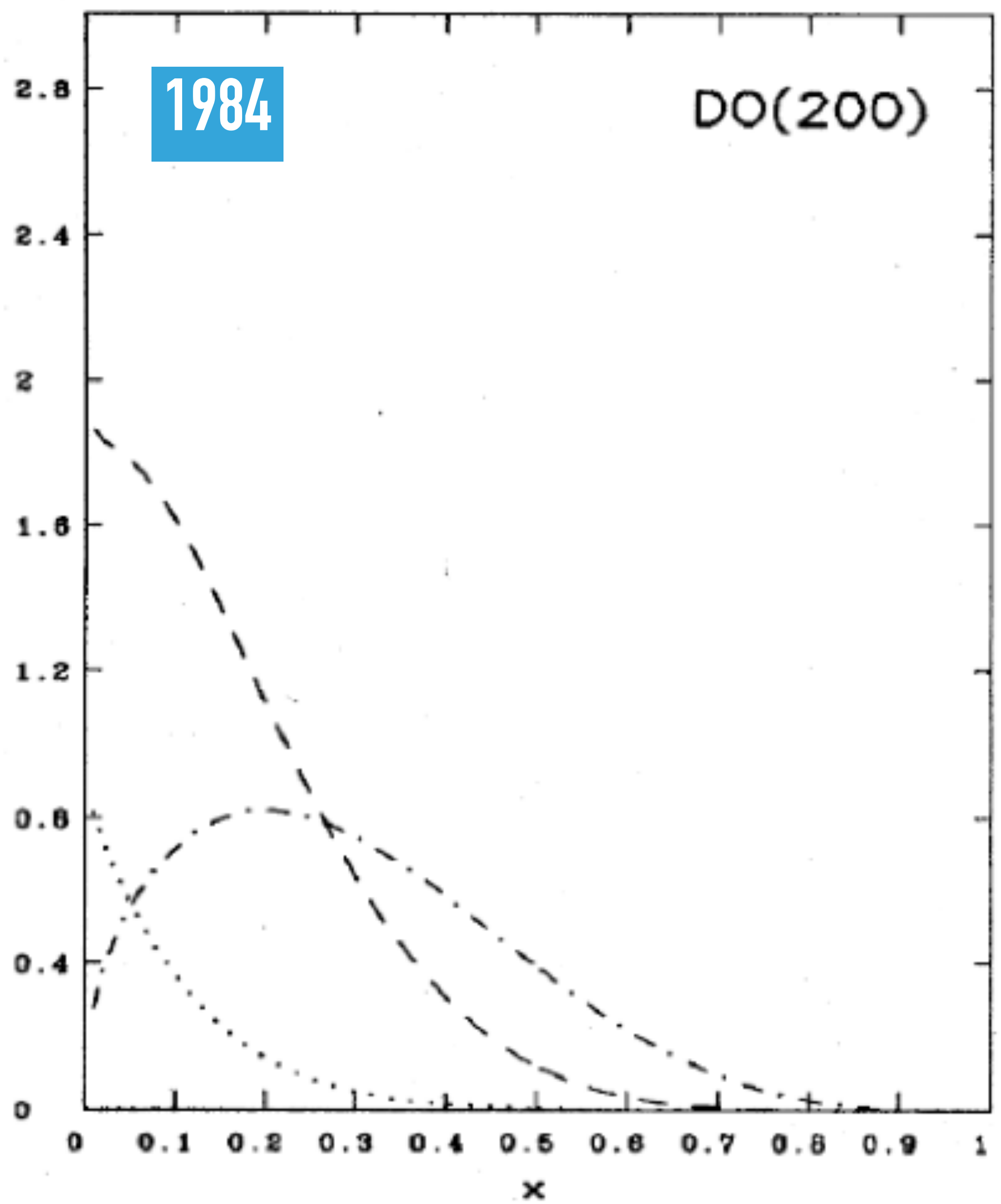
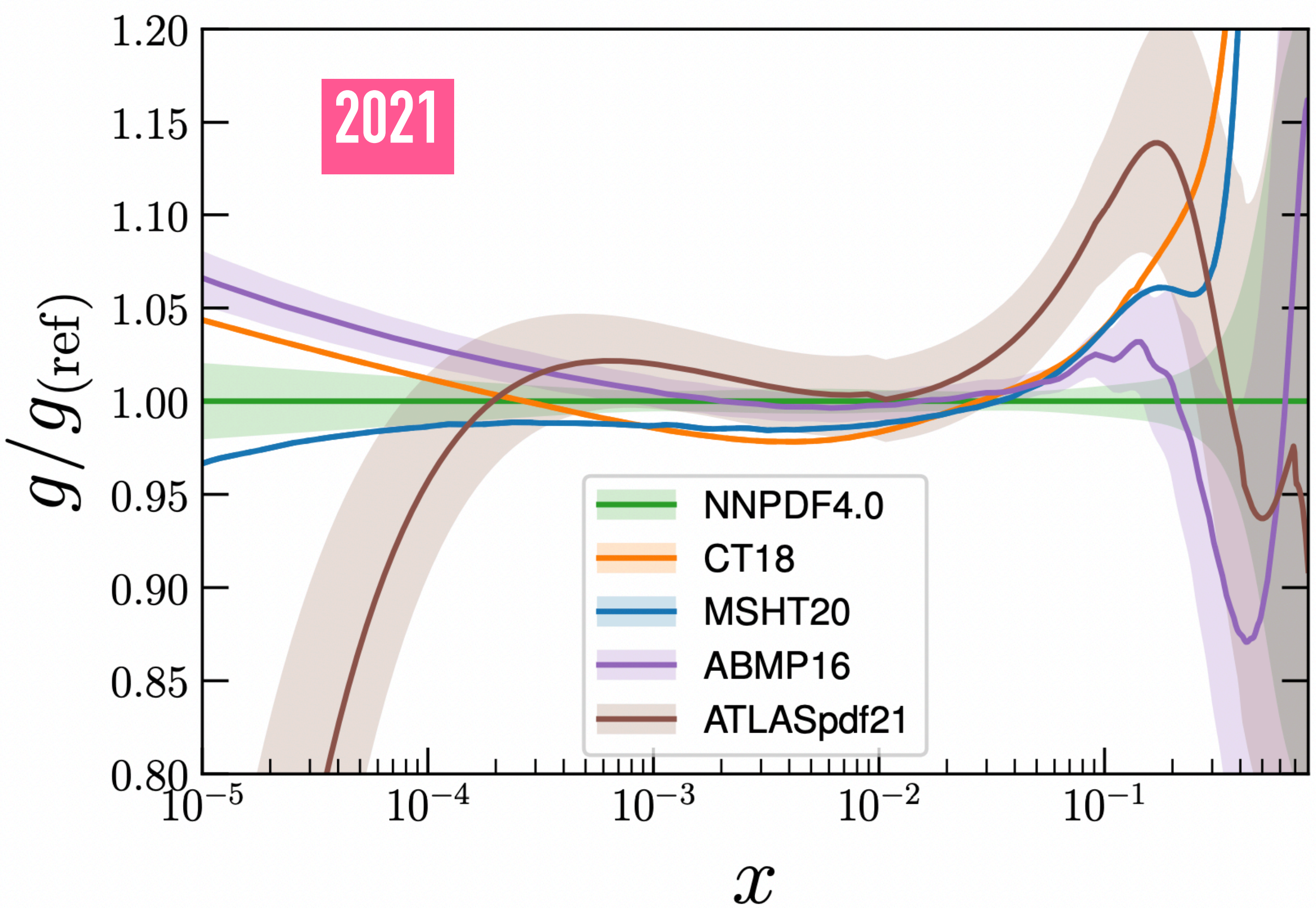


FIG. 27. "Soft-gluon" ($\Lambda = 200$ MeV) parton distributions of Duke and Owens (1984) at $Q^2 = 5$ GeV^2: valence quark distribution $x[u_v(x)+d_v(x)]$ (dotted-dashed line), $xG(x)$ (dashed line), and $q_v(x)$ (dotted line).

Rev.. Mod. Phys. 1984

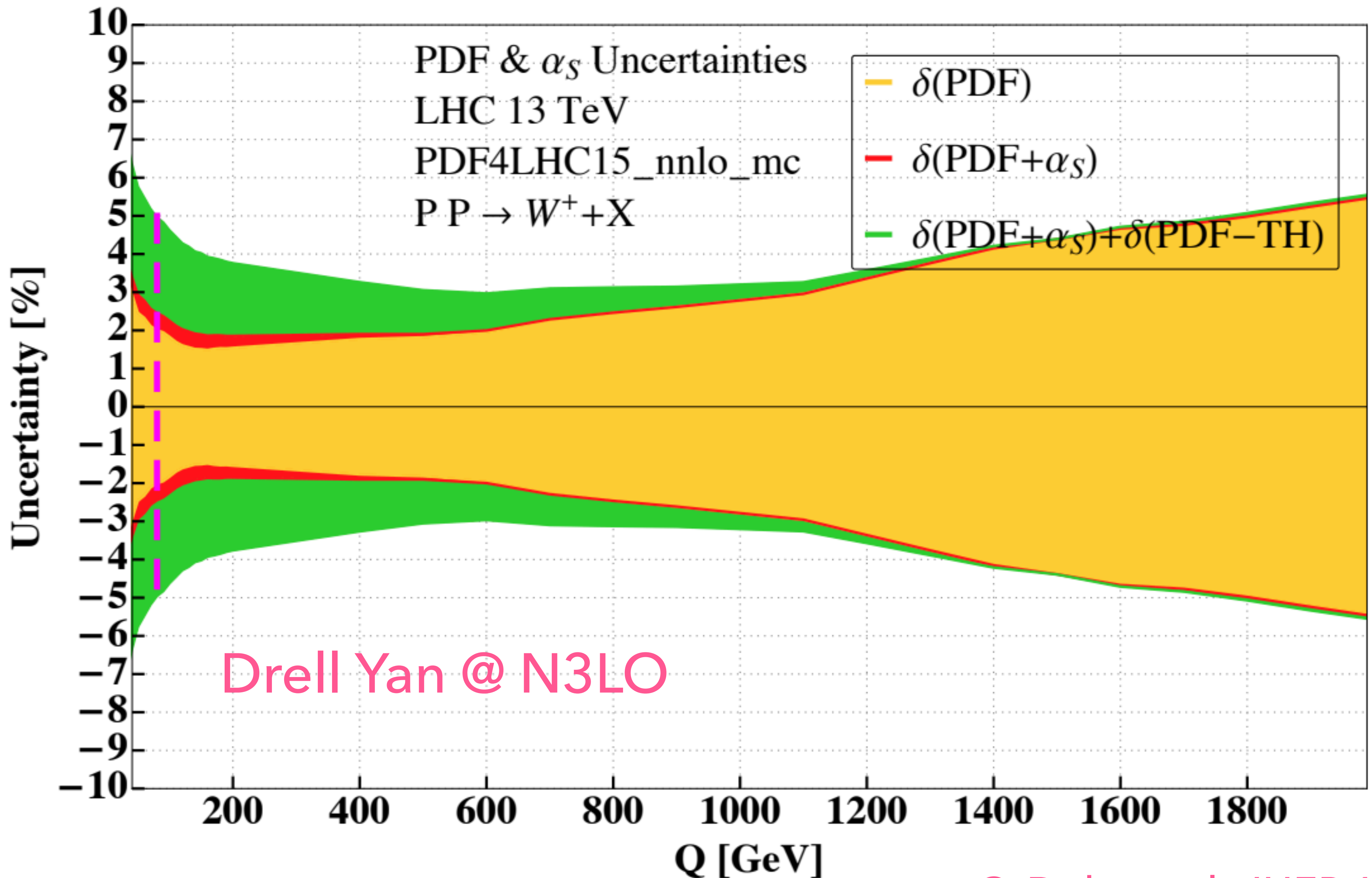


Snowmass 2021 Whitepaper

While at the beginning PDF analyses were performed as a test of QCD collider factorisation and no uncertainties were associated to functions, now PDF uncertainties and differences among different determinations are crucial

PDF uncertainties

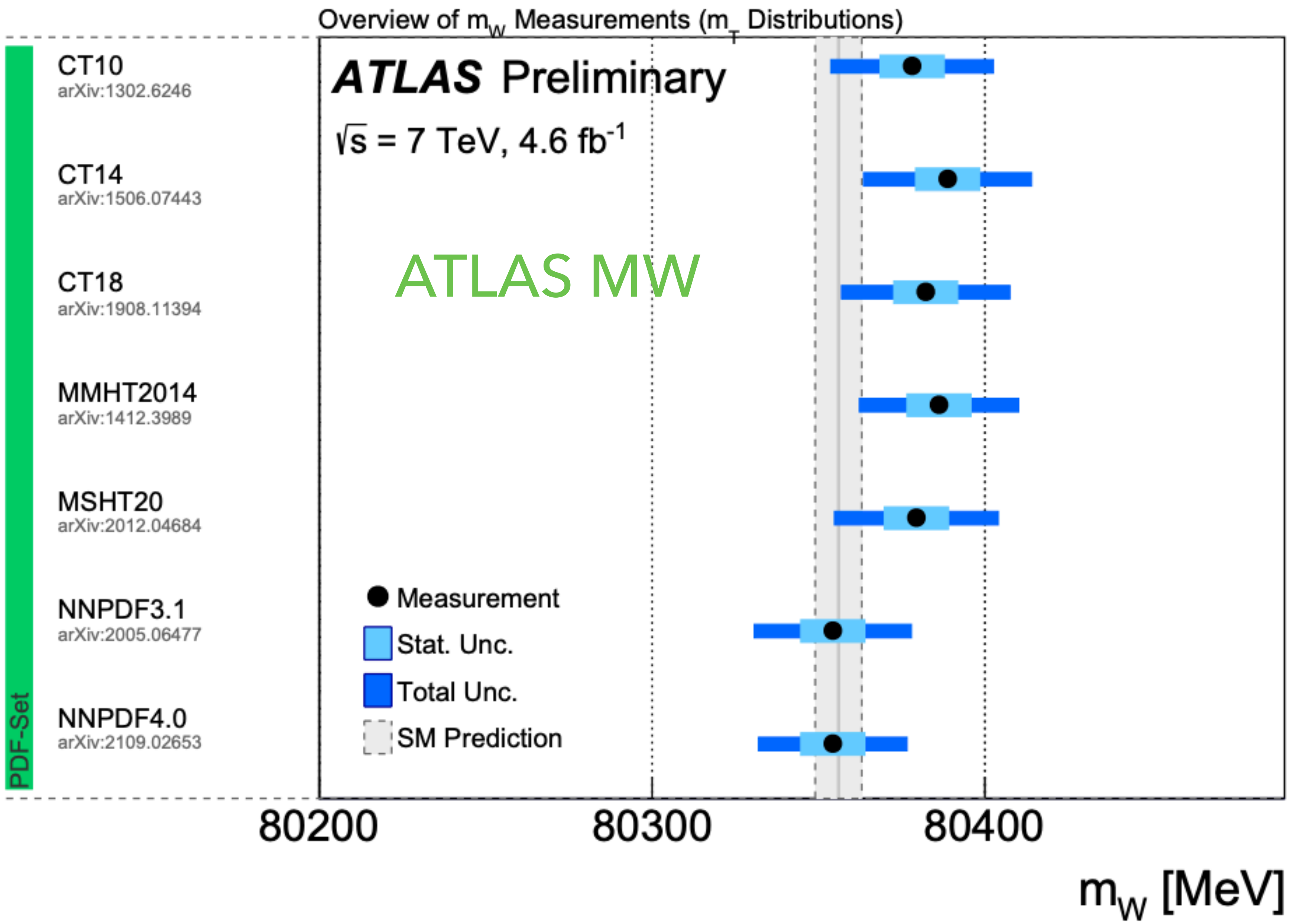
#1: Theory uncertainty of SM predictions



C. Duhr et al, *JHEP* 11 (2020) 143

PDF uncertainties

#2: Determination of SM parameters



ATLAS-CONF-2023-004

PDF uncertainties

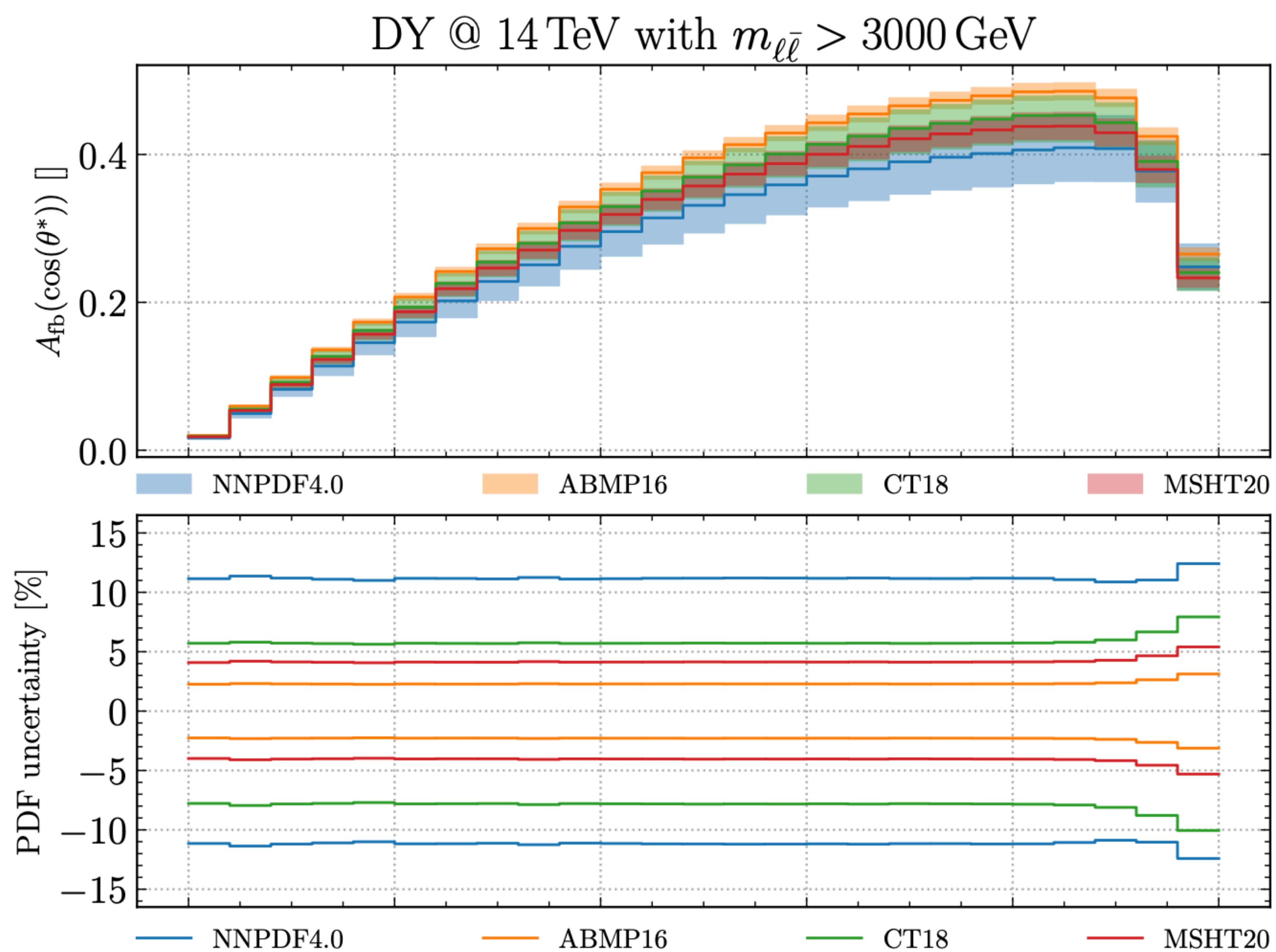
#3: Direct searches: heavy Z′ search in Drell-Yan

$$x \approx \frac{M}{\sqrt{S}}$$

High-mass final states

⇔

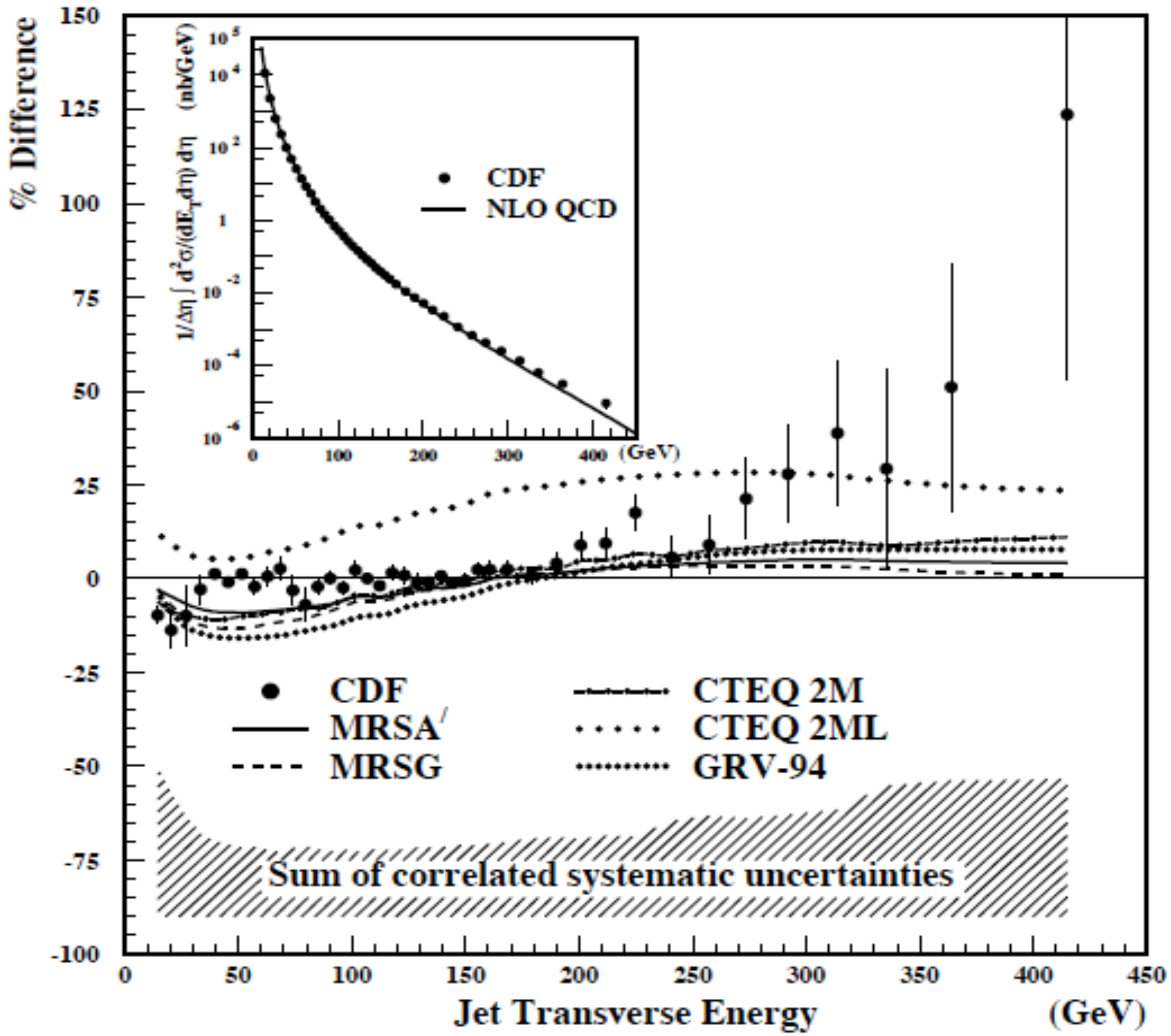
Large-x PDFs



Ball et al arXiv: 2209.08115

PDF uncertainties

#4: Indirect BSM searches



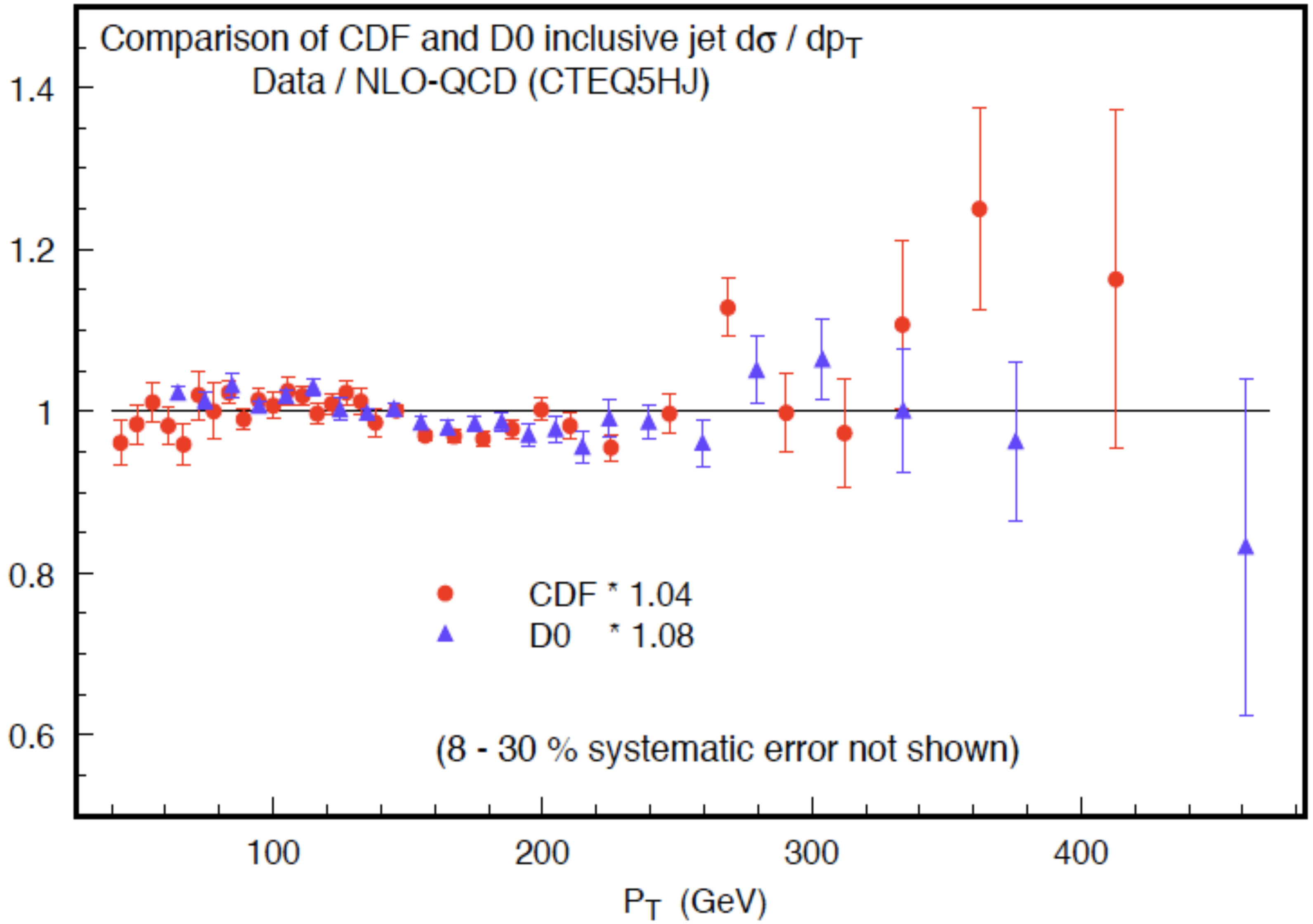
Discrepancy between QCD calculations and CDF jet data (1995)

At that time there was no information on PDF uncertainties
and the theoretical prediction strongly depends on gluon shape at x>0.1

Historia magistra vitae est

PDF uncertainties

FINAL CTEQ FIT (1998)



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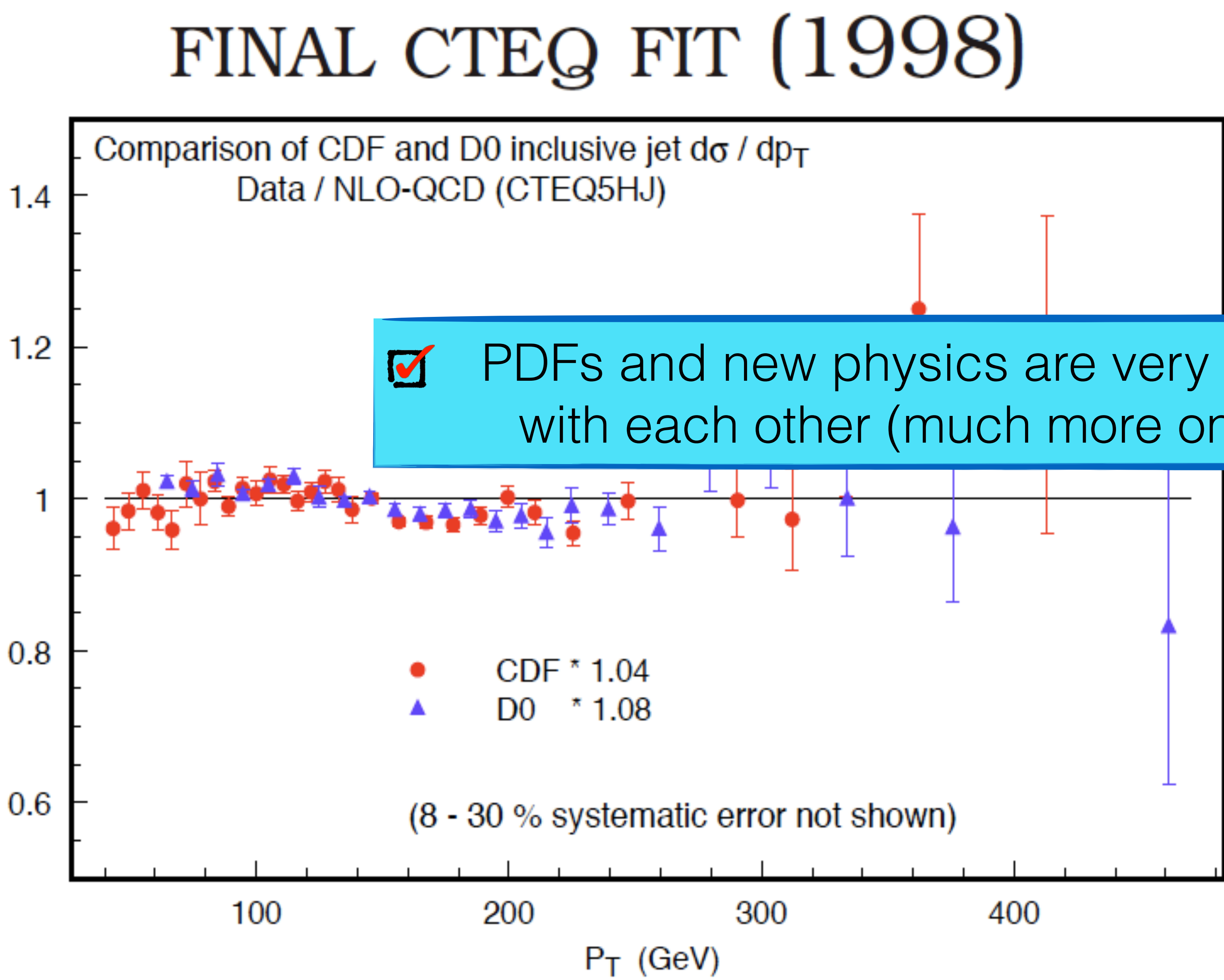
CTEQ re-performed the parton fit by including the jet data and the discrepancy was removed.

PDF uncertainties



PDFs and new physics are very much related with each other (much more on lecture 3)

Discrepancy between QCD calculations and CDF jet data

At that time there was no information on PDF uncertainties and the theoretical prediction strongly depends on gluon shape at x>0.1

CTEQ re-performed the parton fit by including the jet data and the discrepancy was removed.

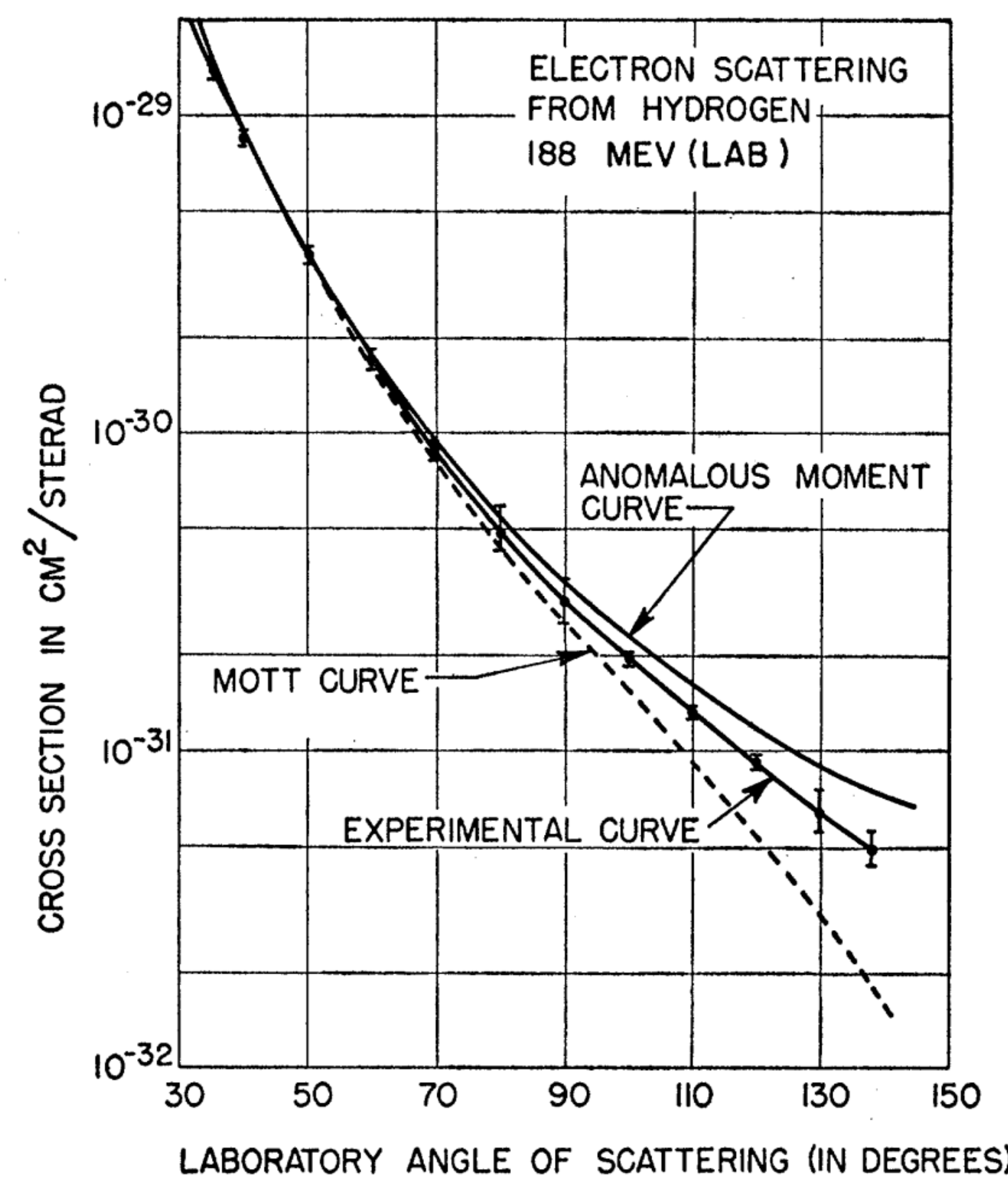
Parton model and QCD

Historic overview

- **1955**: Hofstadter et al observed first deviations in scattering of electron off proton from simple point-like Mott scattering ➜ Finite radius of proton ~ 0.7 fm

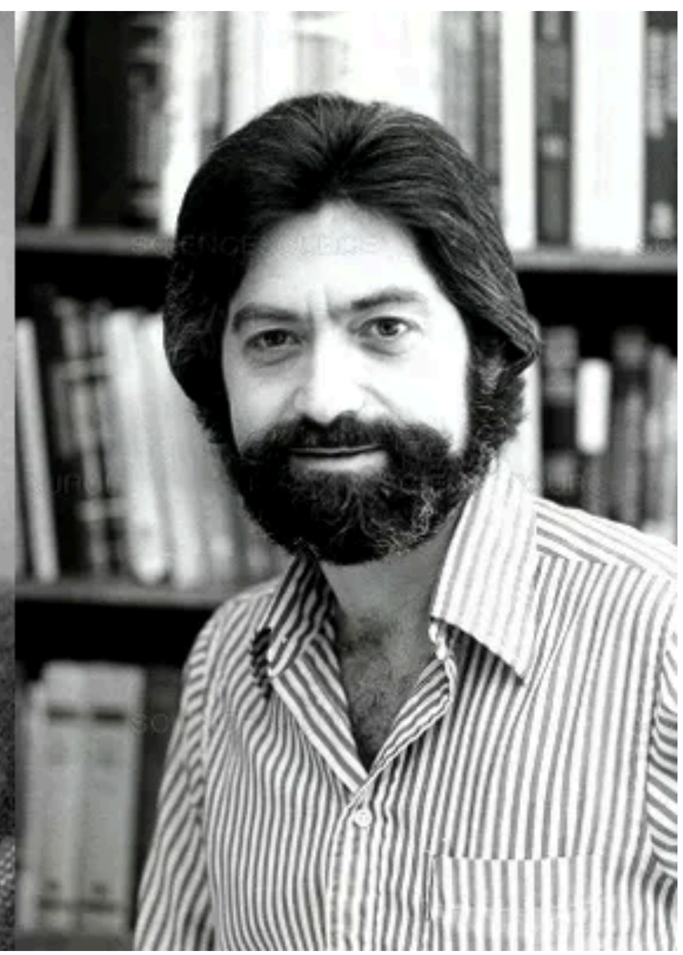
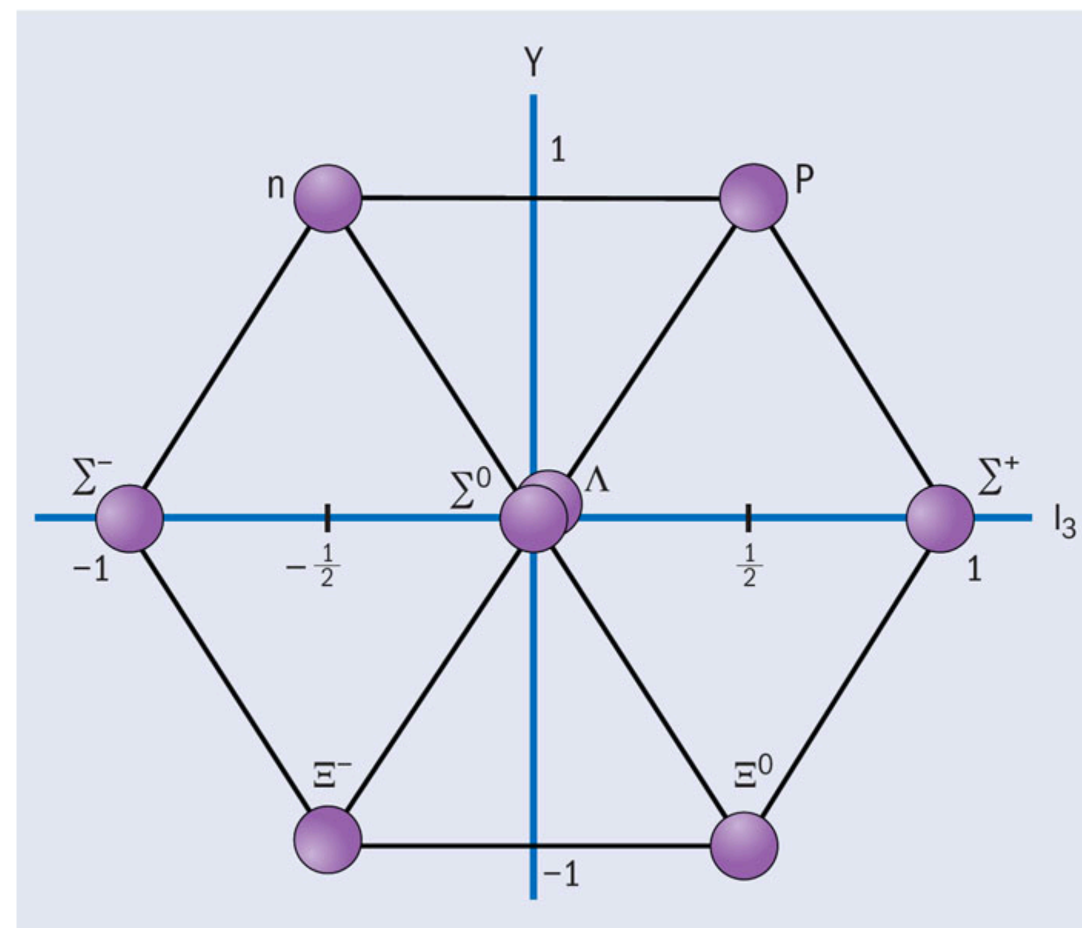
Electron Scattering from the Proton*†‡

Robert Hofstadter and Robert W. McAllister
Department of Physics and High-Energy Physics Laboratory, Stanford University, Stanford, California
(Received January 24, 1955)



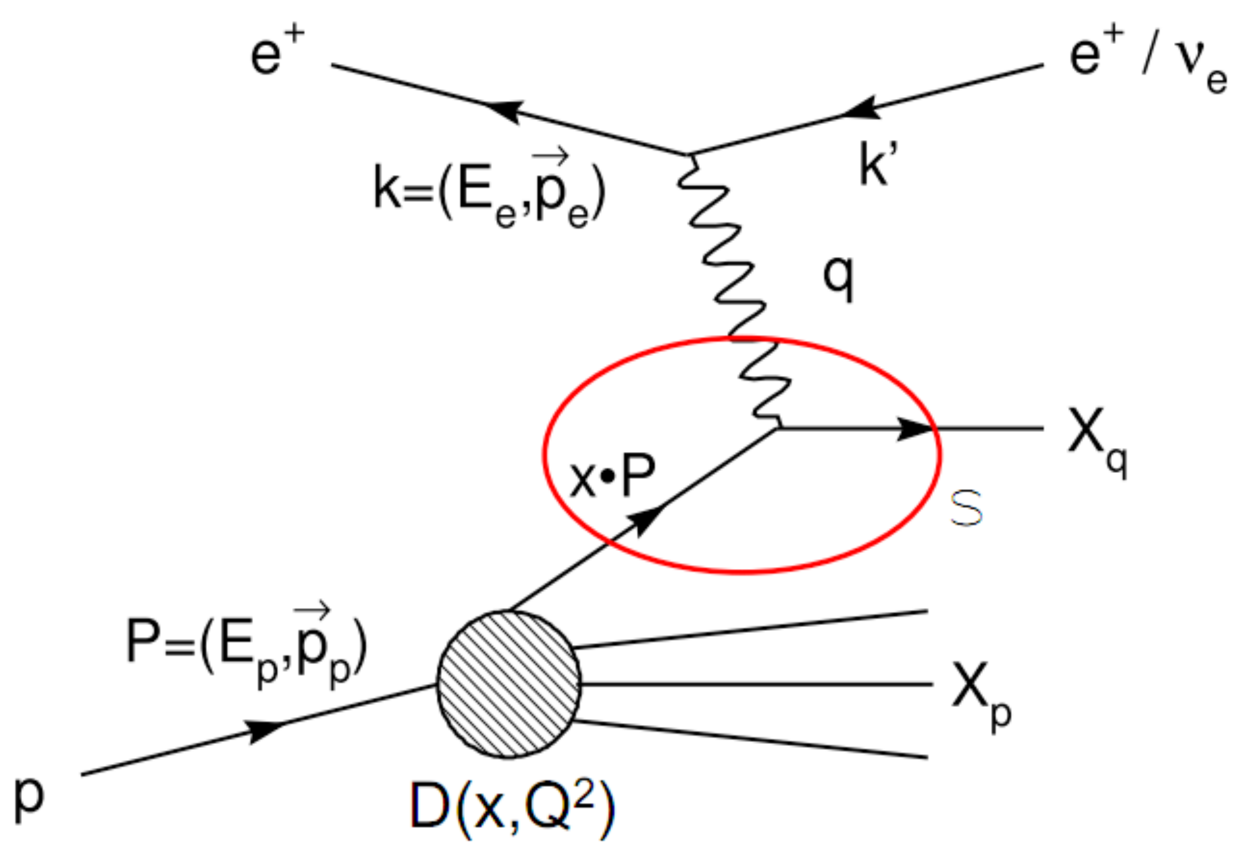
Historic overview

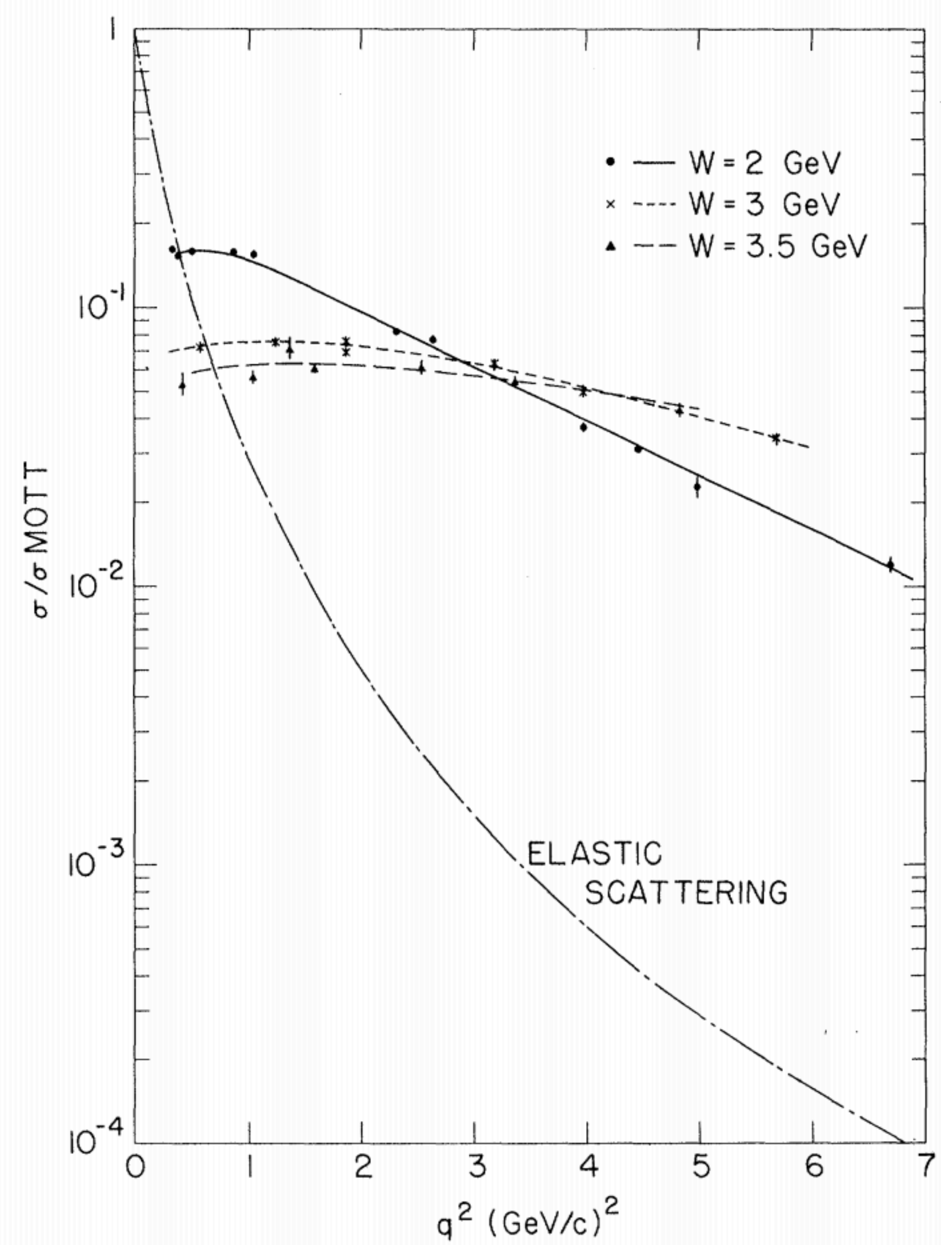
- **1964**: Zweig and Gell-Mann independently postulated existence of three aces (Zweig) or quarks (Gell-Mann) with fractional electric charge and spin–1/2 to explain proliferation of mesons and baryons in nucleon collision experiments. More of a mathematical model rather than particles! How could such objects be bound so tightly together?

Historic overview

- **1967**: First deep-inelastic scattering experiments at SLAC 20 GeV linear collider gave first evidence of point-like elementary constituents of the proton which were later identified as quarks (Bjorken scaling)





DIS Kinematics

Hadronic CoM energy

$$s = (k + P)^2$$

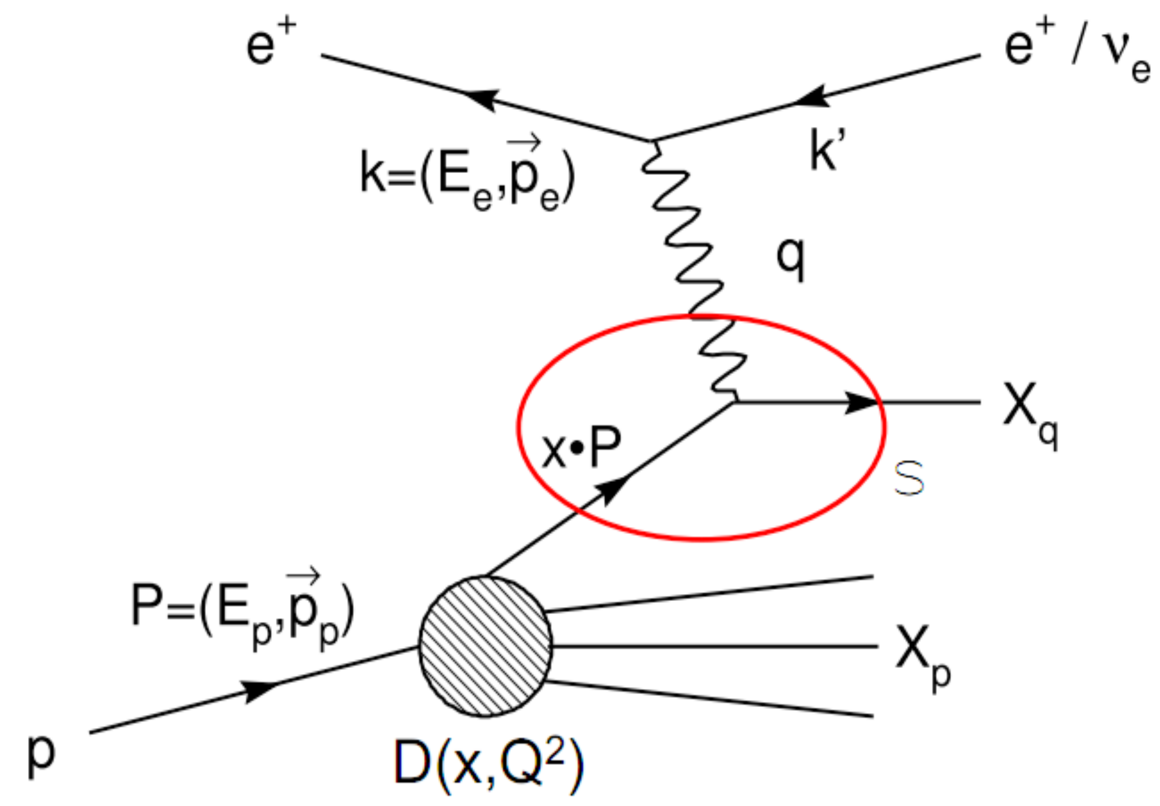
Hadronic invariant mass

$$M_X = (P + q)^2$$

Photon virtuality

$$q = k - k'$$

Leading order QED diagram



DIS Kinematics

Hadronic CoM energy

$$s = (k + P)^2$$

Hadronic invariant mass

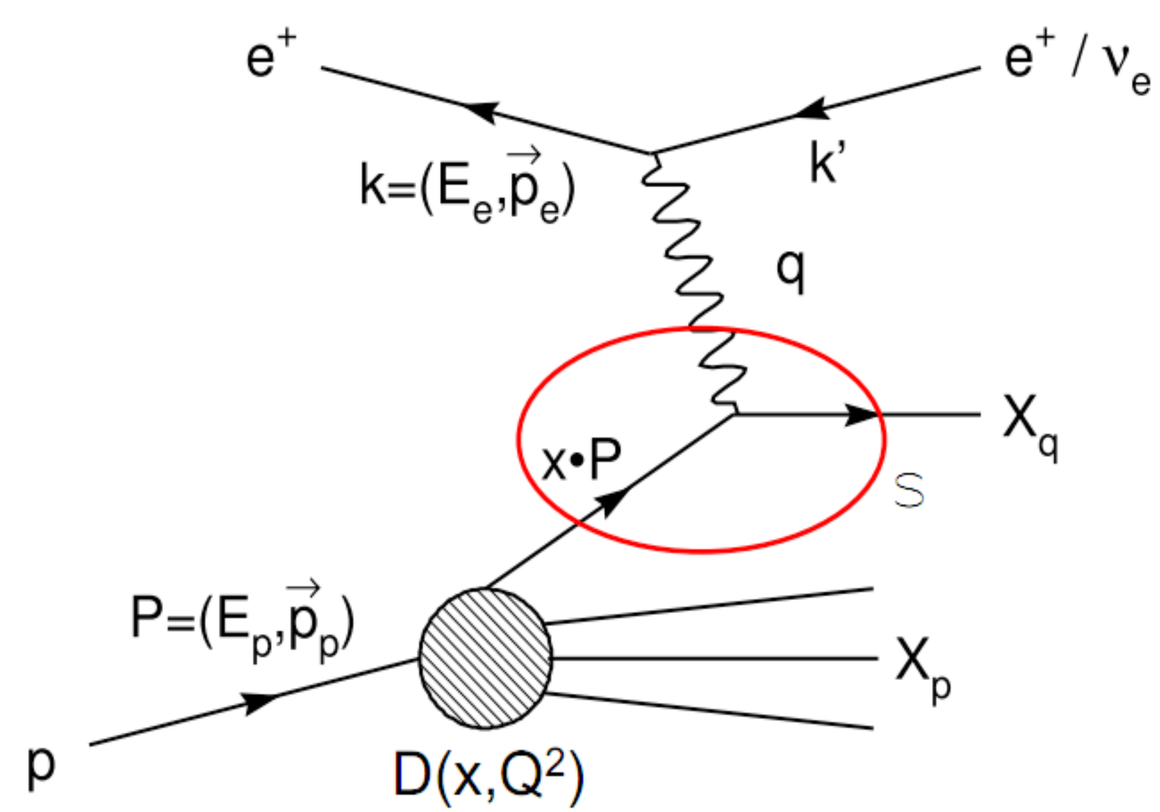
$$M_X = (P + q)^2 = \frac{Q^2(1 - x_B)}{x_B}$$

Photon virtuality

$$q = k - k'$$

In terms of the "standard" invariants

Leading order QED diagram



Q^2

$$Q^2 = -q^2 = -(k - k')^2 \approx 2k \cdot k' = 2E_e E'_e(1 - \cos\theta) \geq 0$$

Bjorken x

$$x_B = \frac{Q^2}{2P \cdot q}$$

Inelasticity

$$y = \frac{P \cdot q}{P \cdot k} \approx \frac{E_e - E'_e}{E_e}$$

$$0 \leq x_B, y \leq 1$$

Deep inelastic regime Mx ≪ Mp, large Q^2 for fixed x_B

DIS Kinematics

Hadronic CoM energy

$$s = (k+P)^2$$

Hadronic invariant mass

$$M_X = (p+q)^2 = \frac{Q^2(1-x_B)}{x_B}$$

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$$q = k - k'$$

In terms of the "standard" invariants

Q²

$$Q^2 = -q^2 = -(k-k')^2 \approx 2k\cdot k' = 2E_eE'_e(1-\cos\theta) \geq 0$$

Bjorken x

$$x_B = \frac{Q^2}{2P\cdot q}$$

Inelasticity

$$y = \frac{P\cdot q}{P\cdot k} \approx \frac{E_e - E'_e}{E_e} \qquad 0 \leq x_B, y \leq 1$$

Deep inelastic regime Mx ≪ Mp, large Q^2 for fixed x_B

Only two out of three of invariants (x_B, y, Q^2) are independent.

In the limit of $m_e \to 0$

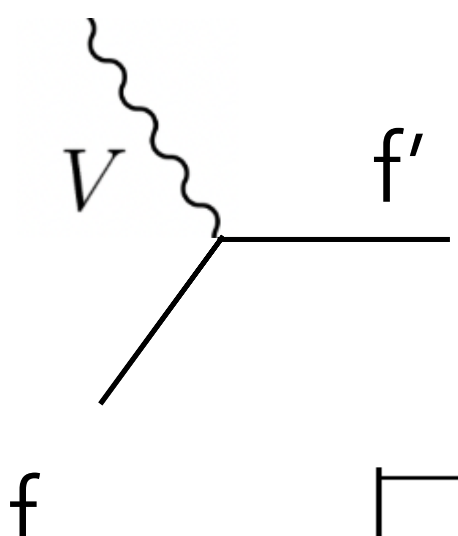
$$x_B = \frac{Q^2}{y\,s}$$

DIS cross section

$$\mathcal{M}(e^{\pm}p \to e^{\pm}X) = \underbrace{< e^{\pm}|J^{\mu}|e^{\pm} >}_{\text{QED current}} g_{e\gamma}\frac{-g_{\mu\nu}}{q^2}g_{p\gamma} \underbrace{< X|J^{\nu}|p >}_{\text{hadron current}}$$

Generalised to the exchange of a vector boson V (Z or W or BSM), with mass M_V
$f, f' = e^{\pm}, \nu_e, \bar{\nu}_e$

$$\mathcal{M}(fp \to f'X) = \underbrace{< f|J^{\mu}|f' >}_{\text{EW current}} g_{fV}\frac{-g_{\mu\nu}}{q^2 - M_V^2}g_{pV} < X|J^{\nu}|p >$$



$$-iek_V\gamma_\mu(v_{lV} + \gamma_5 a_{lV})$$

$$g_{lV} = k_V\sqrt{v_{lV}^2 + a_{lV}^2}$$

	k_V	v_{lV}	a_{lV}
γ	e_l	1	0
Z	$1/(2\sin\theta_W\cos\theta_W)$	$I_3^l - 2e_l\sin^2\theta_W$	-I_3^l
$W^{\pm}$	$V_{ll'}/(2\sqrt{2}\sin\theta_W)$	1	-1

DIS cross section

$$d\sigma(ep \to e'X) = \frac{1}{2s}\frac{(g_{lV}g_{hV})^2}{(Q^2+M_V^2)^2} L_{\mu\nu}W^{\mu\nu}(4\pi)\frac{d^3k'}{(2\pi)^3 2E_e'}$$

② Leptonic tensor **③ Hadronic tensor**

① Final lepton phase space

In the proton rest frame, after integrating over the azimuthal angle

① $$\frac{d^3k'}{(2\pi)^3 2E_e'} = \frac{E_e' dE_e' d\cos\theta}{8\pi^2}$$

In terms of invariant

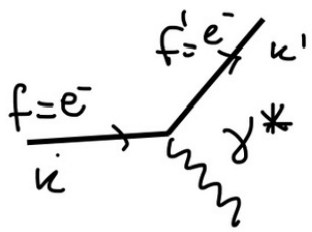
$$Q^2 = 2E_eE_e'(1-\cos\theta) \quad \Rightarrow \quad dQ^2 = 2E_eE_e' d\cos\theta$$

$$2P\cdot q = \frac{S}{2E_e}[2E_e - E_e'(1+\cos\theta)] = \frac{S}{2E_e}\left[2(E_e - E_e') + \frac{Q^2}{2E_e}\right] \quad \Rightarrow dx_B = \frac{x_B^2}{Q^2}\frac{s}{E_e}dE_e'$$ for fixed Q^2

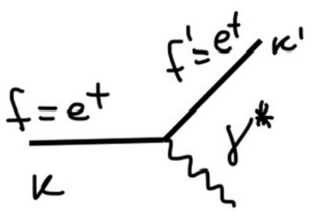
$$E_e'\, dE_e'\, d\cos\theta = \frac{Q^2}{2sx_B^2}dx_B\, dQ^2 = dy\, dQ^2$$

DIS cross section

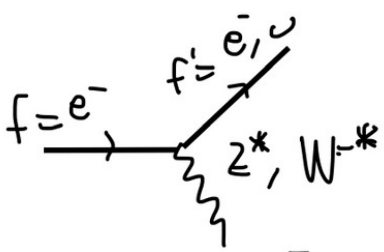
② $$L_{\mu\nu} = \frac{1}{2} < e^- | J_\mu | e^- > < e^- | J_\nu | e^- >$$



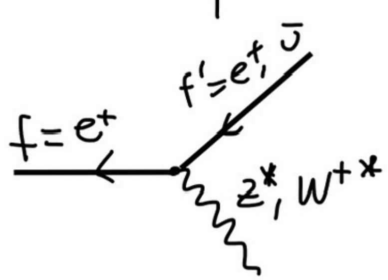
$\bar{u}(k')(-ie\gamma_\mu)u(k)$



$\bar{v}(k)(-ie\gamma_\mu)v(k')$



$\bar{u}(k')[-iek_V\gamma_\mu(v_{eV}+a_{eV}\gamma_5)]u(k)$



$\bar{v}(k)[-iek_V\gamma_\mu(v_{eV}+a_{eV}\gamma_5)]v(k')$

+ for e⁻
– for e⁺

$$c_{lV} = \frac{2a_{lV}v_{lV}}{v_{lV}^2 + a_{lV}^2}$$

Take the trace on the spinor indices

$$L_{\mu\nu} = 2\left[k_\mu k'_\nu + k_\nu k'_\mu - \frac{Q^2}{2}g_{\mu\nu} \pm i\, c_{lV}\, \varepsilon^{\tau\sigma}_{\mu\nu}\, k_\sigma k'_\tau\right]$$

DIS cross section

③ $$W^{\mu\nu} = \frac{1}{2}\frac{1}{4\pi e^2}\sum_X < p|J^\mu|X> < X|J^\nu|p > \delta^{(4)}\left(p+q-P_X\right)$$

Most general form that respects Pointcare invariance, time-reversal symmetry and EM current conservation

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1(x_B, Q^2) + \frac{\tilde{P}^\mu \tilde{P}^\nu}{P\cdot q} F_2(x_B, Q^2) + \frac{i\varepsilon^{\mu\nu}_{\alpha\beta}P^\alpha q^\beta}{2P\cdot q} F_3(x_B, Q^2)$$

$$\boxed{\tilde{P}^\mu = P^\mu - \frac{P\cdot q}{q^2}q^\mu}$$

F_1, F_2 and F_3 structure functions to be determined experimentally.
If we require parity conservation (QED, photon exchange) $F_3 = 0$

② + ③ $$L_{\mu\nu}W^{\mu\nu} = 2Q^2 F_1(x_B, Q^2) + \frac{4(P\cdot k)(P\cdot k')}{(P\cdot q)} F_2(x_B, Q^2) \mp c_{lV}\frac{P\cdot(k+k')Q^2}{(P\cdot q)} F_3(x_B, Q^2)$$

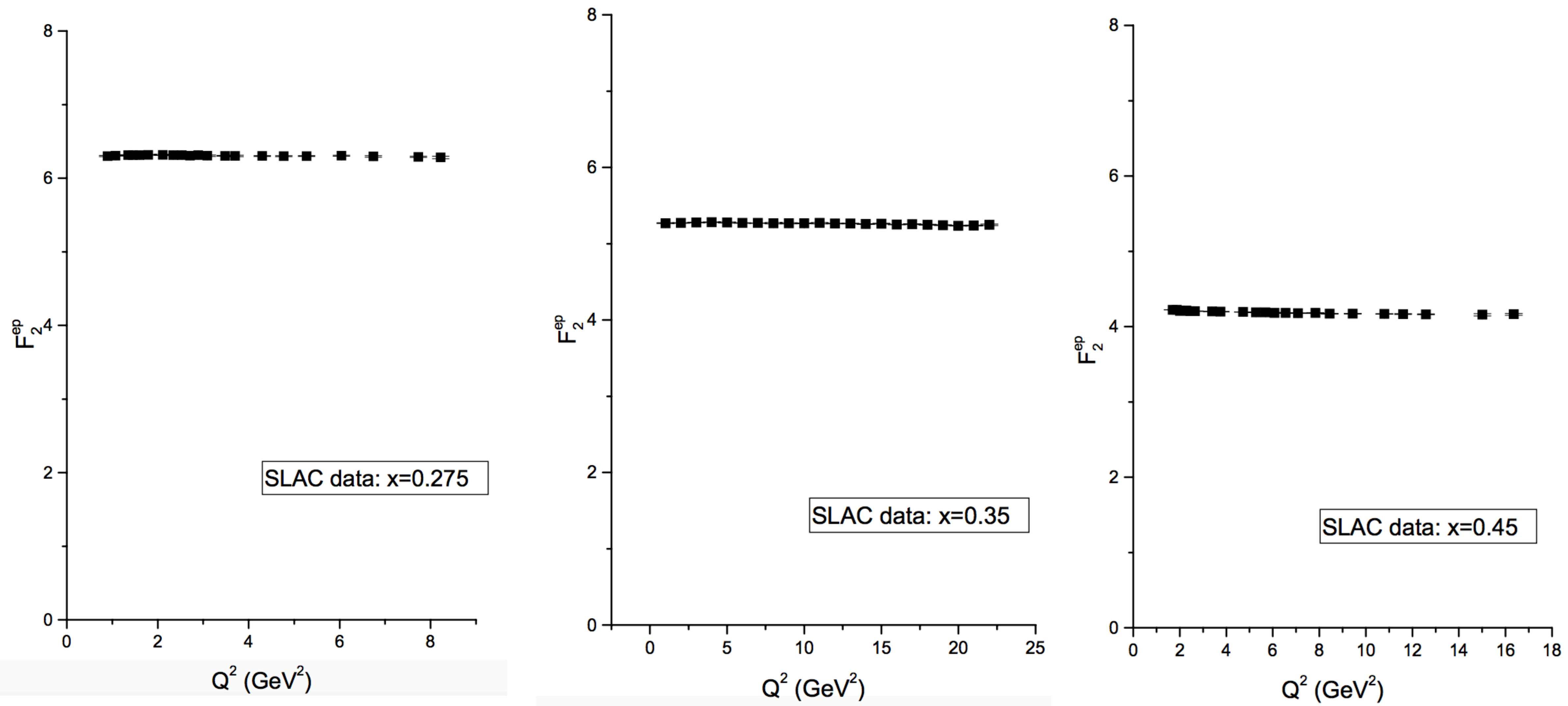
$$L_{\mu\nu}W^{\mu\nu} = \frac{2Q^2}{y^2}\left[y^2 F_1(x_B, Q^2) + (1-y) F_2(x_B, Q^2) \mp c_{lV}x_B\left(y - \frac{y^2}{2}\right) F_3(x_B, Q^2)\right]$$

① + ② + ③

$$\boxed{\frac{d^2\sigma}{dx_B dQ^2} = \frac{4\pi}{x_B}\frac{\alpha_{lV}\alpha_{hV}}{(Q^2+M_V^2)^2}\left[x_B y^2 F_1(x_B, Q^2) + (1-y) F_2(x_B, Q^2) \mp c_{lV}x_B\left(y - \frac{y^2}{2}\right) F_3(x_B, Q^2)\right]}$$

Bjorken scaling

The surprising results at SLAC was that $F_{1,2}$ did not vanish as Q^2 increased, rather they remained finite and constant and depended only on x_B - Bjorken scaling (1969)



Such scaling demonstrated that the exchanged vector boson (photon) scatters off point-like objects that have no mass or scale associated. The lepton scatters off charged spin 1/2 constituents (partons) that carry a fraction x of proton momentum

The Parton Model

Basic assumption:

Parton Distribution Functions

$$d\sigma(P) = \sum_{i \in \text{partons}} \int_0^1 dz\, f_i(z)\, d\hat{\sigma}_i(Pz)$$

z: fraction of proton's momentum carried by parton i

Scattering of massless spin 1/2 parton $\mathcal{Q}_i$ with charge e_i (in units of proton charge) and momentum $\hat{p} = zP$

$$\mathcal{Q}_i(\hat{p}) + \gamma^*(q) \rightarrow \mathcal{Q}_i(\hat{p}')$$

γ exchange

One can compute explicitly the hadronic tensor (exactly as we did with leptonic tensor)

$$W_{\mu\nu}(P,q) = \sum_{i \in \text{partons}} e_i^2 \int_0^1 \frac{dz}{z} f_i(z)\, \hat{W}^i_{\mu\nu}(zP,q)$$

where

$$\hat{W}^i_{\mu\nu}(\hat{p},q) = \frac{e_i^2}{4\pi}\frac{1}{2}\int \frac{d^3\hat{p}'}{(2\pi)^3 2\hat{p}'_0}(2\pi)^4\delta^{(4)}(\hat{p}+q-\hat{p}')\sum_{\text{spin}}[\bar{u}(\hat{p}')\gamma_\mu u(\hat{p})][\bar{u}(\hat{p})\gamma_\nu u(\hat{p}')]$$

And from there the structure functions

$$\boxed{\begin{aligned} F_1(x_B,Q^2) &= \frac{1}{2}\sum_i e_i^2\, f_i(x_B) \\ F_2(x_B,Q^2) &= \sum_i e_i^2\, x_B\, f_i(x_B) \end{aligned}}$$

The Parton Model

Basic assumption:

- The structure functions in the parton model do not depend on Q^2
- Microscopic x ≡ Macroscopic x_B
- Callan-Gross relation typical of spin 1/2 $F_2 = 2xF_1$
- $F_L = F_2 - 2xF_1 \approx 0$ longitudinally polarised virtual photon much smaller than transversally polarised virtual photon

Parton Distribution Functions

on of proton's momentum carried by parton i

Sc e_i (in units of proton charge) and

mo $(\hat{p}')$

γ exchange

On tly as we did with leptonic tensor)

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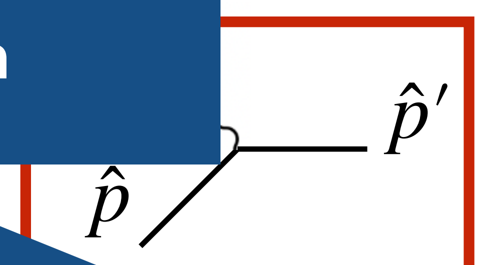


where

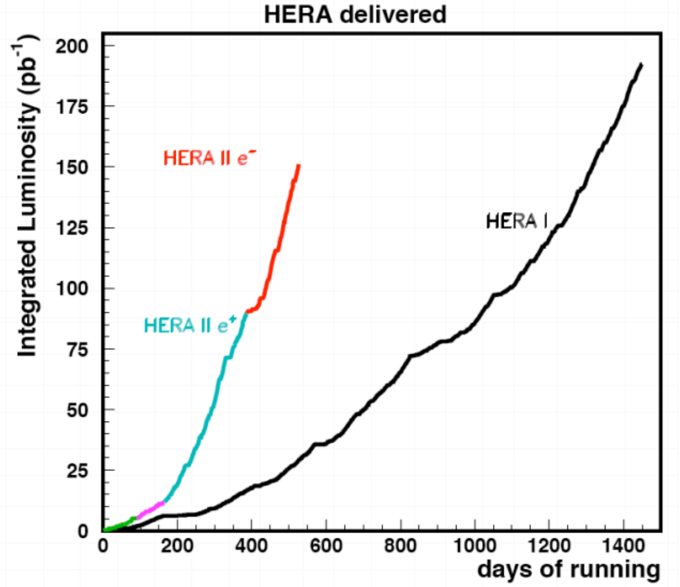
$$\hat{W}^i_{\mu\nu}(\hat{p}, q) = \frac{e_i^2}{4\pi}\frac{1}{2}\int \frac{d^3\hat{p}'}{(2\pi)^3 2\hat{p}'_0}(2\pi)^4\delta^{(4)}(\hat{p} + q \quad \hat{p} \quad \sum_{\text{spin}} [\bar{u}(\hat{p}')\gamma_\mu u(\hat{p})][\bar{u}(\hat{p})\gamma_\nu u(\hat{p}')]$$

And from there the structure functions

$$F_1(x_B, Q^2) = \frac{1}{2}\sum_i e_i^2 f_i(x_B)$$

$$F_2(x_B, Q^2) = \sum_i e_i^2 x_B f_i(x_B)$$

The HERA collider



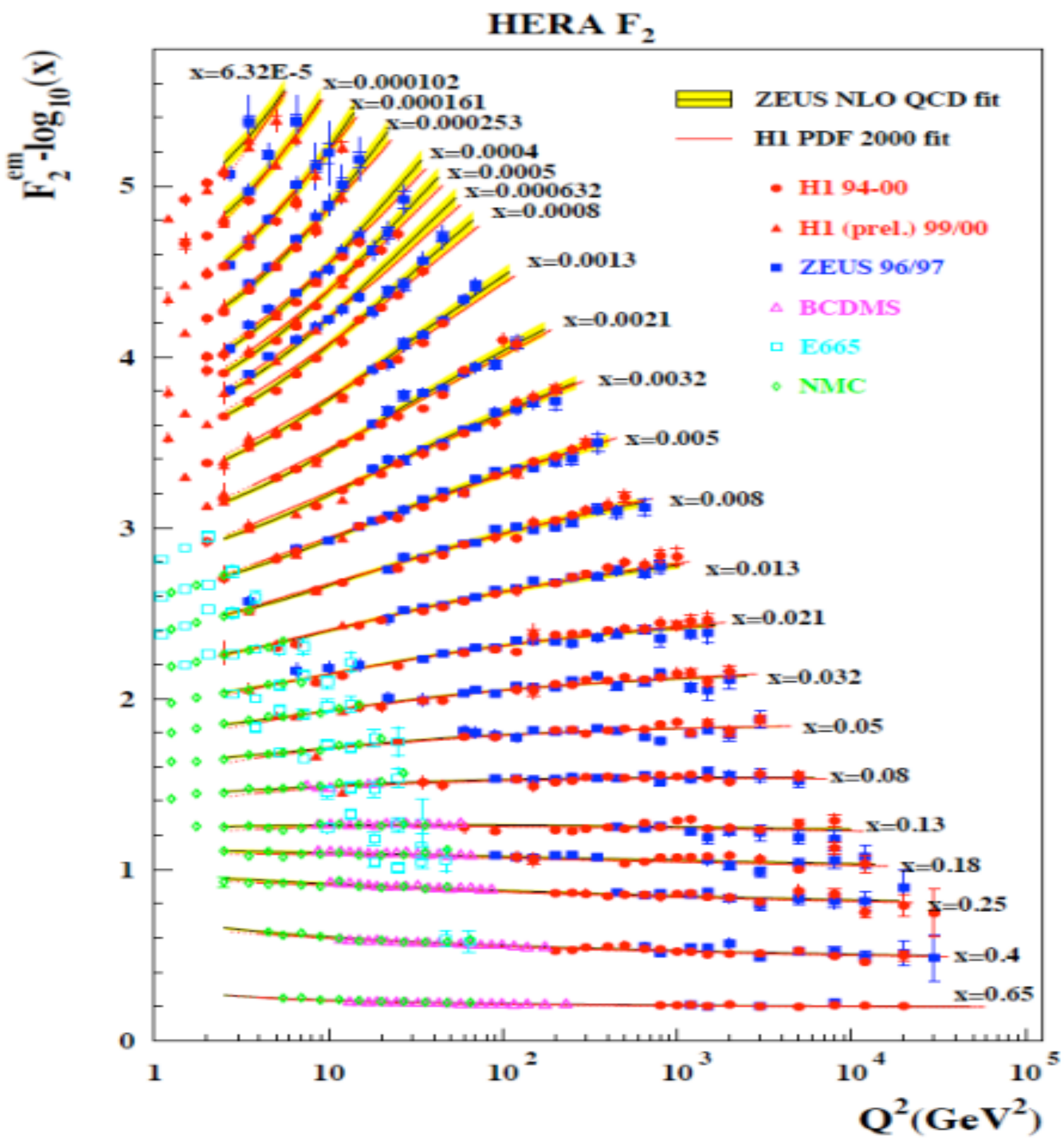
1992-2007

$$\sqrt{S} = 318\,\mathrm{GeV}$$

$$E_e = 27.5\,\mathrm{GeV}$$

$$E_p = 920\,\mathrm{GeV}$$

Scaling violation

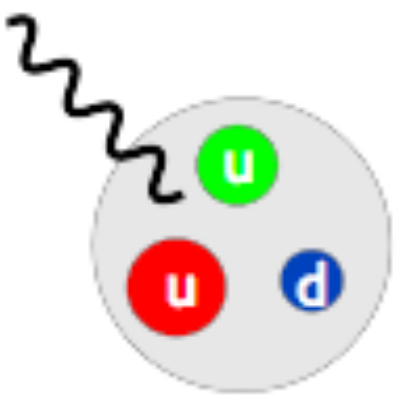


scaling <u>violation</u>

approx. scaling

QCD and improved parton model

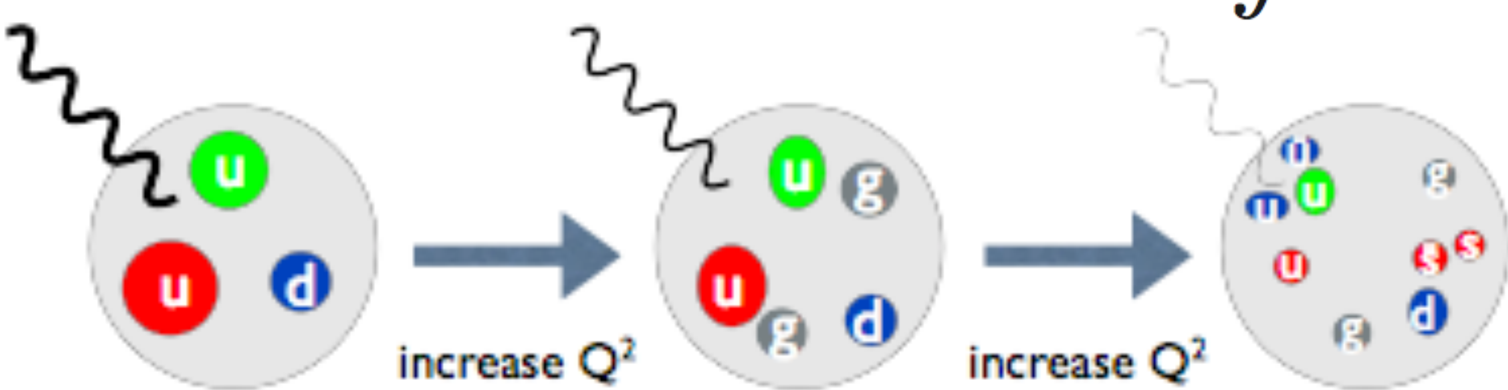
Parton model



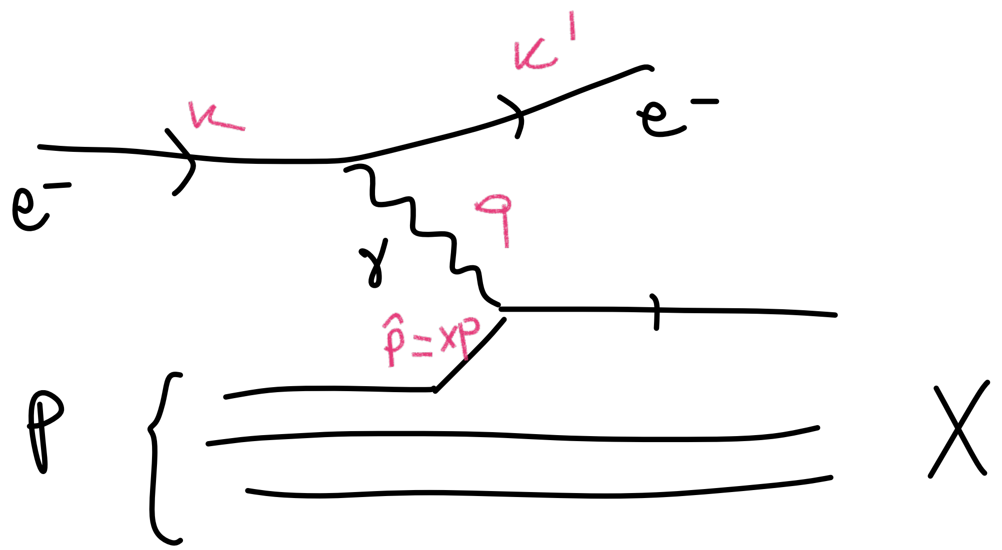
$$\sigma = \int dx\, f_i^{(p)}(x)\, \sigma^{(0)}(xp)$$

QCD-improved Parton model

$$\sigma = \int dx\, f_i^{(p)}(x, \mu_F^2)\, \sigma(xp, \mu_F^2)$$



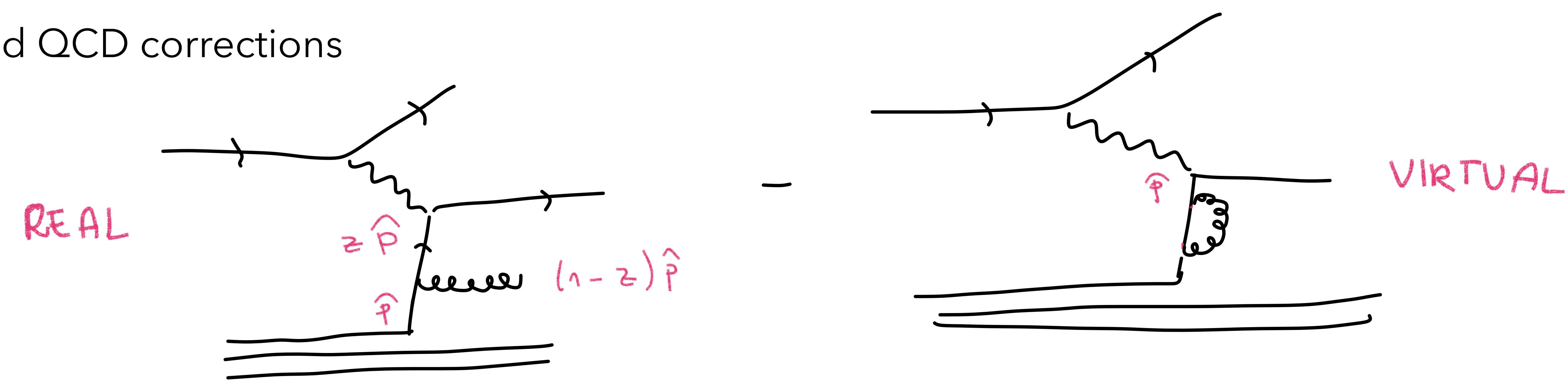
QCD and improved parton model



Parton model

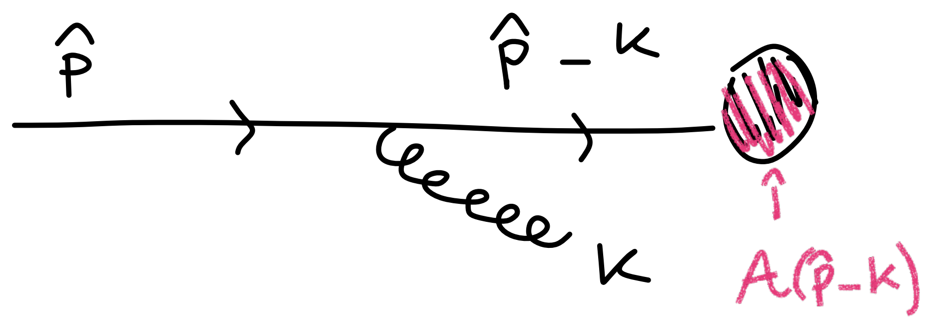
$$\sigma(e^- p \to e^- X) \; = \sum_{i \in \text{partons}} \int_0^1 dx\, f_i(x)\, \hat{\sigma}_i(xP)$$

Add QCD corrections



QCD and improved parton model

Real emission



$$\hat{p} = \hat{p}_0 (1,0,0,1)$$

$$\eta = \frac{1}{4\hat{p}_0}(1,0,0,-1)$$

$$k_T = (0, k_{T,1}, k_{T,2}, 0)$$

Sudakov parametrization

$$k = (1-z)\hat{p} + k_T + \xi\eta$$

with
$$\hat{p} \cdot k_T = 0$$
$$\eta \cdot k_T = 0$$
$$2\hat{p} \cdot \eta = 1$$
$$\eta^2 = 0$$

From gluon on-shell condition $\quad \xi = -\dfrac{k_T^2}{1-z}$

From $\quad (p-k)^2 < 0 \quad \Rightarrow \quad z < 1$

From $\quad k_3 = 0 \quad \Rightarrow \quad |k_T^2| < 4\hat{p}_0^2(1-z)$

QCD and improved parton model

Parametrise phase space and integrate over azimuthal angle

$$\frac{d^3k}{(2\pi)^3 2k_0} = \frac{1}{16\pi^2}\frac{d|k_T|^2\,dz}{1-z}$$

Consider singular part of the amplitude

$$\mathcal{M}^{(1)}_{q,\mathrm{sing}}(\hat{p},k) = g_s A_i(\hat{p}-k)\frac{\hat{\not p}-\not k}{(\hat{p}-k)^2}\not\varepsilon(k)\, t^a_{ij}\, u_j(\hat{p})$$

$$= g_s \frac{(1-z)}{k_T^2} A_i(\hat{p}-k)\,(z\hat{\not p}-\not k_T)\not\varepsilon(k)\, t^a_{ij}\, u_j(\hat{p})$$

$$= -\frac{g_s}{k_T^2} A_i(\hat{p}-k)\left[2z\, k_T\, \varepsilon(k) + (1-z)\not k_T\not\varepsilon(k)\right] t^a_{ij}\, u_j(\hat{p})$$

$$\Rightarrow \quad |\mathcal{M}^{(1)}_{\mathrm{q,sing}}|^2 = -\frac{2g_s^2}{|k_T|^2} C_F \frac{1+z^2}{z} |\mathcal{M}^{(0)}(z\hat{p})|^2$$

QCD and improved parton model

$$\hat{\sigma}^{(1)}_{q,R}(\hat{p}) = \frac{\alpha_S}{2\pi} C_F \int_0^1 \frac{dz}{1-z} \int_0^{|k_T^2|_{\max}} d|k_T^2| \, \frac{1+z^2}{|k_T^2|} \, \hat{\sigma}^{(0)}_q(z\hat{p})$$

The partonic cross section at NLO in α_S (associated with real emission of a gluon off a quark displays two singularities:

- **SOFT** singularity (z → 1), which we regulate with parameter **ε** (→ 0)
- **COLLINEAR** singularity ($|k^2_T|$ → 0), which we regulate with parameter λ^2 (→ 0)

$$\hat{\sigma}^{(1)}_{q,R}(\hat{p}) = \frac{\alpha_S}{2\pi} C_F \int_0^{1-\epsilon} \frac{dz}{1-z} \int_{\lambda^2}^{|k_T^2|_{\max}} d|k_T^2| \, \frac{1+z^2}{|k_T^2|} \, \hat{\sigma}^{(0)}_q(z\hat{p})$$

QCD and improved parton model

Add virtual corrections, which also have soft and collinear singularities

$$\hat{\sigma}_V^{(1)}(\hat{p}) = -\hat{\sigma}_q^{(0)}(\hat{p})\frac{\alpha_S}{2\pi}C_F \int_0^{1-\epsilon} \frac{dz}{1-z} \int_{\lambda^2}^{|k_T^2|_{\max}} d|k_T^2| \frac{1+z^2}{|k_T^2|}$$

$$\hat{\sigma}_{q,R+V}^{(1)}(\hat{p}) = \frac{\alpha_S}{2\pi}C_F \int_0^1 \frac{dz}{1-z} \int_{\lambda^2}^{|k_T^2|_{\max}} d|k_T^2| \frac{1+z^2}{|k_T^2|} \left[\hat{\sigma}_q^{(0)}(z\hat{p}) - \hat{\sigma}_q^{(0)}(\hat{p})\right]$$

Pqq

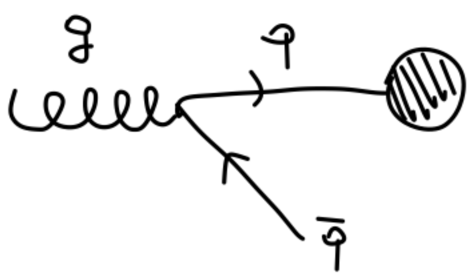
The soft singularity cancels between real and virtual contributions but the collinear singularity still there. Trick: split integration

$$\int_{\lambda^2}^{|k_T^2|_{\max}} \frac{d|k_T^2|}{|k_T^2|} \quad \rightarrow \quad \int_{\lambda^2}^{\mu_F^2} \frac{d|k_T^2|}{|k_T^2|} + \int_{\mu_F^2}^{|k_T^2|_{\max}} \frac{d|k_T^2|}{|k_T^2|}$$

Singular **Non-singular**

QCD and improved parton model

$$\hat{\sigma}_q(\hat{p}) = \hat{\sigma}_q^{(0)}(\hat{p}) + \hat{\sigma}_q^{(1)}(\hat{p}) + \dots$$
$$= \hat{\sigma}_q^{(0)}(\hat{p}) + \frac{\alpha_S(Q^2)}{2\pi} \int_0^1 dz P_{qq}(z)\, \hat{\sigma}_q^{(0)}(z\hat{p}) \log \frac{\mu_F^2}{\lambda^2} + \mathrm{non-singular\,terms}$$



Add LO and NLO correction from initial quark + NLO contribution from gluon-initiated process with gluon splitting into quark-antiquark pair

$$\hat{\sigma}_g(\hat{p}) = \frac{\alpha_S(Q^2)}{2\pi} \int_0^1 dz P_{qg}(z)\, \hat{\sigma}_q^{(0)}(z\hat{p}) \log \frac{\mu_F^2}{\lambda^2} + \mathrm{non-singular\,terms}$$

$P_{qq}(z)$ and $P_{qg}(z)$ are universal functions associated to quark and gluon splittings, called splitting functions

QCD and improved parton model

Plug into the parton model equation:

$$\sigma(P) = \int_0^1 dx\, f_q(x)\, \hat{\sigma}_q \underbrace{(xP)}_{\hat{p}} + f_g(x)\, \hat{\sigma}_g (xP)$$

$$\Downarrow$$

$$\sigma(P) = \int_0^1 dx \Bigg[f_q(x) \left(\hat{\sigma}_q^{(0)}(xP) + \frac{\alpha_s}{2\pi} \int_0^1 dz\, P_{qq}(z)\, \hat{\sigma}_q^{(0)}(xzP) \log\frac{\mu_F^2}{\lambda^2} \right)$$
$$+ f_g(x) \left(\frac{\alpha_s}{2\pi} \int_0^1 dz\, P_{qg}(z)\, \hat{\sigma}_q^{(0)}(xzP) \log\frac{\mu_F^2}{\lambda^2} \right) \Bigg]$$
$$+ \int_0^1 dx \left[f_q(x)\, \hat{\sigma}^{(1)}_{q,\mathrm{reg}}(xP, \mu_F^2) + f_g(x)\, \hat{\sigma}^{(1)}_{g,\mathrm{reg}}(xP, \mu_F^2) \right]$$

QCD and improved parton model

Plug into the parton model equation:

$$\sigma(P) = \int_0^1 dx\, f_q(x)\, \hat{\sigma}_q \underbrace{(xP)}_{\hat{P}} + f_g(x)\, \hat{\sigma}_g (xP)$$

$$\Downarrow$$

$$\sigma(P) = \int_0^1 dx \left[f_q(x) \left(\hat{\sigma}_q^{(0)}(xP) + \frac{\alpha_s}{2\pi} \int_0^1 dz\, P_{qq}(z)\, \hat{\sigma}_q^{(0)}(xzP) \log \frac{\mu_F^2}{\lambda^2} \right) + f_g(x) \left(\frac{\alpha_s}{2\pi} \int_0^1 dz\, P_{qg}(z)\, \hat{\sigma}_q^{(0)}(xzP) \log \frac{\mu_F^2}{\lambda^2} \right) \right]$$

$$+ \int_0^1 dx \left[f_q(x)\, \hat{\sigma}^{(1)}_{q,\mathrm{reg}}(xP, \mu_F^2) + f_g(x)\, \hat{\sigma}^{(1)}_{g,\mathrm{reg}}(xP, \mu_F^2) \right]$$

Absorb collinear divergences into a redefinition of the parton distribution functions, which now depend on the factorisation scale

$$f_q(x,\mu^2) = \int_x^1 \frac{dz}{z} \left[f_q(z) \left(\delta(1-x/z) + \frac{\alpha_S(\mu^2)}{2\pi} P_{qq}\left(\frac{x}{z}\right) \log \frac{\mu^2}{\lambda^2} \right) + f_g(z) \left(\frac{\alpha_S(\mu^2)}{2\pi} P_{qg}\left(\frac{x}{z}\right) \log \frac{\mu^2}{\lambda^2} \right) \right]$$

QCD and improved parton model

With this redefinition of the PDFs, both PDFs and partonic cross section are finite and they both acquired dependence on arbitrary factorisation scale

$$\sigma(P) = \int_0^1 dx\, f_q(x, \mu_F^2)\, \hat{\sigma}_{\mathrm{q,reg}}(xP, \mu_F^2) \; + \; f_g(x, \mu_F^2)\, \hat{\sigma}_{\mathrm{g,reg}}(xP, \mu_F^2)$$

From PDF redefinition (similar to renormalisation) note that the dependence of PDFs on the scale is totally fixed by perturbation theory.

DGLAP evolution equation, similar to renormalisation group equations for α_S

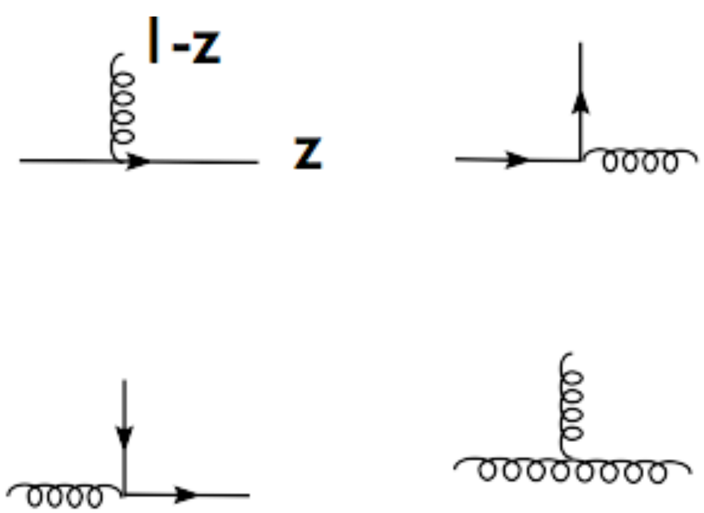
$$\mu^2 \frac{\partial f_q(x, \mu^2)}{\partial \mu^2} = \frac{\alpha_S(\mu^2)}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}\left(\frac{x}{z}\right) f_q(z, \mu^2) + P_{qg}\left(\frac{x}{z}\right) f_q(z, \mu^2)$$

DGLAP evolution equations

When you put all flavours in, get 13 coupled integro-differential equations, which can be reduced to 11 decoupled (non-singlet) and 2 coupled equations (singlet and gluon) with a change of basis in the space of PDFs

Dokshitzer, Gribov, Lipatov, Altarelli, Parisi equations

$$\mu_F \frac{\partial}{\partial \mu_F} \begin{pmatrix} \mathbf{f_q}(x,\mu_F) \\ \mathbf{f_{\bar{q}}}(x,\mu_F) \\ f_g(x,\mu_F) \end{pmatrix} = \int_x^1 \frac{dz}{z} \begin{pmatrix} P_{qq}(z,\alpha_s(\mu_F)) & P_{q\bar{q}}(z,\alpha_s(\mu_F)) & P_{qg}(z,\alpha_s(\mu_F)) \\ P_{\bar{q}q}(z,\alpha_s(\mu_F)) & P_{\bar{q}\bar{q}}(z,\alpha_s(\mu_F)) & P_{\bar{q}g}(z,\alpha_s(\mu_F)) \\ P_{gq}(z,\alpha_s(\mu_F)) & P_{g\bar{q}}(z,\alpha_s(\mu_F)) & P_{gg}(z,\alpha_s(\mu_F)) \end{pmatrix} \begin{pmatrix} \mathbf{f_q}(x/z,\mu_F) \\ \mathbf{f_{\bar{q}}}(x/z,\mu_F) \\ f_g(x/z,\mu_F) \end{pmatrix}$$



- Splitting functions known up to NNLO:

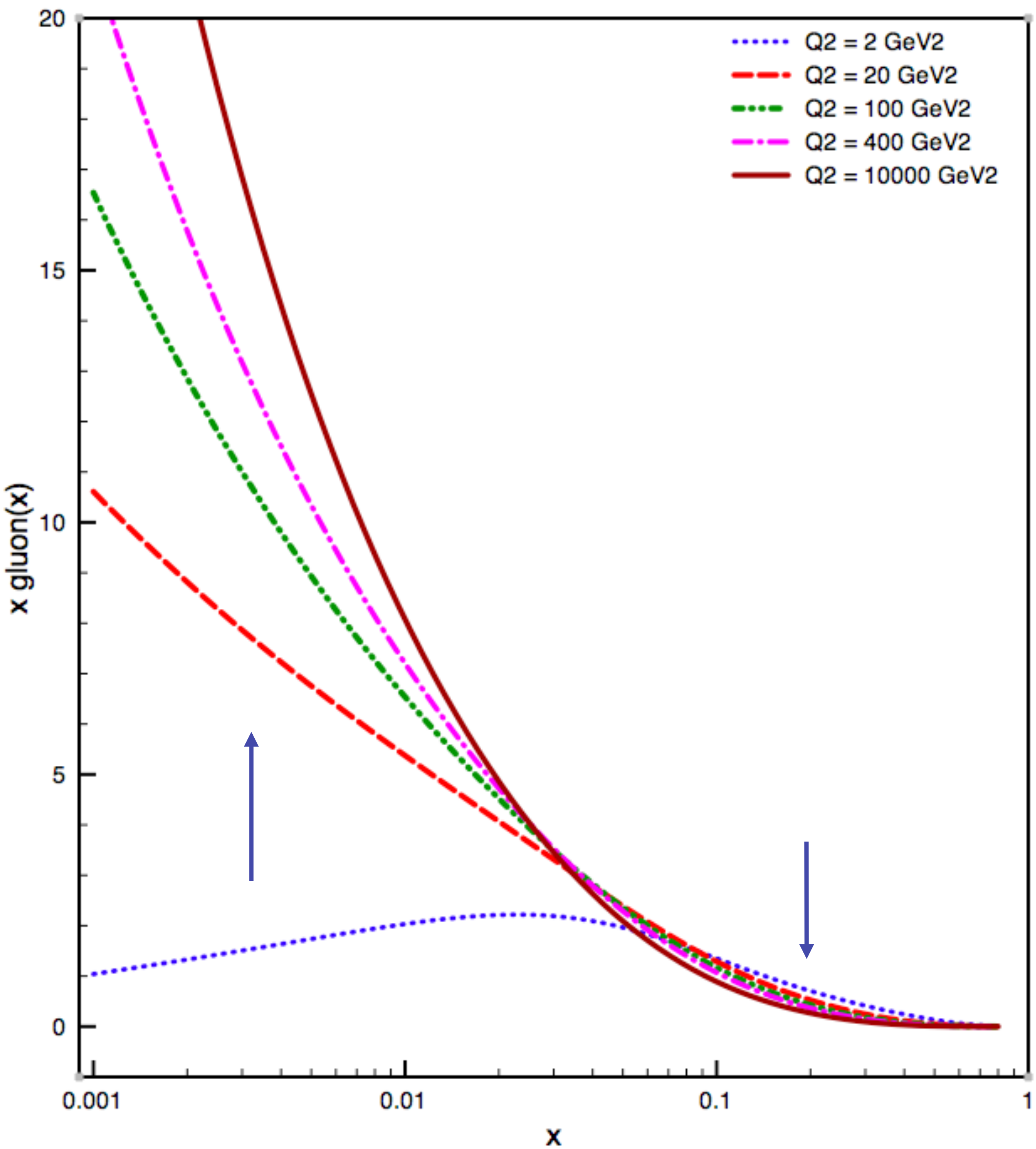
LO Dokshitzer; Gribov, Lipatov; Altarelli, Parisi (1977)

NLO Floratos,Ross,Sachrajda; Floratos, Lacaze, Kounnas, Gonzalez-Arroyo,Lopez,Yndurain; Curci,Furmanski Petronzio, (1981)

NNLO - Moch, Vermaseren, Vogt (2004)

aNNNLO - Blumlein, Moch, Gehrmann, von Manteufel, Sotnikov, Yang, Davies, Vogt, Bonvini, Marzani,… (2022 - ongoing)

DGLAP evolution equations



Gluon evolution

$$g(x,\mu^2) = \Gamma_{gq} \otimes \Sigma(x,\mu_0^2) + \Gamma_{gg} \otimes g(x,\mu_0^2)$$

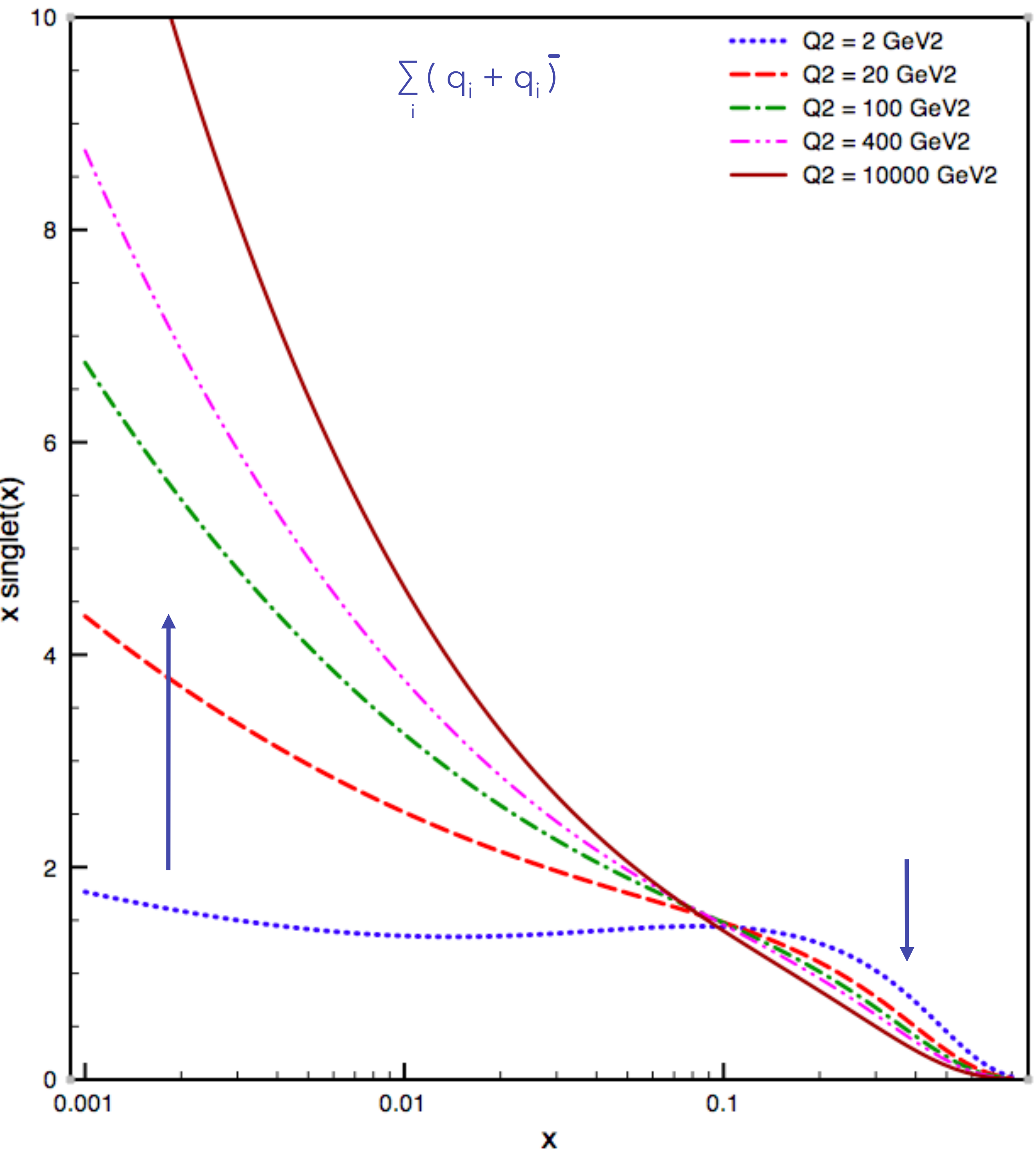
$$P_{gq}^{(0)}(x) = C_F\left[\frac{1+(1-x)^2}{x}\right]$$

$$P_{gg}^{(0)}(x) = 2N\left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x)\right] + \delta(1-x)\frac{(11N - 4n_f T_R)}{6}$$

- Both P_{gg} and P_{gq} diverge for x → 0

- Gluon is depleted at large x

DGLAP evolution equations



Singlet evolution

$$\Sigma(x,\mu^2) = \Gamma_{qq} \otimes \Sigma(x,\mu_0^2) + \Gamma_{qg} \otimes g(x,\mu_0^2)$$

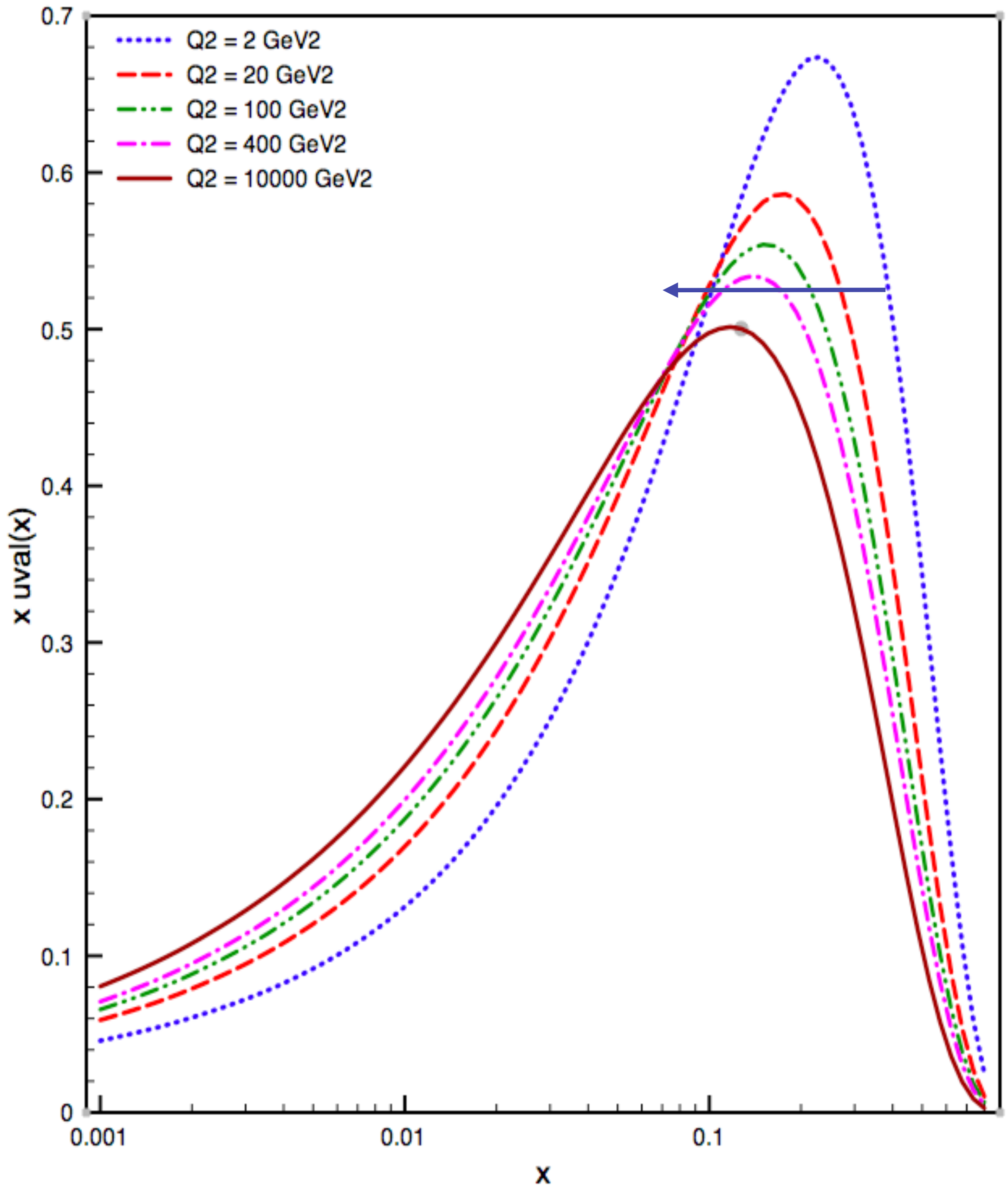
$$P_{qq}^{(0)}(x) = C_F \left[\frac{(1+x^2)}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right]$$

$$P_{qg}^{(0)}(x) = T_R \left[x^2 + (1-x)^2 \right]$$

- High-x gluon feeds growth of small-x gluon and quark

- Gluons can be seen because they help drive the quark evolution

DGLAP evolution equations



Non-singlet valence evolution

$$u_v(x,\mu^2) = \Gamma^v_{NS} \otimes u_v(x,\mu_0^2)$$

$$P^{(0),v}_{NS} = P^{(0)}_{qq}(x) = C_F\left[\frac{(1+x^2)}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right]$$

- As Q^2 increases partons lose longitudinal momentum; distributions all shift to lower x

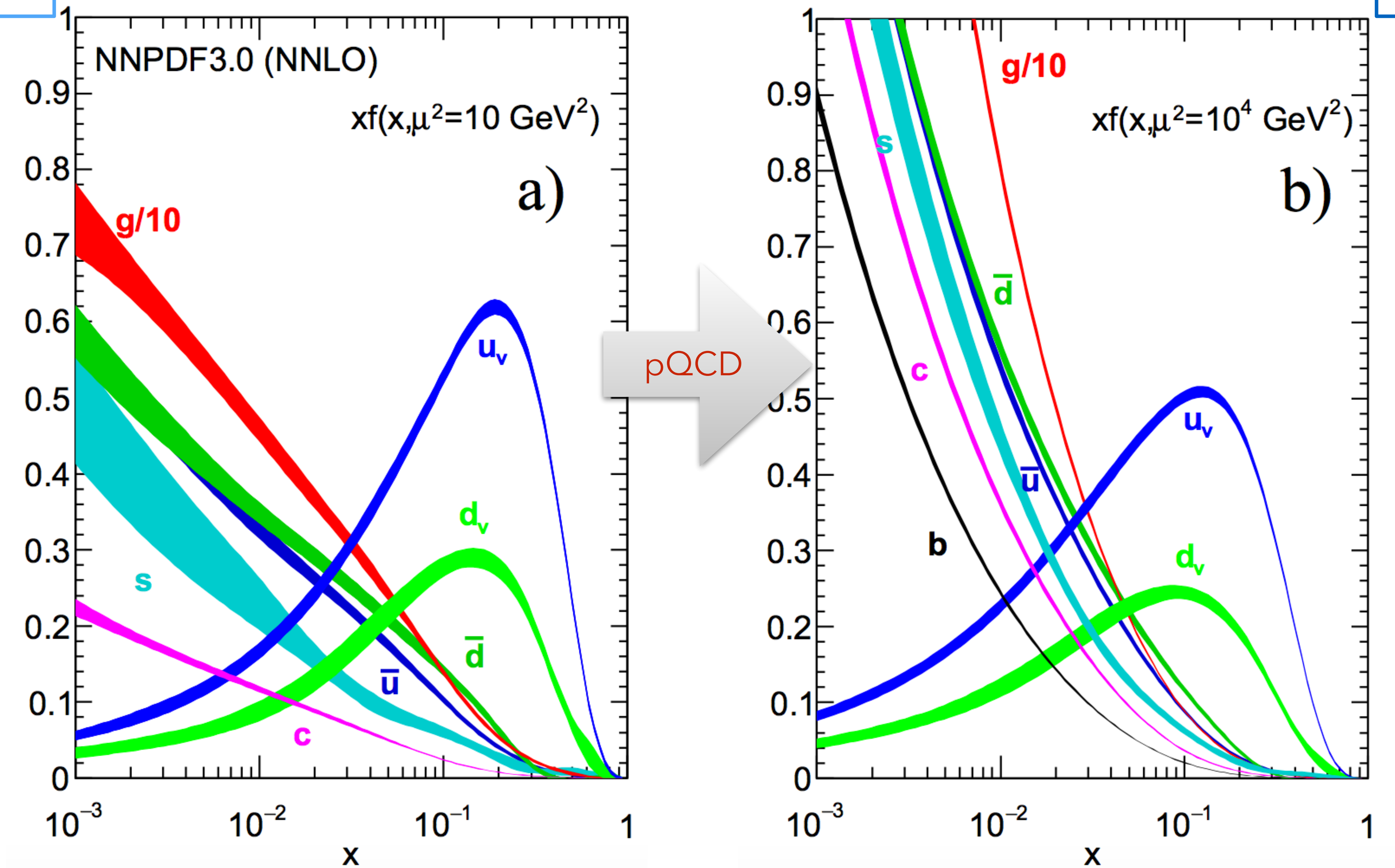
- Gluons can be seen because they help drive the quark evolution

DGLAP evolution equations

Functional dependence of PDFs on the scale is totally predicted up to NNLO accuracy by solving DGLAP evolution equations

Hadronic scale: global fit of PDFs

High scale: input to the LHC



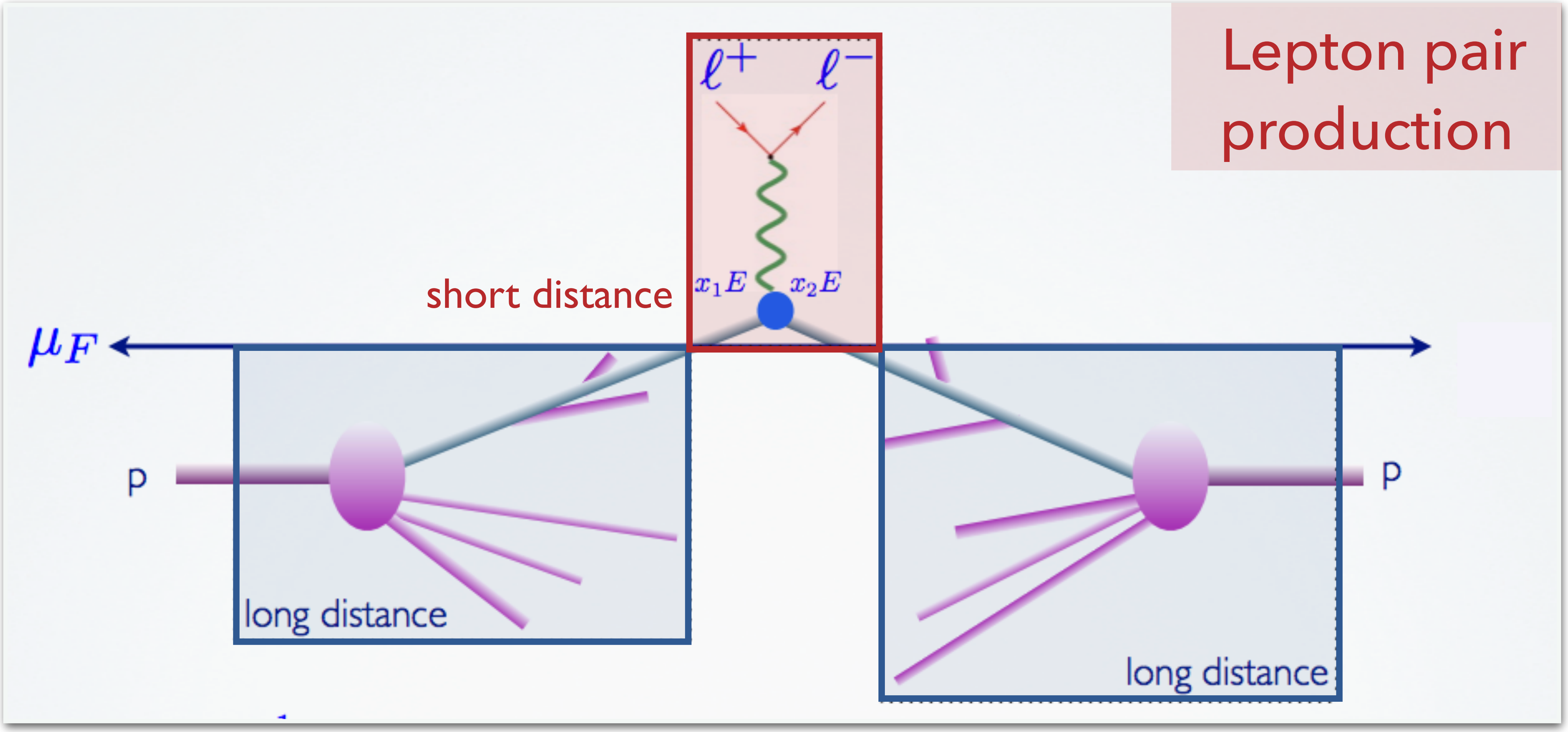
PDG 2016

Collinear factorisation

Collinear Factorisation Theorem

$$\frac{d\sigma_H^{ep\to ab}}{dX} = \sum_{i=-n_f}^{+n_f} \int_{x_B}^{1} \frac{dz}{z} f_i(z,\mu_F) \frac{d\hat{\sigma}_i^{ei}}{dX}(zS, \alpha_s(\mu_R), \mu_F) + \mathcal{O}\left(\frac{\Lambda^n}{S^n}\right)$$

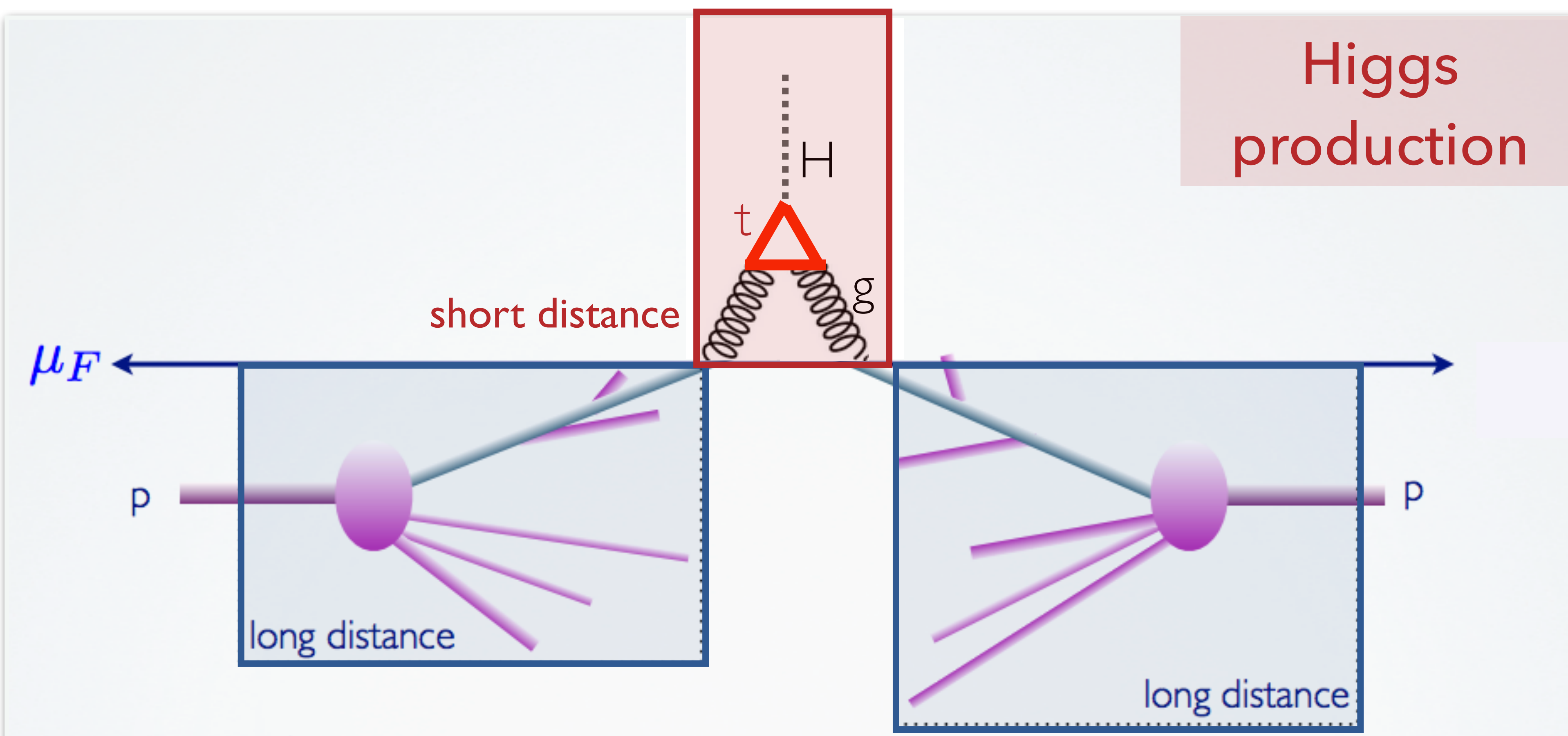
$$\frac{d\sigma_H^{pp\to ab}}{dX} = \sum_{i,j=-n_f}^{+n_f} \int_{\tau_0}^{1} \frac{dz_1}{z_1} \frac{dz_2}{z_2} f_i(z_1,\mu_F) f_j(z_2,\mu_F) \frac{d\hat{\sigma}_i^{ij}}{dX}(zS, \alpha_s(\mu_R), \mu_F) + \mathcal{O}\left(\frac{\Lambda^n}{S^n}\right)$$



Collinear Factorisation Theorem

$$\frac{d\sigma_H^{ep\to ab}}{dX} = \sum_{i=-n_f}^{+n_f} \int_{x_B}^{1} \frac{dz}{z} f_i(z,\mu_F) \frac{d\hat{\sigma}_i^{ei}}{dX}(zS, \alpha_s(\mu_R), \mu_F) + \mathcal{O}\left(\frac{\Lambda^n}{S^n}\right)$$

$$\frac{d\sigma_H^{pp\to ab}}{dX} = \sum_{i,j=-n_f}^{+n_f} \int_{\tau_0}^{1} \frac{dz_1}{z_1} \frac{dz_2}{z_2} f_i(z_1,\mu_F) f_j(z_2,\mu_F) \frac{d\hat{\sigma}_i^{ij}}{dX}(zS, \alpha_s(\mu_R), \mu_F) + \mathcal{O}\left(\frac{\Lambda^n}{S^n}\right)$$



Wrap-up

- ➡ The structure of the proton has been a crucial ingredient to test and verify perturbative QCD and it is now key to the precision challenge that we are facing at the LHC

- ➡ Today's lecture

  - ✓ Parametrisation of the proton in terms of structure functions
  - ✓ Parton model picture
  - ✓ QCD - Improved parton model
  - ✓ DGLAP evolution equations
  - ✓ Collinear Factorisation Theorem

Extra material

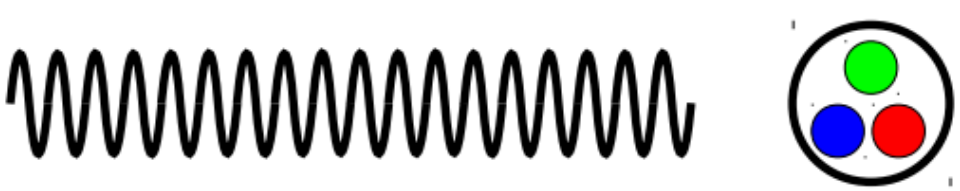
Deep Inelastic Scattering

Slide from F Olness lectures
CTEQ school 2017

$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times 1$$

$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times F\left(\frac{Q^2}{\Lambda^2}\right)$$

Λ of order of the proton mass scale



$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times \sum_i e_i^2$$

Exercise I: Z contribution

- Show that, in the Parton model, considering also the contribution of a virtual Z boson and its interference with the photon one obtains:

$$F_2^{\gamma,Z}(x) = x\sum_{i=1}^{n_f} c_i[q_i(x) + \bar{q}_i(x)]$$

$$F_3^{\gamma,Z}(x) = \sum_{i=1}^{n_f} d_i[q_i(x) - \bar{q}_i(x)]$$

Where

$$c_i = e_i^2 - 2e_i V_{eZ} V_{iZ} P_Z + (V_{eZ}^2 + A_{eZ}^2)(V_{iZ}^2 + A_{iZ}^2) P_Z^2$$

$$d_i = -2e_i A_{eZ} A_{iZ} P_Z + 4 V_{eZ} A_{eZ} V_{iZ} A_{iZ} P_Z^2$$

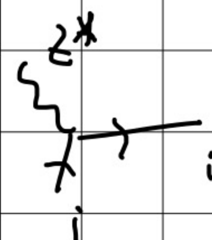
$$P_Z = \frac{Q^2}{(Q^2 + M_Z^2)(4s_w^2 c_w^2)} \longrightarrow \begin{aligned} c_w &= \cos\theta_w \\ s_w &= \sin\theta_w \end{aligned}$$

Solution I

I In the lectures we considered

$$\frac{1}{2}\sum_{pol}\Big|$$  $$\Big|^2 = \frac{e_q^2}{2}\,\mathrm{Tr}\left[\hat{\not p}\,\gamma^\nu\,\hat{\not p}'\,\gamma^\mu\right]$$

Add to hadronic tensor

$$\frac{1}{2}\sum_{pol}\Big|$$  $$\Big|^2 = \frac{e^2 k_Z^2}{2}\,\mathrm{Tr}\left[\hat{\not p}\,\gamma^\nu\left(v_{iZ}+\gamma^5 a_{iZ}\right)\hat{\not p}'\gamma^\mu\left(v_{iZ}+\gamma_5 a_{iZ}\right)\right]$$

$$= \frac{e^2}{8 s_W^2 c_W^2}\left[\left(v_{iZ}^2+a_{iZ}^2\right)\underbrace{\mathrm{Tr}\left[\hat{\not p}\,\gamma^\nu\,\hat{\not p}'\,\gamma^\mu\right]}_{4\left[\hat p^\nu \hat p'^\mu+\hat p^\mu \hat p'^\nu - g^{\mu\nu}\,\hat p\cdot\hat p'\right]} + 2 v_{iZ}\, a_{iZ}\,\underbrace{\mathrm{Tr}\left[\hat{\not p}\,\gamma^\nu\,\hat{\not p}'\gamma^\mu\gamma^5\right]}_{4i\,\epsilon^{\alpha\beta\nu\mu}\,\hat p_\alpha \hat p'_\beta}\right]$$

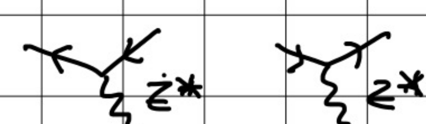
$\neq 0$ when contracted with anti-symm tensor.

Similar contribution in leptonic tensor

$v_{iZ} \to v_{eZ}$

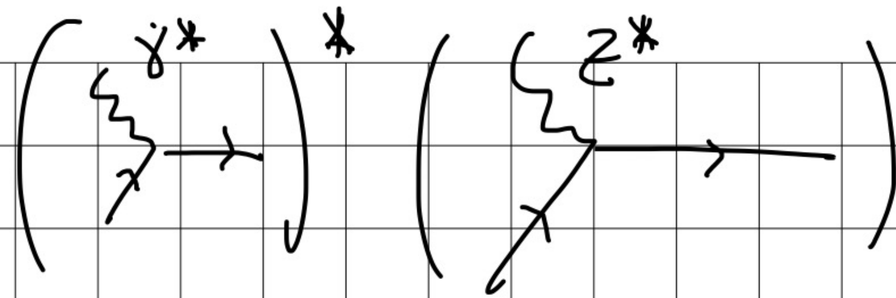
$a_{iZ} \to a_{eZ}$

and propagator $\overbrace{\frac{Q^4}{(Q^2+M_Z^2)^2}}^{P_Z^2}\cdot\frac{1}{Q^4}$



have different sign in front of anti symm. part

Solution I

$\frac{1}{2}\sum_i$ 

$$= -\frac{1}{2}\, e^2 k_\gamma k_Z \,\mathrm{T_R}\left[\hat{\not p}\,\gamma^\nu (v_{iZ} + a_{iZ}\gamma^5)\,\hat{\not p}'\,\gamma^\mu\right]$$

$$= -\frac{e^2 e_J}{4 s_W c_W}\left[v_{iZ}\,\mathrm{T_R}\left[\hat{\not p}\,\gamma^\nu\,\hat{\not p}'\,\gamma^\mu\right] + a_{iZ}\,\mathrm{T_R}\left[\hat{\not p}\,\gamma^\nu\,\hat{\not p}'\,\gamma^\mu\gamma^5\right]\right]$$

Similar contribution in leptonic tensor

$v_{iZ} \to v_{eZ}$

$a_{iZ} \to a_{eZ}$

and propagator $\underbrace{\frac{\cancel{Q^4}^{\,2}}{(Q^2+M_Z^2)\,\cancel{Q^2}}}_{P_Z}$ $\qquad \frac{1}{Q^4}$

Exercise II: Paschos-Wolfenstein relation

- Show that, in the Parton model, considering a (anti)neutrino-initiated DIS process on a deuteron target – assuming SU(2) isospin symmetry $u_n(x)=d_p(x)$ and $d_n(x) = u_p(x)$ – the ratio R

$$R = \frac{\sigma_{\mathrm{NC}}(\nu) - \sigma_{\mathrm{NC}}(\bar{\nu})}{\sigma_{\mathrm{CC}}(\nu) - \sigma_{\mathrm{CC}}(\bar{\nu})}$$

NC (mediated by Z) and CC (mediated by $W^{+/-}$), assuming strange and anti-strange to be equal in the target, is independent of Parton Distribution Functions and can be used to determined the Weinberg angle θ_W

$$R = \frac{1}{2}\left(\frac{1}{2} - \sin\theta_w^2\right)$$

You may use (without deriving it) the result (and set c, cbar = 0)

$$\begin{aligned}
F_2^{W^-} &= 2x\left(u + \bar{d} + \bar{s} + c\right), \\
F_3^{W^-} &= 2x\left(u - \bar{d} - \bar{s} + c\right), \\
F_2^{W^+} &= 2x\left(d + \bar{u} + \bar{c} + s\right), \\
F_3^{W^+} &= 2x\left(d - \bar{u} - \bar{c} + s\right),
\end{aligned}$$

Exercise II: Paschos-Wolfenstein relation

July 1972

TESTS FOR NEUTRAL CURRENTS IN NEUTRINO REACTIONS

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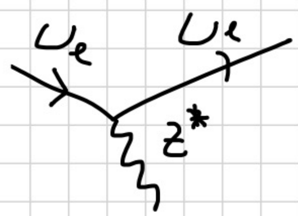
ABSTRACT

Neutral currents predicted by weak interaction models of the type discussed by Weinberg may be detected in neutrino reactions. Limits on the ratio R of $\sigma(\nu + N \to \nu + x)$ to $\sigma(\nu + N \to \mu^- + x)$ are obtained independent of any dynamical assumption. For the total cross-section for high energy neutrinos, we find $R \geq 0.18$, provided the Weinberg mixing angle satisfies $\sin^2 \theta_W \leq 0.33$. For the production of a single π^0 we find $R' \geq 0.50$ contrasted with the experimental result $R' \leq 0.14$ using only the assumption of (3, 3) resonance dominance. Applications are also given to anti-neutrino reactions.

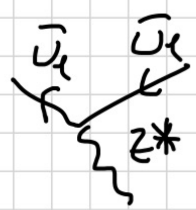
Solution II

II



$$= \bar{u}'(k)\,\frac{-ig}{4c_W}\,\gamma^\mu(1-\gamma_5)\,u(k)$$

$$L_{\mu\nu}(\nu) = \frac{1}{8c_W^2}\left(\mathrm{Tr}\left[\not k\,\gamma^\nu \not k'\,\gamma^\mu\right] - \mathrm{Tr}\left[\not k\,\gamma^\nu \not k'\,\gamma^\mu\gamma^5\right]\right)$$



$$= \bar{v}(k)\,\frac{-ig}{4c_W}\,\gamma^\mu(1-\gamma_5)\,v(k')$$

$$L_{\mu\nu}(\bar\nu) = \frac{1}{8c_W^2}\left(\mathrm{Tr}\left[\not k'\,\gamma^\nu \not k\,\gamma^\mu\right] - \mathrm{Tr}\left[\not k'\,\gamma^\nu \not k\,\gamma^\mu\gamma_5\right]\right)$$

Because

only difference in $\nu, \bar\nu$ incoming lepton

$$d\sigma = \frac{1}{2s}\,\frac{g_{ev}^2\, g_{\nu v}^2}{\left(Q^2+M_V^2\right)^2}\, L_{\mu\nu} W^{\mu\nu}(4\pi)\,\frac{d^3k'}{(2\pi)^3 2E'}$$

$\hookrightarrow M_V^4$ for $Q^2 \ll M_V^2$

$$d\sigma(\nu) - d\sigma(\bar\nu) \propto \Big[\underbrace{L(\nu) - L_{\mu\nu}(\bar\nu)}_{\text{only asymm. i.e. } \frac{i}{c_W^2} k_\alpha k'_\beta \varepsilon^{\alpha\beta\mu\nu}}\Big] \supset \frac{g_{\nu e}^2\, g_{\nu v}^2}{M_V^4}\, W^{\mu\nu}(4\pi)\,\frac{d^3k'}{(2\pi)^3 2E'}$$

Only non-zero contribution when contracted with asymm $W_{\mu\nu}$

Solution II

$$[L_{\mu\nu}(\nu) - L_{\mu\nu}(\bar{\nu})]\, W^{\mu\nu} = -\frac{1}{2(\tilde{p}\cdot q)c_W^2}\, k_\alpha k'_\beta\, \varepsilon^{\alpha\beta\mu\nu}\, \varepsilon_{\rho\sigma\mu\nu}\, \tilde{p}^\rho q^\sigma F_3(x)$$

$$= \frac{ME}{c_W^2}\, \cancel{\;}\, (2-y)\, x F_3^{Z}(x)$$

$$\Rightarrow \frac{d^2\sigma}{dx\,dy} = \frac{G_F^2 c_W^2 M_n}{2\pi^2} E_\nu \; x\, y\, (2-y)\; F_3^{NC}(x)$$

$$\int dy\,dx\,.. = \frac{G_F^2 c_W^2}{3\pi^2}\, E_\nu M_n \int dx\; x\, F_3(x)$$

with $\quad F_3(x) \propto \frac{1}{2}\left(\frac{1}{2} - \frac{4}{3}s_W^2\right)(u - \bar{u} + c - \bar{c}) + \frac{1}{2}\left(\frac{2}{3}s_W^2 - \frac{1}{2}\right)(d - \bar{d} + s - \bar{s})$

using $\quad \begin{array}{l} u^P = d^N \\ d^P = u^N \end{array} \Rightarrow \quad u^P + d^P = u^N + d^N \quad \Rightarrow \begin{array}{l} u = d \\ \bar{u} = \bar{d} \end{array}$

$s = \bar{s}$

$c = \bar{c} \cong 0$

$\rightarrow F_3^{NC}(x) \propto \left(\frac{1}{2} - s_W^2\right)[u(x) - \bar{u}(x)]$

Solution II

Charged current

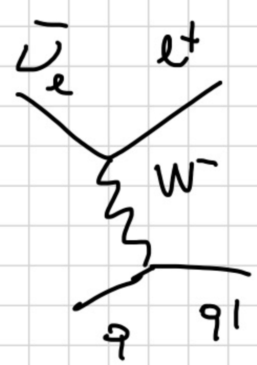
$$\bar{u}_k(p')\left[-\frac{ig}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\, U_{kj}\right]u_j(p)$$

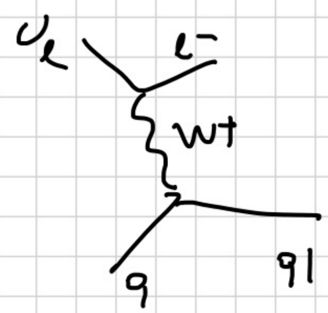
$\downarrow$

CKM Matrix (consider only 2 flavor)

$$U_{kj} = \begin{array}{c c} & \begin{array}{cc} d & s \end{array} \\ \begin{array}{c} u \\ c \end{array} & \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \end{array} = \begin{pmatrix} 1+\theta_c^2 & \theta_c \\ -\theta_c & 1+\theta_c^2 \end{pmatrix}$$

$\downarrow$ expand in θ_c and keep only $O(1)$





$$\Rightarrow F_3^{W^+} = 2x\,(d - \bar{u} + s - \bar{c})$$

$$F_3^{W^-} = 2x\,(u - \bar{d} - \bar{s} + c)$$

Analogously to NC

$$\sigma_{cc}(\nu) - \sigma_{cc}(\bar{\nu}) = \frac{2G_F^2}{3\pi^2} E_\nu M_N \int_0^1 x\, F_3^{cc}(x)\, dx$$

$$F_3^{cc}(x) = \frac{1}{2}\left(u + d + s + c - \bar{u} - \bar{d} - \bar{s} - \bar{c}\right) \longrightarrow (u - \bar{u})$$ same hypothesis

$$= \frac{2G_F^2}{3\pi^2} E_\nu M_N \int dx\, (u(x) - \bar{u}(x))$$

Solution II

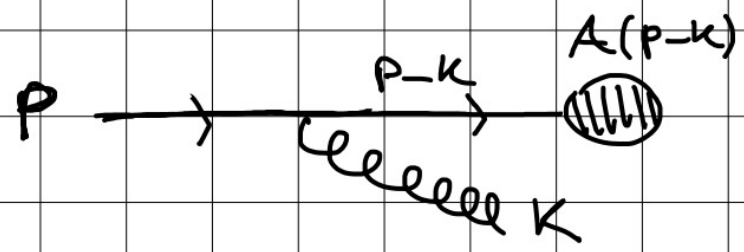
Taking ratio

$$R = \frac{\frac{1}{2}(1-2s^2_W)\int_0^1 dx\,[u(x)-\bar{u}(x)]\,dx}{2\int_0^1 dx\,[u(x)-\bar{u}(x)]}$$

$$= \frac{1}{2}\left(\frac{1}{2}-\sin^2\theta_W\right)$$

QCD and improved parton model

REAL EMISSION



Sudakov parametrization

$$k = (1-z)\,p + k_T + \xi\,\eta$$

where η such that

$$p\cdot k_T = 0$$
$$\eta\cdot k_T = 0$$
$$\eta^2 = 0$$
$$2p\cdot\eta = 1$$

For example

$$p = p^0(1,0,0,1)$$
$$\eta = \frac{1}{4p^0}(1,0,0,-1)$$
$$k_T = (0, k_T^1, k_T^2, 0)$$

From $k^2 = 0$ (on-shell condition)

$$\Rightarrow\ k_T^2 + 2p\cdot\eta\,\xi(1-z) = 0 \quad \Rightarrow\ \xi = -\frac{k_T^2}{1-z}$$

From $(p-k)^2 < 0$

$$\Rightarrow\ z < 1$$

From $k_3 = 0 \quad \Rightarrow \quad |k_T^2| < 4p_0^2(1-z)$

QCD and improved parton model

$$\frac{d^3k}{(2\pi)^3 2k^0} = \frac{1}{16\pi^2} \frac{d|k_T^2|\, dz}{(1-z)} \qquad \text{(integrating over } d\varphi\text{)}$$

Singular part of amplitude:

$$\mathcal{M}^{(1)}_{q,\text{sing}}(p,k) = g_s\, A_i(p-k) \frac{\overbrace{\not p - \not k}^{\to\, z\not p - \not k_T - \xi \not\eta \;\to\; O(k_T^2)}}{\underbrace{(p-k)^2}_{\to\, \frac{1-z}{k_T^2}}} \not\epsilon(k)\, t^A_{ij}\, u_j(p)$$

$t^A_{ij} \to$ SU(3) gen fund. representation

$$= g_s \frac{(1-z)}{k_T^2} A_i(p-k)\,(z\not p - \not k_T)\,\not\epsilon(k)\, t^A_{ij}\, u_j(p)$$

But $\not p\, u(p) = 0$
$k_T \cdot \epsilon(k) = 0$
$$\Rightarrow \quad (1-z)\not p\, \not\epsilon(k)\, u(p) = -\,2\epsilon_\mu(k)\, k_T^\mu\, u(p)$$

$$\Rightarrow \quad \mathcal{M}^{(1)}_{q,\text{sing}}(p,k) = -\frac{g_s}{k_T^2} A_i(p-k)\left[2z\, k_T\cdot\epsilon(k) + (1-z)\not k_T\, \not\epsilon(k)\right] t^A_{ij}\, u_j(p)$$

Detailed version 2/7

QCD and improved parton model

$$|M^{(1)}_{q g^* q}|^2 = -\frac{2 g_s^2 C_F}{k_T^2} \frac{(1+z)^2}{z} |M^{(0)}(zp)|^2$$

where we have used $\sum \epsilon_\mu(k) \epsilon^*_\nu(k) = -g^T_{\mu\nu} = -g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k \cdot n}$

$$t^A_{ij} t^A_{jk} = \delta_{ik} C_F \qquad C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}$$

$$\Rightarrow \hat{\sigma}_q^{(1)}(p) = \frac{1}{16\pi^2} \int_0^1 \frac{dz}{1-z} \int_0^{|k_T^2|_{max}} d|k_T^2| \frac{1}{p \cdot p'} |M_q^{(1)}(p,k)|^2$$

$$\alpha_s = \frac{g_s^2}{4\pi} \qquad = \frac{\alpha_s C_F}{2\pi} \int_0^1 \frac{dz}{1-z} \int_0^{|k_T^2|_{max}} d|k_T^2| \frac{1+z^2}{|k_T^2|} \frac{1}{z(p \cdot p')} |M_q^{(0)}(zp)|^2 + \text{regular terms}$$

$$= \frac{\alpha_s C_F}{2\pi} \int_0^1 \frac{dz}{1-z} \int_0^{|k_T^2|_{max}} d|k_T^2| \frac{1+z^2}{|k_T^2|} \hat{\sigma}_q^{(0)}(zp)$$

divergences! SOFT $z \to 1$ Regulator $\epsilon \to 0$

COLLINEAR $|k_T^2| \to 0$ Regulator $\lambda \to 0$

QCD and improved parton model

$$\hat{\sigma}_q^{(1)}(p) = \frac{\alpha_s}{2\pi} C_F \int_0^{1-\varepsilon} \frac{dz}{1-z} \int_{\lambda^2}^{|k_T^2|_{max}} \frac{d|k_T^2|}{|k_T^2|} (1+z^2)\, \hat{\sigma}_q^{(0)}(zp)$$

Adding virtual corrections $\quad - \hat{\sigma}_q^{(0)}(p) \frac{\alpha_s C_F}{2\pi} \int_0^{1-\varepsilon} \frac{dz}{1-z} \int_{\lambda^2}^{|k_T^2|_{max}} \frac{d|k_T^2|}{|k_T^2|} (1+\varepsilon)$

the soft singularity cancels

$\Rightarrow \varepsilon \to 0$

$$\hat{\sigma}_q^{(1)}(p) = \frac{\alpha_s}{2\pi} C_F \int_0^{1} \frac{dz}{1-z} \int_{\lambda^2}^{|k_T^2|_{max}} \frac{d|k_T^2|}{|k_T^2|} (1+z^2)\left[\hat{\sigma}_q^{(0)}(zp) - \hat{\sigma}_q^{(0)}(p)\right]$$

Still left with COLLINEAR divergence!

Introduce μ_F to split integration

$$\int_{\lambda^2}^{|k_T^2|_{max}} \frac{d|k_T^2|}{|k_T^2|} \rightarrow \underbrace{\int_{\lambda^2}^{\mu_F^2} \frac{d|k_T^2|}{|k_T^2|}}_{\text{singular}} + \underbrace{\int_{\mu_F^2}^{|k_T^2|_{MAX}} \frac{d|k_T^2|}{|k_T^2|}}_{\text{finite}}$$

Detailed version 4/7

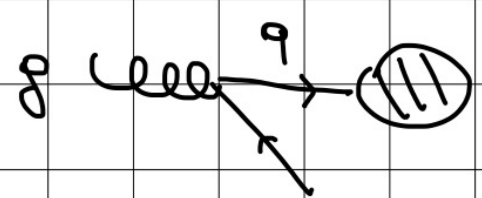
QCD and improved parton model

$$\Rightarrow \quad \hat{\sigma}_q(p) = \hat{\sigma}_q^{(0)}(p) + \hat{\sigma}_q^{(1)}(p)$$

universal function $P(q \to q)$

$$= \hat{\sigma}_q^{(0)}(p) + \frac{\alpha_s}{2\pi}\int_0^1 dz\, P_{qq}(z)\, \hat{\sigma}_q^{(0)}(zp) \log\frac{\mu_F^2}{\lambda^2} + \hat{\sigma}^{(1)}_{q,\mathrm{reg}}(p, \mu_F^2)$$

Of course quark can come from gluon



Doing the whole calculation, we get

$$\hat{\sigma}_g(p) = \hat{\sigma}_g^{(1)}(p)$$

$$= \frac{\alpha_s}{2\pi}\int_0^1 dz\, P_{qg}(z)\, \hat{\sigma}_q^{(0)}(zp) \log\frac{\mu_F^2}{\lambda^2} + \hat{\sigma}^{(1)}_{g,\mathrm{reg}}(p, \mu_F^2)$$

$\hookrightarrow$ In the parton model formula

$$\sigma(p) = \int_0^1 dy \left[f_q(y)\, \hat{\sigma}_q(yp) + f_g(y)\, \hat{\sigma}_g(yp) \right]$$

QCD and improved parton model

$$\sigma(p) = \int_0^1 dy \left[f_q(y)\, \hat{\sigma}_q^{(0)}(yp) \right.$$

$$+ \frac{\alpha_s}{2\pi} \int_0^1 dy\, f_q(y) \int_0^1 dz\, \hat{\sigma}_q^{(0)}(yz\,p)\, P_{qq}(z) \log \frac{\mu_F^2}{\lambda^2}$$

$$+ \frac{\alpha_s}{2\pi} \int_0^1 dy\, f_g(y) \int_0^1 dz\, \hat{\sigma}_g^{(0)}(yzp)\, P_{qg}(z) \log \frac{\mu_F^2}{\lambda^2}$$

$$+ \int_0^1 dy\, f_q(y)\, \hat{\sigma}^{(1)}_{q,\mathrm{reg}}(yp, \mu_F^2) + \int_0^1 dy\, f_g(y)\, \hat{\sigma}^{(1)}_{g,\mathrm{reg}}(yp, \mu_F^2)$$

terms $\propto \log \frac{\mu_F^2}{\lambda^2}$ can be reabsorbed into redefinition of f_q

$x = yz$

$$f_q(x, \mu_F^2) = \int_x^1 \frac{dy}{y} \left\{ f_q(y) \left[\delta\left(1 - \frac{x}{y}\right) + \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{y}\right) \log \frac{\mu_F^2}{\lambda^2} \right] \right.$$

$$\left. + f_g(y) \left[\frac{\alpha_s}{2\pi} P_{qg}\left(\frac{x}{y}\right) \log \frac{\mu^2}{\lambda^2} \right] \right\}$$

QCD and improved parton model

So that

$$\sigma(P) = \int_0^1 dx \; f_q(x, \mu_F^2)\, \hat{\sigma}_q(xP, \mu_F^2) + f_g(x)\, \hat{\sigma}_g(xP, \mu_F^2)$$

both depend on arbitrary FACTORIZATION scale

Note that however the dependence of $f_{q,g}(x, \mu_F^2)$ is totally fixed by perturbation theory

$$\mu^2 \frac{\partial f_q}{\partial \mu^2}(x, \mu^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} \left[P_{qq}\left(\frac{x}{y}\right) f_q(y, \mu^2) + P_{qg}\left(\frac{x}{y}\right) f_g(y, \mu^2) \right]$$

DGLAP evolution equations

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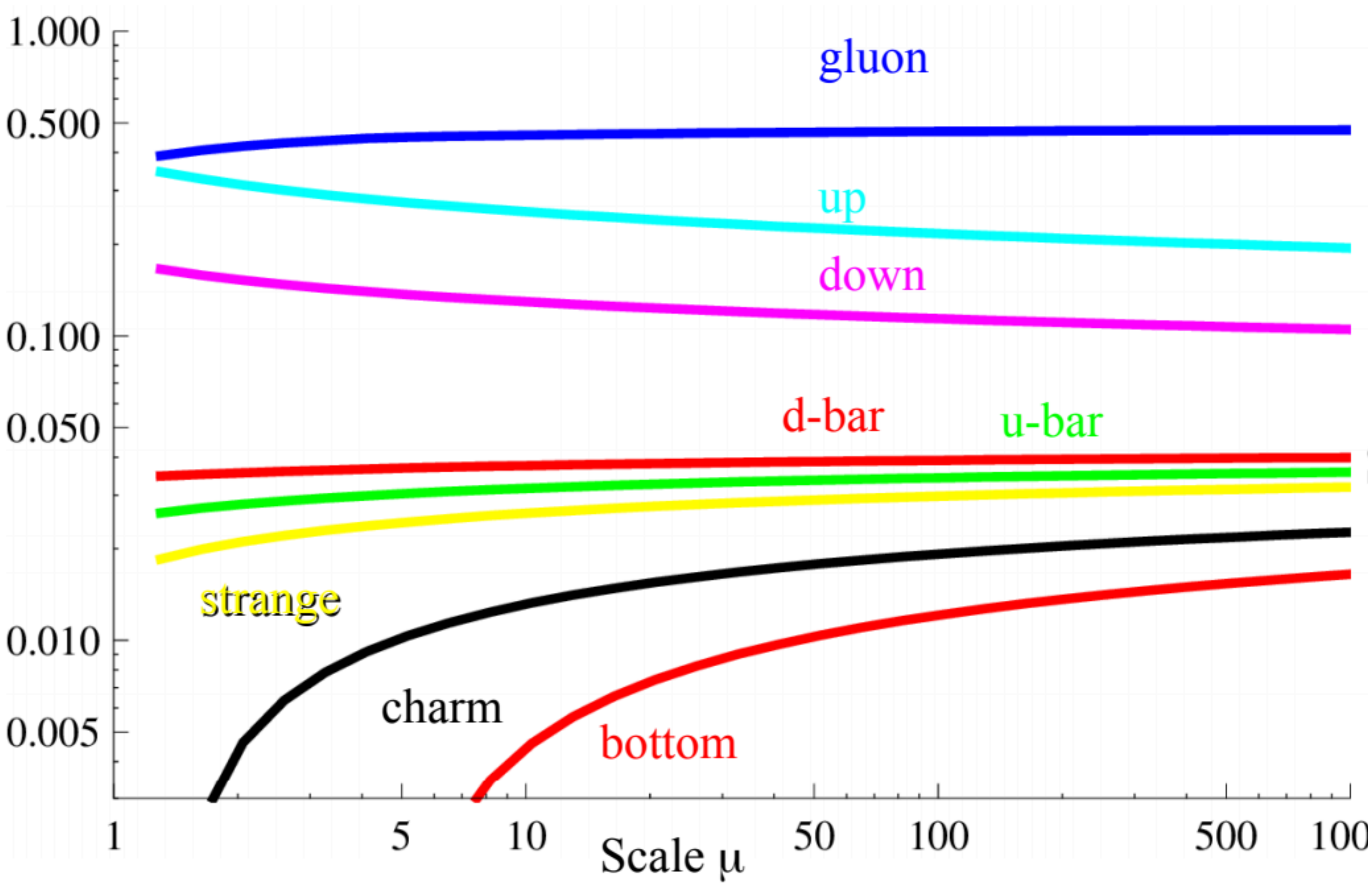


The Parton Model

$$= e_i^2 \int d^4\hat{p}' \, \delta(\hat{p}'^2) \, \delta^{(4)}(\hat{p} + q - \hat{p}') \left(\hat{p}_\mu \hat{p}'_\nu + \hat{p}_\nu \hat{p}'_\mu - g_{\mu\nu} \hat{p}\cdot\hat{p}' \right)$$

$$= e_i^2 \, \delta\left((\hat{p}+q)^2\right) \left[\hat{p}_\mu (\hat{p}+q)_\nu + \hat{p}_\nu (\hat{p}+q)_\mu - g_{\mu\nu} \hat{p}\cdot(\hat{p}+q) \right]$$

$$= e_i^2 \, \delta(2\hat{p}\cdot q + q^2) \Big[2\hat{p}_\mu \hat{p}_\nu - 2\frac{\hat{p}\cdot q}{q^2} (\hat{p}_\mu q_\nu + \hat{p}_\nu q_\mu) + 2\left(\frac{\hat{p}\cdot q}{q^2}\right)^2 q_\mu q_\nu -$$

$$- 2\left(\frac{\hat{p}\cdot q}{q^2}\right)^2 q_\mu q_\nu - g_{\mu\nu} \, \hat{p}\cdot q \Big]$$

$$= \frac{e_i^2}{2\hat{p}\cdot q} \, \delta\left(\frac{2\hat{p}\cdot q}{2\hat{p}\cdot q} - \frac{Q^2}{2\hat{p}\cdot q} \right) \left[2\left(\hat{p}_\mu - \frac{\hat{p}\cdot q}{q^2} q_\mu\right)\left(\hat{p}_\nu - \frac{\hat{p}\cdot q}{q^2} q_\nu\right) + (\hat{p}\cdot q)\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) \right]$$

$$= e^2 \, \delta(1 - \hat{x}) \left[\frac{1}{2}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) + \frac{1}{(\hat{p}\cdot q)} \left(\hat{p}_\mu - \frac{\hat{p}\cdot q}{q^2} q_\mu\right)\left(\hat{p}_\nu - \frac{\hat{p}\cdot q}{q^2} q_\nu\right) \right]$$

with $\hat{x} = \dfrac{Q^2}{2\hat{p}\cdot q}$

Detailed version 1/4

The Parton Model

Applying $\ast$ to DIS cross section we get

$$\frac{4\pi\alpha^2 L_{\mu\nu} W^{\mu\nu}(P,q)}{Q^4}\frac{y^2}{2Q^2}\,dQ^2\,dx = \frac{4\pi\alpha^2}{Q^4} L_{\mu\nu}\frac{y^2}{2Q^2}\,dQ^2 \sum_i \int_0^1 dz\, f_i(z)\,\hat{W}^{\mu\rho}(zP,q)\,d\hat{x}$$

Keeping into account that $\hat{x} = x_B/z$, we get

$$W_{\mu\nu}(P,q) = \sum_{i \in \text{partons}} \int_0^1 dz\, f_i(z)\,\frac{1}{z}\,\hat{W}^i_{\mu\nu}(zP,q)$$

A since $(zP+q)^2 \geq 0 \Rightarrow z \geq x_B$

$$\Rightarrow W_{\mu\nu}(P,q) = \sum_i \int_{x_B}^1 \frac{dz}{z}\, e_i^2 f_i \left[\frac{1}{2}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) + \frac{z}{P\cdot q}\left(P_\mu - \frac{P\cdot q}{q^2}q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2}q_\nu\right)\right]$$

$\delta(1 - x_B/z)$

$dz\ \delta(z - x_B)$

$$\Rightarrow W_{\mu\nu}(P,q) = \sum_i e_i^2 f_i(x)\left[\frac{1}{2}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) + \frac{x}{P\cdot q}\left(P_\mu - \frac{P\cdot q}{q^2}q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2}q_\nu\right)\right]$$

Detailed version 2/4

The Parton Model

Compared to general formula we get

$$\boxed{\begin{aligned} F_1(x,Q^2) &= \frac{1}{2}\sum_i e_i^2 f_i(x) \\ F_2(x,Q^2) &= \sum_i e_i^2\, x\, f_i(x) \end{aligned}}$$

Explicit scaling! Also note that in PARTON MODEL

$$F_2(x) = 2x\, F_1(X)$$ Callan – Gross relation

typical of spin 1/2

Indeed $F_1 \rightarrow$ absorption of transversally polarized virtual photon

$F_L = F_2 - 2xF_1 \rightarrow$ " " longitudinally polarized " "

$F_L \ll F_1$ confirms spin 1/2 of partons

proof $\varepsilon^\mu\, \varepsilon^{*\nu}\, W_{\mu\nu}(p,q) \propto \sigma(p+\gamma^* \rightarrow X)$

Take longitudinally polarized photon

Detailed version 3/4

The Parton Model

- long. polarized γ

$$\varepsilon = \varepsilon_L = \lambda \left(p - \frac{p\cdot q}{q^2}\, q \right) \qquad |\lambda|^2 = \frac{q^2}{(p\cdot q)^2}$$

such that

$$q\cdot \varepsilon_L = 0$$
$$\varepsilon_L \cdot \varepsilon_L^* = -1$$

$$\sigma_L \propto \varepsilon_L^{\mu}\, \varepsilon_L^{*\nu}\, W_{\mu\nu}(p,q) = F_1(x,Q^2) - \frac{F_2(x,Q^2)}{2x}$$

- transv. polarized γ

$$p\cdot \varepsilon_T = 0$$
$$q\cdot \varepsilon_T = 0$$
$$\varepsilon_T \cdot \varepsilon_T^* = -1$$

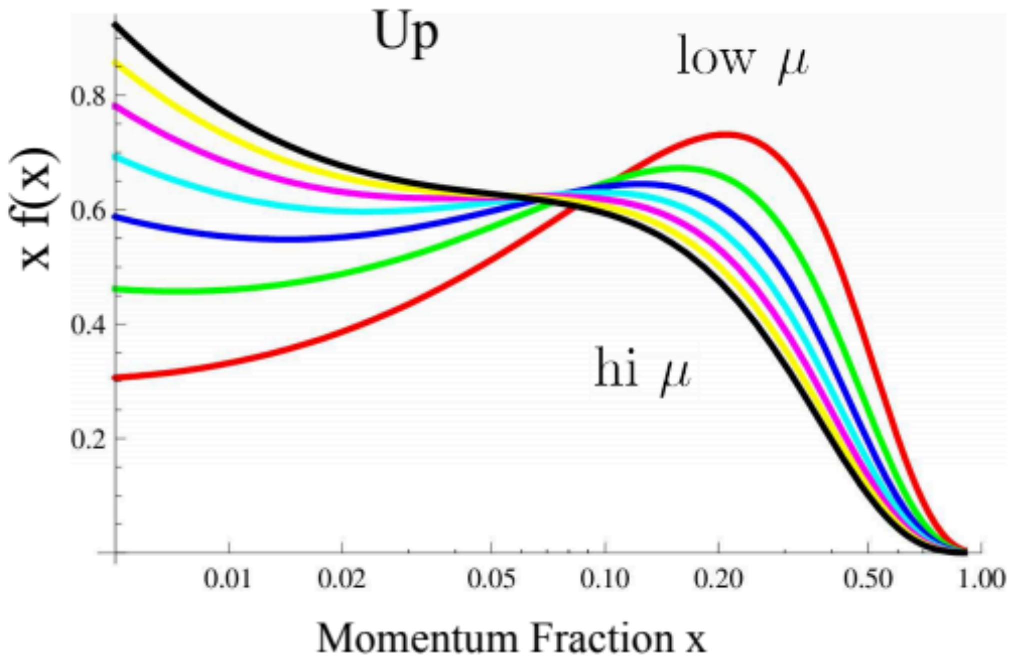
$$\sigma_T \propto \varepsilon_T^{\mu}\, \varepsilon_T^{*\nu}\, W_{\mu\nu}(p,q) = F_1(x,Q^2)$$

$$\Rightarrow \frac{\sigma_L}{\sigma_T} = 1 - \frac{F_2(x,Q^2)}{2x F_1(x,Q^2)} \to = 1 \text{ parton model} \Rightarrow \sigma_L = 0$$

Note that it would be $\neq 0$ if partons were spin 0.

Detailed version 4/4

DGLAP evolution equations



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