A new setup for isospin breaking corrections to the HVP from lattice QCD

Sebastian Lahrtz

Simon Kuberski, Andreas Risch and Hartmut Wittig

Johannes Gutenberg-University, Institute of Nuclear Physics

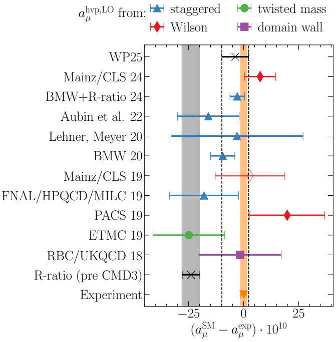
19.09.2025

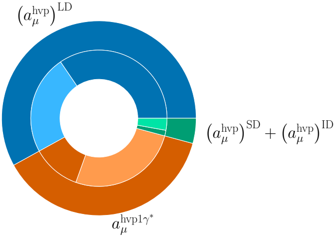
MPA Summer school, Frauenchiemsee

Overview

- Current status for g-2
- Isospin breaking correction
- New setup
- Results for the test on A654
- Summary and Outlook

Current status of g-2 calculations





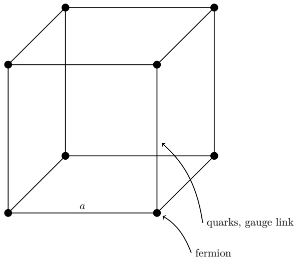
- good agreement between experiment and theory in the white paper of the theory initiative[1]
- LD window and isospin breaking corrections as main error source[2]

[1] Aliberti et al., "The anomalous magnetic moment of the muon in the Standard Model: an update."

[2] Djukanovic et al., "The hadronic vacuum polarization contribution to the muon g − 2 at long distances."

Lattice QCD

- low energy regime of QCD ⇒ large coupling ⇒ Lattice QCD as non-perturbative tool
- lattice QCD is an euclidean and discretized quantum field theory in finite volume
- lattice spacing as ultraviolet cut-off and finite volume as infrared cut-off
- use Monte-Carlo techniques: generate an ensemble with many configurations via a Markov chain



- average of a single ensemble, then extrapolate to the physical point

CLS ensembles

- tree-level Symanzik improved Lüscher-Weisz gauge action and non-perturbatively $\mathcal{O}(a)$ improved Wilson fermions[1]
- large range of pion masses down to the physical point
- six different lattice spacings $[0.097\,\text{fm}, 0.085\,\text{fm}, 0.075\,\text{fm}, 0.064\,\text{fm}, 0.049\,\text{fm}, 0.039\,\text{fm}]$

[1] Bruno et al., "Simulation of QCD with N f = 2 + 1 flavors of non-perturbatively improved Wilson fermions."

CLS ensembles

- tree-level Symanzik improved Lüscher-Weisz gauge action and non-perturbatively $\mathcal{O}(a)$ improved Wilson fermions[1]
- large range of pion masses down to the physical point
- six different lattice spacings [$0.097\,\text{fm}$, $0.085\,\text{fm}$, $0.075\,\text{fm}$, $0.064\,\text{fm}$, $0.049\,\text{fm}$, $0.039\,\text{fm}$]
- A654 as test ensemble for new setup: $a = 0.097\,\text{fm}$, $m_\pi = 338\,\text{MeV}$, $m_K = 462\,\text{MeV}$, periodic boundary conditions, $L^3 \times T = 24^3 \times 48$

[1] Bruno et al., "Simulation of QCD with N f = 2 + 1 flavors of non-perturbatively improved Wilson fermions."

Getting the HVP from the lattice

- compute correlation function on the lattice
- time momentum representation: integral over two-point correlation function with a kernel[1]

$$a_\mu^{HVP} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt \tilde{K}(t; m_\mu) G(t)$$

$$G(t)_{\mu\nu} = \langle J_\mu(t) J_\nu(0) \rangle$$

$$J_\mu(x) = \sum_f q_f(x) \gamma_\mu q_f(x)$$

[1] Djukanovic et al., "The hadronic vacuum polarization contribution to the muon g − 2 at long distances."

Getting the HVP from the lattice

- compute correlation function on the lattice
- time momentum representation: integral over two-point correlation function with a kernel[1]

$$a_\mu^{HVP} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt \tilde{K}(t; m_\mu) G(t)$$
$$G(t)_{\mu\nu} = \langle J_\mu(t) J_\nu(0) \rangle$$
$$J_\mu(x) = \sum_f q_f(x) \gamma_\mu q_f(x)$$

- other lattice methods exists, e.g. the CCS-method

[1] Djukanovic et al., “The hadronic vacuum polarization contribution to the muon g − 2 at long distances.”

RM123 method

- ensembles are isoQCD ($m_u = m_d = m_{ud}$ and no QED effects)
- introduces systematic error of $\mathcal{O}(1\%)$

[1] Divitiis et al., "Leading isospin breaking effects on the lattice."
[2] Divitiis et al., "Isospin breaking effects due to the up-down mass difference in Lattice QCD."

RM123 method

- ensembles are isoQCD ($m_u = m_d = m_{ud}$ and no QED effects)
- introduces systematic error of $\mathcal{O}(1\%)$
- treat these additional effects as small perturbation and expand in $\Delta m_d = m_d - m_{ud}$, $\Delta m_u = m_u - m_{ud}$ and in e[1][2]

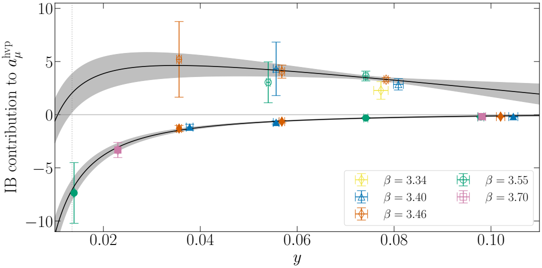
$$\langle \mathrm{O} \rangle = \frac{\int D\phi \mathrm{O} e^{-S_{full}}}{\int D\phi e^{-S_{full}}} = \frac{\int D\phi \mathrm{O} \left(1 + \sum_i \Delta\epsilon_i S_i\right) e^{-S_0}}{\int D\phi \left(1 + \sum_i \Delta\epsilon_i S_i\right) e^{-S_0}}$$

- this expansion leads to new operators and subsequently to new diagrams

[1] Divitiis et al., "Leading isospin breaking effects on the lattice."

[2] Divitiis et al., "Isospin breaking effects due to the up-down mass difference in Lattice QCD."

Current status of isospin breaking computations



Chiral extrapolation of the isospin breaking corrections[1]

- connected part is not well constrained, neither in the physical pion mass region nor in at the SU(3) symmetric point
- missing disconnected mass insertion diagram
- methods for con. and disc. QED diagrams differ substantial

[1] Djukanovic et al., "The hadronic vacuum polarization contribution to the muon g − 2 at long distances."

Diagrams of interest

Diagram 1: Disconnected mass insertion for the valence quark

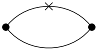


Diagram 2: Connected mass insertion diagram for the valence quark

Diagram 3: Photon self-energy diagrams for the valence quark

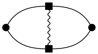


Diagram 4: Photon exchange diagram for the valence quark

We are also interested in the photon disconnected diagrams, which are not shown here.

Calculation of correlation function from lattice QCD

- lattice QCD enables us to calculate correlation function by employing statistical physics concept
- all-to-all propagator to expensive
- make use of translational invariance ⇒ sample it via point sources or stochastic sources
- stochastic source corresponds to insertion of stochastic one

Calculation of correlation function from lattice QCD

- lattice QCD enables us to calculate correlation function by employing statistical physics concept
- all-to-all propagator to expensive
- make use of translational invariance ⇒ sample it via point sources or stochastic sources
- stochastic source corresponds to insertion of stochastic one

$$C(t_1, t_2) = \sum_{\vec{x},\vec{y}} \mathrm{tr}\langle D^{-1}(x;y)\gamma_5 D^{-1}(y;x)\gamma_5\rangle$$

$$= \sum_{\vec{x},\vec{y},\vec{y'}} \langle \mathrm{tr}\langle D^{-1}(x;y)\eta(y)\eta^\dagger(y')\gamma_5 D^{-1}(y';x)\gamma_5\rangle\rangle \text{ (stochastic source)}$$

$$= \sum_{\vec{x},\vec{y},\vec{y'}} \langle \mathrm{tr}\langle D^{-1}(x;y)\eta(y)\eta^\dagger(y') D^{\dagger -1}(x;y')\rangle\rangle$$

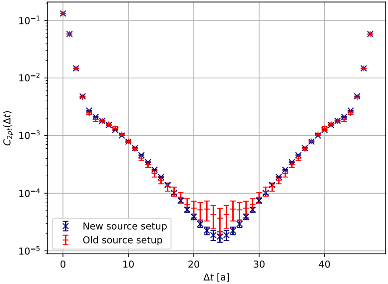
$$\langle \eta(y)\eta^\dagger(y')\rangle = \delta(y-y')$$

New Source setup

- idea: use different source setup (different stochastic estimator) to improve quality of the signal for the same cost
- the standard setup uses 3D time slice sources at the source time of the correlation function
- new setup: 4D volume sources at the operator insertion
- this is particularly useful in this case as the operator insertion is summed over $\Rightarrow$ stochastic all-to-all propagator
- get disconnected for "free"

Improved mass insertion correlation function for A654

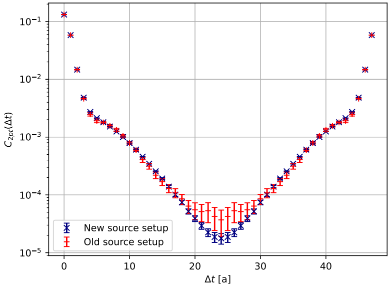
- reduction of statistical noise and reduced errors
- signal "stays" longer ⇒ A654 is actually to small to see loss of signal with new setup



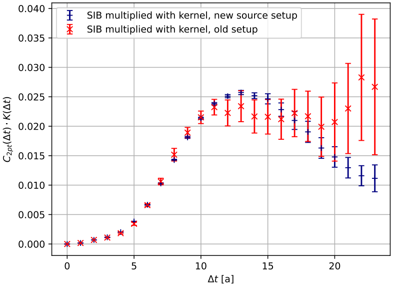
Two-point correlation function

Improved mass insertion correlation function for A654

- reduction of statistical noise and reduced errors
- signal "stays" longer ⇒ A654 is actually to small to see loss of signal with new setup



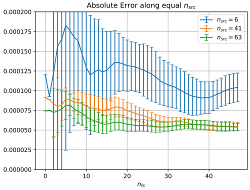
Two-point correlation function



TMR integrand

Error analysis for the connected mass insertion diagram

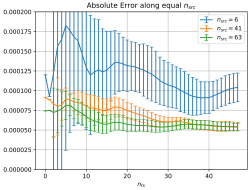
- for gauge noise, roughly 40 to 50 sources and 30 times-slices are needed
- error more or less constant at line of equal cost (at least within error bars)



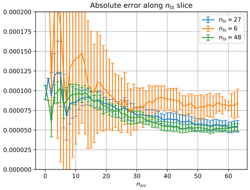
Behavior of the error with respect to the number of time-slices for the sequential propagator

Error analysis for the connected mass insertion diagram

- for gauge noise, roughly 40 to 50 sources and 30 times-slices are needed
- error more or less constant at line of equal cost (at least within error bars)



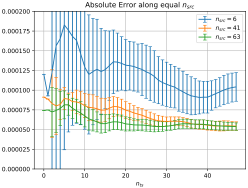
Behavior of the error with respect to the number of time-slices for the sequential propagator



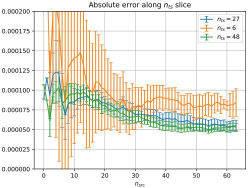
Behavior of the error with respect to the number of 4D sources

Error analysis for the connected mass insertion diagram

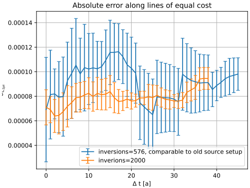
- for gauge noise, roughly 40 to 50 sources and 30 times-slices are needed
- error more or less constant at line of equal cost (at least within error bars)



Behavior of the error with respect to the number of time-slices for the sequential propagator



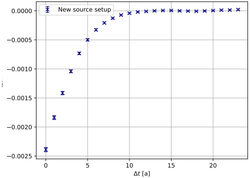
Behavior of the error with respect to the number of 4D sources



Behavior along line of equal cost

Disconnected mass insertion diagram

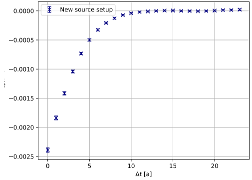
- due to the better estimator, the disconnected diagram has a signal
- gauge noise is reached fairly soon with only $\mathcal{O}(50)$ sources
- for large pion masses, the disconnected diagram is subdominant, expected to be larger for smaller pion masses



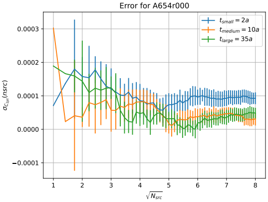
Two-point disconnected mass insertion correlation function for A654

Disconnected mass insertion diagram

- due to the better estimator, the disconnected diagram has a signal
- gauge noise is reached fairly soon with only $\mathcal{O}(50)$ sources
- for large pion masses, the disconnected diagram is subdominant, expected to be larger for smaller pion masses



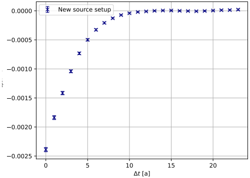
Two-point disconnected mass insertion correlation function for A654



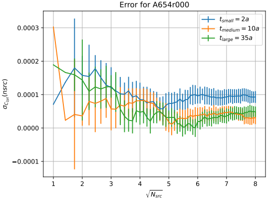
Source dependency of the disconnected mass insertion diagram for A654

Disconnected mass insertion diagram

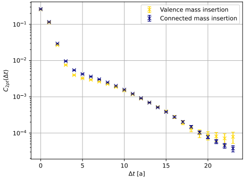
- due to the better estimator, the disconnected diagram has a signal
- gauge noise is reached fairly soon with only $\mathcal{O}(50)$ sources
- for large pion masses, the disconnected diagram is subdominant, expected to be larger for smaller pion masses



Two-point disconnected mass insertion correlation function for A654



Source dependency of the disconnected mass insertion diagram for A654



Combined connected and disconnected mass insertion correlation function

Summary and Outlook

- study on A654 revealed good improvement for the mass insertion
- currently working on including QED diagrams
- perform calculations on many ensembles specifically on small pion masses and at the SU(3)-symmetric point to constrain the extrapolation to physical pion mass, we do not expect a large dependency on the lattice spacing

Backup: Different kinds of connected diagrams

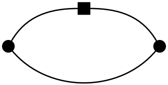


Figure: Connected diagram with operator on one quark line only

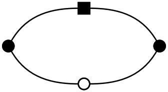
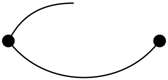


Figure: Connected diagram with operator on both quark lines

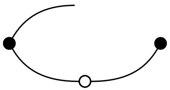
Backup: Operator on one quark line only



Sequential propagator, $N_{inv} = N_{seq} \cdot (N_\Gamma N_{ts} + 1)$

Operator propagator, $N_{inv} = N_{seq} \cdot N_{operators}$

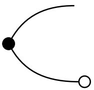
Backup: Operator on both quark lines



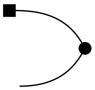
Double Sequential propagator

Operator propagator

In addition to the inversion of the prior diagram, we need to add the following number of inversions $N_{inv,DoubelSeq} = N_{seq} N_{\Gamma} N_{ts} N'_{operators}$.



Left-hand site



Right-hand site

No additional inversions needed, however, due to additional stochastic noise, more stochastic sources are required.