When ε-Factorisation Goes Bananas:

Peeling Back the Geometry

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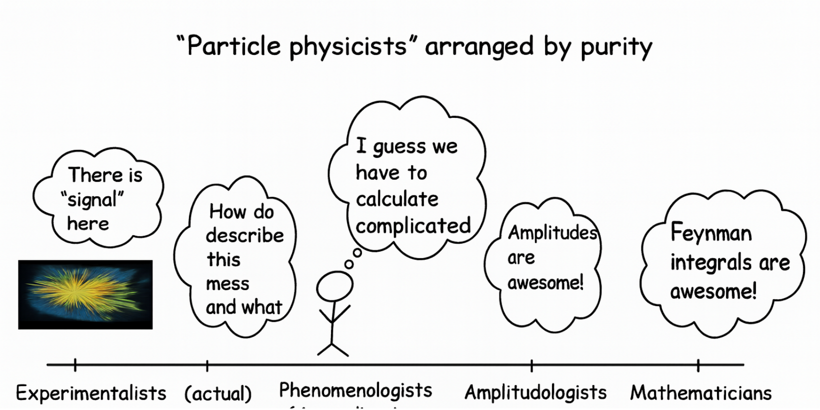
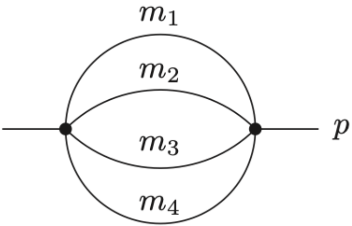
Based on:
The Unequal-Mass Three-Loop Banana Integral [arXiv:2507.23594]

Setup


"Particle physicists" arranged by purity
There is "signal" here
How do describe this mess and what
I guess we have to calculate complicated
Amplitudes are awesome!
Feynman integrals are awesome!
Experimentalists
(actual)
Phenomenologists
Amplitudologists
Mathematicians

1. Introduction



- DE

$$dI(\varepsilon, y) = A(\varepsilon, y)\, I(\varepsilon, y)$$

- Objective: Find a master basis K with ε-factorised DE:

$$dK(\varepsilon, y) = \varepsilon\, \tilde{A}(y)\, K(\varepsilon, y).$$

- This work: First treatment of the 3-unequal-loop banana
- Method: Loop-by-loop Baikov + Hodge-inspired filtrations

Notation & setup — unequal-mass 3-loop banana

- Family (in $D = 2 - 2\varepsilon$).

$$I_{\nu_1 \ldots \nu_9} = e^{3\gamma_E \varepsilon} \left(\mu^2\right)^{\nu - \frac{3}{2}D} \int \left(\prod_{a=1}^{3} \frac{d^D k_a}{i\pi^{D/2}} \right) \frac{1}{\prod_{b=1}^{9} \sigma_b^{\nu_b}},$$

- $\sigma_{1,2,3,4}$: propagators; $\sigma_{5,6}$ loop-by-loop Baikov variables; $\sigma_{7,8,9}$: IBP only.
- loop-by-loop Baikov:

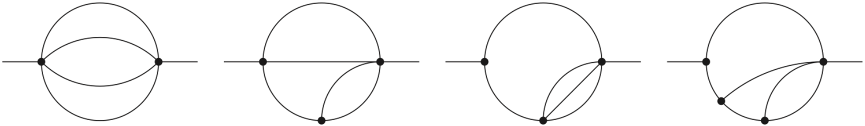
$$I_{\boldsymbol{\nu}} = \int U(\boldsymbol{z})\, \widehat{\Phi}(\boldsymbol{z})\, \mathrm{d}\boldsymbol{z}. \tag{1}$$

- Maximal cut: $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$ to put all propagators on shell.

$$\mathrm{Res}_{\sigma_{1..4}} I_{1111\,\nu_5\,\nu_6\,000}$$

- Restricting the integration to a two-dimensional Baikov integral in σ_5, σ_6 that retains the essential geometry.

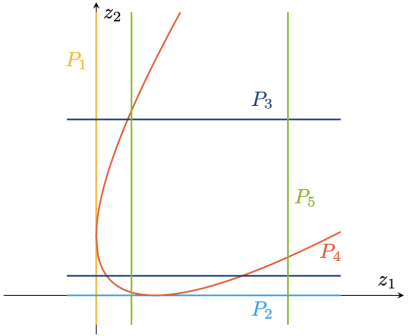
Sectors and Super-Sectors



Sector 15 Sector 31 Sector 47 Sector 63

- Sector of interest: id 15 (3-loop banana).
- Masters basis I; 15 Masters in sector 15.
- Super-sectors $31, 47, 63$, Think embedding space, sectors 31 and 47 have one master each sector 63 has no masters.

Twisted-cohomology setup on the maximal cut



- Map the Feynman integral to twisted cohomology:

$$\iota : V^2 \hookrightarrow H^2_\omega .$$

- In $\mathbb{CP}^2$ the twist reads
$U = P_0^\varepsilon P_1^\varepsilon P_2^\varepsilon \, P_3^{-1/2-\varepsilon} \, P_4^{-1/2-\varepsilon} \, P_5^{-1/2-\varepsilon}$

- The arrangement governs the branch locus and the geometry:

$$\mathbb{CP}^{N-1} \setminus \bigcup_i \{P_i = 0\}.$$

Objects

$$\Psi_{\mu_0\ldots\mu_5}[Q] = C_{\text{Baikov}} C_{\text{abs}} C_{\text{rel}} C_{\text{clutch}}\, U(z) \hat{\Phi}_{\mu_0\ldots\mu_5}[Q]\, \eta$$

- $\hat{\Phi}_{\mu_0\ldots\mu_5}[Q]$ encodes the analytic and geometric structure of propagators
- η is the projective 2-form on $\mathbb{CP}^2$, ensuring scale invariance.
- $C_{\text{clutch}} = \varepsilon^{-|\mu|}$ with $|\mu| = \sum_{i=1}^{5} \mu_i$ ensures that each $+1$ increase in pole order from derivatives is paid for by an extra factor of ε
- $\Psi_{\mu_0\ldots\mu_5}[Q] \in \Omega^2_\omega$, the space of twisted differential two-forms.
- H^2_ω is the space of independent Feynman integrands
- $|\mu|$: combinatorial complexity.
- o: pole order
- r: non-zero residues

Filtrations

- $W_\bullet$: $\Psi \in W_w\Omega^2_\omega$ taking r-fold residue gives weight $2 + r \leq w$
- $F^\bullet_{\text{geom}}$: $\Psi \in F^p_{\text{geom}}\Omega^2_\omega$, for r residues and total pole order o, $2 + r - o \geq p$.
- $F^\bullet_{\text{comb}}$, for combinatorial complexity $|\mu|$ one has $2 - |\mu| \geq p'$.
- $H^{p,q}_{\text{geom}}$ be generated by all geometrically ordered master integrals.
- $H^{p',q'}_{\text{comb}}$ be generated by all combinatorial ordered master integrands.

$$\begin{matrix} & & h^{2,2}_{\text{geom}} & & \\ & h^{2,1}_{\text{geom}} & & h^{1,2}_{\text{geom}} & \\ h^{2,0}_{\text{geom}} & & h^{1,1}_{\text{geom}} & & h^{0,2}_{\text{geom}} \end{matrix} \quad = \quad \begin{matrix} & & 7 & & \\ & 0 & & 0 & \\ 1 & & 4 & & 1 \end{matrix}$$

Weight-4 masters

- Two residues:recole $2 + r \leq w$, for $r = 2$ we have weight $w = 4$ in $H^{2,2}_{\text{geom}}$.
- Candidates: $|\mu| \leq 2$ gives 9 options; 7 independent.
- Filtration detail: First six: $|\mu| = 1$, in $F^1_{\text{comb}}\Omega^2_\omega$. Last three: $|\mu| = 2$, in $F^0_{\text{comb}}\Omega^2_\omega$.
- Generators of $H^{2,2}_{\text{geom}}$:

$$\Psi_{100000}[z_1], \Psi_{100000}[z_2], \Psi_{010000}[z_0], \Psi_{010000}[z_2]$$
$$\Psi_{001000}[z_0], \Psi_{001000}[z_1],\ \Psi_{011000}[z_0^2]$$

- Super-sectors: 63: none; 31 & 47: one master each (reps $\Psi_{010000}[z_0]$, $\Psi_{001000}[z_0]$).
- Three vanishing combinations on the Feynman integral side: $\Psi_{010000}[z_2 + y^2 z_0]$, $\Psi_{001000}[z_1 + y^2 z_0]$, $\Psi_{011000}[y^2 z_0^2 + z_0(z_1 + z_2)]$.
- Lower weights: no weight-3 masters; weight-2 masters have no residues.

Pole Order

Pole order 0

- $\mu_0 = \cdots = \mu_5 = 0 \Rightarrow \widehat{\Phi} = 1, \Rightarrow \Psi_{000000}[1] = C_{\text{Baikov}}\,\varepsilon^3\,U(z)\,\eta$

Pole order 1

- Raise one of μ_3, μ_4, μ_5 to 1.
- Four independent choices:

$$\frac{y_1}{\varepsilon}\,\frac{\partial}{\partial y_1}\Psi_{000000}[1],\ \frac{y_2}{\varepsilon}\,\frac{\partial}{\partial y_2}\Psi_{000000}[1],$$
$$\frac{y_3}{\varepsilon}\,\frac{\partial}{\partial y_3}\Psi_{000000}[1],\ \frac{y_4}{\varepsilon}\,\frac{\partial}{\partial y_4}\Psi_{000000}[1].$$

Pole order 2

- $H_\omega^{0,2}$ is one-dimensional, generated by second derivatives.
- Symmetric generator: $\displaystyle \frac{1}{16\,\varepsilon^2}\sum_{i=1}^{4}\frac{\partial^2}{\partial y_i^2}\Psi_{000000}[1].$

A basis for the twisted cohomology group H^2_ω

Labeling: generators $\Psi_5, \ldots, \Psi_{17}$

$$\Psi_5 = \Psi_{000000}[1], \qquad \Psi_{10} = \Psi_{100000}[z_1], \qquad \Psi_{11} = \Psi_{100000}[z_2],$$

$$\Psi_6 = \frac{y_1}{\varepsilon}\frac{\partial}{\partial y_1}\Psi_{000000}[1], \qquad \Psi_{12} = \Psi_{010000}\big[z_2 + y_2 z_0\big], \qquad \Psi_{16} = \Psi_{010000}[z_0],$$

$$\Psi_7 = \frac{y_2}{\varepsilon}\frac{\partial}{\partial y_2}\Psi_{000000}[1], \qquad \Psi_{13} = \Psi_{001000}\big[z_1 + y_2 z_0\big], \qquad \Psi_{17} = \Psi_{001000}[z_0],$$

$$\Psi_8 = \frac{y_3}{\varepsilon}\frac{\partial}{\partial y_3}\Psi_{000000}[1], \qquad \Psi_{14} = \Psi_{011000}\big[y_2 z_0^2 + z_0(z_1 + z_2)\big],$$

$$\Psi_9 = \frac{y_4}{\varepsilon}\frac{\partial}{\partial y_4}\Psi_{000000}[1], \qquad \Psi_{15} = \frac{1}{16\,\varepsilon^2}\sum_{i=1}^{4}\frac{\partial^2}{\partial y_i^2}\Psi_{000000}[1].$$

going form $\Psi \to J$ System

$$dJ(\varepsilon, y) = \hat{A}(\varepsilon, y)\, J(\varepsilon, y) = \sum_{k=-2}^{1} \varepsilon^k\, \hat{A}^{(k)}(y)\, J(\varepsilon, y), \qquad \hat{A}^{(k)}(y) \tag{2}$$

Construction of the basis K

- Goal is epsilon–factorised DE.

$$K = R_2^{-1} J, \quad dK(\varepsilon, y) = \varepsilon\, \widetilde{A}(y)\, K(\varepsilon, y), \quad R_2 = R_2^{(-2)} R_2^{(-1)} R_2^{(0)}.$$

- block-wise Filtration along the $F^{\bullet}_{\text{comb}}$:

$$J_5 \in F^2_{\text{comb}} \Omega^2_{\omega},\ J_6, \ldots, J_{13} \in F^1_{\text{comb}} \Omega^2_{\omega},\ J_{14}, J_{15} \in F^0_{\text{comb}} \Omega^2_{\omega}.$$

- Ansatz for each $R_2^{(j)}$ with ε-independent functions of y.
- Enforce cancellations $\Rightarrow$ first-order ε-independent DEs, triangular by $F^{\bullet}_{\text{comb}}$.

Differential Equation Martix and R_2 Transformation

$$
\left(\begin{array}{cccc|c|cccccccc|cc}
1 & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & 1 & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & 1 & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & 1 & - & - & - & - & - & - & - & - & - & - & - \\
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- & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
\hline
- & - & - & - & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & - & - & - & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & - & - & - & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & - & - & - & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - \\
\hline
1 & - & 1 & 1 & 0 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - \\
1 & 1 & 1 & 1 & -2 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\cdot
\left(\begin{array}{cccc|c|cccccccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & R_{55}^{(-2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \frac{1}{\varepsilon}R_{65}^{(-2)} & R_{66}^{(-2)} & R_{67}^{(-2)} & R_{68}^{(-2)} & R_{69}^{(-2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\varepsilon}R_{75}^{(-2)} & R_{76}^{(-2)} & R_{77}^{(-2)} & R_{78}^{(-2)} & R_{79}^{(-2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\varepsilon}R_{85}^{(-2)} & R_{86}^{(-2)} & R_{87}^{(-2)} & R_{88}^{(-2)} & R_{89}^{(-2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\varepsilon}R_{95}^{(-2)} & R_{96}^{(-2)} & R_{97}^{(-2)} & R_{98}^{(-2)} & R_{99}^{(-2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
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\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\varepsilon^2}R_{F5}^{(-2)} & \frac{1}{\varepsilon}R_{F6}^{(-2)} & \frac{1}{\varepsilon}R_{F7}^{(-2)} & \frac{1}{\varepsilon}R_{F8}^{(-2)} & \frac{1}{\varepsilon}R_{F9}^{(-2)} & 0 & 0 & 0 & 0 & R_{FE}^{(-2)} & R_{FF}^{(-2)}
\end{array}\right)
$$

- Definition keep only k-order:

$$\left[(R_2^{(k)})^{-1}\, \tilde{A}^{(k)}\, R_2^{(k)} - (R_2^{(k)})^{-1} dR_2^{(k)} \right]\Big|_k = 0.$$

- Implications:
  - ε-independent, first-order DEs for the ansatz functions of $R_2^{(k)}$.
  - After solving, $\tilde{A}^{(k+1)}$ has only orders $\{k{+}1, \dots, 1\}$ oder.
- After $R_2^{(0)}$: $\tilde{A}^{(1)}$ contains only B-order 1 terms.

Summary

- Start:

$$dI(\varepsilon, y) = A(\varepsilon, y)\, I(\varepsilon, y)$$

- Method: Loop-by-loop Baikov + Hodge-inspired filtrations $(W_\bullet, F^\bullet_{\text{geom}}, F^\bullet_{\text{comb}})$ to order Masters
- For a master basis J:

$$dJ(\varepsilon, y) = \hat{A}(\varepsilon, y)\, J = \sum_{k=-2}^{1} \varepsilon^k \hat{A}^{(k)}(y)\, J.$$

- Transform J with

$$K = R_2^{-1} J,\; R_2 = R_2^{(-2)} R_2^{(-1)} R_2^{(0)},$$

imposing at each B-order k:

$$\left[(R_2^{(k)})^{-1} \hat{A}^{(k)} R_2^{(k)} - (R_2^{(k)})^{-1} dR_2^{(k)} \right]\Big|_k = 0,$$

which cancels $\varepsilon^{-2}, \varepsilon^{-1}, \varepsilon^0$ and yields

$$dK(\varepsilon, y) = \varepsilon\, \tilde{A}(y)\, K(\varepsilon, y)$$

Thank you